Research Article

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Stability Analysis of Double Diffusive Convection in Local Thermal Non-equilibrium Porous Medium with Internal Heat Source and Reaction Effects

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Abstract: The internal heat source and reaction effects on the onset of thermosolutal convection in a local thermal non-equilibrium porous medium are examined, where the temperature of the fluid and the solid skeleton may differ. The linear instability and nonlinear stability theories of Darcy–Brinkman type with fixed boundary condition are carried out where the layer is heated and salted from below. The \(D^2\) Chebyshev tau technique is used to calculate the associated system of equations subject to the boundary conditions for both theories. Three different types of internal heat source function are considered, the first type increases across the layer, while the second decreases, and the third type heats and cools in a nonuniform way. The effect of different parameters on the Rayleigh number is depicted graphically. Moreover, the results detect that utilizing the internal heat source, reaction, and non-equilibrium have pronounced effects in determining the convection stability and instability thresholds.

Keywords: stability analysis, double diffusive convection, internal heat source effect, reaction effect, Darcy–Brinkman model, local thermal non-equilibrium, porous media

1 Introduction

Thermal convection in a saturated porous material is a subject of increasing interest. Recent research has given special interest on thermal convection in a porous medium where the fluid temperature \(T_f\) may differ from the solid skeleton temperature \(T_s\). Local thermal non-equilibrium is the term used to describe the situation where the two temperatures may differ. There are many practical applications of this subject area, which may be the main factor contributing to the rising interest in local thermal non-equilibrium flows in porous medium; see, e.g., Straughan [1], Celli et al. [2], Choudhary et al. [3], Straughan [4, 5], Celli et al. [6], Mankhi and Haddad [7], Capone and Gianfrani [8]. Another interesting study on the stability theory that considers the convection in fluid-saturated porous medium induced by internal heating has been given by Nouri-Borujerdi et al. [9, 10], Kuznetsov and Nield [11], Mahajan and Nandal [12, 13], Siddabasappa and Sakshath [14]. Mahajan and Nandal [13] analyzed the onset of convection with local thermal non-equilibrium porous media for the Darcy–Brinkman model. Four types of internal heat-generating function were considered. The effect of various parameters on the onset of convection was discussed and illustrated numerically by using the Chebyshev pseudospectral method. Siddabasappa and Sakshath [16] investigated the stability analysis of Darcy–Brinkman convection with local thermal non-equilibrium porous medium in the presence of internal heat source in two ways, in a liquid phase and in a solid phase; they discussed the effect of different coefficients on the onset of convection and depicted them graphically.

The thermosolutal (double-diffusive) convection in porous media is a topic of increasing attention due to its prevalence in a wide range of real situations. It has attracted many researchers; see, e.g., Sharma et al. [15], Pritchard and Richardson [16], Wang and Tan [17], Liu and Umavathi [18], Mahajan and Tripathi [19],

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and Noon and Haddad [20]. For the case of a local thermal non-equilibrium flow in a porous medium, see, e.g., Malashetty et al. [21], Malashetty and Heera [22], Chen et al. [23], Malashetty et al. [24], Nield et al. [25], Kuznetsov et al. [26], Altawallbeh et al. [27], Kumar et al. [28], and Hemanth Kumar et al. [29]. Chen et al. [23] studied the onset of double-diffusive convection in a porous medium with local thermal non-equilibrium and reaction effect for Darcy model; the effects of various coefficients on the onset of convection was discussed numerically. Kuznetsov et al. [26] analyzed the onset of thermosolutal convection in a local thermal non-equilibrium porous medium for Darcy model composed of two horizontal layers that were internally heated; the effects of parameter variations were discussed numerically using Galerkin method.

In this paper, the onset of thermosolutal convection with the internal heat source and reaction effects in a fluid-saturated porous media under the condition of a local thermal non-equilibrium for the Darcy–Brinkman model is studied with a fixed boundary condition where the layer is heated and salted from below. Three different types of internal heat source function are considered, the first type increases across the layer, while the second decreases, and the third type heats and cools in a nonuniform way. Our goal is here to investigate the effects of the internal heat source, reaction, and local thermal non-equilibrium on the stability of the system. The $D^2$ Chebyshev tau method is used to solve the eigenvalue problem and analyze the linear and nonlinear stability thresholds.

To this end, the governing equations and appropriate steady-state solutions are presented in Section 2. In Sections 3 and 4, the linear instability and the nonlinear stability theories are presented. The numerical method is generated by using the $D^2$ Chebyshev tau technique for fixed surface boundaries in Section 5. The effects of the internal heat source, reaction terms, and local thermal non-equilibrium on the Rayleigh number are graphically represented in Section 6. In the final Section 7, the conclusion is discussed.

## 2 Governing equations

Let us consider a layer of fluid saturated a porous material that is bounded by two horizontal planes and a vertical axis $z$, $(x, y) \in \mathbb{R}^2 \times \{z \in (0, d)\}$. The upper plane $z = d$ is maintained at a constant temperature $T_U$ and salt concentration $C_U$, while the lower plane $z = 0$ is maintained at constant temperature $T_L$ and salt concentration $C_L$. The Darcy–Brinkman equation in double diffusive convection is considered with an internal heat source, allowing gravity $g$ to be determined by the vertical coordinate $z$ as shown in Figure 1. The governing equations can be represented as follows:

