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Research Article

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Pointwise density topology

Abstract: The paper presents a new type of density topology on the real line generated by the pointwise convergence, similarly to the classical density topology which is generated by the convergence in measure. Among other things, this paper demonstrates that the set of pointwise density points of a Lebesgue measurable set does not need to be measurable and the set of pointwise density points of a set having the Baire property does not need to have the Baire property. However, the set of pointwise density points of any Borel set is Lebesgue measurable.

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1 Introduction

Let $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$ denote the sets of real numbers, rational numbers, integers and positive integers, respectively. Throughout this paper, we denote by λ the Lebesgue measure on the real line, by $\mathcal{L}$ the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$ and by $\mathcal{N}$ the σ-ideal of Lebesgue null sets on $\mathbb{R}$. Let $\mathcal{B}a$ and $\mathcal{M}$ be the σ-algebra of sets having the Baire property and the σ-ideal of first category sets on the real line with respect to the Euclidean topology, respectively. Let $\mathcal{B}$ be the σ-algebra of Borel sets on $\mathbb{R}$. We will use the symbol χ_A to denote the characteristic function of set $A \subset \mathbb{R}$. We write Span A for the linear space over $\mathbb{Q}$ generated by elements from a set $A \subset \mathbb{R}$.

Recall that $x_0 \in \mathbb{R}$ is a density point of set $A \in \mathcal{L}$ if and only if

$$\lim_{h \to 0^+} \frac{\lambda(A \cap [x_0 - h, x_0 + h])}{2h} = 1. \tag{1}$$

Observe that condition (1) is equivalent to the following one

$$\lim_{n \to \infty} \frac{\lambda(A \cap [x_0 - \frac{1}{n}, x_0 + \frac{1}{n}])}{\frac{2}{n}} = 1$$

and if we put $nA = \{na : a \in A\}$ and $A - x_0 = \{a - x_0 : a \in A\}$ for $n \in \mathbb{N}$, then the above condition can be written in the following form $\lim_{n\to\infty} \lambda(n(A - x_0) \cap [-1, 1]) = 2$. The last condition means that the sequence of characteristic functions $\{\chi_{n(A-x_0)\cap[-1,1]}\}_{n\in\mathbb{N}}$ converges in measure λ to $\chi_{[-1,1]}$, (cf. [7]).

For all $A \in \mathcal{L}$ denote by $\Phi_d(A)$, the set $\{x \in \mathbb{R} : x$ is a density point of $A\}$. In this way we obtain the density operator $\Phi_d : \mathcal{L} \to \mathcal{L}$. It is well known that the family $\mathcal{T}_d = \{A \in \mathcal{L} : A \subset \Phi_d(A)\}$ is a topology on the real line called the density topology (cf. Chapter 22 in [6] or [1]). Clearly $\mathcal{T}_d$ is stronger than the Euclidean topology $\mathcal{T}$ on the real line. Since the set $\Phi_d(A)$ of density points of a set $A \in \mathcal{L}$ is characterized by the above mentioned convergence in measure, we shall say that density topology $\mathcal{T}_d$ is generated by the convergence in measure or by the operator Φ_d. One can obtain different density-type topologies on the real line by using various types of convergence of sequences of measurable functions and choosing different domains for density operators (cf. [9]). Thus, for example, the density-type topology generated by convergence almost everywhere is called the simple

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density topology and denoted by $\mathcal{T}_s$ (cf. [8]). The topology generated by convergence everywhere except for a finite set is called the finite density topology and denoted by $\mathcal{T}_{fin}$ (cf. [2]). It is easy to see that the Euclidean topology $\mathcal{T}$ can be considered as a density-type topology generated by uniform convergence. One can prove that $\mathcal{T} \subsetneqq \mathcal{T}_{fin} \subsetneqq \mathcal{T}_s \subsetneqq \mathcal{T}_d$, (cf. [2, 8, 9]).

In this paper we define a pointwise density point by a slight modification of the definition density point, replacing the convergence in measure with the poinwise convergence. In Section 2, we study the properties of the pointwise density operator Φ_p assigning to a set $A \subset \mathbb{R}$ the set of its pointwise density points. Section 3 defines the pointwise density topology $\mathcal{T}_p$ which turns out to be generated by the operator Φ_p for Lebesque measurable sets. The pointwise topology is significantly different from the ones mentioned above. It is known that if $A \in \mathcal{L}$, then the set $\Phi_d(A)$ and the set $\Phi_s(A)$ of all points of simple density points of A are both Lebesgue measurable (cf. [7, 8]). We prove in Theorem 2.10 that one can construct a Lebesgue measurable set for which the set of all pointwise density points is nonmeasurable. Theorem 2.11 is a duality result which asserts that there exists a set having the Baire property such that the set of all pointwise density points is not the set having the Baire property.

2 Pointwise density point

First, we introduce the concept of pointwise density point (p-density point for short) for every subset of $\mathbb{R}$. Then we study the properties of the operator Φ_p assigning to a set $A \subset \mathbb{R}$ the set of its pointwise density points.

