Open Mathematics Open Access

Research Article

Liang Zhang and Ze-Hua Zhou*

Dynamics of differentiation operators on generalized weighted Bergman spaces

Abstract: The chaos of the differentiation operator on generalized weighted Bergman spaces of entire functions has been characterized recently by Bonet and Bonilla in [CAOT 2013], when the differentiation operator is continuous. Motivated by those, we investigate conditions to ensure that finite many powers of differentiation operators are disjoint hypercyclic on generalized weighted Bergman spaces of entire functions.

Keywords: Disjoint hypercyclic, Differentiation operator, Generalized weighted Bergman spaces

MSC: 47A16, 46E15

DOI 10.1515/math-2015-0013
Received June 15, 2013; accepted June 30, 2014.

1 Introduction

A weigh v on the complex plane $\mathbb{C}$ is a strictly positive continuous function which is radial, i.e. $v(z) = v(|z|)$, $z \in \mathbb{C}$, such that $v(r)$ is non-increasing on $[0, \infty)$ and satisfies $\lim_{r\to\infty} r^m v(r) = 0$ for each $m \in \mathbb{N}$. For an entire function f and $1 \le p < \infty$, we set

$$M_p(f, r) := \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{it}\right)\right|^p dt \right)^{\frac{1}{p}}, \quad r > 0$$

and

$$M_\infty(f, r) := \sup_{|z|=r} |f(z)|, \quad r > 0.$$

Then, for $1 \le p \le \infty$ and a weight function v, we define the following generalized weighted Bergman spaces:

$$B_{p,\infty} = B_{p,\infty}(\mathbb{C}, v) := \{f \in H(\mathbb{C}) : \sup_{r>0} v(r) M_p(f, r) < \infty\}$$

and

$$B_{p,0} = B_{p,0}(\mathbb{C}, v) := \{f \in H(\mathbb{C}) : \lim_{r\to\infty} v(r) M_p(f, r) = 0\}.$$

These spaces are Banach spaces with the norms as in [16]:

$$\|f\|_{p,v} = \|f\|_{p,\infty,v} := \sup_{r>0} v(r) M_p(f, r).$$

Note that for $p = \infty$, the spaces $B_{p,\infty}$ and $B_{p,0}$ are usually denoted by $H_v(\mathbb{C})$ and $H_{v_0}(\mathbb{C})$, respectively; see [2, 14] for example.

Liang Zhang: Department of Mathematics, Tianjin University, Tianjin 300072, P.R. China, E-mail: 168zhangliang2011@163.com
***Corresponding Author: Ze-Hua Zhou:** Department of Mathematics, Tianjin University, Tianjin 300072, P.R. China, and Center for Applied Mathematics, Tianjin University, Tianjin 300072, P.R. China, E-mail: zehuazhoumath@aliyun.com, zhzhou@tju.edu.cn

Besides, the polynomials are contained and dense in $B_{p,0}$ for all $1 \leq p \leq \infty$, see [15, Proposition 2.1]. Obviously, $B_{p,0}$ is separable. And the inclusion $B_{p,\infty} \subset H(\mathbb{C})$ is continuous and the inclusion $B_{p,0} \subset H(\mathbb{C})$ is continuous, too, see [3].

The continuity of the differentiation operator $Df = f'$ on generalized weighted Bergman spaces of entire functions was studied by Harutyunyan and Lusky [15, 16]. Besides, J. Bonet and A. Bonilla [3] obtained that if v is a weight function satisfying

$$\sup_{r>0} \frac{v(r)}{v(r+1)} < \infty,$$

then $D : B_{p,0} \to B_{p,0}$ is continuous.

A bounded linear operator T on a separable infinite dimensional Fréchet space X is called hypercyclic provided there is some $x \in X$ such that the orbit $orb(T, x) = \{T^n x : n = 0, 1, \cdots\}$ is dense in X. Such a vector x is said to be hypercyclic for T. Besides, for every pair U, V of nonempty open subsets of X, if there is a non-negative integer m, such that $T^m(U) \cap U \neq \emptyset$, then we call T topologically transitive. It is well known that an operator T is hypercyclic if and only if it is topologically transitive. A stronger condition is the following: the operator T on X is called topologically mixing if for every pair of non-empty open subsets U and V of X, there is $m \in \mathbb{N}$ such that $T^n(U)$ meets V for each $n \geq m$. For more examples and background about linear dynamics, we refer the readers to [5, 12] and two books [6] by Bayart and Matheron and [13] by Grosse-Erdmann and Peris.