$$
\begin{align*}
p_j &= -\frac{\mu_j}{K} v_j - \rho_0 [1 - \alpha_T(T^f - T_L) + \alpha_C(C - C_L)] g k_i + \lambda \Delta v_j, \\
v_{ij} &= 0, \\
\sqrt{\mathbf{K}}(\rho c) - T_j \nabla + (\rho c_j) v_j \nabla_j &= 0, \\
(1 - \phi)(\rho c_s) T_j \nabla_s &= (1 - \phi) k_s \Delta T^s + \bar{h}(T^f - T^s) + (1 - \bar{\phi}) Q_j(z), \\
\bar{\phi} C_j + v_i C_i &= \phi k_c \Delta C + \bar{k}[C_{eq}(T^f) - C],
\end{align*}
$$

where $v_j$, $p_j$, $T$, $C$, $C_{eq}$ are the velocity, pressure, temperature, salt concentration, and equilibrium concentration of solute at a given temperature, respectively, the subscript $f$ and $s$ refer to the fluid and solid phase properties, respectively, $\mu$, $\lambda$ are viscosities, $\bar{h}$ is the inter phase heat transfer coefficient, $g$, $\rho_0$, $\alpha_T$, $\alpha_C$, $k$, $k_s$, $k_c$, $\phi$ are gravity, density of fluid at the bottom, thermal expansion coefficient, solute expansion coefficient, the thermal conductivity, the molecular diffusivity of the solute through the fluid, the lumped effective reaction rate, the internal heat source, and the matrix porosity, respectively. The equilibrium solute concentration is assumed as a linear function of temperature as in Pritchard and Richardson [16] such that $C_{eq}(T^f) = f_1(T^f - T_L) + f_0$, where $f_1$ and $f_0$ are constants. The boundary conditions are

$$
\begin{align*}
v_j &= 0, & \text{on } z = 0, d, \\
T^f &= T^s = T_L, & C = C_L, & \text{on } z = 0, \\
T^f &= T^s = T_U, & C = C_U, & \text{on } z = d.
\end{align*}
$$
Here $T_L > T_U$ and $C_L > C_U$ for the heated and salted below layer case. Three types of heat-generating function are considered in this study: linearly increasing, decreasing, and nonuniform, see Straughan [30]:

Case A: $Q(z) = Q_0 \left( \frac{1}{2} + \frac{1}{d} z \right)$.
Case B: $Q(z) = Q_0 \left( 2 + \frac{3}{2d^2} z^2 - \frac{3}{d} z \right)$.
Case C: $Q(z) = Q_0 \left( 1 + \sin \frac{2\pi}{d} z + \sin \frac{4\pi}{d} z \right)$.

where $Q_0 > 0$. In the steady state solution, we look for

$$\bar{v}_l = 0, \quad \bar{T}' = \bar{T}^s = \bar{T}(z), \quad \bar{C} = \bar{C}(z).$$

(3)

From eq. (1)_{3-4}, using cases A–C, and eq. (3), we get

$$k_L \Delta T' = -Q_l(z), \quad k_s \Delta T^s = -Q_s(z)$$

and, by using eq. (2), then

$$\bar{v}_l = 0,$$

$$\bar{T}' = \begin{cases} -\frac{Q_0}{k_L} \left[ \frac{1}{4} z^2 + \frac{1}{6d} z^3 - \frac{5d}{12} z \right] - \beta_T z + T_L, \\ -\frac{Q_0}{k_L} \left[ z^2 + \frac{1}{8d^2} z^4 - \frac{1}{2d} z^3 - \frac{5d}{8} z \right] - \beta_T z + T_L, \\ -\frac{Q_0}{k_L} \left[ \frac{1}{2} z^2 - \frac{d^2}{4\pi^2} \sin \frac{2\pi}{d} z - \frac{d^2}{16\pi^2} \sin \frac{4\pi}{d} z - \frac{d}{2} z \right] - \beta_T z + T_L, \end{cases}$$

(4)

$$\bar{T}^s = \begin{cases} -\frac{Q_0}{k_s} \left[ \frac{1}{4} z^2 + \frac{1}{6d} z^3 - \frac{5d}{12} z \right] - \beta_T z + T_L, \\ -\frac{Q_0}{k_s} \left[ z^2 + \frac{1}{8d^2} z^4 - \frac{1}{2d} z^3 - \frac{5d}{8} z \right] - \beta_T z + T_L, \\ -\frac{Q_0}{k_s} \left[ \frac{1}{2} z^2 - \frac{d^2}{4\pi^2} \sin \frac{2\pi}{d} z - \frac{d^2}{16\pi^2} \sin \frac{4\pi}{d} z - \frac{d}{2} z \right] - \beta_T z + T_L, \end{cases}$$

where $\beta_T = \frac{T_U - T_L}{d}$. Also, $\dot{q} k_e \Delta C + \dot{k}(C_{eq}(T') - \bar{C}) = 0$. As presented by Pritchard and Richardson [16], assume that $\bar{C}(z) = C_{eq}(T'(z))$. Hence,
\[ \dot{\phi}k_C \Delta \bar{C} = 0, \]

which gives

\[ \bar{C}(z) = a T'(z) + b, \]

where \( a \) and \( b \) are constants of integration. Now, using eq. (2) gives

\[ \dot{\bar{C}} = \begin{cases} \frac{-\beta_C Q_0}{\beta_T k_f} \left[ \frac{1}{2} \frac{z^2}{6d^2} + \frac{1}{12d^2} - \frac{5d}{12} \right] - \beta_C z + C_L, \\ \frac{-\beta_C Q_0}{\beta_T k_f} \left[ \frac{z^2}{2} - \frac{1}{8d^2} \sin \frac{2\pi z}{d} - \frac{4\pi}{16d^2} \sin \frac{4\pi z}{d} \right] - \beta_C z + C_L, \\ \frac{-\beta_C Q_0}{\beta_T k_f} \left[ \frac{1}{2} - \frac{d^2}{4\pi^2} - \frac{2\pi^2}{12} \sin \frac{2\pi z}{d} - \frac{d^2}{12} \right] - \beta_C z + C_L, \end{cases} \]