Definition 2.1. *Let $A \subset \mathbb{R}$. We shall say that*
(i) *0 is a p-density point of A if and only if the sequence $\{\chi_{nA\cap[-1,1]}\}_{n\in\mathbb{N}}$ is convergent everywhere to the function $\chi_{[-1,1]}$,*
(ii) *$x \in \mathbb{R}$ is a p-density point of A if and only if 0 is a p-density point of $A - x$,*
(iii) *$x \in \mathbb{R}$ is a p-dispersion point of A if and only if x is a p-density point of $\mathbb{R} \setminus A$.*

Directly from the above definition we derive the following characterisation of p-density points.

Proposition 2.2. *Let $A \subset \mathbb{R}$. Then*
(i) *0 is a p-density point of A if and only if $[-1,1] \subset \liminf_{n\to\infty} nA$,*
(ii) *$x \in \mathbb{R}$ is a p-density point of A if and only if $[-1,1] \subset \liminf_{n\to\infty} n(A-x)$,*
(iii) *$x \in \mathbb{R}$ is a p-dispersion point of A if and only if $[-1,1] \subset \liminf_{n\to\infty} n((\mathbb{R}\setminus A)-x)$,*
where $\liminf_{n\to\infty} A_n = \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} A_n$ is the limit inferior for sequence of sets $A_n \subset \mathbb{R}$.

Here and subsequently, for every $A \subset \mathbb{R}$, $\Phi_p(A)$ denotes the set $\{x \in \mathbb{R} : x \text{ is a p-density point of } A\}$. The operator Φ_p has the following properties.

Theorem 2.3. *Let $A, B \subset \mathbb{R}$ and $y \in \mathbb{R}$. Then*
(i) *$\Phi_p(\emptyset) = \emptyset$ and $\Phi_p(\mathbb{R}) = \mathbb{R}$,*
(ii) *if $A \subset B$ then $\Phi_p(A) \subset \Phi_p(B)$,*
(iii) *$\Phi_p(A \cap B) = \Phi_p(A) \cap \Phi_p(B)$,*
(iv) *$\Phi_p(A) \subset A$,*
(v) *$\Phi_p(A) + y = \Phi_p(A + y)$.*

Proof. Properties (i), (ii), (v) are the consequences of the definition of pointwise density point and properties of lower limit. Property (iii) follows immediately from the equality $\liminf_{n\to\infty} n((A\cap B)-x) = \liminf_{n\to\infty} n(A-x) \cap \liminf_{n\to\infty} n(B-x)$. In order to prove (iv), let us observe that if $x \in \Phi_p(A)$, then $0 \in \Phi_p(A-x)$. Hence in particular there exists $k \in \mathbb{N}$ such that for $n \geq k$, we have $0 \in n(A-x)$, which gives $x \in A$. □

Proposition 2.4. *The operator Φ_p has the following properties:*
(i) *if $A \in \mathcal{N}$ or $A \in \mathcal{M}$, then $\Phi_p(A) = \emptyset$,*
(ii) *there exists a set $A \in \mathcal{L}$ of full Lebesgue measure such that $\Phi_p(A) = \emptyset$ and there exists simultaneously a residual set $B \in \mathcal{B}a$, such that $\Phi_p(B) = \emptyset$,*
(iii) *there exists a set $A \in \mathcal{L}$ such that $\lambda(\Phi_p(A) \triangle A) > 0$.*

Proof. Notice that the family of null sets and the family of first category sets form shift and scale invariant σ-ideals. Thus, if $A \in \mathcal{N}$ or $A \in \mathcal{M}$, then $\liminf_{n\to\infty} n(A - x) \in \mathcal{N}$ or $\liminf_{n\to\infty} n(A - x) \in \mathcal{M}$, for all $x \in \mathbb{R}$, respectively. Hence, we conclude that (i). The proof of parts (ii), (iii) follows from part (i) and the fact that there exist a set A of full Lebesgue measure which is also of first category on $\mathbb{R}$, and a residual null set B on $\mathbb{R}$ (cf. Theorem 1.6 in [6]). □

Note the part (iii) of Proposition 2.4 implies that the Lebesgue Density Theorem cannot be improved by using p-density points in the place of density points.

Now we shall show that the set of all p-density points of a Borel set is a coanalytic set. Let us first recall that a set $A \subset \mathbb{R}$ is called analytic if there exists a Borel set $B \subset \mathbb{R}$ and a continuous function $f : B \to \mathbb{R}$ such that $f(B) = A$. A coanalytic set is a complement of an analytic set. Obviously, all analytic and coanalytic sets are Lebesgue measurable (cf. Theorem 3.1.8 in [4]).

