Research Article

Cameron C. Parker*, Efstathios Paparoditis and Dimitris Politis Tapered Block Bootstrap for Unit Root Testing

Abstract: A new bootstrap procedure for unit root testing based on the tapered block bootstrap is introduced. This procedure is similar to previous tests that were based on the block bootstrap and stationary bootstrap, but it has the advantage of the tapering procedure that has been previously shown to reduce the bias of the variance estimator by an order of magnitude. In this paper, the procedure is defined including a specific data-driven method for choosing the block size. Both theoretical results for the asymptotic behavior of the test and simulations that address the small-sample properties and are used for comparison to other methods are given.

Keywords: integrated time series, tapered block bootstrap, unit root testing, resampling

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1 Introduction and notation

In the analysis of many time series in finance and macroeconomics, it is crucial to first determine if the data are coming from a process that is stationary or integrated to determine if standard techniques should be applied to the original series or to the differenced series. Hence, if $\{X_t, t \in \mathbb{Z}\}$ is a time series, then having a powerful test of the hypothesis

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$$\begin{aligned}
 H_0 &: \{X_t\} \text{ is } I(1) \\
 &\text{versus} \\
 H_1 &: \{X_t\} \text{ is stationary}
 \end{aligned}
 \tag{1}$$

is of great importance. There are many such tests in the literature; see for example Fuller (1996), Dickey and Fuller (1979), Dickey, Bell, and Miller (1986), and Phillips and Perron (1988). More recently in Swensen (2003), Paparoditis and Politis (2003), and Parker, Paparoditis, and Politis (2006), bootstrap-based tests for a unit root were proposed. The latter two papers use a bootstrap procedure (block and stationary bootstraps, respectively) on the residuals, while the former applies the stationary bootstrap on the differenced series. This paper will use the same set-up as those two papers and for more details on the construction see Parker, Paparoditis, and Politis (2006, 602–03).

As in the resampling schemes mentioned earlier, we choose a parameter ρ with the property that

$$\rho = 1 \text{ if and only if } H_0 \text{ holds.} \tag{2}$$

Then define $\{U_t\}$ by

$$U_t = X_t - \rho X_{t-1} - \beta \tag{3}$$

for $t = 1, 2, \dots$ where

$$\beta = E(X_t - \rho X_{t-1}), \tag{4}$$

and hence $E(U_t) = 0$. Thus the new series $\{U_t\}$ is *always* stationary – under H_0 and under H_1 . One particular choice for a parameter that satisfies eq. [2] that we will focus on in this paper is

$$\rho = \rho_{LS} = \lim_{t \rightarrow \infty} \frac{EX_t X_{t+1}}{EX_t^2}. \tag{5}$$

The fact that ρ_{LS} satisfies eq. [2] is guaranteed by condition [8] below.

We make no “model” assumption for the $\{X_t\}$ series; the necessary technical assumptions placed on $\{X_t\}$ are moment and mixing conditions and are discussed in detail in Section 2.¹ Thus, while many of the competing methods assume linearity of the X_t process, the proposed method also works for nonlinear processes as well. This is especially important in finance since many of the time series in that field are nonlinear, see for example Franses and Van Dijk (2000) or Tong (1990). Thus the methods of this paper are applicable for unit

¹ We do not consider the possibility of trend breaks; however, some work in this direction has been done in Ioannidis (2005) and could be extended to this case.

root testing in the context of financial time series whereas methods that rely on linearity are not.

The details of the proposed test are given in Section 2. In Section 3, we examine the tapered bootstrap's superior properties in estimating σ_∞ and demonstrate why we can expect these properties to carry over to unit root testing. Since the test will be sensitive to the choice of a tapering window, a discussion of possible windows is given in Section 4. In Section 5, we give a version of the functional central limit theorem which will be used to show the test's consistency. Using a particular choice of a parameter and different estimators, we look at the behavior of the proposed test under the null and the alternative in Sections 6.1 and 6.2, respectively. In Section 7, we give a data-driven method for choosing the block size for the tapered block bootstrap. In Section 8, we set forth the results of a small-sample simulation study. We use the simulations both to investigate the performance of the block size choice and to compare the tapered bootstrap procedure to previously performed bootstrap procedures. Finally, all technical proofs are presented in Section 10.

2 Unit root test based on tapered block bootstrap

In this section we describe the proposed test. To test 1 we assume that we have some parameter, ρ , of $\{X_t\}$ satisfying eq. [2] and we define a new process $\{U_t\}$ by eq. [3]. Also let β be defined by eq. [4] and we will assume that we have an estimator $\hat{\rho}$ of ρ satisfying

$$\begin{aligned} \hat{\rho}_n &= \rho + o_P(1), & \text{if } \rho \neq 1 \\ \hat{\rho}_n &= \rho + O_P(n^{-(2+\delta(\beta))/2}), & \text{if } \rho = 1, \end{aligned} \quad [6]$$

where $\delta(\beta) = 1$ if $\beta \neq 0$ and $\delta(0) = 0$. As will be seen in Section 6.1, the test behaves quite differently when $\beta = 0$ and when $\beta \neq 0$. Finally define

$$\hat{U}_t = (X_t - \hat{\rho}_n X_{t-1}) - \frac{1}{n-1} \sum_{\tau=2}^n (X_\tau - \hat{\rho}_n X_{\tau-1}) \quad \text{for } 2 \leq t \leq n. \quad [7]$$

For most of the paper we will use $\rho = \hat{\rho}_{LS}$ as defined in eq. (5) and two particular estimators of ρ : $\hat{\rho}_{LS}$ and $\hat{\rho}_{LS,C}$, where $\hat{\rho}_{LS}$ is the least square solution to the model

$$X_t = \rho X_{t-1} + U_t,$$

and $\hat{\rho}_{LS,C}$ and $\hat{\beta}_{LS,C}$ are least squares solutions to the model

$$X_t = \rho X_{t-1} + \beta + U_t.$$

The fact that $\hat{\rho}_{LS}$ and $\hat{\rho}_{LS,C}$ both satisfy eq. (6) is known; see Brockwell and Davis (1991) for the stationary case, and Fuller (1996) or Phillips (1987a) for the integrated case.

The essential idea of the test is to use the tapered block bootstrap procedure on the series $\{\hat{U}_t\}$ and then integrate these pseudo-series to create B approximately integrated pseudo-series. Then by computing $\hat{\rho}$ for each of these pseudo-series, we create an estimate of the distribution of $\hat{\rho}$ under the null hypothesis and then use this estimate to decide whether or not the null should be rejected. For a discussion of the difference between this and a difference-based approach (i.e. as is done in Swensen (2003); see Parker, Paparoditis, and Politis (2006, 605). The tapered bootstrap is used because it was shown in Paparoditis and Politis (2001) that a good portion of the bias inherent in the block bootstrap had to do with the end effects: where one block and the other begins. It was found that “tapering” the edges of each block before collating them together in a block bootstrap series results in a reduction of the bias by an order of magnitude. Of course, one has to maintain the correct scale, i.e. variance, for the marginal distribution; hence, any tapering comes hand-in-hand with an appropriate renormalization via the L_2 norm of the tapering function ω_b . The benefits of tapering are discussed in Section 4.

The proposed test is non-parametric so no model assumptions are made; however the following mixing and moment conditions are imposed

$$\begin{aligned}
 (i) \quad & E|U_1|^{2+\varepsilon} < \infty \\
 (ii) \quad & E|X_1|^{2+\varepsilon} < \infty \\
 (iii) \quad & \sum_k [a_U(k)]^{\frac{\varepsilon}{2+\varepsilon}} < \infty \\
 (iv) \quad & \text{Under } H_1: \sum_k [\alpha_X(k)]^{\frac{\varepsilon}{2+\varepsilon}} < \infty \\
 (v) \quad & \text{Under } H_1: f_X(0) > 0.
 \end{aligned} \tag{8}$$

Here, α_Y is the usual strong mixing coefficients for the time series $\{Y_t\}$, and $f_X(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\omega} \gamma(h)$ is the spectral density of $\{X_t\}$. Assumption (v) implies that the series $\{X_t\}$ is not the difference of a stationary time series.

The tapered block bootstrap procedure consists of applying a tapering function that downweights the endpoints of the resampled blocks, thus reducing the discontinuity between block.

We then have the following algorithm which describes the steps of the residual-based tapered bootstrap test (RTB):

1. Choose a positive integer b_n , let $k_n = \lfloor n/b_n \rfloor$ and generate $i_{1,n}, i_{2,n}, \dots, i_{k_n,n}$ i.i.d. from the uniform distribution on $\{1, \dots, n - b + 1\}$. When there is no confusion, we will denote $i_{m,n}, b_n$, and k_n by i_m, b , and k .