(5)

where \( \beta_C = \frac{C_C - C_U}{d} \). To analyze the stability of the solutions in Eqs. (4)–(5), we define perturbations \((u, \pi, \theta', \theta^s, \phi)\) such that

\[ v_i = \nu_i + u, \quad p = p + \pi, \quad T' = T + \theta', \quad T^s = T^s + \theta^s, \quad C = \bar{C} + \phi. \]

(6)

Substituting eq. (6) into eq. (1), we obtain the equations governing \((u, \pi, \theta, \phi)\) as

\[ \begin{align*}
\pi_j &= -\frac{\mu}{K} u_i + \rho_0 \sigma_T g_k \theta' + \rho_0 \alpha_c g_k \phi + \lambda \Delta u_i, \\
u_{ij} &= 0, \\
\dot{\phi}(pc)_j \theta_j' + (pc)_j u_i \theta_j' &= - (pc)_j \frac{d\theta_j'}{dz} w + \dot{\phi}_k \Delta \theta^s + \dot{h}(\theta' - \theta'), \\
(1 - \dot{\bar{\phi}})(pc)_j \theta_j' &= (1 - \dot{\bar{\phi}}) k_s \Delta \theta^s + \dot{h}(\theta' - \theta^s), \\
\dot{\bar{\phi}} \pi_j + \nu_i \phi_j &= - \frac{\mu}{\rho c} \frac{dC}{dz} w + \dot{\phi}_k \Delta \phi + \dot{k}_f \theta_j - \dot{k}_f, 
\end{align*} \]

(7)

where \( \mathbf{u} = (u, v, w) \). Using nondimensionalization, we use the length scale \( d \), time scale \( \tau = \frac{(pc)_j d^2}{\mu} \), velocity scale \( U = \frac{\dot{\phi}_k}{(pc)_j d} \), pressure scale \( P = \frac{\dot{\phi}_k \mu}{(pc)_j R} \), and the temperature \( T^4 = \frac{q_0 d^2}{\beta_T k_f k_s} \), and the concentration scale \( C^4 = \frac{C_C d^2}{\beta_T k_f k_s} \).

Here, \( \epsilon = \frac{Le}{(pc)_j d} \), \( Le = \frac{K_f}{K_s} \) is the Lewis number, and \( \dot{\gamma} = \frac{\lambda K_f}{\rho c} \) is the Brinkman coefficient. The thermal Rayleigh and solutal Rayleigh numbers are defined as follows:

\[ R^2 = \frac{d^3 K Q_0 \sigma_T \rho_0 q_0 \phi}{\dot{\phi}_k \mu K}, \quad R_s^2 = \frac{\beta_C d^3 K Q_0 \alpha_c \rho_0 q_0 s}{\dot{\phi}_k \mu K}. \]

After substituting the nondimensionalization scalings into eq. (7), we have

\[ \begin{align*}
\pi_j &= -u_i + R_k \theta' - R_s k_f \phi + \gamma \Delta u_i, \\
u_{ij} &= 0, \\
\theta_j' + u_i \theta_j' &= RN(z) w + \Delta \theta' + \dot{H}(\theta' - \theta'), \\
\Delta \theta_j' &= \Delta \theta^s + \dot{H}(\theta' - \theta^s), \\
\epsilon \phi_j + \nu_i \phi_j &= \epsilon R_n(z) w + \Delta \phi + \dot{h}(\theta' - \eta \phi), 
\end{align*} \]

(8)

where

\[ N(z) = \begin{cases} \\
\frac{1}{2} z^2 + \frac{1}{2} z^2 - \frac{5}{12} + \zeta, \\
\frac{1}{2} z^3 - \frac{3}{2} z^2 + 2z - \frac{5}{8} + \zeta, \\
z - \frac{1}{2\pi} \cos 2\pi z - \frac{1}{4\pi} \cos 4\pi z - \frac{1}{2} + \zeta. 
\end{cases} \]
Here, the heat parameter is $\zeta = \frac{k_y}{\rho c_x}$. Furthermore, the interphase heat transfer parameter is $\tilde{H} = \frac{\tilde{h} d^2}{\tilde{F} k_f}$, $\tilde{\lambda} = \frac{k_y (\rho c_x)}{k_y (\rho c_x)}$ is the diffusivity ratio, the porosity modified conductivity ratio is $\tilde{\lambda} = \frac{\tilde{\lambda}}{(1-\tilde{\eta})k_f}$, while $\tilde{h}$ and $\eta$ are the reaction terms given by

$$h = \frac{\tilde{k}_f T^4}{\tilde{\phi}_k c^4}, \quad \eta = \frac{\tilde{k}_f}{\tilde{\phi}_k}.$$