Theorem 2.5. *If $A \subset \mathbb{R}$ is a Borel set, then the set $\Phi_p(A)$ is coanalytic.*

Proof. Let $A \subset \mathbb{R}$ be a Borel set. By the definition, $\Phi_p(A) = \{x \in \mathbb{R} : \forall_{y\in[-1,1]}\exists_{k\in\mathbb{N}}\forall_{n\geq k}, x + \frac{y}{n} \in A\}$. Hence its complement $\mathbb{R} \setminus \Phi_p(A) = \{x \in \mathbb{R} : \exists_{y\in[-1,1]}\forall_{k\in\mathbb{N}}\exists_{n\geq k}, x + \frac{y}{n} \notin A\}$ is analytic, since it is the projection onto the first coordinate of the Borel set $\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\{(x, y) \in \mathbb{R} \times [-1, 1] : x + \frac{y}{n} \notin A\}$. Therefore $\Phi_p(A)$ is coanalytic. □

Corollary 2.6. *If $A \subset \mathbb{R}$ is a Borel set, then the set $\Phi_p\,(A)$ is Lebesgue measurable set.*

A major difference between the density operator Φ_d and the operator Φ_p is that the set of density points of a Lebesgue measurable set is always Lebesgue measurable as a consequence of the Lebesgue density theorem, whereas the set of p-density points of a Lebesgue measurable set can be a nonmeasurable. In the construction of the set with the last property we use a Hamel base which is a sum of two Berstein sets.

Definition 2.7 (cf.[6]). *A subset B of the real line $\mathbb{R}$ is a Bernstein set if both B and $\mathbb{R}\setminus B$ have nonempty intersection with every perfect subset of the real line.*

Definition 2.8 (cf. [5]). *A subset $H \subset \mathbb{R}$ is a Burstin set if H is a Bernstein set and simultaneously a Hamel base for the linear space* $(\mathbb{R}; \mathbb{Q}; +; \cdot)$.

Lemma 2.9. *There exists a Hamel base H of $\mathbb{R}$ such that $H = C \cup D$, where C, D are disjoint Bernstein sets.*

Proof. Let H be a Burstin set (for the existence see Theorem 11.4.3 in [4]). We will denote by $\{F_\alpha : \alpha < \omega_c\}$ a family of all nonempty perfect sets on the real line. The proof is based on the transfinite inductive construction of two injective sequences $\{x_\alpha : \alpha < \omega_c\}$ and $\{y_\alpha : \alpha < \omega_c\}$ with the following properties $x_\alpha \neq y_\alpha$ and $x_\alpha, y_\alpha \in [F_\alpha\setminus\bigcup_{\beta<\alpha}(\{x_\beta\} \cup \{y_\beta\})] \cap H$ for all $\alpha < \omega_c$. Define x_1 to be a nonzero element belonging to $F_1 \cap H$. The set $F_1\setminus\{x_1\}$ is an uncountable Borel set. Therefore, it has a perfect subset. Hence $(F_1\setminus\{x_1\}) \cap H$ is uncountable. Let y_1 be a nonzero element belonging to $(F_1\setminus\{x_1\}) \cap H$. Obviously, the elements x_1, y_1 are different. Assume that for $\alpha < \omega_c$ we have found injective sequences $\{x_\beta : \beta < \alpha\}$ and $\{y_\beta : \beta < \alpha\}$, which satisfy the required conditions. Since every perfect subset of $\mathbb{R}$ contains continuum pairwise disjoint perfect sets, the intersection of any Bernstein set and a perfect set is the set of the power of continuum. Therefore, $\left[F_\alpha\setminus\bigcup_{\beta<\alpha}(\{x_\beta\} \cup \{y_\beta\})\right] \cap H$ is uncountable. Let $x_\alpha \in \left[F_\alpha\setminus\bigcup_{\beta<\alpha}(\{x_\beta\} \cup \{y_\beta\})\right] \cap H$ and $y_\alpha \in \left[F_\alpha\setminus\left(\bigcup_{\beta<\alpha}(\{x_\beta\} \cup \{y_\beta\}) \cup \{x_\alpha\}\right)\right] \cap H$.

Then $x_\alpha \neq y_\alpha$ and for all $\beta < \alpha$ we have $x_\beta \neq x_\alpha$ and $y_\beta \neq y_\alpha$. Define $C = \bigcup_{\alpha<\omega_c} \{x_\alpha\}$ and $D = H \backslash C$. From the above construction it follows that the sets C, D are disjoint Bernstein sets and $H = C \cup D$. □

Theorem 2.10. *There exists a set $A \in \mathcal{L}$ such that $\Phi_p(\mathbb{R} \setminus A) \notin \mathcal{L}$.*

Proof. Let H be a Burstin set on $\mathbb{R}$ constructed in the proof of Lemma 2.9. Therefore, we have $H = C \cup D$, where $C, D \subset \mathbb{R}$ are disjoint Bernstein sets. We define $E \subset \mathbb{R}$ to be a null set and residual set. It is easily seen that for all $d \in D$, $\bigcap_{n=1}^{\infty} (nE - (n-1)d)$ is a residual set, therefore it contains a perfect set. Hence, for every $d \in D$ we have $D \cap \bigcap_{n=1}^{\infty} (nE - (n-1)d) \neq \emptyset$ and there exists $e_d \in D \cap \bigcap\limits_{n=1}^{\infty} (nE - (n-1)d)$. Let $A = \bigcup_{d \in D} A_d$, where $A_d = \bigcup_{n \in \mathbb{N}} \left\{d + \frac{1}{n}(e_d - d)\right\}$. It is clear that $A \subset E$. Then A is a null set. We now prove that both sets $\Phi_p(\mathbb{R} \backslash A)$ and its complement $\mathbb{R} \backslash \Phi_p(\mathbb{R} \backslash A)$ contain Bernstein sets C and D.

