Research Article

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Testing for Multiple Structural Changes with Non-Homogeneous Regressors

Abstract: This paper investigates tests for multiple structural changes with non-homogeneous regressors, such as polynomial trends. We consider exponential-type, supremum-type and average-type tests as well as the corresponding weighted-type tests suggested in the literature. We show that the limiting distributions depend on regressors in general, and we need to tabulate critical values depending on them. Then, we focus on the linear trend case and obtain the critical values of the test statistics. The Monte Carlo simulations are conducted to investigate the finite sample properties of the tests proposed in the paper, and it is found that the specification of the number of breaks is an important factor for the finite sample performance of the tests. Since it is often the case that we cannot prespecify the number of breaks under the alternative but can suppose only the maximum number of breaks, the weighted-type tests are useful in practice.

Keywords: multiple breaks, exp-type test, sup-type test, avg-type test, mean-type test

JEL Classification: C12, C22

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1 Introduction

This paper proposes tests for multiple structural changes with non-homogeneous regressors. In particular, we focus on trending regressors. Tests for structural changes have long been investigated in the econometric and statistical literature, and the most commonly used tests in empirical analysis for a one-time break are the supremum-type (sup-type) test by Andrews (1993) in the GMM framework and the exponential-type (exp-type) and the average-type (avg-type) tests by Andrews and Ploberger (1994) in linear regression models. The

latter two tests have an optimal property, which was investigated by Andrews and Ploberger (1994) and Sowell (1996) under the Pitman-type alternative, while Kim and Perron (2009) compared these tests in a framework based on the Bahadur slope.

Although these tests are often used in practice to test for a one-time change, we need to take into account the possibility of multiple structural changes when economic data in long sample periods are available. Bai and Perron (1998) extended the sup-type test to the case of multiple structural changes in univariate regressions, while the multivariate case was considered by Qu and Perron (2007). Andrews, Lee, and Ploberger (1996) investigated the optimality of the exp-type and avg-type tests. Note that these tests are designed for the null hypothesis of no change against the alternative of the fixed number of breaks. On the other hand, Bai and Perron (1998) and Qu and Perron (2007) proposed double maximum tests against the alternative under which only the maximum number of breaks is prespecified, while Bai and Perron (1998), Bai (1999), and Qu and Perron (2007) considered tests for the null hypothesis of ℓ breaks against the alternative of $\ell + 1$ breaks. The multiple structural change tests have an advantage over the single structural change tests in that the former tests are more powerful than the latter when multiple breaks have actually occurred, as shown by Bai and Perron (2006).

The practical difficulty in the multiple structural change tests is that we need to take into account all permissible change points when constructing the test statistics. That is, for the sup-type, the exp-type, and the avg-type tests, we need to construct either the Wald, the likelihood ratio (LR), or the Lagrange multiplier test statistic for all permissible sets of change points, the number of which is proportional to T^m , where m indicates the number of breaks under the alternative. Then, the direct calculation of these test statistics is computationally very expensive when m is large. To overcome this problem, Bai and Perron (2003a) proposed an efficient algorithm for the sup-type test, which requires only the $O(T^2)$ calculations for any number of breaks. Critical values for the sup-type test are tabulated in Andrews (1993) for a one-time break and Bai and Perron (1998, 2003b) for multiple changes, and those for the exp-type and the avg-type tests are given in Andrews and Ploberger (1994) for a one-time change, while asymptotic p -values of these tests can be calculated by the method proposed by Hansen (1997). However, critical values for the exp-type and the avg-type tests with multiple breaks are not yet available.

Most of the above tests assume that regressors are homogeneous in the whole sample period, or at least in each regime under the null hypothesis. However, we sometimes include non-homogeneous regressors, such as trending variables. In this case, most of the above tests are not available in practical

analysis. The exception is the LR test, denoted by $\sup F(\ell + 1|\ell)$, for the null hypothesis of ℓ breaks against the alternative of $\ell + 1$ breaks proposed by Bai (1999). This test allows for polynomial trends, and hence the null hypothesis of no break can be tested using $\sup F(1|0)$. However, as pointed out by Bai and Perron (2006), this test may be less powerful than tests for multiple structural changes when multiple breaks have actually occurred.

In this paper, we develop tests for multiple breaks with non-homogeneous regressors, including trending regressors. We consider sup-type, exp-type, and avg-type tests, as in the literature, and derive the concise expressions of the limiting distributions. It is shown that in general, the limiting distributions depend on non-homogeneous regressors, and then we need to tabulate critical values depending on the case. For this reason, we focus on the linear trend case and tabulate critical values. Since we need to calculate the Wald test statistics for all permissible break points for the exp-type tests, which is computationally very expensive in the case of more than three breaks, we tabulate the critical values of the exp-type test for at most three breaks. On the other hand, we propose a computationally efficient method for obtaining critical values for the avg-type test, which requires only $O(T^2)$ operations for any given number of breaks. According to this efficient algorithm and Bai and Perron's (2003a) method, critical values of the sup-type and the avg-type tests are calculated for up to five breaks. Finite sample properties are investigated by Monte Carlo simulations, and it is confirmed that the tests that assume the maximum number of breaks but not the specific number of breaks are useful in practical analysis.

Note that this paper is related to another strand of the time series literature, which deals with tests for trend breaks robust to the stationary/unit root property in the stochastic errors. See, for example, Roy, Falk, and Fuller (2004), Harvey, Leybourne, and Taylor (2009), Perron and Yabu (2009; 2012), Kejriwal and Perron (2010), Saiginsoy and Vogelsang (2011) and Chun and Perron (2013). These papers focus on testing for only the trend coefficient with possibly integrated errors, whereas our paper considers tests for all (a part) of the regression coefficients associated with both trends and stationary regressors, although we do not allow for integrated regressors/errors. Thus, the main purpose of this paper is different from that of those robust tests.

For example, the international terms of trade between primary and manufactured commodities have been investigated in the literature and it has been pointed out that they follow a downward secular trend (the Prebisch–Singer hypothesis). However, this effect has recently been lessened by the growing demand from emerging market countries and then this hypothesis has been re-investigated by taking structural change into account in such as Kellard and Wohar (2006), Harvey et al. (2010), Ghoshray (2011) and Arezki et al. (2012),

among others. These papers focus only on the coefficient associated with a linear trend, but as modelled by Bloch and Sapsford (1996), the relative prices are explained by such as manufacturing output and relative wages as well as a linear trend and thus our framework with a multiple regression is useful for the full investigation of the relative prices.¹

The rest of this paper is organized as follows. Section 2 explains a model and assumptions. The test statistics are given in Section 3, and their limiting distributions are derived. The computational problem of the test statistics is discussed in Section 4. A model is extended to partial structural changes and serially correlated errors in Section 5, and the finite sample properties are investigated in Section 6. Section 7 gives concluding remarks.

2 Model and assumptions

Let us consider the following regression with m structural changes ($m + 1$ regimes):

$$y_t = x_t' \beta_j + \varepsilon_t \quad (j = 1, \dots, m + 1 \quad \text{and} \quad t = T_{j-1} + 1, \dots, T_j), \quad [1]$$

where x_t is p -dimensional regressors, including a constant, and ε_t is an error term. We set $T_0 = 0$ and $T_{m+1} = T$ so that the total number of observations is T . The testing problem we consider is given by

$$H_0 : \beta_1 = \dots = \beta_{m+1} \quad \text{vs.} \quad H_1 : \beta_i \neq \beta_j \quad \text{for some } i \neq j,$$

and we then consider the null of no structural change. The following assumptions are made throughout the paper.

Assumption A1 For some normalizing matrix D_T , (a) $D_T^{-1} \sum_{t=k}^l x_t x_t' D_T^{-1}$ is invertible for $l - k > k_0$ for some $0 < k_0 < \infty$. (b) $D_T^{-1} \sum_{t=1}^{[Tr]} x_t x_t' D_T^{-1} \xrightarrow{p} \Omega_r$ uniformly over $0 < r \leq 1$, where Ω_r is a $p \times p$ positive definite matrix for $0 < r \leq 1$ with $\Omega_0 = 0$, $[k]$ signifies the largest integer less than k , and $\xrightarrow{p}$ signifies convergence in probability. (c) $\Omega_s - \Omega_r$ is positive definite for all $0 \leq r < s \leq 1$.

Assumption A2 (a) $\{\varepsilon_t\}$ is a martingale difference sequence with respect to $\mathcal{F}_t = \sigma(\varepsilon_t, \varepsilon_{t-1}, \dots, x_{t+1}, x_t, \dots)$ with $E[\varepsilon_t^2 | \mathcal{F}_{t-1}] = \sigma^2$ for all t . (b) $\sup_t E|\varepsilon_t|^{2+\delta} < \infty$ for some $\delta > 0$. (c) For the same normalizing matrix D_T in Assumption A2, $D_T^{-1} \sum_{t=1}^{[Tr]} x_t \varepsilon_t \Rightarrow \sigma G(r)$ for $0 \leq r \leq 1$, where $G(r)$ is a p -dimensional Gaussian

¹ Kurozumi, Yamazaki, and Hadri (2013) have found that the relative prices of many primary commodities are characterized as trend stationary processes and thus trend stationary regressions should be used for testing the Prebisch–Singer hypothesis for some primary commodity prices.

process with mean zero and $E[G(r)G'(s)] = \Omega_{r \wedge s}$, and $\Rightarrow$ signifies weak convergence of the associated probability measures.

Assumption A1(a) is made for the identification of the coefficient. A1(b) allows for non-homogeneous regressors because the second moment of x_t is not asymptotically proportional to the sample fraction r , but possibly depends on r in a complicated way. Note that integrated regressors are not allowed for because Ω_r is non-stochastic. A1(c) implies that the probability limit of $D_T^{-1} \sum_{t=T_{j-1}+1}^{T_j} x_t x_t' D_T^{-1}$ is positive and invertible for $(T_j - T_{j-1})/T > 0$, so that the limiting distribution of $D_T(\beta_j - \beta)$ is well defined. Assumption A2 is standard in linear regressions, but we do not allow for serial correlation for a while. The case with serially dependent errors will be discussed in the later section.

Exactly speaking, Assumptions A1 and A2 are required only under the null hypothesis in order to derive the null limiting distributions of the test statistics and they can be relaxed under the alternative in order for the tests to be consistent. See, for example, Assumptions made in Bai and Perron (1998) for the case of (regime-wise) stationary regressors.