Equation (8) hold in the domain $\{(x, y) \in \mathbb{R}^2\} \times \{z \in (0, 1)\} \times \{t > 0\}$. The boundary conditions become

$$u_l = \theta = \phi = 0 \quad \text{on} \quad z = 0, 1.$$

### 3 Linear instability theory

To perform the linear instability analysis, we first remove the nonlinear terms of Eqs. (8) and then remove the $\pi$ term by taking the third component of the double curl of eq. (8). We seek for solutions of the form

$$w(x, t) = w(x) e^{\sigma t}, \quad \theta'(x, t) = \theta'(x) e^{\sigma t}, \quad \theta(x, t) = \theta(x) e^{\sigma t}, \quad \phi(x, t) = \phi(x) e^{\sigma t},$$

where $\sigma$ is the time-dependent growth rate. Thus, the linearized equations arising from eq. (8) are

$$\Delta w - R \Delta \theta' + R_s \Delta \phi - \tilde{\gamma} \Delta^2 w = 0, \quad \sigma \theta' = RN(z)w + \Delta \theta' + \tilde{H}(\theta' - \theta), \quad \lambda \sigma \theta = \Delta \theta^s + \tilde{\lambda} \theta^s, \quad \tilde{c} \sigma \phi = cR_s N(z)w + \Delta \phi + h \theta' - \eta \phi,$$

where $\Delta^* = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the horizontal Laplacian operator. To analyze equations in eq. (9), we use standard methods, cf. Chandrasekhar [31]. Then we assume the normal mode with the representations for $w$, $\theta'$, $\theta^s$, and $\phi$ in the form of

$$w = W(z) f(x, y), \quad \theta' = \theta'(z) f(x, y), \quad \theta^s = \Theta^s(z) f(x, y), \quad \phi = \Phi(z) f(x, y),$$

where $f$ is the horizontal plane form which satisfies $\Delta^* f = -a^2 f$, $D = \frac{d}{dz}$, $a$ is a wave number, and $\Delta = D^2 - a^2$. The system in eq. (9) may be written in the form

$$(D^2 - a^2) W - \tilde{\gamma} (D^2 - a^2)^2 W + a^2 R \Theta' + a^2 R_s \Phi = 0, \quad \sigma \Theta' = RN(z)W + (D^2 - a^2) \Theta' + \tilde{H}(\Theta' - \Theta), \quad \lambda \sigma \Theta^s = (D^2 - a^2) \Theta^s + \tilde{\lambda} \Theta^s, \quad \tilde{c} \sigma \Phi = cR_s N(z)W + h \Theta' + (D^2 - a^2) \Phi - \eta \Phi.$$

Equation (10) represents an eigenvalue problem for the eigenvalue $\sigma$ to be found subject to the boundary conditions

$$W = DW = \Theta = \Phi = 0 \quad \text{on} \quad z = 0, 1.$$  

We determine the critical Rayleigh number by

$$Ra = \min_{a^2} R^2(a^2),$$

where for all $R^2 > Ra$ the system is unstable.
4 Nonlinear stability theory

In this section, the nonlinear energy stability analysis is provided to give a threshold in which the system is stable. To this end, let \( V \) be a period cell for the disturbance solution defined by \( \Gamma \times (z \in (0, d)) \), where \( \Gamma \) denotes the cell shape in the horizontal plane \((x, y)\) and let \( \| \cdot \| \) and \( \cdot \) be the norm and inner product on \( L^2(V) \). Then, by multiplying eqs. (8), by \( w, \theta' , \theta'' \), and \( \phi \), respectively, and integrating each over \( V \), we find

\[
-\|u\|^2 + R(\theta', w) - R_s(\phi, w) - \bar{y}\|\nabla u\|^2 = 0,
\]

\[
\frac{1}{2} \frac{d}{dt} \|\theta'\|^2 = RN(z)\langle w, \theta' \rangle - \| \nabla \theta' \|^2 + \hat{H}(\theta', \theta') - \hat{H}\|\theta'\|^2,
\]

\[
\frac{A}{2} \frac{d}{dt} \|\theta''\|^2 = -\|\nabla\theta''\|^2 + \tilde{\lambda}\hat{H}(\theta', \theta'') - \tilde{\lambda}\hat{H}\|\theta''\|^2,
\]

\[
\frac{c}{2} \frac{d}{dt} \|\phi\|^2 = cR_sN(z)\langle w, \phi \rangle - \| \nabla \phi \|^2 + h(\theta', \phi) - \eta\|\phi\|^2.
\]

Then we form the combination of the equations in eq. (12) as \( \lambda_1(12)_1 + (12)_2 + (12)_3 + \lambda_2(12)_4 \), where \( \lambda_1 \) and \( \lambda_2 \) are positive numbers to be chosen. This leads to the energy equation

\[
\frac{dE}{dt} = I - D = -D\left(1 - \frac{I}{D}\right),
\]

where

\[
E = \frac{1}{2}\|\theta'\|^2 + \frac{A}{2}\|\theta''\|^2 + \frac{\lambda_1\epsilon}{2}\|\phi\|^2,
\]

\[
I = R(\lambda_1 + N(z))\langle \theta', w \rangle + R_s(\epsilon\lambda_2N(z) - \lambda_2)\langle \phi, w \rangle + \hat{H}(1 + \tilde{\lambda})(\theta', \theta') + \lambda_2 h(\theta', \phi),
\]

\[
D = \lambda_1\|u\|^2 + \lambda_1\bar{y}\|\nabla u\|^2 + \| \nabla \theta' \|^2 + \| \nabla \theta'' \|^2 + \tilde{\lambda}\hat{H}\|\theta''\|^2 + \lambda_2\|\nabla \phi\|^2 + \lambda_2\eta\|\phi\|^2.
\]