First, we show that $D \subset \mathbb{R} \backslash \Phi_p(\mathbb{R} \backslash A)$. For this inclusion it suffices to prove that for all $d \in D$ there exists $\alpha \in [-1, 1]$ which belongs to $\limsup_{n\to\infty} n(A - d)$. Let $d \in D$ and we can choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0}(e_d - d) \in [-1, 1]$. For $\alpha = \frac{1}{n_0}(e_d - d)$ we obtain $\frac{\alpha}{n} + d \in A_d \subset A$ for every $n \in \mathbb{N}$. Finally, $D \subset \mathbb{R} \backslash \Phi_p(\mathbb{R} \backslash A)$.

Now we show that $C \subset \Phi_p(\mathbb{R} \backslash A)$. Let $c \in C$ and first consider $\alpha = 0$. Since $A \subset \operatorname{Span} D = \{q_1 d_1 + ... + q_n d_n : d_1, ..., d_n \in D, q_1, ..., q_n \in \mathbb{Q} \backslash \{0\}, n \in \mathbb{N}\}$ and $\operatorname{Span} C \cap \operatorname{Span} D = \{0\}$, $\frac{\alpha}{n} + c \notin A$ for every $n \in \mathbb{N}$, and consequently $c \notin \Phi_p(\mathbb{R} \setminus A)$. Now, let us take $\alpha \in [-1, 1] \setminus \{0\}$. There exist $q_1, ..., q_l \in \mathbb{Q} \setminus \{0\}$, $h_1, ..., h_l \in H$ such that $\alpha = q_1 h_1 + ... + q_l h_l$. Suppose contrary that $c \notin \Phi_p(\mathbb{R} \setminus A)$, i.e. for all $k \in \mathbb{N}$ there exists $n \geq k$ such that $\frac{\alpha}{n} + c \in A$. By the definition of set A and the form of number α we obtain the following equality

$$\frac{q_1}{n} h_1 + \cdots + \frac{q_l}{n} h_l + c = d_n + \frac{1}{m_n}(e_{d_n} - d_n), \tag{2}$$

for some $m_n \in \mathbb{N}$, $d_n \in D$ and some $e_{d_n} \in D \cap \bigcap_{n=1}^{\infty} (nE - (n-1)d)$. Choose $n_0 > \max\{|q_i| : i = 1, 2, ..., l\}$. Then, for each $i = 1, 2, ..., l$ and for all $n \geq n_0$ we have that $\frac{-q_i}{n} h_i \neq c$, thus $\frac{q_1}{n} h_1 + \cdots + \frac{q_l}{n} h_l + c \notin \operatorname{Span} D$. On the other hand, elements on the right hand side of (2) belongs to Span D, which contradicts the equality (2). As a consequence, $c \in \Phi_p(\mathbb{R} \setminus A)$. Thus $C \subset \Phi_p(\mathbb{R} \setminus A)$. In this way we have proved that the set $\Phi_p(\mathbb{R} \backslash A)$ is a Bernstein set. This implies that $\Phi_p(\mathbb{R} \backslash A)$ is Lebesgue nonmeasurable. □

The duality Theorem 2.10 in the aspect of the Baire property is also true.

Theorem 2.11. *There exists a set $A \in \mathcal{B}a$ such that $\Phi_p(\mathbb{R} \setminus A) \notin \mathcal{B}a$.*

Proof. Let H be a Bustin set on $\mathbb{R}$ defined in the assertion of Lemma 2.9. Therefore, we have $H = C \cup D$, where $C, D \subset \mathbb{R}$ are disjoint Bernstein sets. Let $E \subset \mathbb{R}$ be a set of full Lebesgue measure and of first category. Then, for every $d \in D$ there exists $e_d \in D \cap \bigcap_{n=1}^{\infty} (nE - (n-1)d)$. Define $A_d = \bigcup_{n \in \mathbb{N}} \left\{d + \frac{1}{n}(e_d - d)\right\}$ and let $A = \bigcup_{d \in D} A_d$. It is easily seen that $A \subset E$ and $A \in \mathcal{M}$. Repeating arguments from the proof of Theorem 2.10 we conclude that $\Phi_p(\mathbb{R} \backslash A)$ is a Bernstein set, which completes the proof. □

The next theorem shows another difference between the operator Φ_p and the classical density operator Φ_d. It is well known that if $A \in \mathcal{L}$ and $x \in \Phi_d(A)$, then there exists a Borel set $B \subset A$ such that $x \in \Phi_d(B)$. This property does not hold for the operator Φ_p.