Hypercyclic operators $T_1, \cdots, T_N$, $N \geq 2$, acting on the same space X are said to be disjoint or d-hypercyclic provided there is some $x \in X$ for which the vector $(x, \cdots, x) \in X^N$ is hypecyclic for the direct sum operator $\oplus_{i=1}^N T_i$ acting on the product spaces X^N endowed with the product topology. Recently, there have been an increasing interest in studying the disjoint hypercyclicity acting on different spaces of holomorphic functions. For example, disjoint hypercyclicity was studied in [1, 7, 8, 17, 18]. Besides, disjoint hypercyclic powers of weighted backward shifts were also characterized in [10].

On the basis of above foundations, we will show that $T_1, T_2, \cdots, T_N$ which are hereditarily densely d-hypercyclic with respect to some syndetic sequence must be disjoint mixing. And we will discuss the equivalent conditions of disjoint hypercyclic and disjoint mixing properties of powers of differentiation operators on generalized weighted Bergman space $B_{p,0}$.

2 Main results

Note that $B(X)$ denotes the space of bounded linear operators on a separable infinite dimensional Fréchet space X. As we all know, for $N \geq 2$, operators $T_1, \cdots, T_N$ are d-hypercyclic, if

$$\{(z, z, \cdots, z), (T_1 z, T_2 z, \cdots, T_N z), (T_1^2 z, T_2^2 z, \cdots, T_N^2 z), \cdots\}$$

is dense in X^N endowed with the product topology for some $z \in X$. Next, we list some related definitions in details.

Definition 2.1 ([10, Definition 2.1]). *We say that $N \geq 2$ sequences of operators $(T_{1,n})_{n=1}^\infty, \cdots, (T_{N,n})_{n=1}^\infty$ in $B(X)$ are d-topologically transitive (respectively d-mixing) provided for any non-empty open subsets $V_0, \cdots, V_N$ of X there exists $m \in \mathbb{N}$ such that*

$$V_0 \bigcap T_{1,m}^{-1}(V_1) \bigcap \cdots \bigcap T_{N,m}^{-1}(V_N) \neq \emptyset$$

(respectively, so that $V_0 \bigcap T_{1,j}^{-1}(V_1) \bigcap \cdots \bigcap T_{N,j}^{-1}(V_N) \neq \emptyset$ for all $j \geq m$). Also, we say that $N \geq 2$ operators $T_1, \cdots, T_N$ in $B(X)$ are d-topologically transitive (respectively d-mixing) provided $(T_1^n)_{n=1}^\infty, \cdots, (T_N^n)_{n=1}^\infty$ are d-topologically transitive sequences (respectively d-mixing sequences).

Definition 2.2 ([10, Definition 2.5]). *Let (n_k) be a strictly increasing sequence of positive integers. We say that $T_1, T_2, \cdots, T_N \in B(X)$ satisfy the d-Hypercyclicity Criterion with respect to (n_k) provided there exist dense subsets $X_0, X_1, \cdots, X_N$ of X and mappings $S_{l,k} : X_l \to X (1 \leq l \leq N, k \in \mathbb{N})$ satisfying*

$$(i)\ T_l^{n_k} \underset{k\to\infty}{\to} 0 \ \text{pointwise on}\ X_0, \tag{1}$$

$$(ii)\ S_{l,k} \underset{k\to\infty}{\to} 0 \text{ pointwise on } X_l, \text{ and} \tag{2}$$

$$(iii)\ (T_l^{n_k} S_{i,k} - \delta_{i,l} Id_{X_l}) \underset{k\to\infty}{\to} 0 \text{ pointwise on } X_l (1 \le i \le N). \tag{3}$$

In general, we say that $T_1, T_2, \cdots, T_N \in B(X)$ satisfy the d-Hypercyclicity Criterion if there exists some sequence (n_k) for which the above is satisfied.