One of the interesting non-homogeneous regressors is a polynomial trend. For example, when x_t is given by

$$x_t' = [1, t, t^2, \dots, t^d, x_{1t}, \dots, x_{qt}],$$

where $x_{1t}, \dots, x_{qt}$ are stationary regressors, we can choose $D_T = \text{diag}\{T^{1/2}, T^{3/2}, T^{5/2}, \dots, T^{d+1/2}, T^{1/2}I_q\}$ and Assumptions A1(b) and A2(c) then become

$$D_T^{-1} \sum_{t=1}^{[Tr]} x_t x_t' D_T^{-1} \xrightarrow{p} \begin{bmatrix} r & \frac{r^2}{2} & \cdots & \frac{r^{d+1}}{d+1} & r\mu_x' \\ \frac{r^2}{2} & \frac{r^3}{3} & \cdots & \frac{r^{d+2}}{d+2} & \frac{r^2}{2}\mu_x' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{r^{d+1}}{d+1} & \frac{r^{d+2}}{d+2} & \cdots & \frac{r^{2d+1}}{2d+1} & \frac{r^{d+1}}{d+1}\mu_x' \\ r\mu_x & \frac{r^2}{2}\mu_x & \cdots & \frac{r^{d+1}}{d+1}\mu_x & r\Gamma_x \end{bmatrix} \quad \text{and}$$

$$D_T^{-1} \sum_{t=1}^{[Tr]} x_t \varepsilon_t \Rightarrow \sigma \begin{bmatrix} \int_0^r dW_1(s) \\ \int_0^r s dW_1(s) \\ \vdots \\ \int_0^r s^d dW_1(s) \\ \Gamma_x^{1/2} W_2(r) \end{bmatrix},$$

where μ_x and Γ_x are $q \times 1$ and $q \times q$ and consist of the first and second moments of $x_{1t}, \dots, x_{qt}$, respectively, $W_1(r)$ and $W_2(r)$ are 1- and q -dimensional standard Brownian motions on $[0, 1]$ and $[0, 1]^q$, respectively, and they are independent of each other. Apparently, the second moment of x_t is not proportional to r , and hence we cannot use the exiting results for multiple structural changes. We also note that a model considered by Vogelsang (1997) is a special case of our model with no stationary regressors ($q = 0$) and thus his tests are equivalent to ours if $q = 0$.

3 Tests for multiple structural changes

In this section, we define the test statistics for multiple structural changes and derive their limiting distributions. Let $\hat{\beta} = [\hat{\beta}'_1, \dots, \hat{\beta}'_{m+1}]'$ be the least squares estimator of the coefficients for a given number of breaks m with change points $\{T_1, \dots, T_m\}$, $\hat{\Sigma} = \text{diag}\{\hat{\Sigma}_1, \dots, \hat{\Sigma}_{m+1}\}$ be an $(m+1)p \times (m+1)p$ block-diagonal matrix where $\hat{\Sigma}_j = (\sum_{t=T_{j-1}+1}^{T_j} x_t x_t')^{-1}$, and $\hat{\sigma}^2$ be a consistent estimator of σ^2 . Typically, $\hat{\sigma}^2 = T^{-1} \sum_{t=1}^T \hat{\varepsilon}_t^2$, where $\hat{\varepsilon}_t$ is the regression residual. Then, the Wald test statistic for the null hypothesis of H_0 is given by

$$W_T(\Lambda_m) = (R\hat{\beta})' (\hat{\sigma}^2 R\hat{\Sigma}R')^{-1} (R\hat{\beta}), \text{ where } R = \begin{bmatrix} I_p & -I_p & & & 0 \\ & I_p & -I_p & & \\ & & \ddots & \ddots & \\ 0 & & & I_p & -I_p \end{bmatrix}$$

and $\Lambda_m = \{\lambda_1, \dots, \lambda_m\}$ with $\lambda_j = T_j/T$ for $j = 1, \dots, m$ being break fractions. We set $\lambda_0 = 0$ and $\lambda_{m+1} = 1$ for convention because $T_0 = 0$ and $T_{m+1} = T$. Using $W_T(\Lambda_m)$, we construct the exp-type, the sup-type, and the avg-type tests, as in the literature.

$$\text{exp-}W_T(m, \epsilon) = \log \left[\frac{1}{T^*} \sum_{\Lambda_m \in \Lambda_m^\epsilon} \exp \left(\frac{1}{2} W_T(\Lambda_m) \right) \right], \quad [2]$$

$$\text{sup-}W_T(m, \epsilon) = \max_{\Lambda_m \in \Lambda_m^\epsilon} W_T(\Lambda_m), \quad [3]$$

$$\text{avg-}W_T(m, \epsilon) = \frac{1}{T^*} \sum_{\Lambda_m \in \Lambda_m^\epsilon} W_T(\Lambda_m), \quad [4]$$

where $\Lambda_m^\epsilon = \{(\lambda_1, \dots, \lambda_m) : \lambda_j - \lambda_{j-1} \geq \epsilon \text{ for } j = 1, \dots, m+1\}$ for a given trimming parameter ϵ and T^* is the number of permissible sets of break fractions included in Λ_m^ϵ . The trimming parameter ϵ should be small; $\epsilon = 0.05, 0.1$, and

0.15 have often been considered in the literature. See Bai and Perron (2006) for the discussion on how to choose ϵ ; roughly speaking, the empirical sizes of the tests become closer to the nominal one for larger ϵ , and the large value of the trimming parameter is required for complicated models. As discussed in Andrews, Lee, and Ploberger (1996), the exp-type test is optimal against the alternative of the large magnitude of structural changes, whereas the avg-type test is asymptotically most powerful against the alternative of small changes.

The above three tests require the specific number of breaks m under the alternative before constructing the test statistics, but, if we do not want to prespecify the number of breaks, then we may set only the maximum number of breaks given by M and consider the following weighted exp-type and avg-type tests² suggested by Andrews, Lee, and Ploberger (1996) as well as the weighted double maximum test proposed by Bai and Perron (1998):

$$\text{Wexp-}W_T(M, \epsilon) = \sum_{m=1}^M \frac{c_{\text{exp}}(p, \alpha, 1)}{c_{\text{exp}}(p, \alpha, m)} \text{exp-}W_T(m, \epsilon),$$

$$\text{WDmax-}W_T(M, \epsilon) = \max_{1 \leq m \leq M} \frac{c_{\text{sup}}(p, \alpha, 1)}{c_{\text{sup}}(p, \alpha, m)} \text{sup-}W_T(m, \epsilon),$$

$$\text{Wavg-}W_T(M, \epsilon) = \sum_{m=1}^M \frac{c_{\text{avg}}(p, \alpha, 1)}{c_{\text{avg}}(p, \alpha, m)} \text{avg-}W_T(m, \epsilon),$$

where $c_i(p, \alpha, m)$ for $i = \text{exp}, \text{sup}, \text{and avg}$ are the critical values of eqs [2]–[4] for a given m with significance level α . These weights are suggested by Bai and Perron (1998).

The limiting distributions of these test statistics are given by the following theorem.

Theorem 1 *Assume that Assumptions A1 and A2 hold. Then, under the null hypothesis,*

² As suggested by Bai and Perron (1998), we can also consider the uniformly weighted exp-type and avg-type tests as well as the double maximum test. Since the critical values of each type of tests for a fixed q increase as m increases from Tables 1 to 3, the uniform weight implies that the marginal p -values decrease with m decreases, which then implies that the uniformly weighted tests tend to lose power for small m . In fact, our preliminary simulations show that the weighted tests proposed in the paper are more powerful than the corresponding uniformly weighted tests for $m = 1$ and 2, while these tests have almost the same power for larger m . Because of this superiority of the weighted tests, we do not give the results of the uniformly weighted tests to save space.

$$\begin{aligned}
& \exp-W_T(m, \epsilon) \xrightarrow{d} \log \int_{\Lambda_m \in \Lambda_m^\epsilon} \exp\left(\frac{1}{2} W(\Lambda_m)\right) d\Lambda_m, \\
& \sup-W_T(m, \epsilon) \xrightarrow{d} \sup_{\Lambda_m \in \Lambda_m^\epsilon} W(\Lambda_m), \quad \text{avg-}W_T(m, \epsilon) \xrightarrow{d} \int_{\Lambda_m \in \Lambda_m^\epsilon} W(\Lambda_m) d\Lambda_m, \\
& W_{\exp}-W_T(M, \epsilon) \xrightarrow{d} \sum_{m=1}^M \frac{c_{\exp}(p, \alpha, 1)}{c_{\exp}(p, \alpha, m)} \log \int_{\Lambda_m \in \Lambda_m^\epsilon} \exp\left(\frac{1}{2} W(\Lambda_m)\right) d\Lambda_m, \\
& W_{D\max}-W_T(M, \epsilon) \xrightarrow{d} \max_{1 \leq m \leq M} \frac{c_{\sup}(p, \alpha, 1)}{c_{\sup}(p, \alpha, m)} \sup_{\Lambda_m \in \Lambda_m^\epsilon} W(\Lambda_m), \\
& W_{\text{avg}}-W_T(M, \epsilon) \xrightarrow{d} \sum_{m=1}^M \frac{c_{\text{avg}}(p, \alpha, 1)}{c_{\text{avg}}(p, \alpha, m)} \int_{\Lambda_m \in \Lambda_m^\epsilon} W(\Lambda_m) d\Lambda_m,
\end{aligned}$$

where

$$W(\Lambda_m) = Q_{1,m} \tag{5}$$

$$\equiv \sum_{j=1}^m \left(\Omega_{\lambda_{j+1}}^{-1} G(\lambda_{j+1}) - \Omega_{\lambda_j}^{-1} G(\lambda_j) \right)' \left(\Omega_{\lambda_j}^{-1} - \Omega_{\lambda_{j+1}}^{-1} \right)^{-1} \left(\Omega_{\lambda_{j+1}}^{-1} G(\lambda_{j+1}) - \Omega_{\lambda_j}^{-1} G(\lambda_j) \right).$$

Remark 1 Theorem 1 shows that although the Wald test statistic for a given set of break points Λ_m is asymptotically chi-square distributed, $W(\Lambda_{m,1})$ is correlated with $W(\Lambda_{m,2})$ in a complicated way for $\Lambda_{m,1} \neq \Lambda_{m,2}$ ($\Lambda_{m,1}, \Lambda_{m,2} \in \Lambda_m^\epsilon$), and the test statistics for unknown breaks are then nonstandard.

Remark 2 When the regressors are homogeneous with $\Omega_r = r\Omega$, we have $G(r) = \Omega^{1/2}B(r)$, where $B(r)$ is a p -dimensional standard Brownian motion on $[0, 1]^p$, and it is then not difficult to show that

$$W(\Lambda_m) = Q_{2,m} \equiv \sum_{j=1}^m \frac{\| \lambda_j B(\lambda_{j+1}) - \lambda_{j+1} B(\lambda_j) \|^2}{\lambda_j \lambda_{j+1} (\lambda_{j+1} - \lambda_j)}, \tag{6}$$

which is given by Bai and Perron (1998) for the case of stationary regressors. Thus, Theorem 1 includes the existing result as a special case.