Theorem 2.12. *There exists a set $Z \subset \mathbb{R}$ such that*

(i) $\mathbb{R} \setminus Z \in \mathcal{N}$,

(ii) *all subsets of Z with the Baire property are of the first category,*

(iii) $\Phi_p(Z) \neq \emptyset$,

(iv) $\Phi_p(B) = \emptyset$ *for every Borel set $B \subset Z$.*

Proof. By Lemma 2.9 there exists a Hamel basis H which can be decomposed onto two Bernstein sets C and D. Let $E \in \mathcal{N}$ be a residual set. Let $\{G_\alpha : \alpha < \omega_c\}$ be a sequence of all Borel sets of the second category. Note that $E \cap G_\alpha \notin \mathcal{M}$ for each $\alpha < \omega_c$. Fix $d \in D$ and $\alpha < \omega_c$. Since the set $E_d = \bigcap_{n=1}^{\infty} (nE - (n-1)d)$ is residual, the set $R_{d,\alpha} = G_\alpha \cap E_d$ is of the second category with the Baire property and, consequently, $D \cap R_{d,\alpha} \neq \emptyset$. Choose

$e_{d,\alpha} \in D \cap R_{d,\alpha}$. Set $A_0 = \{e_{d,\alpha} : d \in D, \alpha < \omega_c\}$ and $A = \{d + \frac{1}{n}(e_{d,\alpha} - d) : n \in \mathbb{N}, d \in D, \alpha < \omega_c\}$. Observe that $A \subset E$, so $A \in \mathcal{N}$, and $A_0 \subset A$. Let $Z = \mathbb{R} \setminus A$. Since $A \in \mathcal{N}$, condition (i) is fulfilled. Since the set A_0 meets all Borel sets of the second category, we have part (ii). To prove condition (iii), we only need to show that $C \subset \Phi_p(Z)$. The proof is the same as in Theorem 2.10. Finally, let $B \subset Z$ be a Borel set. By applying property (ii) we have that $B \in \mathcal{M}$. Thus, from part (i) of Proposition 2.4 we obtain $\Phi_p(B) = \emptyset$ and, consequently, we get (iv). □

Corollary 2.13. *There exists a Lebesgue measurable set $Z \subset \mathbb{R}$ and a point $x \in \Phi_p(Z)$ such that $x \notin \Phi_p(B)$ for all Borel subsets B of Z.*

Theorem 2.14. *There exists a Lebesgue nonmeasurable set $A \subset \mathbb{R}$, which does not have the Baire property and such that $\Phi_p(A) = A$.*

Proof. Let B be a Burstin set. For every $x \in \mathbb{R}$ let us denote by $x = \alpha_1 b_1 + \alpha_2 b_2 + ... + \alpha_n b_n$ the representation of x in the base B, where $b_i \in B, \alpha_i \in \mathbb{Q} \setminus \{0\}$ for every $i = 1, ..., n$, $n \in \mathbb{N}$. Define

$$A = \{x \in \mathbb{R} : x = \alpha_1 b_1 + \alpha_2 b_2 + ... + \alpha_n b_n \ \wedge \ \sum_{i=1}^{n} |\alpha_i| < p \ \wedge \ n \in \mathbb{N}\},$$

where $p \in \left(\frac{1}{2}, 1\right) \cap \mathbb{Q}$. Obviously, $B \cap 2P \neq \emptyset$ and $\frac{1}{2}B \cap P \neq \emptyset$ for all nonempty perfect sets P. By the definition of set A we know that $\frac{1}{2}B \subset A$. Hence the set A intersects every nonempty perfect set. Moreover, $B \subset \mathbb{R}\setminus A$. Thus, the complement of A also intersects every nonempty perfect set. This implies that both A, $\mathbb{R} \setminus A$ are neither Lebesgue measurable, nor have the Baire property. We now prove that $A \subset \Phi_p(A)$. Let $a \in A$ and $x \in [-1, 1]$ be such that $x = \beta_1 b_1 + \beta_2 b_2 + ... + \beta_k b_k$ and $a = \alpha_1 b_1 + \alpha_2 b_2 + ... + \alpha_k b_k$, where $b_i \in B$, $\beta_i \in \mathbb{Q} \setminus \{0\}$ for all $i = 1, ..., k$, $\sum\limits_{i=1}^{k} |\alpha_i| < p$ and $p \in \left(\frac{1}{2}, 1\right)$. Fix $n_0 \in \mathbb{N}$ such that $\sum_{i=1}^{k} |\alpha_i| + \frac{1}{n_0} \sum_{i=1}^{k} |\beta_i| < p$. Then, for every natural number $n \geq n_0$ we have $x \in n(A - a)$, and the proof of $A \subset \Phi_p(A)$ is complete. By Theorem 2.3 (iv) we have the opposite inclusion. Hence $A = \Phi_p(A)$. □

3 Pointwise density topology

Consider the family of Lebesque measurable sets:

$$\mathcal{T}_p = \{A \in \mathcal{L} : A \subset \Phi_p(A)\}.$$

Since the pointwise convergence implies convergence in measure, we have $\mathcal{T}_p \subset \mathcal{T}_d$. Recall that by Theorem 2.3 it follows that $A \in \mathcal{T}_p$ if and only if $\Phi_p(A) = A$.