By Proposition 2.6 in [10], it can be seen that if $T_1, \cdots, T_N$ satisfy the d-Hypercyclicity Criterion with respect to a sequence (n_k), then $T_1, \cdots, T_N$ are d-hypercyclic. Indeed, if $(n_k) = (k)$, then $T_1, \cdots, T_N$ are d-mixing.

It was shown by J. Bès and A. Peris [10] that hereditarily densely d-hypercyclicity is equivalent to satisfying the d-Hypercyclicity Criterion. So it can be easily seen that operators $T_1, \cdots, T_N$ satisfying d-Hypercyclicity Criterion with respect to (n_k) is not only disjoint hypercyclic but also hereditarily densely d-hypercyclic with respect to (n_k). It is a question to know whether the converse is true. We will solve this problem for finitely many powers of differentiation operators on $B_{p,0}$.

Recall that a sequence (n_k) of positive integers is called syndetic, if

$$\sup_k \{n_{k+1} - n_k\} < \infty.$$

We say that $T_1, T_2, \cdots, T_N$ satisfy the d-Hypercyclicity Criterion for a syndetic sequence if the sequence (n_k) is syndetic in the above criterion.

It was shown in [4, 11] that if T satisfies the Hypercyclicity Criterion with respect to a syndetic sequence (n_k), then T satisfies the Kitai Criterion.

Now we are ready to state the main theorem.

Theorem 2.3. *Let $T_1, T_2, \cdots, T_N$ be operators on a separable, infinite dimensional Fréchet space X, where $N \ge 2$, and let (n_k) be a syndetic sequence of positive integers. Then*

(a) *If $T_1, T_2, \cdots, T_N$ are hereditarily densely d-hypercyclic with respect to (n_k), then they are d-mixing.*

(b) *If $T_1, T_2, \cdots, T_N$ commute and they satisfy the d-Hypercyclicity Criterion with respect to (n_k), then they satisfy this criterion with respect to the full sequence of positive integers.*

Proof. (a) If $T_1, T_2, \cdots, T_N$ are hereditarily densely d-hypercyclic with respect to (n_k) and we have nonempty open sets $V_0, \cdots, V_N$, then there is $n \in \mathbb{N}$ such that

$$V_0 \cap T_1^{-n_k}\left(T_1^{-i}(V_1)\right) \cap \cdots \cap T_N^{-n_k}\left(T_N^{-i}(V_N)\right) \neq \emptyset,\ i = 0, \cdots, m-1,$$

for every $k \ge n$, where $m = \sup_k (n_{k+1} - n_k)$. Then we get that $T_1, T_2, \cdots, T_N$ are d-mixing.

(b) Assume that $T_1, T_2, \cdots, T_N$ commute and they satisfy the d-Hypercyclicity Criterion with respect to (n_k). Let $m = \sup_k (n_{k+1} - n_k)$ and $\sigma : \mathbb{N} \to \mathbb{N}$ be a non-decreasing map such that

$$k = n_{\sigma(k)} + i_k = n_{1+\sigma(k)} - j_k,$$

for some $i_k, j_k \in [0, m-1]$ and for any $k \in \mathbb{N}$ $(w.l.o.g., n_1 = 1)$. And we set the maps

$$\tilde{S}_{l,k} = T_l^{j_k} \circ S_{l,1+\sigma(k)},\ k \in \mathbb{N},\ l = 1, \cdots, N.$$

With these new maps, it is easy to check that $T_1, T_2, \cdots, T_N$ satisfy the disjoint mixing criterion (i.e., with respect to the full sequence of positive integers). We omit the details. □

Remark 2.4. *The above proof of (a) was provided by an anonymous referee and we thank the referee for simplifying the original one. Besides, in the proof of part (b), we need the assumption that $T_1, T_2, \cdots, T_N$ commute, but if this assumption is removed, we do not know whether the part (b) of Theorem 2.3 still holds.*

The following is an immediate result of Theorem 2.3 and the proof is trivial.

Corollary 2.5. *Let $T_1, T_2, \cdots, T_N$ be acting on a separable, infinite dimensional Fréchet space X. Assume that $T_1, T_2, \cdots, T_N$ satisfy the d-Hypercyclicity Criterion with respect to some syndetic sequence (n_k). Then $T_1, T_2, \cdots, T_N$ are d-mixing.*

The following result is a particular case of Proposition 1.7 in [9].