2. For all m and j , $1 \leq m \leq k$ and $0 \leq j < b$ set

$$\hat{U}_{mb+j}^* = \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \hat{U}_{im+j},$$

where

$$\|\omega_b\|_2 = \sqrt{\sum_{t=1}^b \omega_b^2(t)},$$

and $\omega_b(t)$ is a tapering window whose properties are described in detail in Section 4.

3. Set $X_1^* = X_1$ and $X_t^* = X_t^* + \hat{U}_{t-1}^* + \tilde{\beta}$ for each $t = 2, \dots, n$. Typical choices for $\tilde{\beta}$ include $\tilde{\beta} \equiv \hat{\beta}$ where $\hat{\beta}$ is some consistent estimator of β , or $\tilde{\beta} \equiv 0$.
4. Let $\hat{\rho}_n^* = \hat{\rho}_n(X_1^*, \dots, X_n^*)$.
5. Repeat the above B times to form an empirical (based on B replications) distribution of $\hat{\rho}_n^*$ which will be our estimator of the true distribution of $\hat{\rho}_n$. Use the $1 - \alpha$ quantile of this estimated distribution to test H_0 at the α level.

The above algorithm is performed conditional on the data $\{X_1, X_2, \dots, X_n\}$. We will denote the conditional probability measure that generates $\{X_1^*, X_2^*, \dots, X_n^*\}$ as P^* . Similarly, we will denote quantities (expectation, variance, etc.) taken with respect to P^* with an asterisk (*); for example, note that $E^* \hat{U}_t^* = 0$ by the way that $\hat{U}_t$ is defined in eq. [7].

We will let $\tilde{\beta} = 0$ when $\hat{\rho}_{LS}$ is used but will consider $\hat{\rho}_{LS,C}$ both with $\tilde{\beta} = 0$ and $\tilde{\beta} = \hat{\rho}_{LS,C}$. Clearly, $\hat{\rho}_{LS,C}^*$ depends on the choice of $\tilde{\beta}$ but it will be clear from context which value of $\tilde{\beta}$ is being used. As will be seen in Section 6.1, the asymptotic behavior of $\hat{\rho}_n$ depends greatly on whether or not $\beta = 0$. As West (1988) notes it is

surprising that the asymptotic distributions of $[\hat{\rho}_n]$ are so sensitive to whether the means of the first differences are zero. In a stationary environment, means in general do not affect the asymptotic distribution of estimates of parameters ... this is not at all true in the nonstationary environment.

3 Motivation

The motivation behind our proposal is the following. In the case where $\beta = 0$, it is known (Phillips 1987a; Hamilton 1994) that the asymptotic distribution of $n(\hat{\rho}_{LS} - 1)$ is given by

$$\frac{W(1)^2 - \sigma_U^2 / \sigma^2}{2 \int_0^1 W^2(r) dr}.$$

where

$$\sigma_U^2 = EU_1^2,$$

$$\sigma^2 = \sigma_\infty^2 = 2\pi f_U(0)$$

and $f_U(\cdot)$ is the spectral density of the process $\{U_t\}$. Also if we let

$$R(s) = E(U_t U_{t+s})$$

then

$$\sigma_\infty^2 = \sum_{s=-\infty}^{\infty} R(s).$$

Thus in estimating the distribution of $n(\hat{\rho}_{LS} - 1)$ under the null, the bootstrap procedure must (and does) implicitly estimate both the parameters σ_U^2 and σ_∞^2 . The parameter σ_U^2 is easily estimable at a $\sqrt{n}$ -rate but σ_∞^2 is more problematic. However, it is known that the MSE of the tapered bootstrap estimate of σ_∞^2 has a lower order of magnitude than the corresponding estimators given by either the stationary or block bootstrap methods.

To be exact, under conditions detailed in the next section, the block bootstrap estimate of σ_∞^2 , denoted $\hat{\sigma}_{b,n,\text{block}}^2$ has the following properties (Politis, Romano, and Wolf 1999; Künsch 1989):

$$\text{Bias}(\hat{\sigma}_{b,n,\text{block}}^2) = E\hat{\sigma}_{b,n,\text{block}}^2 - \sigma_\infty^2 = -\frac{1}{b} \sum_{k=-\infty}^{\infty} |k| R(k) + O(b/n) + o(1/b) \quad [9]$$

$$\text{Var}(\hat{\sigma}_{b,n,\text{block}}^2) = \frac{4b}{3n} \sigma_\infty^4 + o(b/n). \quad [10]$$

Thus, plugging in the optimal b , which is proportional to $n^{1/3}$ we get

$$\text{MSE}(\hat{\sigma}_{b,n,\text{block}}^2) = O(n^{-2/3}).$$

This is also the same order of magnitude shared by the stationary bootstrap. However, as will be seen in Section 4, under the right conditions and appropriate block size choice, the tapered block bootstrap gives an estimate, $\hat{\sigma}_{b,\text{TBB}}^2$ of σ_∞^2 with

$$\text{MSE}(\hat{\sigma}_{b,\text{TBB}}^2) = O(n^{-4/5}). \quad [11]$$

We therefore expect a better performance from the tapered block bootstrap procedure in finite-sample size situations.

4 The tapering function ω

For purposes of this section we will deal with tapering functions ω_b , which are derived from a single positive function, ω , in the following way:

$$\omega_b = \omega\left(\frac{t - 0.5}{b}\right) \quad [12]$$

where ω is a function defined on $[0, 1]$ satisfying the following conditions:

- (i) $0 \leq \omega(t) \leq 1$, for all $t \in [0, 1]$
 - (ii) ω is symmetric about $t = 1/2$
 - (iii) ω is nondecreasing on $[0, 1/2]$
 - (iv) There exists an $\varepsilon > 0$ such that $\omega(t) > 0$ on $(1/2 - \varepsilon, 1/2 + \varepsilon)$.
- [13]

There are many examples of tapering functions satisfying eq. [13] in the literature for spectral estimation, see for example Brillinger (1981), Welch (1967), Priestley (1981), Dahlhaus (1990), Dahlhaus (1985).

One way to satisfy eq. [13] is simply to use, $\omega(t) = 1$. In this case however, the procedure above reduces to the block bootstrap without tapering and is identical to the procedure in Paparoditis and Politis (2003). To get the added benefits of using the tapering function we need to place additional conditions on the function ω as in Paparoditis and Politis (2001); see also Künsch (1989) for a similar condition for the tapered jackknife. Namely, we need to choose ω such that

$$(\omega * \omega)(t) \text{ is twice continuously differentiable at } t = 0, \quad [14]$$

where

$$(\omega * \omega)(t) = \int_{x=-1}^{x=1} \omega(x)\omega(x + |t|) dx = \int_{x=0}^{x=1-|t|} \omega(x)\omega(x + |t|) dx$$

is the self-convolution of ω .

Using a tapering function that satisfies eqs [13] and [14] gives an estimate of σ_∞^2 that has improved accuracy as evidenced by the following result of Paparoditis and Politis (2001):

$$E(\sigma_{b,\text{TBB}}^2) = \sigma_\infty^2 + \frac{\Gamma}{b^2} + o\left(\frac{1}{b^2}\right),$$

and

$$\text{Var}(\sigma_{b,\text{TBB}}^2) = \Delta \frac{b}{N} + o\left(\frac{b}{N}\right),$$

where

$$\Gamma = \frac{(\omega * \omega)''(0)}{2(\omega * \omega)(0)} \sum_{k=-\infty}^{\infty} k^2 R(k), \quad [15]$$

and

$$\Delta = 2\sigma_{\infty}^4 \int_{-1}^1 \frac{(\omega * \omega)^2(x)}{(\omega * \omega)^2(0)} dx. \quad [16]$$

Then eq. [11] is achieved when b is set to be proportional to $n^{1/5}$. Although this works asymptotically, it does not give much guidance in selecting b in practice. However, we do give a data-driven method for choosing b in Section 7.

The triangular (tent) window defined by

$$\omega^{\text{TENT}}(t) = \begin{cases} 1 - |1 - 2t| & \text{when } t \in (0, 1) \\ 0 & \text{otherwise} \end{cases}$$

does not satisfy eq. [13] and is suboptimal since

$$\omega^{\text{TENT}} * \omega^{\text{TENT}}(t) = \begin{cases} 1 + 3|t| - 6t^2 - \frac{2}{3}|t|^3 & \text{when } |t| \leq 1/2 \\ \frac{-2(-1+|t|)^3}{3} & \text{when } 1/2 \leq |t| \leq 1 \\ 0 & \text{otherwise,} \end{cases}$$

is not twice differentiable at $t = 0$.

We will focus on two one-parameter families of functions that satisfy both eqs [13] and [14] and are given particular attention in Paparoditis and Politis (2001). Both families have the triangular window as an extreme case but in the other cases do satisfy eq. [13].