Theorem 3.1. *The family $\mathcal{T}_p$ is a topology on the real line.*

Proof. From parts (i) and (iii) of Theorem 2.3 it follows easily that $\emptyset, \mathbb{R} \in \mathcal{T}_p$ and $A \cap B \in \mathcal{T}_p$ for all A, $B \in \mathcal{T}_p$. To prove that $\mathcal{T}_p$ is closed under arbitrary unions, first for any set of indices T observe that $\bigcup_{t \in T} A_t \in \mathcal{T}_d \subset \mathcal{L}$, where $A_t \in \mathcal{T}_p$ for all $t \in T$, and then apply a monotonicity of the operator Φ_p. □

We call the topology $\mathcal{T}_p$ the pointwise density topology.

Proposition 3.2. *The set of irrational numbers is open with respect to $\mathcal{T}_p$.*

Proof. Let $y \in \mathbb{R}\setminus\mathbb{Q}$. Recall that $y \in \Phi_p(\mathbb{R}\setminus\mathbb{Q})$ is equivalent to the condition that for all $x \in [-1, 1]$ there exists $k \in \mathbb{N}$ such that $\frac{x}{n} + y \in \mathbb{R}\setminus\mathbb{Q}$ for all $n \geq k$. Fix $x \in [-1, 1]$. We prove that there exists at most one $n \in \mathbb{N}$ such that $x \notin n\left((\mathbb{R}\setminus\mathbb{Q}) - y\right)$. On the contrary, suppose that there exist two different numbers $n_1, n_2 \in \mathbb{N}$ such that

$x \notin n_1((\mathbb{R}\backslash\mathbb{Q}) - y)$ and $x \notin n_2((\mathbb{R}\backslash\mathbb{Q}) - y)$. Hence

$$\frac{x}{n_1} + y = \frac{p_1}{q_1}, \ \frac{x}{n_2} + y = \frac{p_2}{q_2}, \text{ where } p_1, p_2 \in \mathbb{Z}\, q_1, q_2 \in \mathbb{N}.$$

From the above equalities, we thus obtain $y = \frac{n_2 \frac{p_2}{q_2} - n_1 \frac{p_1}{q_1}}{n_2 - n_1} \in \mathbb{Q}$, which is impossible, since $y \in \mathbb{R}\backslash\mathbb{Q}$. Finally, we get $\mathbb{R}\backslash\mathbb{Q} \subset \Phi_p(\mathbb{R}\backslash\mathbb{Q})$. □

Proposition 3.3. *If $\{u_n\}_{n\in\mathbb{N}}$ is an unbounded sequence of natural number, then the set $A = \mathbb{R} \setminus \left\{\frac{1}{u_i} : n \in \mathbb{N}\right\}$ does not belong to $\mathcal{T}_p$.*

Proof. It is clear that $1 \notin \liminf_{n\to\infty} nA$. Hence $0 \notin \Phi_p(A)$, but $0 \in A$. □

Theorem 3.4. *Topology $\mathcal{T}_p$ is stronger than the Euclidean topology and weaker than the simple density topology.*

Proof. Since uniform convergence implies pointwise convergence and pointwise convergence implies almost everywhere convergence it is obvious that $\mathcal{T} \subset \mathcal{T}_p \subset \mathcal{T}_s$. Both inclusions are proper. Indeed, from Proposition 3.2 the set $\mathbb{R} \setminus \mathbb{Q}$ belongs to $\mathcal{T}_p$ but not to the Euclidean topology. It is shown in [8] that $A = \mathbb{R} \setminus \{\frac{1}{n} : n \in \mathbb{N}\} \in \mathcal{T}_s \setminus \mathcal{T}$. The fact that $A \notin \mathcal{T}_p$ follows from Proposition 3.3. □

In the sequel, we present a few algebraic properties of $\mathcal{T}_p$-open set. By Theorem 2.3 (v) it follows that if $A \in \mathcal{T}_p$ and $x \in \mathbb{R}$, then $A + x \in \mathcal{T}_p$. It is clear from the definition that if $A \in \mathcal{T}_p$, then $mA \in \mathcal{T}_p$ for every $m \geq 1$, as contrasted by Proposition 3.5.

Proposition 3.5. *For every $m \in (0, 1)$ there exists a set $A \in \mathcal{T}_p$ such that $mA \notin \mathcal{T}_p$.*

Proof. Let $m \in (0, 1)$ and define $B = \bigcup_{n=1}^{\infty} \left\{\frac{1}{p_n}\right\}$, where $\{p_n\}_{n\in\mathbb{N}}$ is an increasing sequence of prime numbers. Consider $A = \frac{1}{m}(\mathbb{R}\backslash B)$. First we show that $A \in \mathcal{T}_p$. We know that $A\backslash\{0\} \in \mathcal{T} \subset \mathcal{T}_p$. It is sufficient to prove that $0 \in \Phi_p(A)$, i.e. for all $x \in [-1, 1]$ there exists a natural number n_0 such that $x \in nA$ for all $n \geq n_0$. If $x \in [-1, 0]$, then $x \in nA$ for all $n \in \mathbb{N}$. For all $x \in (0, 1]$, we prove that there exists at most one $n \in \mathbb{N}$ such that $x \notin nA$. On the contrary, suppose that there exist two different natural numbers n_1, n_2 such that $x \notin n_1 A$ and $x \notin n_2 A$. Hence there exist prime numbers p_1, p_2 such that pairs of numbers (n_1, p_1), (n_2, p_2) are relatively prime and $\frac{x}{n_1} = \frac{1}{mp_1}$ and $\frac{x}{n_2} = \frac{1}{mp_2}$. Hence we have the following equation