Remark 3 Another important aspect of Theorem 1 is that when obtaining the critical values by simulations, we need to invert only $p \times p$ matrices Ω_{λ_j} from the expression of $W(\Lambda_m)$, whereas the original definition of the Wald test statistic requires the inverse of the $mp \times mp$ matrix $R\hat{\Sigma}R'$. As a result, we can save the

computational time by using expression [5]. We also develop a computationally efficient method for the avg-type test in the next section.

As we can see from Theorem 1, the limiting distributions of the test statistics depend on the structure of Ω_r , and then, we need to calculate critical values for a given regressor x_t . The dependency of critical values on x_t has sometimes been observed in different situations in the literature. For example, the critical values for unit root tests depend on whether a linear trend is included as a regressor, while the LR tests for cointegrating rank are known to have different distributions depending on the structure of the deterministic term.

In the following, we focus on the case where x_t includes a linear trend, which is widely used in practical analysis. More precisely, let us consider the case where

$$x_t = [1, t, x_{1t}, \dots, x_{qt}]', \quad [7]$$

with $x_{1t}, \dots, x_{qt}$ being stationary variables and lagged dependent variables. In this case, we have the following corollary.

Corollary 1 Assume that Assumptions A1 and A2 hold. Then, under the null hypothesis with x_t given by eq. [7], Theorem 1 holds with $W(\Lambda_m) = Q_{1,m} + Q_{2,m}$, where $Q_{1,m}$ is given by eq. [5] with

$$\Omega_r^{-1} = \begin{bmatrix} \frac{4}{r} & -\frac{6}{r^2} \\ -\frac{6}{r^2} & \frac{12}{r^3} \end{bmatrix} \quad \text{and} \quad (\Omega_r^{-1} - \Omega_s^{-1})^{-1} = \begin{bmatrix} \frac{rs(r^2 + rs + s^2)}{(s-r)^3} & \frac{(rs)^2(r+s)}{2(s-r)^3} \\ \frac{(rs)^2(r+s)}{2(s-r)^3} & \frac{(rs)^3}{3(s-r)^3} \end{bmatrix}$$

for $0 < r < s \leq 1$ and $G(r) = [B_1(r), \int_0^r s dB_1(s)]'$, where $B_1(r)$ is a one-dimensional standard Brownian motion on $[0, 1]$, while $Q_{2,m}$ is given by eq. [6] with $B(r)$ being a q -dimensional standard Brownian motion independent of $B_1(r)$.

The result in Corollary 1 is similar to that given by Bai (1999) for testing the null hypothesis of ℓ breaks against the alternative of $\ell + 1$ breaks; the limiting distribution is the sum of the two independent distributions corresponding to (constant plus) a linear trend and stationary regressors. We can see that the limiting distribution of Bai's (1999) test with $\ell = 0$ is the same as ours with $m = 0$.

4 Computation of critical values

Since the limiting distributions of the test statistics are nonstandard, we obtain the critical values by simulations with $G(r)$ approximated by 1,000 partial sums

of the appropriate pseudo-random variables. For example, if $G(r)$ is a standard Brownian motion, then we approximate $G(r)$ using the normalized partial sums of *i.i.d.* $N(0, 1)$ pseudo-random variables. However, the computation of the critical values is not necessarily easy for large values of m because the number of permissible sets of breaks is proportional to T^m , so the direct calculation of all permissible Wald test statistics is computationally too expensive when $m \geq 4$. For the sup-type test, Bai and Perron (2003a) gives an efficient algorithm for the computation of the test statistics, which requires operations of order $O(T^2)$ for any given number of breaks; we can use this in our case.

For the avg-type test, we can also calculate the critical values computationally efficiently.³ Let $Q_1(T_a, T_b)$ be the summand of eq. [5] approximated by the above method with T observations given $T_a = \lambda_a T$ and $T_b = \lambda_b T$. Since the distance between two consecutive break points must be at least $h = \epsilon T$, the permissible ranges of $T_1, T_2, \dots, T_m$ are $T_1 = h, h+1, \dots, T-hm$, $T_2 = T_1 + h, T_1 + h + 1, \dots, T-h(m-1)$, $\dots$, $T_m = T_{m-1} + h, T_{m-1} + h + 1, \dots, T-h$; then, the limiting distribution of the avg-type test statistic can be approximated by

$$\frac{1}{T^*} \sum_{T_1=h}^{T-hm} \sum_{T_2=T_1+h}^{T-h(m-1)} \cdots \sum_{T_m=T_{m-1}+h}^{T-h} \sum_{j=1}^m Q_1(T_j, T_{j+1}). \quad [8]$$

However, eq. [8] requires the summation operators of order $O(T^m)$, which is computationally expensive as explained above. Instead, we calculate the limiting distributions by noting that each of $Q_1(T_j, T_{j+1})$ appears in eq. [8] many times; if we count them, we can save computational time. For example, $Q_1(T_1, T_2)$ appears as many times as the permissible number of allocations of $T_3, \dots, T_m$ in $[T_2, T]$. Since, in general, the permissible number of combinations of ℓ breaks in $[T_a, T_b]$ with two consecutive breaks' distance being larger than h is given by

$$k_h(T_a, T_b, \ell) = \frac{1}{\ell!} \prod_{i=1}^{\ell} \{(T_a - T_b + 1) - h(\ell + 1) + i\},$$

which is obtained by direct calculations, we can see that $Q_1(T_1, T_2)$ appears $k_h(T_2, T, m-2)$ times in eq. [8]. Similarly, we observe $Q_1(T_m, T_{m+1})$ as many times as the number of allocations of $T_1, \dots, T_{m-1}$ in $[1, T_m]$, which is given by $k_h(1, T_m, m-1)$. For the case of $Q_1(T_j, T_{j+1})$ for $j = 2, \dots, m-1$, there are $j-1$ and $m-j-1$ breaks allocated in $[1, T_j]$ and $[T_{j+1}, T]$, respectively. Then, we can see that

³ Pierre Perron pointed out the existence of the efficient calculation of the avg-type test through personal communications.

$$(8) = \frac{1}{k_h(1, T, m)} \sum_{j=1}^{m-1} \sum_{T_a=jh}^{T-h(m-j+1)} \sum_{T_b=T_a+h}^{T-h(m-j)} k_h(1, T_a, j-1) Q_1(T_a, T_b, j) k_h(T_b, T, m-j-1) \\ + \sum_{T_c=hm}^{T-h} k_h(1, T_c, m-1) Q_1(T_c, T, m), \tag{9}$$

where $k_h(T_a, T_b, 0) = 1$ for convention. We can see that the number of summation operators on the right-hand side of eq. [9] is proportional to $O(T^2)$ for any given number of m .

On the other hand, we cannot find an efficient computational method for the exp-type test. Therefore, we consider the exp-type test only up to $m = 3$.

The critical values in the case of a linear trend are given in Tables 1–3 for $\epsilon = 0.05, 0.10$, and 0.15 and $q = 0$ to 9 , where q is the number of homogeneous regressors. They are obtained by approximating Brownian motions by 1,000 partial sums of *i.i.d.N*(0, 1) pseudo-random variables with 10,000 replications. Because of the above reason, the critical values for the exp-type test are given for only up to $m = 3$ and $M = 3$, whereas those for the sup-type and avg-type

Table 1: Asymptotic critical values of the exp-type test with level α

											q
m	α	0	1	2	3	4	5	6	7	8	9
exp- $W_T(m, 0.05)$ ($\epsilon = 0.05$)											
1	0.10	2.63	3.60	4.49	5.38	6.18	6.98	7.86	8.59	9.36	10.14
	0.05	3.26	4.32	5.28	6.21	7.10	7.99	8.81	9.64	10.42	11.17
	0.01	4.80	5.83	7.10	8.00	9.00	9.93	10.79	11.70	12.61	13.43
2	0.10	4.55	6.32	7.97	9.61	11.16	12.70	14.23	15.71	17.20	18.61
	0.05	5.35	7.18	8.97	10.66	12.26	13.88	15.43	16.99	18.51	19.96
	0.01	7.20	9.28	10.98	12.93	14.77	16.42	18.12	19.66	21.35	22.97
3	0.10	6.29	8.80	11.18	13.49	15.75	17.98	20.20	22.41	24.57	26.60
	0.05	7.21	9.82	12.37	14.79	17.12	19.38	21.62	23.90	26.08	28.27
	0.01	9.37	12.11	14.65	17.28	19.69	22.40	24.67	27.10	29.29	31.68
Wexp- $W_T(3, 0.05)$ ($\epsilon = 0.05, M = 3$)											
	0.10	7.81	10.65	13.33	15.99	18.30	20.77	23.30	25.51	27.84	30.25
	0.05	9.66	12.79	15.60	18.29	21.02	23.59	26.20	28.55	30.78	33.08
	0.01	14.14	17.35	20.81	23.75	26.46	29.21	31.79	34.42	37.02	39.87
exp- $W_T(m, 0.10)$ ($\epsilon = 0.10$)											
1	0.10	2.59	3.56	4.45	5.33	6.11	6.90	7.74	8.48	9.24	10.00
	0.05	3.25	4.27	5.25	6.16	6.98	7.92	8.72	9.52	10.32	11.08
	0.01	4.82	5.90	7.04	8.00	8.98	9.82	10.67	11.60	12.50	13.34
	0.10	4.43	6.18	7.80	9.38	10.84	12.35	13.89	15.32	16.75	18.14

(continued)

Table 1: (Continued)

											<i>q</i>
<i>m</i>	α	0	1	2	3	4	5	6	7	8	9
2	0.05	5.28	7.06	8.80	10.41	11.97	13.54	15.14	16.70	18.13	19.57
	0.01	7.10	9.20	10.89	12.65	14.40	16.27	17.77	19.32	20.85	22.56
	0.10	6.07	8.56	10.84	13.05	15.21	17.39	19.56	21.65	23.70	25.75
3	0.05	7.06	9.60	12.02	14.36	16.59	18.83	20.98	23.24	25.40	27.41
	0.01	9.15	12.00	14.31	16.80	19.30	21.78	23.97	26.20	28.52	30.91
Wexp- $W_T(3, 0.10)$ ($\epsilon = 0.10, M = 3$)											
	0.10	7.70	10.55	13.17	15.81	18.10	20.48	22.97	25.13	27.44	29.78
	0.05	9.63	12.63	15.44	18.11	20.61	23.36	25.79	28.20	30.56	32.86
	0.01	14.18	17.25	20.69	23.48	26.26	28.81	31.47	34.19	36.78	39.25
exp- $W_T(m, 0.15)$ ($\epsilon = 0.15$)											
	0.10	2.56	3.51	4.42	5.25	6.05	6.83	7.61	8.40	9.13	9.86
1	0.05	3.21	4.25	5.20	6.10	6.94	7.84	8.60	9.38	10.23	10.99
	0.01	4.84	5.90	6.92	7.93	8.97	9.79	10.63	11.56	12.34	13.26
	0.10	4.33	6.06	7.68	9.19	10.62	12.12	13.62	14.93	16.37	17.77
2	0.05	5.21	6.99	8.60	10.18	11.75	13.31	14.83	16.28	17.84	19.24
	0.01	7.07	9.14	10.67	12.59	14.07	15.87	17.50	19.08	20.54	22.05
	0.10	5.92	8.34	10.47	12.63	14.77	16.89	18.96	20.96	22.97	24.99
3	0.05	6.87	9.35	11.69	13.90	16.12	18.37	20.45	22.57	24.59	26.63
	0.01	8.96	11.78	14.17	16.57	18.91	21.25	23.38	25.65	27.97	30.07
Wexp- $W_T(3, 0.15)$ ($\epsilon = 0.15, M = 3$)											
	0.10	7.56	10.34	13.09	15.56	17.87	20.22	22.52	24.86	27.00	29.22
	0.05	9.50	12.60	15.28	17.91	20.41	23.04	25.44	27.78	30.34	32.57
	0.01	14.18	17.27	20.33	23.19	26.35	28.65	31.13	33.85	36.18	38.80