The first is simply a trapezoid function

$$\omega_c^{\text{TRAP}}(t) = \begin{cases} t/c & \text{if } t \in [0, c] \\ 1 & \text{if } t \in [c, 1 - c] \\ (1 - t)/c & \text{if } t \in [1 - c, 1] \\ 0 & \text{if } t \notin [0, 1] \end{cases}$$

which is equal to the triangular window when $c = 0$ but when $c \neq 0$ has a flattop which turns out to be enough to satisfy eq. [13]. In fact, when $|t| < \min(c, 1 - 2c)$ we have

$$\omega_c^{\text{TRAP}} * \omega_c^{\text{TRAP}}(t) = \frac{|t|^3}{2c^2} - \frac{t^2}{c} + 1 - \frac{4c}{3}.$$

Since this range includes 0, $(\omega_c^{\text{TRAP}} * \omega_c^{\text{TRAP}})''$ exists at 0.

Another approach is to smooth the triangular window, which can be done in the following way:

$$\omega_a^{\text{SMOOTH}}(t) = 1 - |2t - 1|^a.$$

It is clear that ω_a^{SMOOTH} reduces to the triangular window when $a = 1$.

It is suggested in Paparoditis and Politis (2001) that the best performing members of the ω_c^{TRAP} and ω_a^{SMOOTH} families are when $c = 0.43$ and $a = 1.3$, respectively. These values are used for simulations in Sections 7 and 8.1.

5 Functional limit theorem for the tapered bootstrap partial sum process

The consistency of the RTB test relies on the following version of the functional central limit theorem, which serves as the theoretical linchpin for the rest of the paper. We define the process $\{S_n^*(r), 0 \leq r \leq 1\}$ by

$$S_n^*(r) = \frac{1}{\sqrt{n}\hat{\sigma}_n^*} \left(\sum_{t=1}^{\lfloor nr \rfloor} \hat{U}_t^* \right),$$

where

$$\hat{\sigma}_n^{*2} = \text{Var}^* \left(\frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor nr \rfloor} \hat{U}_j^* \right).$$

Notice that $S_n^*(r) \in D[0, 1]$, where $D[0, 1]$ is the space of real-valued right-continuous functions on $[0, 1]$ that have finite left limits.

The following theorem shows that as long as our original series satisfies eq. [8], and the tapering function ω_b is defined by eq. [12] where ω satisfies eq. [13] (but not necessarily eq. [14]), then it follows that the process $\{S_n^*(r), 0 \leq r \leq 1\}$ defined above converges weakly to the standard Wiener process on $[0, 1]$.

Theorem 1 *Let $\{X_t\}$ be a stochastic process and $\{U_t\}$ be as defined in eq. [3] for some parameter ρ satisfying eq. [2]. Assume condition [8], and that $\hat{\rho}_n$ is an estimator of ρ satisfying eq. [6] and that ω_b is defined by eq. [12] with ω satisfying eq. [13]. Then if $b \rightarrow \infty$ and $b/\sqrt{n} \rightarrow 0$ as $n \rightarrow \infty$, then*

$$S_n^*(\cdot) \xrightarrow{d^*} W \text{ in probability}$$

where W is the standard Wiener process.

6 Application to unit root testing

6.1 Behavior of RTB under the null

From Theorem 1 we can now show the consistency of the RTB procedure for the particular choice of $\rho = \hat{\rho}_{LS}$ and estimators $\hat{\rho}_{LS}$ and $\hat{\rho}_{LS,C}$ under different assumptions on β . Note, however, that Theorem 1 applies to a larger class of parameters and estimators.

Theorem 2 Assume the conditions of Theorem 1, and $b_n \rightarrow \infty$ and $(b_n/\sqrt{n}) \rightarrow 0$ as $n \rightarrow \infty$.

(1) If $\beta = 0$ and $\tilde{\beta} = 0$ then

$$n(\hat{\rho}_{LS}^* - 1) \xrightarrow{P^*} \frac{W(1)^2 - \frac{\sigma_y^2}{\sigma^2}}{2 \int_0^1 W^2(r) dr}$$

$$n(\hat{\rho}_{LS,C}^* - 1) \xrightarrow{P^*} \frac{W(1)^2 - 2W(1) \int_0^1 W(r) dr - \frac{\sigma_y^2}{\sigma^2}}{2 \left(\int_0^1 W^2(r) dr - \left[\int_0^1 W(r) dr \right]^2 \right)}.$$

(2) If $\beta = 0$ and $\tilde{\beta} = \hat{\beta}_{LS,C}$ then

$$n(\hat{\rho}_{LS,C}^* - 1) \xrightarrow{P^*} \frac{C + \frac{1}{2} \left(W^2(1) - \frac{\sigma_y^2}{\sigma^2} \right) - B \cdot W(1)}{B^2 - A}.$$

(3) If $\beta \neq 0$ and $\tilde{\beta} = \hat{\beta}_{LS,C}$ then

$$n^{3/2}(\hat{\rho}_{LS,C}^* - 1) \xrightarrow{P^*} N\left(0, \frac{12}{\beta^2}\right).$$

All convergences above occur in probability and

$$A = \frac{1}{3} \left[\frac{W(1) \int_0^1 (W^2(r) - W(r)) dr}{\int_0^1 W^2(r) dr - \left[\int_0^1 W(r) dr \right]^2} \right]^2$$

$$+ \left[\frac{W(1) \int_0^1 (W^2(r) - W(r)) dr}{\int_0^1 W^2(r) dr - \left[\int_0^1 W(r) dr \right]^2} \right] \cdot \int_0^1 rW(r) dr + \int_0^1 W^2(r) dr,$$

$$B = \frac{1}{2} \left[\frac{W(1) \int_0^1 (W^2(r) - W(r)) dr}{\int_0^1 W^2(r) dr - \left[\int_0^1 W(r) dr \right]^2} \right] + \int_0^1 W(r) dr,$$

$$C = \left[\frac{W(1) \int_0^1 (W^2(r) - W(r)) dr}{\int_0^1 W^2(r) dr - \left[\int_0^1 W(r) dr \right]^2} \right] \cdot \left[\left(W(1) - \int_0^1 W^2(r) dr \right) \right].$$

These results are the same as Parker, Paparoditis, and Politis (2006, 609–10) and are summarized in tables there.

Notice the RTB is consistent in either the case that β is exactly zero and $\tilde{\beta} = 0$ or when $\beta \neq 0$ and $\tilde{\beta} = \hat{\beta}_{LS,C}$. It is assumed that if β is exactly zero, it is so for some theoretical reason and so the practitioner would use the correct model.

6.2 Power properties and local alternatives for the RTB procedure

We will again restrict our attention to the case $\beta = 0$ and $\tilde{\beta} = 0$ and focus on the parameter $\hat{\rho}_{LS}$. To consider the power of the bootstrap procedure proposed, let

$$\beta_{\text{RTB},n}(\rho; \alpha) = P(n(\hat{\rho}_{LS} - 1) \leq C_\alpha^*), \quad \rho \in (-1, 1],$$

where

$$C_\alpha^* = \inf \{ C : P^*(n(\hat{\rho}_{LS}^* - 1) \leq C) \geq \alpha \}.$$

Notice that $\beta_{\text{RTB},n}(\rho; \alpha)$ describes the power of the RTB test. Consistency of the bootstrap-based test follows from the following theorem:

Theorem 3 *Under the assumptions of Theorem 2 with $\beta = 0$, and $\tilde{\beta} = 0$, we have*

$$(i) \quad \beta_{\text{RTB},n}(\rho; \alpha) \rightarrow 1 \text{ for all } \rho \in (-1, 1)$$

$$(ii) \quad \beta_{\text{RTB},n}(1; \alpha) \rightarrow \alpha,$$

as $n \rightarrow \infty$.

To see that favorable local power also occurs in this case, suppose that X_t is a triangular array satisfying:

$$X_t = \rho_n X_{t-1} + U_t, \text{ where } \rho_n = 1 + \frac{c}{n} \text{ with } c < 0, \quad [17]$$

and $\{X_t\}$ and $\{U_t\}$ satisfy eq. [8]. As the following theorem shows, the RTB testing procedure has the same local power properties as those shown for the residual-based block bootstrap procedure (RSB) from Paparoditis and Politis (2003).

Theorem 4 *Let $\{X_t\}$ satisfy conditions [8] and [17]. If $b_n \rightarrow \infty$ as $n \rightarrow \infty$ such that $(b_n/\sqrt{n}) \rightarrow 0$, then*

$$\beta_{\text{RTB},n}(\rho_n; \alpha) \rightarrow P(J \leq C_\alpha - c),$$

in probability, where C_α is the α -quantile of the distribution of

$$\frac{W^2(1) - \sigma_U^2/\sigma^2}{2 \int_0^1 J_c(r)^2 dr},$$

and $J_c(r) = \int_0^1 \exp\{(r-s)c\} dW(s)$ is the Ornstein–Uhlenbeck process generated by the stochastic differential equation $dJ_c(r) = cJ_c(r)dr + dW(r)$ and initial condition $J_c(0) = 0$.