Corollary 2.6. *Let $T_1, T_2, \cdots, T_N$ be acting on a separable, infinite dimensional Fréchet space X. Then $T_1, T_2, \cdots, T_N$ are hereditarily densely d-hypercyclic with respect to the full sequence (k) if and only if $T_1, T_2, \cdots, T_N$ is disjoint mixing.*

Bonet and Bonilla [3] have fully characterized hypercyclicity, weak mixing, and mixing for the derivative operator D acting on the generalized weighted Bergman sapces $B_{p,0}(1 \leq p < \infty)$. We extend this to the setting of disjointness.

Theorem 2.7. *Let $1 \leq r_1 < r_2 < \cdots < r_N$, where $N \geq 2$, $r_i \in \mathbb{N}$, $i = 1, \cdots, N$. Assume that the derivative operator $D : B_{p,0} \to B_{p,0}$ is continuous for some $1 \leq p < \infty$. If D is hypercyclic (respectively, mixing) on $B_{p,0}$, then the operators $D^{r_1}, \cdots, D^{r_N}$ satisfy the d-Hypercyclicity Criterion (respectively, are d-mixing) on $B_{p,0}$.*

Proof. We will only show d-hypercyclic property; the d-mixing case follows similarly.

If D is hypercyclic on $B_{p,0}$, by Theorem 2.8 in [3], there exists an increasing sequence (n_k), such that $\lim_{k\to\infty} \frac{\|z^{n_k}\|_{\infty,v}}{(n_k)!} = 0$. Denote $T_1 = D^{r_1}, \cdots, T_N = D^{r_N}$. Since $D : B_{p,0} \to B_{p,0}$ is continuous, there is a positive constant $C \geq 1$ such that

$$\|f^{(j)}\|_{\infty,v} \leq C^j \|f\|_{\infty,v}, \tag{4}$$

for each $f \in B_{p,0}$ and each $j \in \mathbb{N}$.

Now, we will prove that $T_1, \cdots, T_N$ satisfy d-Hypercyclicity Criterion with respect to the sequence (n_k). Choose $X_0 = X_1 = \cdots = X_N =$ span$\{1, z, z^2, \dots\}$, and take the mappings

$$S_{l,n_k} : X_l \to X$$

for each $1 \leq l \leq N$, defined by

$$S_{l,n_k} z^m := \frac{z^{m+n_k r_l}}{(m+1)(m+2)\dots(m+n_k r_l)}, \ m = 0, 1, 2 \dots. \tag{5}$$

Since the set of all polynomials is dense in $B_{p,0}$, then $X_0, X_1, \cdots, X_N$ are dense in $B_{p,0}$. It is easy to check that $T_l^{n_k} \underset{k\to\infty}{\to} 0$ pointwise on $X_0 (1 \leq l \leq N)$.

Next, fix $m \in \mathbb{N} \cup \{0\}$, we know that

$$S_{l,n_k}(z^m) = \frac{z^{m+n_k r_l}}{(m+1)\dots(m+n_k r_l)} = m!\frac{z^{m+n_k r_l}}{(m+n_k r_l)!},$$

By assumption, it follows that

$$\|S_{l,n_k}(z^m)\|_{\infty,v} = m!\frac{\|z^{m+n_k r_l}\|_{\infty,v}}{(m+n_k r_l)!} \to 0, \ k \to \infty. \tag{6}$$

It is immediate to know that $T_l^{n_k} S_{l,n_k} z^m = z^m, \ m = 0, 1, \dots.$ and for $1 \leq i < l \leq N$, it follows that $r_i < r_l$. Then, we have that

$$\begin{aligned}\|T_l^{n_k} S_{i,n_k} z^m\|_{\infty,v} &= \|T_l^{n_k}\Big(\frac{z^{m+n_k r_i}}{(m+1)(m+2)\dots(m+n_k r_i)}\Big)\|_{\infty,v} \\ &= \|(D^{r_l})^{n_k}\Big(\frac{z^{m+n_k r_i}}{(m+1)(m+2)\dots(m+n_k r_i)}\Big)\|_{\infty,v} \to 0, \ k \to \infty.\end{aligned}$$

and

$$\|T_i^{n_k} S_{l,n_k} z^m\|_{\infty,v} = \|(D^{r_i})^{n_k}\Big(\frac{z^{m+n_k r_l}}{(m+1)(m+2)\dots(m+n_k r_l)}\Big)\|_{\infty,v}$$

$$= m!\frac{\|z^{m+n_k(r_l-r_i)}\|_{\infty,v}}{(m+n_k(r_l-r_i))!} \to 0,\ k \to \infty.$$