Table 2: Asymptotic critical values of the sup-type test with level α

											<i>q</i>
<i>m</i>	α	0	1	2	3	4	5	6	7	8	9
sup- $W_T(m, 0.05)$ ($\epsilon = 0.05$)											
	0.10	12.10	14.25	16.10	17.94	19.79	21.60	23.21	24.86	26.60	28.16
1	0.05	13.70	16.03	17.95	20.00	21.84	23.69	25.48	27.18	28.88	30.56
	0.01	17.50	20.34	22.48	24.58	26.38	28.19	30.44	32.28	33.74	35.65
	0.10	22.41	26.62	30.15	33.64	36.92	40.39	43.62	46.80	49.82	52.85
2	0.05	24.78	28.87	32.69	36.18	39.57	43.07	46.31	49.66	52.68	55.69
	0.01	28.96	33.67	38.03	41.47	45.18	48.70	52.20	55.87	59.03	62.18
	0.10	31.02	36.92	42.31	47.35	52.25	57.16	61.86	66.62	71.06	75.49

(continued)

Table 2: (Continued)

											q
m	α	0	1	2	3	4	5	6	7	8	9
3	0.05	33.62	39.65	45.20	50.35	55.36	60.21	65.27	69.79	74.57	79.06
	0.01	38.72	45.22	50.93	56.43	61.67	66.86	71.76	77.15	81.92	86.45
	0.10	38.89	46.48	53.58	60.33	66.66	72.94	79.16	85.22	91.34	97.06
4	0.05	41.57	49.40	56.67	63.51	69.96	76.38	82.90	89.11	94.90	100.96
	0.01	47.60	56.17	63.02	70.45	77.15	83.76	90.96	97.12	103.79	109.33
	0.10	45.90	55.12	63.92	72.13	79.96	87.85	95.34	102.75	110.20	117.24
5	0.05	48.72	58.51	67.47	75.77	83.77	91.71	99.60	107.03	114.51	121.77
	0.01	55.08	65.82	74.31	83.09	91.16	99.74	108.76	116.48	124.60	131.19
WDmax- $W_T(3, 0.05)$ ($\epsilon = 0.05, M = 3$)											
	0.10	13.13	15.31	17.24	19.16	21.00	22.92	24.57	26.25	28.01	29.60
	0.05	14.68	17.19	19.19	21.29	23.20	25.04	26.87	28.61	30.41	31.92
	0.01	18.77	21.64	23.80	25.99	27.78	29.76	31.96	33.98	35.30	37.40
WDmax- $W_T(5, 0.05)$ ($\epsilon = 0.05, M = 5$)											
	0.10	13.26	15.48	17.43	19.36	21.22	23.09	24.78	26.51	28.21	29.82
	0.05	14.93	17.38	19.35	21.47	23.40	25.24	27.04	28.87	30.66	32.33
	0.01	19.04	21.83	24.01	26.17	28.08	30.04	32.19	34.14	35.53	37.52
sup- $W_T(m, 0.10)$ ($\epsilon = 0.10$)											
	0.10	11.54	13.54	15.52	17.41	19.02	20.71	22.50	24.00	25.61	27.31
1	0.05	13.32	15.51	17.43	19.31	21.10	22.97	24.73	26.51	28.11	29.55
	0.01	16.96	19.41	21.61	23.80	25.75	27.55	29.27	31.18	32.96	34.62
	0.10	20.62	24.33	27.86	31.36	34.61	37.90	41.25	44.25	47.32	50.30
2	0.05	22.66	26.67	30.48	34.00	37.29	40.68	44.05	47.26	50.45	53.37
	0.01	27.30	31.99	35.54	39.27	42.85	46.62	49.79	53.20	56.51	59.75
	0.10	27.70	33.21	38.46	43.38	48.28	53.09	57.60	62.13	66.51	70.83
3	0.05	30.07	36.10	41.48	46.63	51.36	56.22	60.75	65.57	70.15	74.54
	0.01	35.16	41.76	47.25	52.22	57.53	62.85	67.49	72.46	77.27	82.54
	0.10	33.70	40.75	47.42	53.92	60.04	66.34	72.18	78.18	83.90	89.32
4	0.05	36.35	43.93	50.79	57.30	63.64	69.94	76.21	82.23	88.04	93.74
	0.01	42.27	50.28	57.46	64.29	70.72	77.03	83.19	89.61	96.12	101.48
	0.10	38.69	47.19	55.36	63.19	70.58	78.13	85.22	92.66	99.46	106.41
5	0.05	41.64	50.56	58.80	66.86	74.55	82.08	89.39	96.87	104.25	111.19
	0.01	47.39	57.01	66.05	74.41	82.18	90.19	97.62	105.40	113.23	120.14
WDmax- $W_T(3, 0.10)$ ($\epsilon = 0.10, M = 3$)											
	0.10	12.53	14.71	16.72	18.63	20.22	22.01	23.76	25.33	27.08	28.73
	0.05	14.47	16.67	18.68	20.50	22.42	24.31	26.17	27.99	29.52	31.14
	0.01	18.19	20.71	22.78	25.22	27.27	29.07	30.80	32.68	34.62	36.45
WDmax- $W_T(5, 0.10)$ ($\epsilon = 0.10, M = 5$)											
	0.10	12.75	14.95	16.92	18.85	20.51	22.25	24.11	25.63	27.35	29.02
	0.05	14.68	16.93	18.93	20.75	22.75	24.69	26.45	28.26	29.79	31.35
	0.01	18.48	20.87	23.09	25.54	27.55	29.34	31.14	33.10	34.87	36.65

(continued)

Table 2: (Continued)

											q
m	α	0	1	2	3	4	5	6	7	8	9
sup- $W_T(m, 0.15)$ ($\epsilon = 0.15$)											
1	0.10	11.07	13.03	14.97	16.81	18.48	20.16	21.83	23.34	24.92	26.65
	0.05	12.80	14.95	16.91	18.70	20.42	22.35	23.95	25.75	27.51	29.01
	0.01	16.54	18.87	20.81	23.03	25.07	26.88	28.70	30.65	32.28	33.97
2	0.10	18.92	22.71	26.21	29.58	32.78	35.97	39.10	42.04	45.16	47.93
	0.05	21.05	25.03	28.73	32.02	35.33	38.73	42.05	45.15	48.25	51.17
	0.01	25.62	30.32	33.59	37.55	40.88	44.58	47.79	50.84	54.05	57.56
3	0.10	24.66	30.05	34.93	39.72	44.36	49.06	53.52	57.83	62.22	66.33
	0.05	27.02	32.68	37.98	42.84	47.61	52.63	56.94	61.33	65.77	70.13
	0.01	32.29	38.60	43.66	49.08	53.92	59.02	63.64	68.15	73.02	77.19
4	0.10	28.50	35.30	41.37	47.56	53.46	59.31	64.99	70.59	76.16	81.62
	0.05	31.28	38.23	44.75	50.91	56.70	63.22	68.78	74.44	80.19	85.77
	0.01	36.62	44.41	51.11	57.56	64.34	70.51	76.35	82.29	88.67	94.30
5	0.10	28.82	36.30	43.61	50.63	57.41	64.02	70.49	77.03	83.70	89.74
	0.05	31.39	39.52	46.95	54.35	61.10	67.97	74.56	81.42	88.16	94.58
	0.01	36.78	45.68	53.60	61.06	68.40	76.30	82.94	89.34	96.70	103.25
WDmax- $W_T(3, 0.15)$ ($\epsilon = 0.15, M = 3$)											
	0.10	12.12	14.23	16.20	18.07	19.74	21.55	23.19	24.75	26.41	28.18
	0.05	13.85	16.17	18.16	19.98	21.75	23.58	25.41	27.21	28.93	30.55
	0.01	17.74	20.23	22.30	24.11	26.50	28.22	30.29	32.08	33.87	35.82
WDmax- $W_T(5, 0.15)$ ($\epsilon = 0.15, M = 5$)											
	0.10	12.39	14.52	16.51	18.40	20.11	21.93	23.61	25.24	26.87	28.64
	0.05	14.27	16.50	18.52	20.36	22.22	24.10	25.85	27.66	29.41	31.04
	0.01	18.12	20.83	22.62	24.71	26.95	28.65	30.67	32.81	34.45	36.48

Table 3: Asymptotic critical values of the avg-type test with level α

												q
m	α	0	1	2	3	4	5	6	7	8	9	
avg- $W_T(m, 0.05)$ ($\epsilon = 0.05$)												
1	0.10	3.51	4.86	6.24	7.44	8.72	9.93	11.10	12.34	13.52	14.63	
	0.05	4.32	5.77	7.08	8.49	9.84	11.17	12.38	13.63	14.84	16.04	
	0.01	6.32	7.80	9.20	10.51	12.02	13.44	14.90	16.33	17.56	18.76	
2	0.10	6.36	8.94	11.39	13.82	16.16	18.52	20.82	23.10	25.44	27.63	
	0.05	7.49	10.19	12.78	15.25	17.71	20.21	22.66	25.01	27.42	29.70	
	0.01	10.15	12.97	15.63	18.22	20.95	23.64	26.21	28.82	30.97	33.48	
	0.10	9.00	12.78	16.35	19.89	23.27	26.71	30.12	33.48	36.85	40.19	

(continued)