Then

\[
\frac{dE}{dt} \leq -D\left(1 - \max_{\mathcal{H}} \frac{I}{D}\right) = -D\left(1 - \frac{1}{R_E}\right),
\]

where \( \mathcal{H} \) is the space of admissible solutions and

\[
\frac{1}{R_E} = \max_{\mathcal{H}} \frac{I}{D}.
\]

The nonlinear stability follows when \( R_E > 1 \), which implies that \( (1 - \frac{1}{R_E}) > 0 \). From eq. (14), and by using Poincaré’s inequality, one may obtain

\[
D = \lambda_1\|u\|^2 + \lambda_1\bar{y}\|\nabla u\|^2 + \| \nabla \theta' \|^2 + \tilde{\lambda}\hat{H}\|\theta''\|^2 + \lambda_2\|\nabla \phi\|^2 + \lambda_2\eta\|\phi\|^2 \\
\geq (\hat{H} + \tilde{\lambda}\epsilon)(\|\theta'\|^2 + \| \nabla \theta' \|^2 + \| \nabla \theta'' \|^2 + \lambda_2\|\nabla \phi\|^2 + \lambda_2\eta\|\phi\|^2) \\
= (\hat{H} + \tilde{\lambda}\epsilon)(\|\theta'\|^2 + \frac{A}{\lambda}(\tilde{\lambda}\hat{H} + \epsilon\tilde{\lambda}\epsilon^2)(\|\theta''\|^2 + \lambda_2\epsilon\|\phi\|^2) \\
\geq 2k\left(\frac{\|\theta'\|^2 + \| \nabla \theta' \|^2 + \lambda_2\epsilon\|\phi\|^2}{2}\right) = 2kE,
\]

where \( k = \min\{\hat{H} + \tilde{\lambda}\epsilon, \frac{\tilde{\lambda}\hat{H} + \epsilon\tilde{\lambda}\epsilon^2}{\lambda} \} \). Then from eq. (15) one may derive the inequality

\[
\frac{dE}{dt} \leq -2kbE,
\]

where \( b = 1 - \frac{1}{R_E} \).
This may be integrated to see that

\[ E(t) \leq E(0)e^{-2kt}, \quad (18) \]

where \( E(0) \) is constant, and hence we see that \( E(t) \to 0 \) as \( t \to \infty \). Therefore, by the definition of \( E \), one proves that \( ||\theta'|| \to 0, ||\theta''|| \to 0 \), and \( ||\phi|| \to 0 \) as \( t \to \infty \).

To obtain the decay of \( u \), by using the Poincaré and arithmetic–geometric mean inequalities in eq. (12)\(_1\), we can then deduce that

\[
(1 + \gamma^2\pi^2)||u||^2 \leq \frac{R}{2v_1}||\theta'||^2 + \frac{Rv_1}{2}||\theta''||^2 + \frac{R}{2v_2}||\phi||^2 + \frac{Rv_2}{2}||w||^2 \\
\leq \left( \frac{Rv_1}{2} + \frac{Rv_2}{2} \right)||u||^2 + \frac{R}{2v_1}||\theta'||^2 + \frac{R}{2v_2}||\phi||^2,
\]

where \( v_1 \) and \( v_2 \) are constants to be chosen such that \( Rv_1 + Rv_2 = 1 \), which gives \( v_1 = \frac{1}{R} \) and \( v_2 = \frac{1}{R} \), so that inequality (19) will be

\[ ||u||^2 \leq \frac{R^2}{2\gamma\pi^2}||\theta'||^2 + \frac{R^2}{2\gamma\pi^2}||\phi||^2.\]

Therefore we see that \( ||u|| \to 0 \) as \( t \to \infty \), so we have shown the decay of \( u, \theta', \theta'', \) and \( \phi \).

The nonlinear stability threshold is thus given by the solution of the variational problem in eq. (16). The Euler–Lagrange equations for the latter are

\[
R(\lambda_1 + N(z))\theta' k_1 + R_s(\epsilon \lambda_2 N(z) - \lambda_1)\phi k_1 - 2\lambda_1 u_i + 2\lambda_1 \gamma \Delta u_i = \xi, \\
R(\lambda_1 + N(z))w + 2\Delta \theta' - 2H\theta' + H(1 + \hat{\lambda})\theta'' + \lambda_2 h\phi = 0, \\
\hat{H}(1 + \hat{\lambda})\theta'' + 2\Delta \theta'' - 2\lambda \hat{H}\theta'' = 0, \\
R_s(\epsilon \lambda_2 N(z) - \lambda_1)w + \lambda_2 h\theta' + 2\lambda_2 \Delta \phi - 2\lambda_2 \eta \phi = 0,
\]

where \( \xi \) is a Lagrange multiplier. By taking the double curl of eq. (20)\(_1\), and introducing the normal mode representation as presented in Sect. 3, eq. (20) becomes

\[
2\lambda_1(D^2 - a^2)W - 2\lambda_1 \gamma (D^2 - a^2)W + a^2 R(\lambda_1 + N(z))\theta' + a^2 R_s(\epsilon \lambda_2 N(z) - \lambda_1)\phi = 0, \\
R(\lambda_1 + N(z))W + 2(D^2 - a^2)\theta' - 2H\theta' + H(1 + \hat{\lambda})\theta'' + \lambda_2 h\phi = 0, \\
\hat{H}(1 + \hat{\lambda})\theta'' + 2(D^2 - a^2)\theta'' - 2\lambda \hat{H}\theta'' = 0, \\
R_s(\epsilon \lambda_2 N(z) - \lambda_1)W + \lambda_2 h\theta' + 2\lambda_2 (D^2 - a^2)\phi - 2\lambda_2 \eta \phi = 0,
\]

The corresponding boundary conditions are in eq. (11). We can determine the critical Rayleigh number

\[
Ra_E = \max_{\lambda_1, \lambda_2} \min_{a^2} R^2(a^2, \lambda_1, \lambda_2),
\]

where for all \( R^2 < Ra_E \) the system is stable.