7 Block size choice

In this section we give a heuristic data-driven method of choosing the block size b . Recall that one reason we are using the tapered block bootstrap procedure in this application is that it produces a provably more efficient estimator of σ_∞^2 , which is one of the unknown quantities in the asymptotic distribution of $\hat{\rho}$. Thus it is reasonable that we would want to choose our block size in such a way to make $\text{MSE}(\sigma_b^2, \text{TBB})$ of eq. [11] as small as possible.

By a calculation in Paparoditis and Politis (2001) the optimal choice of b in terms of reducing $\text{MSE}(\sigma_b^2, \text{TBB})$ is

$$b_{\text{opt},\text{TB}} = \left(\frac{4\Gamma^2}{\Delta} \right)^{1/5} n^{1/5},$$

where Γ and Δ are defined in eqs [15] and [16], respectively. For the purposes of the discussion we will let $\omega = \omega_{0.43}^{\text{TRAP}}$ and can calculate

$$\Gamma = -5.45 \sum_{k=-\infty}^{\infty} k^2 R(k) \quad \text{and} \quad \Delta = 1.1\sigma_\infty^4.$$

Since

$$\sigma_\infty^4 = \left(\sum_{k=-\infty}^{\infty} R(k) \right)^2,$$

we can estimate $\sum_{k=-\infty}^{\infty} R(k)$ and $\sum_{k=-\infty}^{\infty} k^2 R(k)$ and then obtain plug-in estimates of Γ and Δ from which a plug-in estimate of $b_{\text{opt,TB}}$ can be obtained. We denote the so-obtained estimator by $\hat{b}_{\text{opt,TB}}$ or when clear from context just $\hat{b}_{\text{opt}}$.

We proceed by estimating $\sum_{k=-\infty}^{\infty} R(k)$ by

$$\sum_{k=-M}^M \lambda(K/M) \hat{R}(k),$$

and $\sum_{k=-\infty}^{\infty} k^2 R(k)$ by

$$\sum_{k=-M}^M k^2 \lambda(K/M) \hat{R}(k),$$

where λ is the trapezoidal flattop window (see Politis and Romano (1995)):

$$\lambda(t) = \begin{cases} 1 & \text{if } |t| \in [0, 1/2] \\ 2(1 - |t|) & \text{if } |t| \in [1/2, 1] \\ 0 & \text{otherwise} \end{cases}$$

Finally, we let $M = 2\hat{m}$ where $\hat{m}$ is chosen so that $\hat{R}(k)$ is negligible for all $k > \hat{m}$. One particular algorithm for choosing $\hat{m}$ given in Politis (2003) is to let $\hat{m}$ be the smallest positive integer such that

$$|\hat{\rho}(\hat{m} + k)| < c \sqrt{\frac{\log_{10} n}{n}} \quad \text{for all } k = 1, 2, \dots, K_n$$

where $\hat{\rho}(k) = \hat{R}(k)/\hat{R}(0)$. Politis (2003) also suggests letting $K_n = 5$ and $c = 2$. We further investigate the value of this heuristic in Section 8.

8 Simulations

To investigate the small-sample performance of RTB method, several simulation studies were performed. We look at ARMA(1,1) model including the AR(1) case and also an AR(2) model.

In the first case, 2,000 instances of data were generated from an ARMA(1,1) model:

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1},$$

where $\{Z_t\}$ is an i.i.d. $N(0, 1)$ process, with different values of n (the data size), ϕ (the auto-regressive coefficient), and θ (the moving average coefficient). The null hypothesis occurs when $\phi = 1$, and in each case the nominal level of the test is $\alpha = 0.05$, and the number of bootstrap repetitions is $B = 1,000$. In addition, the tapering function that is used is $\omega_{0.43}^{\text{Trap}}$.

Table 1: Empirical rejection probabilities of the RTB test based on $\hat{\rho}_{LS}$ with $\beta = 0$

n	ϕ	θ	$\hat{b}_{opt}$	$b = 3$	$b = 5$	$b = 7$	$b = 9$	$b = 11$	$b = \hat{b}_{opt}$
25	1	0.8	4.3930	0.0745	0.0910	0.1010	0.1320	0.1785	0.0655
		0	3.7350	0.0760	0.0880	0.1340	0.1420	0.1705	0.0830
		−0.8	4.7010	0.3045	0.2770	0.3340	0.3345	0.3875	0.3035
	0.9	0.8	4.3785	0.1300	0.1610	0.1790	0.2135	0.2540	0.1465
		0	3.7200	0.2315	0.2400	0.2520	0.2630	0.2955	0.2255
		−0.8	3.5355	0.9920	0.9815	0.9700	0.9530	0.9570	0.9805
	0.85	0.8	4.3615	0.1915	0.2160	0.2330	0.2635	0.3130	0.2235
		0	3.6910	0.3105	0.3215	0.3490	0.3635	0.4060	0.2980
		−0.8	3.2910	0.9990	0.9980	0.9895	0.9905	0.9845	0.9985
50	1	0.8	5.5025	0.0635	0.0660	0.0755	0.0720	0.0870	0.0635
		0	3.7715	0.0610	0.0690	0.0690	0.0860	0.0815	0.0655
		−0.8	6.3820	0.3730	0.3185	0.3310	0.3275	0.3710	0.3170
	0.9	0.8	5.3885	0.2570	0.2865	0.3205	0.3100	0.2975	0.2835
		0	3.6215	0.4050	0.3900	0.3845	0.4300	0.3890	0.3865
		−0.8	3.2960	1.0000	0.9995	1.0000	0.9990	1.0000	1.0000
	0.85	0.8	5.3855	0.4155	0.4585	0.4965	0.4610	0.4315	0.4700
		0	3.4940	0.6245	0.6045	0.5950	0.6180	0.6145	0.5935
		−0.8	2.8045	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
100	1	0.8	6.3160	0.0540	0.0425	0.0435	0.0590	0.0655	0.0480
		0	3.5825	0.0615	0.0585	0.0485	0.0595	0.0825	0.0525
		−0.8	8.6905	0.4810	0.3705	0.3795	0.3405	0.3580	0.3810
	0.9	0.8	6.2755	0.6040	0.6900	0.6805	0.6720	0.6680	0.6515
		0	3.4590	0.7945	0.7770	0.7795	0.7655	0.7695	0.7780
		−0.8	3.1925	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	0.85	0.8	6.6145	0.8810	0.9255	0.9085	0.8900	0.8730	0.8885
		0	3.3920	0.9730	0.9700	0.9640	0.9520	0.9445	0.9725
		−0.8	2.5400	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
150	1	0.8	6.4675	0.0450	0.0455	0.0460	0.0550	0.0530	0.0470
		0	3.5160	0.0470	0.0455	0.0510	0.0625	0.0670	0.0525
		−0.8	10.9010	0.5240	0.4155	0.3930	0.3945	0.3970	0.3995
	0.9	0.8	6.4375	0.8805	0.9350	0.9345	0.9095	0.9115	0.9120
		0	3.3435	0.9710	0.9665	0.9620	0.9590	0.9555	0.9695
		−0.8	3.4655	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	0.85	0.8	7.0085	0.9940	0.9960	0.9935	0.9945	0.9890	0.9925
		0	3.3065	1.0000	1.0000	0.9995	0.9995	0.9980	0.9985
		−0.8	2.2980	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Note: Results are based on the test statistic $\hat{\rho}_{LS}$ with nominal level $\alpha = 0.05$ under different settings of the ARMA parameters ϕ and θ , and block size b .

Table 2: Empirical rejection probabilities of the RTB test based on $\hat{\rho}_{LS,C}$ with $\beta = 0$