Table 3: (Continued)

											q
m	α	0	1	2	3	4	5	6	7	8	9
3	0.05	10.35	14.29	18.02	21.60	25.11	28.79	32.26	35.88	39.31	42.80
	0.01	13.46	17.55	21.24	25.13	29.07	32.87	36.41	40.29	43.91	47.48
	0.10	11.50	16.42	21.15	25.78	30.27	34.84	39.25	43.71	48.19	52.57
4	0.05	13.14	18.26	23.10	27.82	32.39	37.13	41.67	46.35	50.95	55.40
	0.01	16.67	21.70	26.69	31.82	36.73	41.87	46.28	51.53	56.34	60.94
	0.10	14.09	20.11	25.91	31.58	37.18	42.86	48.27	53.86	59.46	64.87
5	0.05	15.80	22.11	28.07	33.89	39.64	45.44	50.92	56.71	62.32	68.00
	0.01	19.70	25.90	32.09	38.29	44.45	50.54	56.45	62.38	68.48	74.21
Wavg- $W_T(3, 0.05)$ ($\epsilon = 0.05, M = 3$)											
	0.10	10.48	14.54	18.59	22.20	26.00	29.70	33.16	36.81	40.38	43.77
	0.05	12.94	17.24	21.10	25.29	29.42	33.36	37.08	40.76	44.34	47.90
	0.01	18.66	23.18	27.59	31.42	35.85	40.21	44.85	48.85	52.16	55.89
Wavg- $W_T(5, 0.05)$ ($\epsilon = 0.05, M = 5$)											
	0.10	17.35	24.19	30.91	36.92	43.27	49.33	55.16	61.24	67.18	72.71
	0.05	21.24	28.53	35.01	41.96	48.70	55.31	61.60	67.57	73.90	79.67
	0.01	30.94	38.59	45.79	51.96	59.72	66.38	73.57	80.71	86.74	92.61
avg- $W_T(m, 0.10)$ ($\epsilon = 0.10$)											
	0.10	3.58	4.98	6.40	7.60	8.88	10.11	11.34	12.52	13.77	14.88
1	0.05	4.45	5.97	7.32	8.70	10.10	11.48	12.68	13.92	15.22	16.39
	0.01	6.64	8.15	9.49	10.93	12.36	13.91	15.42	16.79	18.06	19.35
	0.10	6.50	9.19	11.67	14.12	16.48	18.86	21.19	23.47	25.85	28.09
2	0.05	7.74	10.47	13.09	15.63	18.07	20.68	23.08	25.48	27.95	30.19
	0.01	10.67	13.39	16.19	18.76	21.45	24.40	26.89	29.43	31.82	34.33
	0.10	9.21	13.09	16.71	20.33	23.69	27.22	30.65	33.93	37.46	40.84
3	0.05	10.67	14.68	18.48	22.14	25.71	29.39	32.93	36.64	40.03	43.39
	0.01	14.16	18.28	21.96	25.98	30.04	33.72	37.40	41.17	44.99	48.72
	0.10	11.82	16.91	21.58	26.24	30.88	35.52	39.87	44.37	48.96	53.33
4	0.05	13.64	18.78	23.77	28.50	33.18	38.04	42.60	47.43	52.02	56.42
	0.01	17.28	22.72	27.69	32.90	37.79	42.84	47.76	52.73	57.84	62.53
	0.10	14.47	20.66	26.48	32.14	37.88	43.78	49.12	54.65	60.44	65.82
5	0.05	16.35	22.84	28.87	34.83	40.53	46.48	52.09	57.96	63.60	69.29
	0.01	20.44	27.22	33.37	39.73	45.63	51.88	57.75	64.26	70.31	76.11
Wavg- $W_T(3, 0.10)$ ($\epsilon = 0.10, M = 3$)											
	0.10	10.66	14.87	19.01	22.65	26.52	30.19	33.89	37.39	41.10	44.44
	0.05	13.22	17.72	21.75	25.88	30.09	34.25	37.88	41.48	45.42	48.90
	0.01	19.31	24.06	28.36	32.32	36.83	41.53	45.73	50.08	53.43	57.61
Wavg- $W_T(5, 0.10)$ ($\epsilon = 0.10, M = 5$)											
	0.10	17.68	24.64	31.56	37.61	43.95	49.98	56.21	62.08	68.22	73.94
	0.05	21.81	29.27	35.96	42.96	49.75	56.61	62.74	69.00	75.49	81.26
	0.01	32.36	39.74	46.82	53.46	61.04	68.48	76.05	82.83	88.92	95.59

(continued)

Table 3: (Continued)

											q
m	α	0	1	2	3	4	5	6	7	8	9
avg- $W_T(m, 0.15)$ ($\epsilon = 0.15$)											
1	0.10	3.66	5.10	6.53	7.78	9.07	10.31	11.59	12.70	13.99	15.12
	0.05	4.61	6.13	7.54	8.96	10.35	11.72	12.94	14.24	15.50	16.74
	0.01	6.89	8.52	9.87	11.37	12.87	14.30	15.92	17.30	18.60	19.99
2	0.10	6.65	9.36	11.94	14.35	16.78	19.18	21.58	23.85	26.17	28.49
	0.05	7.92	10.74	13.45	15.95	18.45	21.01	23.55	25.95	28.42	30.72
	0.01	10.99	13.80	16.54	19.43	22.14	24.85	27.29	30.07	32.60	35.17
3	0.10	9.41	13.40	17.10	20.66	24.12	27.72	31.09	34.55	37.98	41.44
	0.05	10.99	15.16	18.98	22.55	26.30	30.00	33.66	37.27	40.88	44.22
	0.01	14.63	18.78	22.70	26.87	30.78	34.52	38.21	42.30	46.21	49.68
4	0.10	12.16	17.37	22.11	26.85	31.49	36.15	40.68	45.28	49.82	54.19
	0.05	14.02	19.32	24.36	29.14	33.96	38.86	43.48	48.28	52.87	57.59
	0.01	17.79	23.53	28.92	34.27	39.16	44.09	49.14	54.31	59.61	64.31
5	0.10	15.09	21.38	27.47	33.43	39.16	44.86	50.55	56.14	61.93	67.48
	0.05	17.01	23.80	30.13	36.22	42.17	48.22	54.08	60.00	65.63	71.42
	0.01	21.31	28.98	35.54	41.93	48.24	54.26	60.74	67.02	73.35	79.35
Wavg- $W_T(3, 0.15)$ ($\epsilon = 0.15, M = 3$)											
	0.10	10.89	15.18	19.42	23.12	27.02	30.67	34.56	37.90	41.69	45.04
	0.05	13.62	18.14	22.31	26.61	30.76	34.98	38.49	42.36	46.14	49.76
	0.01	20.07	24.95	29.01	33.45	38.20	42.72	47.18	51.23	54.94	59.15
Wavg- $W_T(5, 0.15)$ ($\epsilon = 0.15, M = 5$)											
	0.10	17.91	25.01	32.01	38.30	44.64	50.68	57.04	62.46	69.05	74.68
	0.05	22.39	29.83	36.81	43.74	50.63	57.41	63.58	69.96	76.41	82.33
	0.01	33.59	41.02	47.67	55.21	62.69	70.18	77.68	84.14	90.41	97.53

tests are obtained for up to $m = 5$ and $M = 3$ and 5. As in the case of homogeneous regressors, the critical values get larger as q and/or m increase.

5 Extension to more general models

5.1 Partial structural changes

In the previous sections, we have proposed the null hypothesis of no structural change against the alternative of pure structural change model [1] in which all the coefficients sustain structural changes. However, in practical analysis, we sometimes assume partial structural change model such that only a part of the coefficients changes and the rest is fixed. This model can be expressed as

$$y_t = x_t' \beta_j + z_t' \gamma + \varepsilon_t \quad (j = 1, \dots, m+1 \quad \text{and} \quad t = T_{j-1} + 1, \dots, T_j), \quad [10]$$

where z_t is p_z -dimensional regressors whose coefficients are stable throughout the sample. Although we may implement the tests proposed in the previous section assuming that all the coefficients sustain structural changes, we should impose the constraint of a fixed γ to increase the efficiency of the estimated regression and hence increase power.

If z_t is a stationary variable and $T^{-1/2} D_T^{-1} \sum_{t=1}^{[Tr]} x_t (z_t - E[z_t])' = o_p(1)$, then we can show that Theorem 1 holds and the test statistics proposed in the previous section are still valid. The reason is that the off-block diagonal elements of the cross-products of regressors suitably normalized are asymptotically negligible, so that the variance-covariance matrix of the estimators are asymptotically block diagonal. For example, if $x_t = [1, t]'$ and z_t is a mean-zero stationary variable,⁴ then we can see that

$$\frac{1}{\sqrt{T}} D_T^{-1} \sum_{t=T_{j-1}+1}^{T_j} x_t z_t = \begin{bmatrix} \frac{1}{T} \sum_{t=T_{j-1}+1}^{T_j} z_t \\ \frac{1}{T^2} \sum_{t=T_{j-1}+1}^{T_j} t z_t \end{bmatrix} = O_p\left(\frac{1}{\sqrt{T}}\right).$$

As a result, we can see that the limiting distribution of $D_T(\hat{\beta}_j - \beta_j)$ is the same as given in the proof of Theorem 1.

However, if z_t is a general non-homogeneous regressor, we cannot proceed with the proof of Theorem 1 because, as is seen in the appendix, the proof of Theorem 1 uses the fact that $\hat{\beta}_j$ is asymptotically independent of $\hat{\beta}_k$ for $j \neq k$; in other words, the variance-covariance matrix of the estimators is asymptotically block diagonal as given by $\hat{\Sigma}$ in the definition of $W_T(\Lambda_m)$. When z_t is non-homogeneous, it may be correlated with x_t in general and hence $\hat{\beta}_j$ for $j = 1, \dots, m+1$ are asymptotically correlated. In this case, we cannot apply the tests proposed in the previous section. Note that in the special case where z_t consists of polynomial trends, whereas x_t is a stationary variable, z_t is uncorrelated with x_t . However, it can be shown that the structure of the variance matrix of $D_T(\hat{\beta}_j - \beta_j)$ is different from that of the pure structural change estimator. Again, in this case, we have to obtain new critical values if we efficiently estimate [10] with a fixed γ .

We finally note that we should use partial structural change model [10] only when we have a strong confidence that the coefficient imposed to be fixed, γ , is actually constant over the sample period. This is because the tests will suffer from over-size distortion if the coefficient associated with z_t sustains structural changes but we impose a constraint in the estimation. Intuitively, this is because

⁴ We assume $E[z_t] = 0$ without loss of generality because a constant is included as a regressor.

the wrong imposition of a fixed parameter of γ leads to spurious structural changes in β even if β does not change over the sample period. Because of this spurious structural changes, the tests tend to reject the null hypothesis.

5.2 Heteroskedasticity and serially dependent errors

In the previous sections, $\{\varepsilon_t\}$ is assumed to be a martingale difference sequence with constant variance by Assumption A2 so that neither heteroskedasticity nor serial correlation is allowed. In this section, we relax this assumption and discuss how to modify the base test statistic $W_T(\Lambda_m)$ following Bai and Perron (2006).