## 5 Numerical method

In this section, the bound for the linear instability theory and the energy theory of Eqs. (10) and (21), respectively, corresponding to boundary conditions eq. (11), are solved numerically by using a \( D^2 \) Chebyshev tau method, see Dongarra et al. [32]. The numerical results for the heated and salted below layer are reported in detail in Sect. 6. We can write Eqs. (10)\(_1\) and (21)\(_1\) as second-order equations by setting
\[ \chi = (D^2 - a^2)W. \]

To this end, we begin by resetting the domain from \((0, 1)\) to \((-1, 1)\), selecting \(\tilde{z} = 2z - 1\). Then, we may rewrite Eqs. (10) and (21), respectively, as the follows (omitting the hat):

\begin{align*}
(4D^2 - a^2)W - \chi &= 0, \\
\chi - \tilde{y}(4D^2 - a^2)\chi + a^2 R\Theta' - a^2 R_2\Phi &= 0,
\end{align*}

where the matrices \(C, B\) are given by

\[ \sigma\Theta' = RN_1 W + (4D^2 - a^2)\Theta' + \tilde{H}(\Theta^s - \Theta'), \]

\[ \lambda\sigma\Theta^s = (4D^2 - a^2)\Theta^s + \lambda\tilde{H}(\Theta' - \Theta^s), \]

\[ c\sigma\Phi = cR_2N_1 W + h\Theta' + (4D^2 - a^2)\Phi - \eta\Phi. \]

where \(N_1 = N(\frac{z + 1}{2})\), \(z \in (-1, 1)\). Then, the functions \(W, \chi, \Theta', \Theta^s, \) and \(\Phi\) are expanded in terms of Chebyshev polynomials

\[ W(z) = \sum_{n=1}^{M} W_n T_n(z), \quad \chi(z) = \sum_{n=1}^{M} \chi_n T_n(z), \quad \Theta'(z) = \sum_{n=1}^{M} \Theta'_n T_n(z), \]

\[ \Theta^s(z) = \sum_{n=1}^{M} \Theta^s_n T_n(z), \quad \Phi(z) = \sum_{n=1}^{M} \Phi_n T_n(z). \]

Therefore, the \(D^2\) Chebyshev tau method requires solving the matrix system of eq. (22), namely

\[ AX = \sigma BX, \]

where \(X = (W_1, W_2, \ldots, W_M, \chi_1, \chi_2, \ldots, \chi_M, \Theta'_1, \Theta'_2, \ldots, \Theta'_M, \Theta^s_1, \Theta^s_2, \ldots, \Theta^s_M, \Phi_1, \Phi_2, \ldots, \Phi_M)\) and the matrices \(A\) and \(B\) are given by

\[ A = \begin{pmatrix}
4D^2 - a^2 I & -I & 0 & 0 & 0 \\
0 & I - \tilde{y}(4D^2 - a^2)I & a^2 R I & 0 & -a^2 R_2 I \\
R N_1 I & 0 & 4D^2 - (a^2 + \tilde{H}) I & \tilde{H} I & 0 \\
0 & 0 & \lambda\tilde{H} I & 4D^2 - (a^2 + \lambda\tilde{H}) I & 0 \\
e R_2 N_1 I & 0 & h I & 0 & 4D^2 - (a^2 + \eta I)
\end{pmatrix}, \]

\[ B = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & \lambda I & 0 \\
0 & 0 & 0 & 0 & e I
\end{pmatrix}. \]

And for eq. (23), the matrix system is

\[ CX = RFX, \]

where the matrices \(C\) and \(F\) are given by
and critical Rayleigh number becomes significantly larger, namely $Ra$ number increases faster than in cases B and C. For example, at grows as the coefficient $\zeta$ parameter and subject to the boundary conditions in eq. (26). The numerical results are discussed for different choices by various parameters with respect to three different types of the internal heat source:

Case A: $N_1(z) = \frac{(z + 1)^2}{8} + \frac{z + 1}{4} - \frac{5}{12} + \zeta$,  
Case B: $N_1(z) = \frac{(z + 1)^3}{16} - \frac{3(z + 1)^2}{8} + z + \frac{3}{8} + \zeta$,  
Case C: $N_1(z) = \frac{z + 1}{2} + \frac{1}{2\pi} \cos \pi z - \frac{1}{4\pi} \cos 2\pi z - \frac{1}{2} + \zeta$,

and subject to the boundary conditions in eq. (26). The numerical results are discussed for different choices of the reaction rates $\eta$, $h$, interphase heat transfer parameter $\hat{H}$, porosity modified conductivity ratio $\hat{\lambda}$, heat parameter $\zeta$, and salt Rayleigh number $Ra_s$. The cases A–C to have encoded the behavior of various physical quantities to verify their effect on the stability of the system.