$$n_1 p_2 = n_2 p_1, \tag{3}$$

and $n_i < p_i$ for $i = 1, 2$. Let $t \in \mathbb{N}$ be the greatest common divisor of numbers n_1, n_2. There exist numbers $l_1, l_2 \in \mathbb{N}$ relatively prime such that $n_1 = tl_1$ and $n_2 = tl_2$. Hence by (3) we obtain

$$p_1 = \frac{l_1 p_2}{l_2}. \tag{4}$$

From (4) we get that $p_1 = l_1$ and $p_2 = l_2$. This contradicts the fact that the number p_1 and n_1 are relatively prime, which completes the proof of $0 \in \Phi_p(A)$. Therefore, $A \in \mathcal{T}_p$. We now show that $mA \notin \mathcal{T}_p$. Indeed, recall that $mA = \mathbb{R} \setminus B$ and $0 \in mA$. By Proposition 3.3, it follows that $0 \notin \Phi_p(\mathbb{R} \setminus B) = \Phi_p(mA)$. □

The following theorem summarizes some of properties of the space of reals with pointwise density topology.

Theorem 3.6. *The space $(\mathbb{R}, \mathcal{T}_p)$ is neither first countable, nor second countable, nor does have Lindelöf property, nor separable.*

In the proof of Theorem 3.6, we need the following lemma, of which the proof is similar to that of Lemma 2.7 in [3], but for the convenience of the reader we include it here.

Lemma 3.7. *There exists a nonempty perfect set $F \subset \mathbb{R}$ such that $\Phi_p((\mathbb{R}\backslash F) \cup \{x\}) = (\mathbb{R}\backslash F) \cup \{x\}$ for each $x \in F$.*

Proof. Let H be any Hamel basis of $\mathbb{R}$ over the field $\mathbb{Q}$, containing a nonempty perfect set F (cf. Theorem 11.7.2 in [4]). Theorem 3.4 gives $\mathbb{R}\backslash F \in \mathcal{T}_p$, which means that $\mathbb{R}\backslash F = \Phi_p(\mathbb{R}\backslash F)$. We shall show that $[-1,1] \subset \liminf_{n\to\infty} n\,[((\mathbb{R}\backslash F)\cup\{x\}) - x]$. Fix $a \in [-1,1]$. Clearly, we may assume that $a \neq 0$. Suppose, contrary that we have $n_1, n_2 \in \mathbb{N}$, $n_1 \neq n_2$ such that $a \notin n_1((\mathbb{R}\backslash F) - x)$ and $a \notin n_2((\mathbb{R}\backslash F) - x)$. Consequently, $\frac{a}{n_1} + x = z_1$ and $\frac{a}{n_2} + x = z_2$, for some $z_1, z_2 \in F$ such that $z_1 \neq z_2 \neq x$. Hence we have $x(n_2 - n_1) - z_2 n_2 + z_1 n_1 = 0$. Since $z_1 \neq z_2 \neq x$ and z_1, z_2, x are elements of a Hamel basis, therefore they are linearly independent. We obtain $n_1 = n_2 = 0$, contrary to the fact that $n_1 \neq n_2$. Finally, $x \in \Phi_p((\mathbb{R}\backslash F)\cup\{x\})$ for each $x \in F$. It means that $(\mathbb{R}\backslash F)\cup\{x\} \subset \Phi_p((\mathbb{R}\backslash F)\cup\{x\})$. By Theorem 2.3 (iv) we have $\Phi_p((\mathbb{R}\backslash F)\cup\{x\}) = (\mathbb{R}\backslash F)\cup\{x\}$ for $x \in F$. □

Proof of Theorem 3.6. On the contrary, suppose that the space $(\mathbb{R}, \mathcal{T}_p)$ is first countable. Let $\{E_n\}_{n\in\mathbb{N}}$ be a sequence of $\mathcal{T}_p$-open neighbourhoods of 0. We now construct the sequence $\{x_n\}_{n\in\mathbb{N}}$ convergent to 0. We first choose $x_1 \in (E_1 \setminus \{0\}) \cap (0,1)$, and next we choose $x_2 \in (E_2 \setminus \{0\}) \cap (0, \frac{1}{2})$ such that $x_1 > x_2$ and $\frac{x_2}{x_1} \notin \mathbb{Q}$. Subsequently, let $x_3 \in (E_3 \setminus \{0\}) \cap (0, \frac{1}{3})$ be such that $x_3 > x_2$ and $\frac{x_3}{x_1}, \frac{x_3}{x_2} \notin \mathbb{Q}$ and so on. In this way we obtain the announced sequence. Put $E = \mathbb{R}\backslash\{x_n : n \in \mathbb{N}\}$. By Proposition 0.2 in [9] we know that $E \in \mathcal{T}_p$. Obviously, $0 \in E$, but there is no E_n such that $E_n \subset E$. This contradicts the fact that $\{E_n\}_{n\in\mathbb{N}}$ is $\mathcal{T}_p$-neighbourhood basis of 0, hence $(\mathbb{R}, \mathcal{T}_p)$ is neither first countable nor second countable.