We first relax the assumption of constant variance to regime-wise heteroskedasticity. In this case, $\hat{\sigma}^2 \hat{\Sigma}$ in $W_T(\Lambda_m)$ should be replaced by

$$\hat{\Sigma}_h = \text{diag}\{\hat{\sigma}_1^2 \hat{\Sigma}_1, \hat{\sigma}_2^2 \hat{\Sigma}_2, \dots, \hat{\sigma}_{m+1}^2 \hat{\Sigma}_{m+1}\} \quad \text{where} \quad \hat{\sigma}_i^2 = \frac{1}{T_j - T_{j-1}} \sum_{t=T_{j-1}+1}^{T_j} \hat{\varepsilon}_t^2. \quad [11]$$

Then, we can see that both Theorem 1 and Corollary 1 hold.

The case with serially dependent errors is more complicated. Suppose the special case where regressors consist of polynomial trends and stationary variables such as $x_t = [x'_{1,t}, x'_{2,t}]'$ where $x_{1,t} = [1, t, t^2, \dots, t^d]'$ and $x_{2,t}$ is a q -dimensional stationary vector with $x_{2,t}$ being uncorrelated with ε_s for all s and t . We assume that $x_{2,t}$ is a mean zero process without loss of generality because a constant term is included as a regressor. In this case, $D_{1,T}^{-1} \sum_{t=T_{j-1}+1}^{T_j} x_{1,t} \varepsilon_t$ with $D_{1,T} = \text{diag}\{T^{1/2}, T^{3/2}, \dots, T^{d+1/2}\}$ is asymptotically independent of $T^{-1/2} \sum_{t=T_{j-1}+1}^{T_j} x_{2,t} \varepsilon_t$. Then, following Bai and Perron (2006, 218), we should replace the estimator of the variance-covariance matrix $\hat{\sigma}^2 \hat{\Sigma}$ by $\hat{\Sigma}_s = \text{diag}\{\hat{\Sigma}_{s,1}, \hat{\Sigma}_{s,2}, \dots, \hat{\Sigma}_{s,m+1}\}$ where $\hat{\Sigma}_{s,i} = \text{diag}\{\hat{\Sigma}_{s,i}^n, \hat{\Sigma}_{s,i}^s\}$ with

$$\hat{\Sigma}_{s,i}^n = \hat{\sigma}_{lr}^2 \left(\sum_{t=T_{j-1}+1}^{T_j} x_{1,t} x'_{1,t} \right)^{-1}$$

where $\hat{\sigma}_{lr}^2$ is the estimator of the long-run variance of ε_t , and

$$\hat{\Sigma}_{s,i}^s = \left(\sum_{t=T_{j-1}+1}^{T_j} x_{2,t} x'_{2,t} \right)^{-1} \hat{W}_j \left(\sum_{t=T_{j-1}+1}^{T_j} x_{2,t} x'_{2,t} \right)^{-1}$$

with $\hat{W}_j$ being the consistent estimator of

$$\lim_{T \rightarrow \infty} \frac{1}{T_j - T_{j-1}} \sum_{s=T_{j-1}+1}^{T_j} \sum_{t=T_{j-1}+1}^{T_j} E \left[x_{2,s} x'_{2,t} \varepsilon_s \varepsilon_t \right], \quad [12]$$

which is obtained by the HAC methods proposed by, for example, Andrews (1991). In this case, we can use critical values in Tables 1–3 when $d = 1$.

However, in a general case where x_t includes general non-homogeneous regressors, the test statistics based on the Wald test statistics taking serial correlation into account, denoted by $\tilde{W}_T(\Lambda_m)$, may not have the same limiting distributions as those based on $W_T(\Lambda_m)$ under the assumption of no serial correlation. Note that although both $W_T(\Lambda_m)$ and $\tilde{W}_T(\Lambda_m)$ have the same limiting chi-square distribution, the correlation between $\tilde{W}_T(\Lambda_m^1)$ and $\tilde{W}_T(\Lambda_m^2)$ for $\Lambda_m^1 \neq \Lambda_m^2$ may be different from that between $W_T(\Lambda_m^1)$ and $W_T(\Lambda_m^2)$, because the limiting variance of $D_T^{-1} \sum_{t=1}^{[Tr]} x_t \varepsilon_t$ may depend on r possibly in a different manner according to whether or not ε_t is serially correlated. Therefore, taking the supremum or weighted averages over the permissible break fractions Λ_m^ϵ results in different distributions. In this case, we have to calculate critical values according to the new limiting variance.

6 Finite sample properties

In this section, we investigate the finite sample properties of the proposed tests via Monte Carlo simulations. We consider two cases where $q = 0$ (DGPO) and $q = 1$ (DGP1). In the case of $q = 0$, the data-generating process under the null hypothesis is given by

$$y_t = \beta_1 + \beta_2 t + \varepsilon_t \quad [13]$$

for $t = 1, \dots, T$ with $\varepsilon_t \sim i.i.d.N(0, 1)$. We set $\beta_1 = \beta_2 = 0$ because all the test statistics are invariant to the true values of β_1 and β_2 under H_0 . On the other hand, the DGP1 has an autoregressive (AR) regressor as follows:

$$y_t = \beta_1 + \beta_2 t + \beta_3 x_t + \varepsilon_t, \quad x_t = \phi x_{t-1} + \varepsilon_t \quad [14]$$

for $t = 1, \dots, T$, where, again, we set $\beta_1 = \beta_2 = \beta_3 = 0$ without loss of generality. The initial value of x_t is set to $x_0 = 0$ while $\phi = -0.8, -0.4, 0, 0.4$, and 0.8 . The sample size T is 120 and 300 and the significance level is set to 0.05. We investigate the case where the maximum number of breaks is three, so that we construct the weighted-type tests with $M = 3$. The trimming parameter ϵ is set to 0.1, and all computations are conducted using the GAUSS matrix language with 2,000 replications.

Table 4 shows the empirical sizes of the tests. For comparison, we also consider the LR test for the null hypothesis of no break against the alternative of a one-time break proposed by Bai (1999). For DGPO with $T = 120$, the empirical sizes of all the tests are close to the nominal one except for the exp-type test with $m = 3$, which tends to over-reject the null hypothesis slightly, while the LR test

Table 4: Empirical sizes of the tests

T ϕ	DGPO								DGP1			
	120		300		120		300		120		300	
	–	–	–0.8	–0.4	0.0	0.4	0.8	–0.8	–0.4	0.0	0.4	0.8
exp(1)	0.058	0.049	0.064	0.062	0.059	0.058	0.060	0.048	0.048	0.050	0.043	0.050
exp(2)	0.064	0.056	0.085	0.080	0.079	0.076	0.077	0.053	0.062	0.060	0.057	0.063
exp(3)	0.071	0.057	0.096	0.095	0.089	0.094	0.092	0.058	0.069	0.064	0.059	0.066
Wexp	0.066	0.053	0.076	0.072	0.071	0.073	0.076	0.051	0.057	0.060	0.053	0.059
sup(1)	0.041	0.044	0.046	0.043	0.041	0.038	0.041	0.035	0.040	0.036	0.035	0.040
sup(2)	0.043	0.047	0.057	0.053	0.050	0.049	0.050	0.041	0.053	0.043	0.043	0.048
sup(3)	0.053	0.052	0.065	0.062	0.058	0.064	0.067	0.044	0.050	0.050	0.049	0.054
WDmax	0.045	0.042	0.056	0.056	0.050	0.055	0.053	0.041	0.045	0.044	0.041	0.050
avg(1)	0.047	0.051	0.055	0.052	0.049	0.051	0.052	0.041	0.043	0.048	0.044	0.047
avg(2)	0.052	0.055	0.058	0.057	0.059	0.053	0.053	0.048	0.052	0.054	0.050	0.057
avg(3)	0.054	0.053	0.060	0.054	0.059	0.052	0.053	0.049	0.055	0.055	0.048	0.056
Wavg	0.051	0.054	0.056	0.056	0.059	0.051	0.051	0.047	0.049	0.053	0.047	0.053
LR	0.037	0.041	0.041	0.040	0.039	0.036	0.037	0.035	0.038	0.034	0.034	0.039

is conservative. However, the size distortions of the tests disappear when the sample size is 300.

On the other hand, when the stationary variable is included as a regressor, almost all the tests reject the null hypothesis more frequently compared to DGPO. As a result, we observe the tendency of the over-rejection for the exponential type test with $m = 2$ and $m = 3$; however, again, this tendency disappears when $T = 300$. As a whole, the empirical sizes of all the tests are not greatly affected by the AR parameter of the stationary regressor as long as ϕ ranges from -0.8 to 0.8 ; its effect seems marginal.

To see the finite sample powers of the tests, we first consider the following DGP with a one-time break for $q = 0$ and $q = 1$, respectively:

$$\begin{aligned} \text{DGPO} \quad y_t &= \begin{cases} \beta_{1,1} + \beta_{2,1}t + \varepsilon_t & : \quad t = 1, \dots, T_1, \\ \beta_{1,2} + \beta_{2,1}t + \beta_{2,2}(t - T_1) + \varepsilon_t & : \quad t = T_1 + 1, \dots, T, \end{cases} \\ \text{DGP1} \quad y_t &= \begin{cases} \beta_{1,1} + \beta_{2,1}t + \beta_{3,1}x_t + \varepsilon_t & : \quad t = 1, \dots, T_1, \\ \beta_{1,2} + \beta_{2,1}t + \beta_{2,2}(t - T_1) + \beta_{3,2}x_t + \varepsilon_t & : \quad t = T_1 + 1, \dots, T, \end{cases} \end{aligned} \tag{15}$$

where $T_1 = 0.5T$, $\beta_{1,1} = \beta_{2,1} = \beta_{3,1} = 0$, while $\beta_{1,2} = \beta_{3,2} = 5\gamma$ and $\beta_{2,2} = \gamma$ with γ taking positive values, which are chosen so that the overall shapes of the power functions can be observed.

Figure 1 shows the size-adjusted powers of the tests when $q = 0$ (DGP0). As expected, the test against the correct number of breaks ($m = 1$) is more powerful than the others among the same type of tests. For example, we observe from (i-a) and (i-b) that the exp-type test with $m = 1$ is most powerful, followed by the tests with $m = 2$ and $m = 3$. The effect of the over-specification of the number of breaks is relatively large for the sup-type tests, whereas the avg-type tests are less affected by m . We can also see that the weighted-type tests are the second-best tests compared to tests with a fixed number of breaks. Figure 1(iv-a) and (iv-b) compare the three weighted-type tests and the LR test by Bai (1999). We observe that the weighted avg-type test is most powerful, and the second-best is the weighted exp-type test. The weighted double maximum test and the LR test are inferior to the others in this case, and the former is slightly less powerful than the latter.