In this section, the internal heat source and the reaction effects on the threshold thermal Rayleigh number are investigated using thermal non-equilibrium Darcy–Brinkman model. The numerical results are obtained by various parameters with respect to three different types of the internal heat source:

$$W(-1) = \sum_{n=1}^{M} (-1)^n W(-1) = 0, \quad W(1) = \sum_{n=1}^{M} W(1) = 0,$$

$$DW(-1) = \sum_{n=1}^{M} (-1)^n W(-1) = 0, \quad DW(1) = \sum_{n=1}^{M} n^2 W(1) = 0,$$

$$\Theta(-1) = \sum_{n=1}^{M} (-1)^n \Theta(-1) = 0, \quad \Theta(1) = \sum_{n=1}^{M} \Theta(1) = 0,$$

$$\Phi(-1) = \sum_{n=1}^{M} (-1)^n \Phi(-1) = 0, \quad \Phi(1) = \sum_{n=1}^{M} \Phi(1) = 0.$$

The eigenvalues of the generalized eigenvalue problems (24) and (25) are found efficiently by using the QZ algorithm.

6 Numerical results and discussion

In this section, the internal heat source and the reaction effects on the threshold thermal Rayleigh number are investigated using thermal non-equilibrium Darcy–Brinkman model. The numerical results are obtained by various parameters with respect to three different types of the internal heat source:

For different values of $\hat{H}$ with fixed $R_s = 5$, $\beta = 1$, $\hat{\lambda} = 1$, $\hat{\lambda} = 0.5$, and $\zeta = 0.6$, when $\eta > h$, $\eta < h$, and $\eta = h$, Table 1 shows that the critical Rayleigh number of the linear instability and nonlinear stability boundaries grows as the coefficient $\hat{H}$ increases, as seen in Figure 2. It is observed that in case A the critical Rayleigh number increases faster than in cases B and C. For example, at $\hat{H} = 10$, in case A, as seen in Table 1, the critical Rayleigh number becomes significantly larger, namely $Ra_e = 4245.037$ and $Ra_E = 4161.283$, at $\eta = 1$ and $h = 1$, whereas in cases B and C, the critical Rayleigh number is $Ra_e = 3781.784$, $Ra_E = 3719.287$ and
In this article, the problem of thermosolutal convection in a porous medium of Darcy–Brinkman model where the layer is heated and salted from below is studied. We have investigated in detail the reaction and thermal non-equilibrium effects with three basic forms of the internal heat source. The linear instability and nonlinear theories is noticeable with the increase of $H$, especially when $\eta > h$.

In Table 2, we list the results for various values of $\Lambda$ with fixed $R^2 = 5$, $y = 1$, $\Lambda = 1$, $H = 1$, and $\zeta = 0.6$ at $\eta > h$, $\eta < h$, and $\eta = h$. When $\Lambda$ increases, the critical Rayleigh number decreases, and we notice a subcritical stability region as there is a difference between the critical Rayleigh numbers of linear and nonlinear theories when $\Lambda = 100$, particularly when $\eta < h$, as seen in Figure 3.

For different values of $\zeta$, with the reaction rates $\eta > h$, $\eta < h$, and $\eta = h$, and fixed values of $R^2 = 5$, $y = 1$, $\Lambda = 1$, $H = 1$, and $\Lambda = 0.5$, the results in Table 3 clearly illustrate that the heat parameter $\zeta$ is destabilizing the system where the values of $R_\ell$ and $R_E$ rapidly decrease as $\zeta$ grows, for instance, as $\eta = 10$, $h = 1$, for case A, $R_\ell = 1895.276$, $R_E = 1882.406$ at $\zeta = 1$, whereas at $\zeta = 10$, $R_\ell = 188.883$, $R_E = 131.078$. The same qualitative behavior is also seen in the other cases and situations considered, e.g., $\eta = 1$, $h = 10$, and $\eta = 10$, $h = 10$, which are depicted in Figure 4. However, a small subcritical stability region is found where there is a difference between $R_\ell$ and $R_E$ boundaries when the value of $h$ is bigger than the value of $\eta$.

In the case of the reaction rates $\eta > h$ and $\eta < h$ and fixed values of $y = 1$, $\Lambda = 0.5$, $H = 1$, $\Lambda = 0.5$, and $\zeta = 0.6$, analyzing the influence of various values of the salt Rayleigh number $R^2$ on the stability boundary demonstrates that increasing the value of salt Rayleigh number $R^2$ leads to an increasing value of the linear instability, and the difference between the linear instability and energy stability values will be considerable, especially as $\eta < h$, the gap between $R_\ell$ and $R_E$ thresholds widens, as shown in Figure 5.

### 7 Conclusions

In this article, the problem of thermosolutal convection in a porous medium of Darcy–Brinkman model where the layer is heated and salted from below is studied. We have investigated in detail the reaction and thermal non-equilibrium effects with three basic forms of the internal heat source. The linear instability and nonlinear theories is noticeable with the increase of $H$, especially when $\eta < h$. In Table 2, we list the results for various values of $\Lambda$ with fixed $R^2 = 5$, $y = 1$, $\Lambda = 1$, $H = 1$, and $\zeta = 0.6$ at $\eta > h$, $\eta < h$, and $\eta = h$. When $\Lambda$ increases, the critical Rayleigh number decreases, and we notice a subcritical stability region as there is a difference between the critical Rayleigh numbers of linear and nonlinear theories when $\Lambda = 100$, particularly when $\eta < h$, as seen in Figure 3.
Figure 2: The critical Rayleigh number of linear instability and nonlinear stability threshold for $\eta > h$, $\eta < h$, $\eta = h$, $R_a^2 = 5$, $\gamma = 1$, $\lambda = 1$, $\lambda = 0.5$, and $\xi = 0.6$, for various values of $\dot{H}$.