<i>n</i>	ϕ	θ	$\hat{b}_{opt}$	<i>b</i> = 3	<i>b</i> = 5	<i>b</i> = 7	<i>b</i> = 9	<i>b</i> = 11	<i>b</i> = $\hat{b}_{opt}$
25	1	0.8	4.3785	0.0280	0.0475	0.0500	0.0725	0.0865	0.0420
		0	3.5420	0.0895	0.0965	0.0940	0.1195	0.1600	0.0860
		−0.8	3.0525	0.9700	0.9555	0.9380	0.9310	0.9135	0.9545
	0.9	0.8	4.3670	0.0475	0.0730	0.0725	0.1085	0.1510	0.0715
		0	3.5185	0.1695	0.1860	0.1935	0.2110	0.2500	0.1450
		−0.8	2.9445	0.9955	0.9905	0.9760	0.9730	0.9570	0.9930
	0.85	0.8	4.3475	0.0535	0.1030	0.1110	0.1270	0.1665	0.0850
		0	3.4590	0.2375	0.2115	0.2505	0.2595	0.3425	0.2060
		−0.8	2.9270	0.9970	0.9965	0.9875	0.9785	0.9685	0.9935
50	1	0.8	5.4970	0.0260	0.0295	0.0380	0.0395	0.0480	0.0425
		0	3.5070	0.0670	0.0775	0.0740	0.0860	0.1010	0.0650
		−0.8	3.2475	0.9910	0.9840	0.9890	0.9860	0.9885	0.9850
	0.9	0.8	5.3815	0.0800	0.1155	0.1280	0.1145	0.1180	0.1070
		0	3.3625	0.2400	0.2410	0.2635	0.2730	0.2990	0.2310
		−0.8	2.4470	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	0.85	0.8	5.3810	0.1280	0.1820	0.1955	0.1810	0.1915	0.1780
		0	3.2940	0.3815	0.3795	0.4260	0.4095	0.4485	0.3725
		−0.8	2.3455	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
100	1	0.8	6.2330	0.0245	0.0295	0.0505	0.0415	0.0410	0.0430
		0	3.4620	0.0635	0.0570	0.0630	0.0665	0.0810	0.0595
		−0.8	5.3005	0.9890	0.9870	0.9910	0.9885	0.9925	0.9935
	0.9	0.8	6.0710	0.2180	0.2970	0.3040	0.3030	0.3165	0.2835
		0	3.3065	0.4755	0.5095	0.5465	0.5545	0.5275	0.4895
		−0.8	2.4390	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	0.85	0.8	6.3060	0.4770	0.5605	0.5635	0.5510	0.5485	0.5240
		0	3.2200	0.7985	0.8075	0.8130	0.8435	0.8440	0.7950
		−0.8	2.0445	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
150	1	0.8	6.4525	0.0200	0.0425	0.0330	0.0435	0.0315	0.0345
		0	3.4870	0.0560	0.0475	0.0605	0.0660	0.0700	0.0535
		−0.8	8.0345	0.9880	0.9820	0.9765	0.9855	0.9905	0.9865
	0.9	0.8	6.4315	0.4515	0.5965	0.5830	0.5635	0.5640	0.5535
		0	3.2795	0.7810	0.8085	0.8110	0.7990	0.8075	0.7960
		−0.8	2.8690	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	0.85	0.8	6.9170	0.8245	0.8810	0.8950	0.8625	0.8500	0.8660
		0	3.2250	0.9775	0.9735	0.9755	0.9840	0.9860	0.9850
		−0.8	2.0745	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Note: Results are based on the test statistic $\hat{\rho}_{LS,C}$ with nominal level $\alpha = 0.05$ under different settings of the ARMA parameters ϕ and θ , and block size *b*.

Tables 1 and 2 give the rejection rates for different values of b , including $b = \hat{b}_{\text{opt}}$ for the test statistics $\hat{\rho} = \hat{\rho}_{LS}$ and $\hat{\rho} = \hat{\rho}_{LS,C}$, respectively. For the purposes of this section we will focus on the results for $\hat{\rho}_{LS}$, although the results for $\hat{\rho}_{LS,C}$ are included as well.

The case where $\phi = 1, \theta = -0.8$ deserves some special attention, since the RTB and the other unit root test do very poorly. This case, however, is known to be problematic since the moving average polynomial has a root close to unity and, although it is integrated, it behaves almost like an i.i.d. series for small samples. In fact, Campbell and Perron (1991) go as far as to claim that a stationary model might be more appropriate for modeling this situation.² Thus it is neither surprising nor all that troublesome that the test does not perform well in this case. It is, however, included in the table for completeness. This case does not contradict the consistency of the test, but the data set needs to be quite large for the test to give anything close to rejecting at the nominal rate. For this reason, we will not look at this case for the remainder of this section.

In the other cases we can see that $\hat{b}_{\text{opt}}$ does quite well. Notice for $\hat{\rho}_{LS}$ under the null, $\hat{b}_{\text{opt}}$ gives a rejection rate closest to the nominal level $\alpha = 0.05$ and at the same time its power is comparable. To be concrete, consider the case where $n = 50$ and $\theta = 0.8$. We see that using $\hat{b}_{\text{opt}}$ gives a rejection rate closer to 0.05 under the null of $\phi = 1$ than any of the fixed block strategies. However, under the alternative of $\phi = 0.9$, the rejection rate is comparable to any of the fixed block sizes that are rejecting more often under the null. In fact, the rejection rate is significantly higher than any of its closest competitors when $\phi = 0.85$.

We also can compare the RTB method to the unit root test given in Phillips and Perron (1988). The results for the same ARMA process are given in tables 5 and 8 of Parker, Paparoditis, and Politis (2006). Notice the RTB gives values significantly closer to nominal value under the null and rejects more often under the alternative. Thus, the RTB is indeed an attractive alternative to this standard unit root test.

We also looked at the AR(2) model:

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = Z_t,$$

which can be written as:

$$(1 - \phi_1 B - \phi_2 B^2)X_t = Z_t,$$

where B is the lag-one operator. We only consider the case where the polynomial $1 - \phi_1 x - \phi_2 x^2$ has two real roots $\varepsilon_1 > \varepsilon_2$. The case where $\varepsilon_1 = 1$ is when the

² For a more general discussion of using a stationary model when the true process is integrated, see Cochrane (1991).

process is $I(1)$, otherwise it is stationary. The results of this simulation are given in Table 3. They are similar to the ARMA model, as, under the null hypothesis, the rejection rate get close to nominal rate as n gets larger. For $n = 100$ it is quite close. The table also shows go power results for $n \geq 100$ even for $\varepsilon = 0.9$. In fact, the test performs well even in the $n = 50$ case when $\varepsilon_2 \leq 0.5$, we see a definitive increase in rejection of the null hypothesis as the ε_1 decreases away from 1. These results are achieved with $b = \hat{b}_{\text{opt}}$ and hence, the results do not rely on the practitioner guessing an appropriate value of b .

Table 3: Empirical rejection probabilities of the RTB test based on $\hat{\rho}_{LS}$ with $\beta = 0$

n	ε_1	$\varepsilon_2 = 0.25$	$\varepsilon_2 = 0.5$	$\varepsilon_2 = 0.75$
25	1	0.0850	0.1075	0.1440
	0.95	0.1055	0.0980	0.1045
	0.9	0.1440	0.1015	0.1060
	0.85	0.2325	0.1380	0.0900
50	1	0.0705	0.0770	0.0980
	0.95	0.1290	0.1115	0.0880
	0.9	0.2910	0.1020	0.1185
	0.85	0.4465	0.3445	0.1535
100	1	0.0530	0.0575	0.0630
	0.95	0.2725	0.2425	0.1500
	0.9	0.6555	0.5435	0.3225
	0.85	0.8980	0.7475	0.4525
150	1	0.0485	0.0465	0.0550
	0.95	0.4675	0.4170	0.2820
	0.9	0.9060	0.8375	0.6165
	0.85	0.9965	0.9705	0.7705

Note: Results are based on the test statistic $\hat{\rho}_{LS}$ with nominal level $\alpha = 0.05$ under different settings with AR(2) process with roots ε_1 and ε_2 the block size is $\hat{b}_{\text{opt}}$.

8.1 Comparison between block bootstrap unit root tests

Together with the Residual-Based Block Bootstrap test (RBB) from Paparoditis and Politis (2003) as well as the Residual-Based Stationary Bootstrap test (RSB) from Parker, Paparoditis, and Politis (2006) and the RTB, we have three residual-based block bootstrap techniques for testing for a unit root. All three have been shown to be asymptotically first-order consistent and to have favorable power properties. Thus, the practitioner has three choices for unit root testing based on this paradigm. However, it would be useful to know which method works best in

which cases. It is an open problem to prove that any of the methods is better than any of the others under some fixed criteria. We suspect that if the criterion is minimizing variance, the tapered block bootstrap would be superior, however this has not yet been proved. Nor is it known whether or not this would also give a test of the most accurate size. These are areas where additional research is required.

Table 4: Comparison table between block bootstrap techniques

ϕ	θ	n	RTB	RBB	RSB	n	RTB	RBB	RSB
1	0.8	25	0.0655	0.0770	0.0935	100	0.0480	0.0495	0.0520
	0		0.0830	0.1160	0.0770		0.0525	0.0985	0.0505
0.9	0.8		0.1465	0.1460	0.1135		0.6515	0.6645	0.4895
	0		0.2255	0.2755	0.1875		0.7780	0.8645	0.7745
0.85	0.8		0.2235	0.2165	0.1420		0.8885	0.8655	0.7680
	0		0.2980	0.3830	0.2590		0.9725	0.9885	0.9655
1	0.8	50	0.0635	0.0835	0.0715	150	0.0470	0.0550	0.0480
	0		0.0655	0.1010	0.0615		0.0525	0.0915	0.0595
0.9	0.8		0.2835	0.2820	0.1855		0.9120	0.9130	0.7900
	0		0.3865	0.5445	0.3555		0.9695	0.9755	0.9675
0.85	0.8		0.4700	0.4495	0.2885		0.9925	0.9930	0.9720
	0		0.5935	0.7515	0.5710		0.9985	1.0000	1.0000

Note: For each method the block sizes are chosen by data-based methods and the test statistic $\hat{\rho}_{LS}$ is used.