Figure 2 corresponds to the case where $q = 1$ (DGP1) and the AR coefficient of x_t is 0. We observe that the relative performance is preserved compared to Figure 1. Regarding the effect of the persistence of x_t , the tests are most powerful when x_t is an *i.i.d.* sequence ($\phi = 0$) and the powers decrease as the absolute values of ϕ get larger. However, the difference between the powers is not substantial, and the effect of the persistence of the regressor is slight for $-0.8 \leq \phi \leq 0.8$ (we do not show the figure to save space).

We next investigate the case where the number of breaks is two. The DGP in this case is given by

$$\begin{aligned} \text{DGP0} \quad y_t &= \begin{cases} \beta_{1,1} + \beta_{2,1}t + \varepsilon_t & : \quad t = 1, \dots, T_1, \\ \beta_{1,2} + \beta_{2,1}t + \beta_{2,2}(t - T_1) + \varepsilon_t & : \quad t = T_1 + 1, \dots, T_2, \\ \beta_{1,3} + \beta_{2,1}t + \beta_{2,2}(t - T_1) + \beta_{2,3}(t - T_2) + \varepsilon_t & : \quad t = T_2 + 1, \dots, T, \end{cases} \\ \text{DGP1} \quad y_t &= \begin{cases} \beta_{1,1} + \beta_{2,1}t + \beta_{3,1}x_t + \varepsilon_t & : \quad t = 1, \dots, T_1, \\ \beta_{1,2} + \beta_{2,1}t + \beta_{2,2}(t - T_1) + \beta_{3,2}x_t + \varepsilon_t & : \quad t = T_1 + 1, \dots, T_2, \\ \beta_{1,3} + \beta_{2,1}t + \beta_{2,2}(t - T_1) + \beta_{2,3}(t - T_2) + \beta_{3,3}x_t + \varepsilon_t & : \quad t = T_2 + 1, \dots, T, \end{cases} \end{aligned}$$

where $T_1 = 0.3T$ and $T_2 = 0.7T$, and we consider two kinds of changes: The first is the case of two successive increases in the coefficients; $\beta_{1,1} = \beta_{2,1} = \beta_{3,1} = 0$ while $\beta_{1,2} = \beta_{3,2} = 5\gamma$ and $\beta_{2,2} = \gamma$ in the second regime and $\beta_{1,3} = \beta_{3,3} = 10\gamma$ and $\beta_{2,3} = \gamma$ in the third regime with γ taking positive values. The second case is such that the first break occurs in the upward directions while the dependent variable crashes down by the second break; $\beta_{1,1} = \beta_{2,1} = \beta_{3,1} = 0$ while $\beta_{1,2} = \beta_{3,2} = 5\gamma$ and $\beta_{2,2} = \gamma$ in the second regime and $\beta_{1,3} = \beta_{3,3} = -5\gamma$ and $\beta_{2,3} = -0.5\gamma$ in the third regime.

Figure 3 shows the size-adjusted powers for $q = 0$. We observe that the test with under-specification of the number of breaks ($m = 1$) is still most powerful than the others in this case for each type of the tests. As in the one-time break

Figure 1: Size-adjusted power ($q = 0$, $m = 1$)

Figure 2: Size-adjusted power ($q = 1, m = 1, \phi = 0$)

Figure 3: Size-adjusted power ($q = 0$, $m = 2$, case a)

case, the second-best are the weighted-type tests, while the avg-type test is most powerful among the weighted-type tests, followed by the exp-type test, as is observed from (iv-a) and (iv-b). Similar property is observed with a stationary regressor from Figure 4, but as far as the avg-type test is concerned, the specification of the number of breaks does not so much affect the differences in powers. In addition, we do not observe the significant differences in powers among the weighted-type tests. On the other hand, the persistence in the AR regressor affects the finite sample powers very much. The case with $\phi = 0$ is most powerful, whereas $\phi = 0.8$ corresponds to the least powerful case. The maximum differences in powers in these two cases with $m = 2$ are 0.490, 0.493, and 0.454 for the exp-type, sup-type and avg-type tests, respectively, when $T = 120$. The similar magnitude of the differences is observed even when $T = 300$ (we do not show the figure to save space).

Figure 5 shows the size-adjusted powers in the second case of the two breaks for $q = 0$. When $T = 120$, the test correctly specifying the number of breaks ($m = 2$) is most powerful in each type of the tests. In particular, the differences in powers are relatively large for the avg-type tests. On the other hand, when $T = 300$, the differences become smaller and as far as the sup-type test is concerned, the test with $m = 1$ becomes most powerful. The differences in powers are more pronounced when a stationary variable is included as a regressor, as is observed from Figure 6. The effect of the persistence of the AR regressor on the powers is mitigated in this case; the largest differences in the powers of the tests with $m = 2$ are 0.169, 0.158, and 0.115 for the exp-type, sup-type and avg-type tests, respectively, when $T = 120$, and the differences become marginal when $T = 300$.

The possible reason for the highest power of the single break tests is that when two successive increases (or decreases) in the coefficients occur as in the first case, the single break model estimates a sharp “V-shape” trend, so that they have good power. On the other hand, when the second break is in the opposite direction to the first break as in the second case, the estimated trend is relatively stable for the under-specified model and then the two break tests are more powerful than the single-break tests.

We also investigate the case when we under-specify the maximum number of breaks. To see the effect of the under-specification, we consider eqs [13] and [14] with β_i replaced by

$$\beta_{i,t} = \beta_{i,t-1} + v_{i,t}, \quad v_{i,t} \sim i.i.d.N(0, \sigma_i^2),$$

for $i = 1$ and 2 in the case of $q = 0$ and $i = 1, \dots, 3$ in the case of $q = 1$, respectively, and $v_{i,t}$ are independent across i . That is, the parameters independently follow a random walk processes, so that these parameters change every

Figure 4: Size-adjusted power ($q = 1$, $m = 2$, $\phi = 0$, case a)

Figure 5: Size-adjusted power ($q = 0$, $m = 2$, case b)

Figure 6: Size-adjusted power ($q = 1$, $m = 2$, $\phi = 0$, case b)

time. Even in this case, all the test statistics have reasonable power for large values of σ_i^2 and there is no large difference among the power of the tests (we omit the figures to save space).

Finally, we investigate the effect of the partial structural change tests considered in Section 5. To demonstrate the power gain by imposing restrictions on parameter, we consider DGP1 [15] with a one-time break in β_1 and β_2 , whereas β_3 is fixed throughout the sample period. From Figure 7, we can see, as expected, that the partial structural change tests are more powerful than the full structural change tests. The power gain is about 5–10%.

Figure 7: Power comparison of the full and partial structural change tests

In summary, the performance of the tests depends on the DGP and none of the tests dominates the others uniformly, while we also observe that the weighted-type tests are the second best in most cases and the powers of those tests are close to those of the best tests.

7 Concluding remarks

In this paper, we have investigated tests for multiple breaks with non-homogeneous regressors. We have derived the limiting distributions of the test statistics in a general case and found that the limiting distributions depend on the regressors. Another contribution of this paper is that we have proposed a new computational method for obtaining the critical values of the avg-type test, which requires only $O(T^2)$ operations for any given number of breaks. By focusing on the linear trend case, we have obtained the critical values for the sup-type and avg-type tests by computationally efficient methods (although we cannot find such a method for the exp-type test) and have obtained the critical values for the exp-type test only up to $m = 3$. By Monte Carlo simulations, we have showed that the correct specification of the number of breaks is very important in order for the tests to have good power. However, since we often cannot specify the specific number of breaks under the alternative but can only suppose the maximum number of breaks, the weighted-type tests would be useful in practice.

Appendix

Proof of Theorem 1: According to Assumptions A1 and A2, we can see that under the null hypothesis,

$$D_T(\hat{\beta}_j - \beta) = \left(D_T^{-1} \sum_{t=T_{j-1}+1}^{T_j} x_t x_t' D_T^{-1} \right)^{-1} D_T^{-1} \sum_{t=T_{j-1}+1}^{T_j} x_t \varepsilon_t$$

$$\xrightarrow{d} \sigma(\Omega_{\lambda_j} - \Omega_{\lambda_{j-1}})^{-1} (G(\lambda_j) - G(\lambda_{j-1})) \equiv \sigma \tilde{\Sigma}_{\lambda_j} \tilde{G}(\lambda_j),$$

where $\tilde{\Sigma}_{\lambda_j} = (\Omega_{\lambda_j} - \Omega_{\lambda_{j-1}})^{-1}$ with $\Omega_{\lambda_0} = 0$ and $\tilde{G}(\lambda_j) = G(\lambda_j) - G(\lambda_{j-1})$ with $G(\lambda_0) = 0$. Then, we can see that

$$W_T(\Lambda_m) \xrightarrow{d} (R \tilde{\Sigma} \tilde{G})' (R \tilde{\Sigma} R')^{-1} (R \tilde{\Sigma} \tilde{G})$$

uniformly over the permissible sets of break fractions, where $\tilde{G} = [\tilde{G}'(\lambda_1), \dots, \tilde{G}'(\lambda_{m+1})]'$ and $\tilde{\Sigma} = \text{diag}\{\tilde{\Sigma}_{\lambda_1}, \dots, \tilde{\Sigma}_{\lambda_{m+1}}\}$. Then, we need to show that

$$(R\tilde{\Sigma}\tilde{G})(R\tilde{\Sigma}R')^{-1}(R\tilde{\Sigma}\tilde{G}) = Q_{1,m}, \quad [16]$$

where $Q_{1,m}$ is defined in eq. [5]. Note that the difficulty is in that the inverse of the variance matrix is given by

$$(R\tilde{\Sigma}R')^{-1} = \begin{bmatrix} \tilde{\Sigma}_{\lambda_1} + \tilde{\Sigma}_{\lambda_2} & -\tilde{\Sigma}_{\lambda_2} & 0 & \cdots & 0 \\ -\tilde{\Sigma}_{\lambda_2} & \tilde{\Sigma}_{\lambda_2} + \tilde{\Sigma}_{\lambda_3} & -\tilde{\Sigma}_{\lambda_3} & \ddots & \vdots \\ 0 & -\tilde{\Sigma}_{\lambda_3} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\tilde{\Sigma}_{\lambda_m} \\ 0 & \cdots & 0 & -\tilde{\Sigma}_{\lambda_m} & \tilde{\Sigma}_{\lambda_m} + \tilde{\Sigma}_{\lambda_{m+1}} \end{bmatrix}^{-1}, \quad [17]$$

which does not have a simple closed-form expression.