Table 2: Critical Rayleigh numbers of linear theory $R_aL$ and nonlinear energy theory $R_aE$, corresponding to critical wave numbers $\alpha_c$ for $\eta > h$, $\eta < h$, $\eta = h$, $R_a^2 = 5$, $\gamma = 1$, $\lambda = 1$, $\dot{H} = 1$, and $\xi = 0.6$, for various values of $\dot{\lambda}$.

<table>
<thead>
<tr>
<th>$\dot{\lambda}$</th>
<th>Case A</th>
<th>Case B</th>
<th>Case C</th>
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<tr>
<td></td>
<td>$R_aL$, $\alpha_cL$, $R_aE$, $\alpha_cE$</td>
<td>$R_aL$, $\alpha_cL$, $R_aE$, $\alpha_cE$</td>
<td>$R_aL$, $\alpha_cL$, $R_aE$, $\alpha_cE$</td>
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<td>$10^{-2}$</td>
<td>3209.055, 3.182, 3171.453, 3.185</td>
<td>2858.382, 3.178, 2833.183, 3.179</td>
<td>2672.864, 3.179, 2647.917, 3.180</td>
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<tr>
<td>$10^{-1}$</td>
<td>3146.135, 3.161, 2955.488, 3.179</td>
<td>2799.081, 3.156, 2708.964, 3.176</td>
<td>2615.621, 3.156, 2565.434, 3.177</td>
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<tr>
<td>10</td>
<td>3215.900, 3.173, 2813.175, 3.088</td>
<td>2867.609, 3.170, 2578.359, 3.086</td>
<td>2683.296, 3.171, 2441.878, 3.087</td>
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<tr>
<td>$10^2$</td>
<td>3138.928, 3.151, 9.95 e $-08$, 3.615</td>
<td>2798.980, 3.148, 7.17 e $-08$, 3.615</td>
<td>2619.135, 3.149, 5.91 e $-08$, 3.657</td>
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<tr>
<td>$10^{-2}$</td>
<td>3247.049, 3.191, 3171.421, 3.185</td>
<td>2894.275, 3.188, 2833.156, 3.179</td>
<td>2707.569, 3.188, 2647.891, 3.180</td>
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<tr>
<td>10</td>
<td>3196.194, 3.162, 3019.413, 3.094</td>
<td>2848.882, 3.159, 2696.780, 3.089</td>
<td>2665.142, 3.159, 2520.527, 3.089</td>
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Table 3: Critical Rayleigh numbers of linear theory $Ra_L$ and nonlinear energy theory $Ra_E$, corresponding to critical wave numbers $a_c$ for $\eta > h, \eta < h, \eta = h, R_\parallel = 5, \varphi = 1, \Lambda = 1, H = 1$, and $\zeta = 0.6$, for various values of $\lambda$.

<table>
<thead>
<tr>
<th>$\zeta$</th>
<th>$Ra_L$</th>
<th>$Ra_E$</th>
<th>$ac_L$</th>
<th>$ac_E$</th>
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<th>$Ra_L$</th>
<th>$Ra_E$</th>
<th>$ac_L$</th>
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<tbody>
<tr>
<td>$\eta = 1, h = 10$</td>
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<tr>
<td>10</td>
<td>175.090</td>
<td>3.090</td>
<td>115.412</td>
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<td>173.828</td>
<td>3.090</td>
<td>114.060</td>
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<td>3.090</td>
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<td>$\eta = 10, h = 10$</td>
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<td>1721.839</td>
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<td>1683.096</td>
<td>3.173</td>
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</table>

Figure 3: The critical Rayleigh number of linear instability and nonlinear stability threshold for $\eta > h, \eta < h, \eta = h, R_\parallel = 5, \varphi = 1, \Lambda = 1, H = 1$, and $\zeta = 0.6$, for various values of $\lambda$. 

Table 3: Critical Rayleigh numbers of linear theory $Ra_L$ and nonlinear energy theory $Ra_E$, corresponding to critical wave numbers $a_c$ for $\eta > h, \eta < h, \eta = h, R_\parallel = 5, \varphi = 1, \Lambda = 1, H = 1$, and $\zeta = 0.6$, for various values of $\lambda$. 

$Ra_L = \eta \frac{h}{\Lambda}, Ra_E = h L a_c = \eta L a_c, Ra_L = \eta L a_c, Ra_E = h L a_c, Ra_L = \eta L a_c, Ra_E = h L a_c$.
stability theories were applied to investigate the effect of various parameters on the stability of the system. Numerical results were achieved by using the $D^2$ Chebyshev tau method. The following conclusions can be stated:
1. It could be argued based on the results that the interphase heat transfer parameter $\tilde{H}$ for fixed values of $R^2_s$, $\gamma$, $\tilde{A}$, $\tilde{\lambda}$, and $\zeta$, when $\eta > h$, $\eta < h$, and $\eta = h$, for all cases of the internal heat source, has a stabilizing effect on the system. Also, it is observed that a small subcritical stability area may arise for $h$ larger than $\eta$.

2. For all cases of the internal heat source, the porosity modified conductivity ratio $\tilde{\lambda}$ and heat parameter $\zeta$, for fixed values of $R^2_s$, $\gamma$, $\tilde{A}$, and $\tilde{H}$ at $\eta > h$, $\eta < h$, and $\eta = h$, play important roles in the stability and instability regions.

3. The salt Rayleigh number $R^2$, for fixed values of $\gamma$, $\tilde{A}$, $\tilde{H}$, $\tilde{\lambda}$, and $\zeta$ whenever $\eta > h$ or $\eta < h$, for all cases of the internal heat source, is an important factor in determining whether a region is stable or unstable.

References