However, the three tests can be compared through simulation. To this end, Table 4 compares the three methods, where the block size (or average block size in the case of the RSB) is chosen by the data-driven methods described in Section 7 for RTB as well as the corresponding methods used to pick the best block size suggested in Paparoditis and Politis (2003) for the RBB and the best average block size for the RSB described in Parker, Paparoditis, and Politis (2006).

The simulations do suggest that the RTB is in fact consistently superior in approaching the correct size of the test in nearly every case (only when $n = 100$, $\phi = 1$ and $\theta = 0.8$ does the simulated RBB have a rejection slightly closer to the nominal rate). Thus it appears the tapering process does improve the test's ability to give the correct size.

But giving the correct size is not enough to say the TBB is a superior test. We must also consider the power. Power and size must be considered together, because a test that rejects too often under the null will also tend to have higher power. We thus follow the recommendation of Lloyd (2005) and adjust the

power by according to how much it under- or overestimates the nominal size of the test. For example, consider the case $n = 150$ and $\theta = 0.0$. When $\phi = 1.0$ we get an estimate of size of the test $\hat{\alpha}_{\text{RTB}} = 0.0525$ and $\hat{\alpha}_{\text{RBB}} = 0.0915$. When $\phi = 0.9$ we get an estimate of the power for this alternative $\hat{\beta}_{\text{RTB}} = 0.9695$ and $\hat{\beta}_{\text{RBB}} = 0.9755$. Using the suggestion of the above paper, we let the adjusted power be

$$R(\alpha) = \Phi\left(\Phi^{-1}(\hat{\beta}) - \Phi^{-1}(\hat{\alpha}) + \Phi^{-1}(\alpha)\right),$$

where α is the nominal level, so for our simulations $\alpha = 0.05$. So in the above example $R_{\text{RTB}}(0.05) = 0.9678$ and $R_{\text{RBB}}(0.05) = 0.9511$. Thus, even though the RTB gives a smaller unadjusted power, its adjusted power is larger than the RBB in this case. The adjusted power in the other cases is given in Table 5.

Table 5: Adjusted comparison table between block bootstrap techniques

ϕ	θ	n	RTB	RBB	RSB	n	RTB	RBB	RSB
0.9	0.8	25	0.1138	0.1015	0.0626	100	0.6588	0.6662	0.4819
	0		0.1554	0.1478	0.1343		0.7709	0.7722	0.7730
0.85	0.8		0.1854	0.1578	0.0812		0.8922	0.8666	0.7621
	0		0.2148	0.2275	0.1933		0.9710	0.9725	0.9651
0.9	0.8	50	0.2447	0.2005	0.1412	150	0.9167	0.9054	0.7900
	0		0.3361	0.3983	0.3181		0.9678	0.9511	0.9607
0.85	0.8		0.4231	0.3985	0.2303		0.9931	0.9920	0.9732
	0		0.5406	0.6218	0.5304		0.9984	1.0000	1.0000

The table gives clear evidence that the RTB does give better results than the RSB; however the comparison between the RTB and RBB is more mixed. The RTB does better when $\theta = 0.8$ in all but one case. Notice the $\theta = 0.8$ is also the case where the dependency structure on X_t is the greatest (i.e. $R(h)$ is larger when $\theta = 0.8$ for either value of ϕ then it is for $\theta = 0.0$). Conversely, in all but one case when $\theta = 0.0$ the RBB does better. Thus it appears that the RTB does the best when the long-range dependency is greater, which is exactly when a unit root test is most often used. Taken together with the RTB’s better size properties, the RTB seems to perform the best of the three methods. One must keep in mind that performance of each test in this simulation depends not only on the value of the test but also on the value of the heuristic for choosing the block size.

9 Conclusions

The success of the tapered bootstrap in estimating the variance of the sample mean suggests that a residual-based test using this procedure would be a good alternative to the RBB and RSB methods previously proposed and thus a useful non-parametric unit root test. As shown in the paper, the RTB is in fact a first-order consistent test that shares the favorable power properties of the RBB and RSB methods. In addition, using the suggestion for picking a block size presented in Section 7, we are able to compare the three techniques and conclude that in many finite-sample situations the RTB appears to be a superior alternative.

10 Technical proofs

Lemma 5 *For all taper functions ω satisfying eq. [13], then for some constant C_ω (depending on ω but not b)*

$$\omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \leq C_\omega.$$

PROOF: Fix a tapering function ω satisfying eq. [13]. Then there exists an $\varepsilon > 0$ and $\delta > 0$ so that

$$\omega(t) \geq \delta \cdot \mathbf{1}_{(1/2-\varepsilon, 1/2+\varepsilon)}(t).$$

Thus

$$\omega_b(t) \geq \delta \cdot \mathbf{1}_{((1/2-\varepsilon)b+1/2, (1/2+\varepsilon)b+1/2)}.$$

Since there are at least $2\varepsilon b - 1$ integers in $((1/2 - \varepsilon)b + 1/2, (1/2 + \varepsilon)b + 1/2)$ it follows that

$$\sum_{t=1}^b \omega_b^2(t) \geq \delta^2(2\varepsilon b - 1),$$

and so

$$\|\omega_b\|_2 \geq \delta\sqrt{2\varepsilon b - 1}.$$

This gives

$$\omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \leq \frac{\sqrt{b}}{\|\omega_b\|_2} \leq \frac{\sqrt{b}}{\delta\sqrt{2\varepsilon b - 1}} = \frac{1}{\delta\sqrt{2\varepsilon - 1/b}} \leq \frac{1}{\delta\sqrt{2\varepsilon - 1}}.$$

Hence $C_\omega = \frac{1}{\delta\sqrt{2e-1}}$ is the desired constant.

△

For the proof of Theorem 1 define the following process:

$$R(r) = \frac{1}{\hat{\sigma}_n^* \sqrt{n}} \sum_{m=1}^{\lfloor \frac{nr}{b} \rfloor} \sum_{j=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \tilde{U}_{i_m+j-1}.$$

We can thus show that this process is asymptotically equivalent to the process S_n^* .

Lemma 6 *Uniformly in r the following holds in probability*

$$E^* \left(\hat{\sigma}_n^* R(r) - \hat{\sigma}_n^* S_n^*(r) \right)^2 \rightarrow 0.$$

PROOF: Define

$$k_{r,n} = \left\lfloor \frac{\lfloor nr \rfloor}{b} \right\rfloor$$

hence

$$k_{r,n} = O\left(\frac{n}{b}\right).$$

Notice that

$$\begin{aligned} \hat{\sigma}_n^* S_n^*(r) &= n^{-1/2} \sum_{m=1}^{k_{r,n}} \sum_{j=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \hat{U}_{i_m+j-1} \\ &\quad + n^{-1/2} \sum_{j=1}^{\lfloor nr \rfloor - bk_{r,n}} \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \hat{U}_{i_{k_{r,n}+1}+j-1}. \end{aligned}$$

First we show that uniformly in r

$$E^* \left(\hat{\sigma}_n^* R(r) - n^{-1/2} \sum_{m=1}^{k_{r,n}} \sum_{j=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \hat{U}_{i_m+j-1} \right)^2 \xrightarrow{P} 0. \quad [18]$$

To see eq. [18], using Lemma 5

$$\sum_{m=1}^{k_{r,n}} \sum_{s=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \hat{U}_{i_m+s}$$

$$\begin{aligned}
&= \sum_{m=1}^{k_{r,n}} \sum_{s=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \tilde{U}_{i_m+s} \\
&\quad - (\hat{\rho}_n - \rho) \sum_{m=1}^{k_{r,n}} \sum_{s=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \left(X_{i_m+s+1} - \frac{1}{n-1} \sum_{\tau=1}^{n-1} X_\tau \right) \\
&\quad - \sum_{m=1}^{k_{r,n}} \sum_{s=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \left(\frac{1}{n-1} \sum_{\tau=2}^n U_\tau - \mathbb{E}^* U_{i_m+s} \right) \\
&\quad - (\hat{\rho}_n - \rho) \sum_{m=1}^{k_{r,n}} \sum_{s=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \left(\frac{1}{n-1} \sum_{\tau=1}^{n-1} X_\tau - \mathbb{E}^* X_{i_m+s} \right)
\end{aligned} \tag{19}$$