In order to treat [17], we introduce a nonsingular matrix H and transform the left-hand side of eq. [16] using H as $(HR\tilde{\Sigma}\tilde{G})'(HR\tilde{\Sigma}R'H')^{-1}(HR\tilde{\Sigma}\tilde{G})$ and evaluate the transformed expression. More precisely, we define an $mp \times mp$ lower triangular matrix H and decompose the restriction matrix R as

$$H = \begin{bmatrix} -I_p & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ -I_p & \cdots & \cdots & -I_p \end{bmatrix}, \quad R = \begin{bmatrix} 0 & -I_p & 0 & \cdots & 0 \\ \vdots & I_p & -I_p & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & I_p & -I_p \end{bmatrix} + \begin{bmatrix} I_p & 0 & \cdots & \cdots & 0 \\ 0 & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

$$\text{so that } HR = \begin{bmatrix} 0 & I_p & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & I_p \end{bmatrix} - \begin{bmatrix} I_p & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ I_p & 0 & \cdots & \cdots & 0 \end{bmatrix}.$$

Let us decompose $\tilde{\Sigma}$ as $\tilde{\Sigma} = \text{diag}\{\tilde{\Sigma}_{\lambda_1}, \tilde{\Sigma}_{2,m+1}\}$, where $\tilde{\Sigma}_{2,m+1} = \text{diag}\{\tilde{\Sigma}_{\lambda_2}, \dots, \tilde{\Sigma}_{\lambda_{m+1}}\}$. Then, we can see that

$$\begin{aligned} (HR\tilde{\Sigma}R'H')^{-1} &= (\tilde{\Sigma}_{2,m+1} + F_p\tilde{\Sigma}_{\lambda_1}F_p')^{-1} \\ &= \tilde{\Sigma}_{2,m+1}^{-1} - \tilde{\Sigma}_{2,m+1}^{-1}F_p(\tilde{\Sigma}_{\lambda_1}^{-1} + F_p'\tilde{\Sigma}_{2,m+1}^{-1}F_p)^{-1}F_p'\tilde{\Sigma}_{2,m+1}^{-1} \\ &= \tilde{\Sigma}_{2,m+1}^{-1} - \tilde{\Sigma}_{2,m+1}^{-1}F_p(\tilde{\Sigma}_{\lambda_1}^{-1} + \cdots + \tilde{\Sigma}_{\lambda_{m+1}}^{-1})^{-1}F_p'\tilde{\Sigma}_{2,m+1}^{-1}, \end{aligned} \quad [18]$$

where $F_p = [I_p, \dots, I_p]'$ is an $mp \times p$ matrix, while

$$HR\tilde{\Sigma}\tilde{G} = \begin{bmatrix} \tilde{\Sigma}_{\lambda_2}\tilde{G}(\lambda_2) - \tilde{\Sigma}_{\lambda_1}\tilde{G}(\lambda_1) \\ \vdots \\ \tilde{\Sigma}_{\lambda_{m+1}}\tilde{G}(\lambda_{m+1}) - \tilde{\Sigma}_{\lambda_1}\tilde{G}(\lambda_1) \end{bmatrix}. \quad [19]$$

Then, since $\tilde{\Sigma}_{\lambda_j} = (\Omega_{\lambda_j} - \Omega_{\lambda_{j-1}})^{-1}$ and $\tilde{G}(\lambda_j) = G(\lambda_j) - G(\lambda_{j-1})$, we can see from eq. [19] that

$$\begin{aligned} (HR\tilde{\Sigma}\tilde{G})'\tilde{\Sigma}_{2,m+1}^{-1}(HR\tilde{\Sigma}\tilde{G}) &= \sum_{j=1}^m \tilde{G}'(\lambda_{j+1})\tilde{\Sigma}_{\lambda_{j+1}}\tilde{G}(\lambda_{j+1}) - 2\sum_{j=1}^m \tilde{G}'(\lambda_1)\tilde{\Sigma}_{\lambda_1}\tilde{G}(\lambda_{j+1}) \\ &\quad + \tilde{G}'(\lambda_1)\tilde{\Sigma}_{\lambda_1}(\tilde{\Sigma}_{\lambda_2}^{-1} + \dots + \tilde{\Sigma}_{\lambda_{m+1}}^{-1})\tilde{\Sigma}_{\lambda_1}\tilde{G}(\lambda_1) \\ &= \sum_{j=1}^m (G(\lambda_{j+1}) - G(\lambda_j))'(\Omega_{\lambda_{j+1}} - \Omega_{\lambda_j})^{-1}(G(\lambda_{j+1}) - G(\lambda_j)) \\ &\quad - 2\left\{G'(\lambda_1)\Omega_{\lambda_1}^{-1}G(\lambda_{m+1}) - G'(\lambda_1)\Omega_{\lambda_1}^{-1}G(\lambda_1)\right\} \\ &\quad + G'(\lambda_1)\Omega_{\lambda_1}^{-1}\Omega_{\lambda_{m+1}}\Omega_{\lambda_1}^{-1}G(\lambda_1) - G'(\lambda_1)\Omega_{\lambda_1}^{-1}G(\lambda_1). \end{aligned} \quad [20]$$

Similarly, since $F_p'\tilde{\Sigma}_{2,m+1}^{-1} = [\tilde{\Sigma}_{\lambda_2}^{-1}, \dots, \tilde{\Sigma}_{\lambda_{m+1}}^{-1}]$, we have

$$\begin{aligned} F_p'\tilde{\Sigma}_{2,m+1}^{-1}HR\tilde{\Sigma}\tilde{G} &= \sum_{j=1}^m \tilde{G}(\lambda_{j+1}) - (\tilde{\Sigma}_{\lambda_2}^{-1} + \dots + \tilde{\Sigma}_{\lambda_{m+1}}^{-1})\tilde{\Sigma}_{\lambda_1}\tilde{G}(\lambda_1) \\ &= G(\lambda_{m+1}) - G(\lambda_1) - (\Omega_{\lambda_{m+1}} - \Omega_{\lambda_1})\Omega_{\lambda_1}^{-1}G(\lambda_1) \\ &= G(\lambda_{m+1}) - \Omega_{\lambda_{m+1}}\Omega_{\lambda_1}^{-1}G(\lambda_1), \end{aligned}$$

so that

$$\begin{aligned} (HR\tilde{\Sigma}\tilde{G})'\tilde{\Sigma}_{2,m+1}^{-1}F_p\left(\tilde{\Sigma}_{\lambda_1}^{-1} + \dots + \tilde{\Sigma}_{\lambda_{m+1}}^{-1}\right)^{-1}F_p'\tilde{\Sigma}_{2,m+1}^{-1}(HR\tilde{\Sigma}\tilde{G}) \\ = \left(G(\lambda_{m+1}) - \Omega_{\lambda_{m+1}}\Omega_{\lambda_1}^{-1}G(\lambda_1)\right)'\Omega_{\lambda_{m+1}}^{-1}\left(G(\lambda_{m+1}) - \Omega_{\lambda_{m+1}}\Omega_{\lambda_1}^{-1}G(\lambda_1)\right) \\ = G'(\lambda_{m+1})\Omega_{\lambda_{m+1}}^{-1}G(\lambda_{m+1}) - 2G'(\lambda_1)\Omega_{\lambda_1}^{-1}G(\lambda_{m+1}) + G'(\lambda_1)\Omega_{\lambda_1}^{-1}\Omega_{\lambda_{m+1}}\Omega_{\lambda_1}^{-1}G(\lambda_1). \end{aligned} \quad [21]$$

Then, by combining eqs [18], [20] and [21], we have

$$\begin{aligned}
& (HR\tilde{\Sigma}\tilde{G})'(HR\tilde{\Sigma}R'H')^{-1}(HR\tilde{\Sigma}\tilde{G}) \\
&= \sum_{j=1}^m (G(\lambda_{j+1}) - G(\lambda_j))' (\Omega_{\lambda_{j+1}} - \Omega_{\lambda_j})^{-1} (G(\lambda_{j+1}) - G(\lambda_j)) \\
&\quad + G'(\lambda_1)\Omega_{\lambda_1}^{-1}G(\lambda_1) - G'(\lambda_{m+1})\Omega_{\lambda_{m+1}}^{-1}G(\lambda_{m+1}) \\
&= \sum_{j=1}^m \left\{ (G(\lambda_{j+1}) - G(\lambda_j))' (\Omega_{\lambda_{j+1}} - \Omega_{\lambda_j})^{-1} (G(\lambda_{j+1}) - G(\lambda_j)) \right. \\
&\quad \left. + G'(\lambda_j)\Omega_{\lambda_j}^{-1}G(\lambda_j) - G'(\lambda_{j+1})\Omega_{\lambda_{j+1}}^{-1}G(\lambda_{j+1}) \right\} \\
&= \sum_{j=1}^m \left(\Omega_{\lambda_{j+1}}^{-1}G(\lambda_{j+1}) - \Omega_{\lambda_j}^{-1}G(\lambda_j) \right)' \left(\Omega_{\lambda_j}^{-1} - \Omega_{\lambda_{j+1}}^{-1} \right)^{-1} \left(\Omega_{\lambda_{j+1}}^{-1}G(\lambda_{j+1}) - \Omega_{\lambda_j}^{-1}G(\lambda_j) \right),
\end{aligned}$$

where the last equality is obtained by using the following three equivalent expressions:

$$\begin{aligned}
(\Omega_{\lambda_{j+1}} - \Omega_{\lambda_j})^{-1} &= \Omega_{\lambda_{j+1}}^{-1} + \Omega_{\lambda_{j+1}}^{-1}(\Omega_{\lambda_j}^{-1} - \Omega_{\lambda_{j+1}}^{-1})^{-1}\Omega_{\lambda_{j+1}}^{-1} \\
&= -\Omega_{\lambda_j}^{-1} + \Omega_{\lambda_j}^{-1}(\Omega_{\lambda_j}^{-1} - \Omega_{\lambda_{j+1}}^{-1})^{-1}\Omega_{\lambda_j}^{-1} \\
&= \Omega_{\lambda_j}^{-1}(\Omega_{\lambda_j}^{-1} - \Omega_{\lambda_{j+1}}^{-1})^{-1}\Omega_{\lambda_{j+1}}^{-1}. \blacksquare
\end{aligned}$$

Proof of Corollary 1: Since a constant term is included as a regressor, the stationary variables can be assumed to be mean zero without loss of generality. Similarly, because the lagged dependent variables can be decomposed into a constant, a linear trend, and stationary components, we can treat the expectation of the lagged dependent variables to be zero because 1 and t are included as regressors. Then, we can see that Ω_r becomes a block diagonal matrix with the first 2-by-2 diagonal block given by

$$\begin{bmatrix} r & \frac{r^2}{2} \\ \frac{r^2}{2} & \frac{r^3}{3} \end{bmatrix}$$

and the last q -by- q block given by $r\Omega_2$, where Ω_2 consists of the second moments of the stationary regressors. The result immediately follows because of the diagonality of Ω_r . $\blacksquare$

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