We now prove that the space $(\mathbb{R}, \mathcal{T}_p)$ does not possess the Lindelöf property. By Lemma 3.7 there exists a nonempty perfect set F such that for every $x \in F$, the set $F_x = (\mathbb{R}\backslash F)\cup\{x\}$ is $\mathcal{T}_p$-open. Therefore, the family $\{F_x\}_{x\in F}$ is a $\mathcal{T}_p$-open covering $\mathbb{R}$, but clearly there is no countable subfamily of $\{F_x\}_{x\in F}$ covering $\mathbb{R}$. Hence $(\mathbb{R}, \mathcal{T}_p)$ is not Lindelöf.

On the contrary, suppose that $(\mathbb{R}, \mathcal{T}_p)$ is separable. Let $P \subset \mathbb{R}$ be a countable dense set. We show that there exists a nonempty set $W \in \mathcal{T}_p$ such that $W \cap P = \emptyset$. If P is a finite set, then $W = \mathbb{R} \setminus P \in \mathcal{T}_p$ and $W \cap P = \emptyset$. We now assume that P is infinite. Let $P = \{x_1, x_2, ...\}$ and B be a Hamel basis of $\mathbb{R}$. For any element $x \in \mathbb{R}$ we have the unique representation $x = q_1 b_1 + q_2 b_2 + \cdots + q_m b_{m_x}$, where $m_x \in \mathbb{N}$, $q_i \in \mathbb{Q}\setminus\{0\}$, $b_i \in B$ for $1 \le i \le m_x$. Let us consider $B(x) = \bigcup_{i=1}^{m_x}\{b_i\}$ and $B(P) = \bigcup_{i\in\mathbb{N}} B(x_i)$. Set $W = \mathbb{R}\setminus \operatorname{Span}(B(P))$. Hence $W \cap P = \emptyset$. It suffices to show that $W \in \mathcal{T}_p$. First, observe that W is complement of a countable set. Thus $W \in \mathcal{B}a$. Let $x \in W$ and $a \in [-1,1]$. Clearly, $x \neq 0$. Since $0 \in W - x$, let us consider the case $a \neq 0$. We only need to show that there exists $k \in \mathbb{N}$ such that for all $n \ge k$ we have $\frac{a}{n} + x \notin \operatorname{Span} B(P)$. We prove that there exists at most one number $n \in \mathbb{N}$ such that $\frac{a}{n} + x \in \operatorname{Span} B(P)$. On the contrary, suppose that there exist two numbers $n_1, n_2 \in \mathbb{N}$, $n_1 \neq n_2$ such that $\frac{a}{n_1} + x \in \operatorname{Span} B(P)$ and $\frac{a}{n_2} + x \in \operatorname{Span} B(P)$. Therefore, there exist $z_1, z_2 \in \operatorname{Span} B(P)$ such that $\frac{a}{n_1} + x = z_1$ and $\frac{a}{n_2} + x = z_2$. Hence, $x = z_2\frac{n_2}{(n_2-n_1)} - z_1\frac{n_1}{(n_2-n_1)}$. Thus, $x \in \operatorname{Span} B(P)$, contrary to the fact that $x \in W$. Finally, $W \in \mathcal{T}_p$, and hence P can not be dense. □

From Theorem 3.4 we infer that the space $(\mathbb{R}, \mathcal{T}_p)$ is Hausdorff.

Proposition 3.8. *The space* $(\mathbb{R}, \mathcal{T}_p)$ *is not normal.*

Proof. Suppose that $(\mathbb{R}, \mathcal{T}_p)$ is normal. Let $B = \mathbb{Q} + \sqrt{2}$. It is easy to check that the sets $\mathbb{Q}$, B are $\mathcal{T}_p$-closed. It is evident that $\mathbb{Q} \cap B = \emptyset$. By the Urysohn lemma, there exists a continuous function from $(\mathbb{R}, \mathcal{T}_p)$ to $(\mathbb{R}, \mathcal{T})$ such that $f(x) = 0$ for $x \in \mathbb{Q}$ and $f(x) = 1$ for $x \in B$. It is clear that f is discontinuous for every $x \in \mathbb{R}$. On the other hand, we have that $\mathcal{T}_p \subset \mathcal{T}_d$, thus f is a continuous function from $(\mathbb{R}, \mathcal{T}_d)$ to $(\mathbb{R}, \mathcal{T})$ (approximately continuous) and the set of discontinuities of f is the set of the first category (cf. [1]). This contradiction completes the proof. □

The space $(\mathbb{R}, \mathcal{T}_p)$ is Hausdorff, but is not normal. The question whether the space $(\mathbb{R}, \mathcal{T}_p)$ is regular remains open.

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