since

$$\begin{aligned}
&\sum_{m=1}^{k_{r,n}} \sum_{s=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \hat{U}_{i_m+s} \\
&= \sum_{m=1}^{k_{r,n}} \sum_{s=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \left[(X_{i_m+s+1} - \rho X_{i_m+s}) - \frac{1}{n-1} \sum_{\tau=1}^{n-1} (X_{\tau+1} - \rho X_\tau) \right] \\
&\quad - (\hat{\rho}_n - \rho) \sum_{m=1}^{k_{r,n}} \sum_{s=1}^b \left(X_{i_m+s} - \frac{1}{n-1} \sum_{\tau=1}^{n-1} X_\tau \right) \\
&= \sum_{m=1}^{k_{r,n}} \sum_{s=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \left(U_{i_m+s} - \frac{1}{n-1} \sum_{\tau=1}^{n-1} U_\tau \right) \\
&\quad - (\hat{\rho}_n - \rho) \sum_{m=1}^{k_{r,n}} \sum_{s=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \left(X_{i_m+s+1} - \frac{1}{n-1} \sum_{\tau=1}^{n-1} X_\tau \right) \\
&= \sum_{m=1}^{k_{r,n}} \sum_{s=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} (U_{i_m+s} - \mathbb{E}^* U_{i_m+s}) \\
&\quad - (\hat{\rho}_n - \rho) \sum_{m=1}^{k_{r,n}} \sum_{s=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \left(X_{i_m+s+1} - \frac{1}{n-1} \sum_{\tau=1}^{n-1} X_\tau \right) \\
&\quad - \sum_{m=1}^{k_{r,n}} \sum_{s=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \left(\frac{1}{n-1} \sum_{\tau=1}^{n-1} U_\tau - \mathbb{E}^* U_{i_m+s} \right) \\
&= \sum_{m=1}^{k_{r,n}} \sum_{s=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} (U_{i_m+s} - \mathbb{E}^* U_{i_m+s})
\end{aligned}$$

$$\begin{aligned}
& -(\hat{\rho}_n - \rho) \sum_{m=1}^{k_{r,n}} \sum_{s=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \left(X_{i_m+s+1} - \frac{1}{n-1} \sum_{\tau=1}^{n-1} X_\tau \right) \\
& - \sum_{m=1}^{k_{r,n}} \sum_{s=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \left(\frac{1}{n-1} \sum_{\tau=1}^{n-1} U_\tau - E^* U_{i_m+s} \right) \\
& - (\hat{\rho}_n - \rho) \sum_{m=1}^{k_{r,n}} \sum_{s=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \left(\frac{1}{n-1} \sum_{\tau=1}^{n-1} X_\tau - E^* X_{i_m+s} \right).
\end{aligned}$$

In order to show eq. [18] it remains to show

$$E^* \left((\hat{\rho}_n - \rho) \sum_{m=1}^{k_{r,n}} \sum_{s=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} (X_{i_m+s+1} - E^* X_{i_m+s+1}) \right)^2 \xrightarrow{P} 0 \quad [20]$$

$$E^* \left(\sum_{m=1}^{k_{r,n}} \sum_{s=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \left(\frac{1}{n-1} \sum_{\tau=1}^{n-1} U_\tau - E^* U_{i_m+s} \right) \right)^2 \xrightarrow{P} 0 \quad [21]$$

Let

$$\tilde{X}_t = X_t - \sum_{\tau=1}^{n-1} X_\tau$$

and for each m define

$$V_m^* \equiv \sum_{j=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \tilde{X}_{i_m+j}.$$

Then $\{V_m\}$ is an i.i.d. sequence with respect to P^* . By eq. [20] it is enough to show that, uniformly in r

$$E^* \left((\hat{\rho}_n - \rho) \frac{1}{\sqrt{n}} \sum_{m=1}^{k_{r,n}} V_m^* \right)^2 \rightarrow 0. \quad [22]$$

In the case where $\rho \neq 1$, it is straightforward to see that eq. [22] holds. This is because here $(\hat{\rho}_n - \rho) = o_P(1)$ and because $\{X_t\}$ is stationary and hence

$$\sum_{\tau=1}^{n-1} X_\tau = O_P(n^{1/2}),$$

and so

$$E^*(V_m)^2 = O_P(b),$$

and hence

$$E^*\left(\hat{\rho}_n - \rho\right) \frac{1}{\sqrt{n}} \sum_{m=1}^{k_{r,n}} V_m^* \Big)^2 = o_P(1) n^{-1} O(n/b) O_P(b) = o_P(1).$$

However in the case where $\rho = 1$ then

$$\begin{aligned} E^*\left(\frac{1}{n-1} \sum_{\tau=1}^{n-1} X_{\tau}\right) &= \frac{1}{n-1} \sum_{\tau=1}^{n-1} \left(\sum_{k=1}^{\tau} U_k + \tau\beta\right) \\ &= \frac{1}{n-1} \sum_{\tau=1}^{n-1} (O_P(\sqrt{\tau}) + \tau\beta) \\ &= O_P(n^{\delta(\beta)}), \end{aligned}$$

and

$$\begin{aligned} E^*(X_{i_m+j}) &= \frac{1}{n-b} \sum_{t=1}^{n-b} (X_{t+j}) = \frac{1}{n-b} \sum_{t=1}^{n-b} (O_P((t+b)^{1/2} + \tau\beta)) \\ &= O_P(n^{\delta(\beta)}), \end{aligned}$$

and

$$E^*(\tilde{X}_{i_m+j}) = O_P(n^{\delta(\beta)})$$

hence, for all $1 \leq j_1 \leq j_2 \leq b$ we have

$$\begin{aligned} E^*(X_{i_m+j_1} X_{i_m+j_2}) &= \frac{1}{n-b} \sum_{t=1}^{n-b} \left[X_{t+j_1} \omega_b(j_2) \frac{\sqrt{b}}{\|\omega_b\|_2} X_{t+j_2} \right] \\ &= \frac{1}{n-b} \sum_{t=1}^{n-b} \left[\left(\sum_{k=1}^{t+j_1} U_k + (t+s_1)\beta \right) \left(\sum_{k=1}^{t+j_2} U_k + (t+s_1)\beta \right) \right] \\ &= \frac{1}{n-b} \sum_{t=1}^{n-b} \left[\sum_{k=1}^{t+j_1} U_k \sum_{k=1}^{t+j_2} U_k \right. \\ &\quad \left. + (t+j_1)\beta \sum_{k=1}^{t+j_2} U_k + (t+j_2)\beta \sum_{k=1}^{t+j_1} U_k + (t+j_1)(t+j_2)\beta^2 \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{n-b} \sum_{t=1}^{n-b} \left[\left(\sum_{k=1}^{t+j_1} U_k \right)^2 + \sum_{k=1}^{t+j_1} U_k \sum_{k=t+j_1+1}^{t+j_2} U_k \right. \\
&\quad \left. + (t+j_1)\beta \sum_{k=1}^{t+j_2} U_k + (t+j_2)\beta \sum_{k=1}^{t+j_1} U_k + (t+j_1)(t+j_2)\beta^2 \right] \\
&= \frac{1}{n-b} \sum_{t=1}^{n-b} \left[O_P((t+b)) + O_P(b(t+b)^{1/2})1_{\beta \neq 0} \right. \\
&\quad \left. + O_P((t+b)^{3/2})1_{\beta \neq 0} + O_P((t+b)^2)1_{\beta \neq 0} \right] \\
&\quad + (t+j_1)\beta \sum_{k=1}^{t+j_2} U_k + (t+j_2)\beta \sum_{k=1}^{t+j_1} U_k + (t+j_1)(t+j_2)\beta^2 \\
&= \frac{1}{n-b} \sum_{t=1}^{n-b} \left[O_P(t+b) + O_P((t+b)^2)1_{\beta \neq 0} \right] \\
&= O_P(n) + O_P(n^2)1_{\beta \neq 0} \\
&= O_P(n^{1+\delta(\beta)}),
\end{aligned}$$

and thus

$$\begin{aligned}
&E^* \left(\omega_b(j_1) \frac{\sqrt{b}}{\|\omega_b\|_2} \tilde{X}_{i_m+j_1} \omega_b(j_2) \frac{\sqrt{b}}{\|\omega_b\|_2} \tilde{X}_{i_m+j_2} \right) \\
&= O_P(n^{1+\delta(\beta)}) + 3O_P(n^{\delta(\beta)})^2 \\
&= O_P(n^{1+\delta(\beta)}).
\end{aligned}$$

In particular

$$E^* \left(\omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \tilde{X}_{i_m+j} \right)^2 = O_P(n^{1+\delta(\beta)}).$$