Therefore, $D^{r_1}, \cdots, D^{r_N}$ satisfy the d-Hypercyclicity Criterion with respect to (n_k). □

Example. *Let* $1 \le r_1 < r_2 < \cdots < r_N$, *where* $N \ge 2$, $r_i \in \mathbb{N}$, $i = 1, \cdots, N$ *and* v *be a weight function such that* $\lim_{r\to\infty} v(r)\frac{e^r}{r^{\frac{1}{2p}}} = 0$. *If the differentiation operator* $D : B_{p,0} \to B_{p,0}$ *is continuous for some* $1 \le p < \infty$, *then* $D^{r_1}, \cdots, D^{r_N}$ *are disjoint mixing.*

Acknowledgement: We would like to thank the referees for useful comments and suggestions which improved the presentation of this paper.

The authors were supported in part by the National Natural Science Foundation of China (Grant Nos. 11371276; 11301373; 11201331).

References

[1] Bernal-González L., Disjoint hypercyclic operators, Studia Math., 2007, 182(2), 113-131.
[2] Bonet J., Dynamics of differentiation operator on weighted spaces of entire functions, Math. Z., 2009, 261, 649-657.
[3] Bonet J., Bonilla A., Chaos of the differentiation operator on weighted Banach spaces of entire functions, Complex Anal. Oper. Theory, 2013, 7, 33-42.
[4] Bermúdez T., Bonilla A., Conejero J. A., Peris A., Hypercyclic, topologically mixing and chaotic semigroups on Banach spaces, Studia Math., 2005, 170, 57-75.
[5] Bonilla A., Grosse-Erdmann K. G., Frequently hypercyclic operators and vectors, Ergodic Theory Dynam. Systems, 2007, 27, 383-404. Erratum: Ergodic Theory Dynam. Systems, 2009, 29, 1993-1994.
[6] Bayart F., Matheron É., Dynamics of linear operators, Cambridge Tracts in Mathematics, 179, Camberidge University Press, Cambridge, 2009.
[7] Bès J., Martin Ö., Peris A., Disjoint hypercyclic linear fractional composition operators, J. Math. Appl., 2011, 381, 843-856.
[8] Bès J., Martin Ö., Peris A., Shkarin S., Disjoint mixing operators, J. Funct. Anal., 2012, 263, 1283-1322.
[9] Bès J., Martin Ö., Sanders R., Weighted shifts and disjoint hypercyclicity, 2012, manuscript.
[10] Bès J., Peris A., Disjointness in hypercyclicity, J. Math. Anal. Appl., 2007, 336, 297-315.
[11] Costakis G., Sambarino M., Topologically mixing hypercyclic operators, Proc. Amer. Math. Soc., 2004, 132(2), 385-389.
[12] Chen R. Y., Zhou Z. H., Hypercyclicity of weighted composition operators on the unit ball of $\mathbb{C}^N$, J. Korean Math. Soc., 2011, 48(5), 969-984.
[13] Grosse-Erdmann K. G., Peris Manguillot A., Linear Chaos, Springer, New York, 2011.
[14] Harutyunyan A., Lusky W., On the boundedness of the differentiation operator between weighted spaces of holomorphic functions, Studia Math., 2008, 184, 233-247.
[15] Lusky W., On generalized Bergman space, Studia Math., 1996, 119, 77-95.
[16] Lusky W., On the Fourier series of unbounded harmonic functions, J. London. Math. Soc., 2000, 61, 568-580.
[17] Salas H. N., Dual disjoint hypercyclic operators, J. Math. Anal. Appl., 2011, 374, 106-117.
[18] Shkarin S., A short proof of existence of disjoint hypercyclic operators, J. Math. Anal. Appl., 2010, 367, 713-715.