From the above calculations we get that

$$E^*(V_m^*) = O_P(b^2 n^{1+\delta(\beta)}).$$

It follows that in this case ($\rho \neq 1$)

$$\sum_{m=1}^{k_{r,n}} \sum_{s=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} (X_{i_m+s+1} - E^* X_{i_m+s+1})^2$$

and thus eq. [20] holds in this case as well since

$$E^* \left((\hat{\rho}_n - \rho) \sum_{m=1}^{k_{r,n}} \sum_{t=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} (X_{i_m+t+1} - E^* X_{i_m+t+1}) \right)^2$$

$$E^* \left((\hat{\rho}_n - \rho) \sum_{m=1}^{k_{r,n}} V_m^* \right)^2$$

$$n^{-2-\delta(\beta)} n^{-1} O_P(bn^{2+\delta(\beta)})$$

$$O_P(bn^{-1}) \xrightarrow{P} 0.$$

Since

$$E^* U_{i_m+s} = \frac{1}{n-b} \sum_{\tau=1}^{n-b} U_{\tau},$$

to prove eq. [21] it is enough to show

$$\left(\sum_{m=1}^{k_{r,n}} \sum_{s=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \left(\frac{1}{n-1} \sum_{\tau=1}^{n-1} U_{\tau} - \frac{1}{n-1} \sum_{\tau=1}^{n-b} U_{\tau} \right) \right)^2 \xrightarrow{P} 0, \quad [23]$$

$$\left(\sum_{m=1}^{k_{r,n}} \sum_{s=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \left(\frac{1}{n-1} \sum_{\tau=1}^{n-b} U_{\tau} - \frac{1}{n-b} \sum_{\tau=1}^{n-b} U_{\tau} \right) \right)^2 \xrightarrow{P} 0. \quad [24]$$

These follow from

$$\begin{aligned} \left(\frac{1}{n-1} \sum_{\tau=1}^{n-1} U_{\tau} - \frac{1}{n-1} \sum_{\tau=1}^{n-b} U_{\tau} \right)^2 &= \frac{1}{(n-1)^2} \left(\sum_{\tau=n-b+1}^{n-1} U_{\tau} \right)^2 \\ &= \frac{1}{(n-1)^2} O_P(b) \\ &= O_P\left(\frac{b}{n^2}\right), \end{aligned}$$

and

$$\begin{aligned} \left(\frac{1}{n-1} \sum_{\tau=1}^{n-b} U_{\tau} - \frac{1}{n-b} \sum_{\tau=1}^{n-b} U_{\tau} \right)^2 &= \left(\frac{1-b}{(n-1)(n-b)} \sum_{\tau=1}^{n-b} U_{\tau} \right)^2 \\ &= O_P \left(\frac{b^2}{n^3} \right). \end{aligned}$$

Thus

$$\begin{aligned} &\left(\sum_{m=1}^{k_{r,n}} \sum_{s=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \left(\frac{1}{n-1} \sum_{\tau=1}^{n-1} U_{\tau} - \frac{1}{n-1} \sum_{\tau=1}^{n-b} U_{\tau} \right) \right)^2 \\ &= n^{-1} O_P \left(\frac{n^2}{b^2} \right) b^2 O_P \left(\frac{b}{n^2} \right) \\ &= O_P(bn^{-1}) \xrightarrow{P} 0, \end{aligned}$$

and

$$\begin{aligned} &\left(\sum_{m=1}^{k_{r,n}} \sum_{s=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \left(\frac{1}{n-1} \sum_{\tau=1}^{n-b} U_{\tau} - \frac{1}{n-b} \sum_{\tau=1}^{n-b} U_{\tau} \right) \right)^2 \\ &= n^{-1} O_P \left(\frac{n^2}{b^2} \right) b^2 O_P \left(\frac{b^2}{n^3} \right) \\ &= O_P(b^2 n^{-2}) \xrightarrow{P} 0, \end{aligned}$$

Which proves eq. [18]. To complete the proof we must show that

$$n^{-1/2} \sum_{j=1}^{\lfloor nr \rfloor - bk_{r,n}} \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \hat{U}_{i_{k_{r,n}+1}+j-1} \xrightarrow{P^*} 0.$$

For this it is enough to show that

$$n^{-1/2} \sum_{j=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \hat{U}_{i_{k_{r,n}+1}+j-1} \xrightarrow{P^*} 0.$$

However, by eq. [18] it is enough to show

$$n^{-1/2} \sum_{j=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} U_{i_{k_{r,n}+1}+j-1} \xrightarrow{P^*} 0,$$

which holds since $U_{i_{k,n+1}+j-1}$ is stationary and $\omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2}$ is bounded. This completes the proof. Δ

Proof of Theorem 1.

From Lemma 6 it is enough to show that:

$$R_n \xrightarrow{d^*} W \quad [25]$$

in probability. Let

$$V_m = \sum_{j=1}^b \omega_b(j) \frac{\sqrt{b}}{\|\omega_b\|_2} \tilde{U}_{im+j-1},$$

where

$$\tilde{U}_t = U_t - \frac{1}{n-1} \sum_{\tau=2}^n U_\tau \quad \text{for } t = 2, 3, \dots, n.$$

Then (with respect to P^*) V_m is i.i.d. with mean 0 and variance b . We can now rewrite R_n as

$$R_n(r) = \frac{1}{\sqrt{n\tilde{\sigma}_n^*}} \sum_{m=1}^{k_{[rn]}} V_m.$$

In order to show eq. [25] we must show the following (see for example Theorem 13.5 on page 142 of Billingsley (1999)):

1. For all sequences $0 \leq s_1 < s_2 < \dots < s_k \leq 1$

$$(R_n(s_1), R_n(s_2), R_n(s_3), \dots, R_n(s_k)) \xrightarrow{d^*} (Z_{s_1}, Z_{s_2}, \dots, Z_{s_k}), \quad [26]$$

where $Z_\alpha \sim N(0, \alpha)$.

2. For all $0 \leq r < s < t \leq 1$,

$$E^* \left[|R_n(t) - R_n(s)|^2 |R_n(s) - R_n(r)|^2 \right] \leq C(t-r)^2, \quad [27]$$

where C is a stochastically bounded random variable independent of s and t .

Notice that the process R_n has independent increments i.e.

$$R_n(t) - R_n(s) \perp R_n(s) - R_n(r). \quad [28]$$

Now to show eq. [26] it is enough to show the equivalent statement

$$(R(s_1), R(s_2) - R(s_1), \dots, R(s_k) - R(s_{k-1})) \xrightarrow{d^*} (Z_{s_1}, Z_{s_2-s_1}, \dots, Z_{s_k-s_{k-1}}).$$

And by eq. [28] it is enough to show that for all s, r such that $0 \leq r < s \leq 1$:

$$R_n(s) - R_n(r) \xrightarrow{d^*} N(0, s - r). \quad [29]$$

Since $R_n(s) - R_n(r)$ and $R_n(s - r)$ have roughly the same number of summands (within 2 of each other) which is asymptotically negligible and thus it is enough to show eq. [29] for the case when $r = 0$ and $s = 1$. Hence we must show that

$$R_n(1) \xrightarrow{d^*} N(0, 1).$$

This however is a consequence of Theorem 2.3 of Paparoditis and Politis (2001). Next compute

$$\begin{aligned} E(R_n(s) - R_n(r))^2 &= \frac{1}{n(\tilde{\sigma}_n^*)^2} \sum_{m=k_{r,n}+1}^{k_{s,n}} E(V_m)^2 \\ &= \frac{1}{n(\tilde{\sigma}_n^*)^2} O\left(\frac{n(s-r)}{b}\right) O_P(b) \\ &= O_P(1)(s-r). \end{aligned}$$

From this and the independent increments eq. [27] follows, and hence eq. [29] holds. Δ

Proof of Theorem 2.

The proof is the same as the proof of Theorems 2–4 in Parker, Paparoditis, and Politis (2006) using Theorem 1. Δ

Proof of Theorem 3.

The second part follows from Theorem 2. The proof of the first part is the same as the proof of Theorem 5 in Parker, Paparoditis, and Politis (2006) or Corollary 5.1 in Paparoditis and Politis (2003). Δ

Proof of Theorem 4:

Notice that since $\rho_n = 1 + O(n^{-1})$:

$$(\rho_n - \hat{\rho}_{LS}) = (\rho_n - 1) + (1 - \hat{\rho}_{LS}) = O_P(n^{-1}).$$

Thus the proof of Theorem 1 still holds in this case as well. Hence

$$n(\hat{\rho}_{LS}^* - 1) \xrightarrow{d^*} \frac{W(1)^2 - \frac{\sigma_U^2}{\sigma^2}}{2 \int_0^1 W^2(r) dr} \quad [30]$$

and thus

$$C_\alpha^* \xrightarrow{P} C_\alpha. \quad [31]$$

By Theorem 1 in Phillips (1987b)

$$n(\hat{\rho}_{LS} - \rho_n) \xrightarrow{d} J. \quad [32]$$

By Slutsky's Theorem the convergences in eqs [31] and [32] are joint convergences. Thus

$$\begin{aligned} P(n(\hat{\rho}_{LS} - 1) \leq C_\alpha^*) &= P(n(\hat{\rho}_{LS} - \rho_n) \leq C_\alpha^* - n(\rho_n - 1)) \\ &= P(n(\hat{\rho}_{LS} - \rho_n) \leq C_\alpha^* - c) \\ &\xrightarrow{d} P(J \leq C_\alpha - c). \end{aligned}$$

Δ

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