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Research Article

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Relative annihilator-preserving congruence relations and relative annihilator-preserving homomorphisms in bounded distributive semilattices

Abstract: In this paper we shall study a notion of relative annihilator-preserving congruence relation and relative annihilator-preserving homomorphism in the class of bounded distributive semilattices. We shall give a topological characterization of this class of semilattice homomorphisms. We shall prove that the semilattice congruences that are associated with filters are exactly the relative annihilator-preserving congruence relations.

Keywords: Bounded distributive semilattices, Relative annihilator, Order-ideals, Congruences

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1 Introduction

It is well known that in a lattice A the *annihilator* of a relative to b is defined as the set $\langle a, b\rangle = \{x \in A : x \wedge a \leq b\}$. A classical result given by Mandelker [12] asserts that a lattice A is distributive if and only if $\langle a, b\rangle$ is an ideal for all $a, b \in A$. Later, Varlet in [14] gives a similar characterization for distributive semilattices. The annihilator or annulet of an element a is the set $a^{\circ} = \langle a, 0\rangle = \{x \in A : x \wedge a = 0\}$ (see [9]). It is clear that if A is an implicative semilattice (also called relatively pseudocomplemented semilattices or Brouwerian semilattices [7]), then $\langle a, b\rangle$ is a principal ideal whose generator is the relative pseudocomplement of a with respect to b, in symbols $a \to b$. Similarly, if A is a pseudocomplemented semilattice, then a° is a principal ideal whose generator is the pseudocomplement of a, written a^*. Thus, the notion of annihilator generalize in semilattices the concept of pseudocomplementation, while the notion of relative annihilator generalize the concept of intuitionistic implication.

In [11] M. F. Janowitz defines the notion of annihilator-preserving congruence relation in a bounded distributive lattice A, called AP-congruence, as a lattice-congruence θ such that for all $a, b \in A$, if $a \wedge b \equiv_{\theta} 0$, then there exists $c \in A$ such that $a \wedge c = 0$ and $c \equiv_{\theta} b$. It is easy to see that if A is a pseudocomplemented bounded distributive lattice, then a lattice-congruence θ is an AP-congruence iff it is a congruence of A. In [4] some new characterizations of this notion for bounded distributive lattices are proved. The main aim of this paper is to study this notion and the notion of annihilator-preserving semilattice homomorphism in the class of bounded distributive semilattices. All results given in this paper can be adapted to the case of bounded distributive lattices.

The paper is organized as follows. In Section 2 we shall provide all the needed information to make the paper self-contained. In Section 3 we shall review some results on relative annihilators in bounded distributive semilattices. In Section 4 we shall introduce and study the notion of (relative) annihilator-preserving semilattice homomorphism.

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In case of implicative semilattices, Heyting algebras or distributive pseudocomplemented lattices we have the usual notion of homomorphism in these classes of algebras. We shall also introduce the notions of $\langle\,\rangle$-relation and $\circ$-relation, and we shall prove that these notions are the dual of $\langle\,\rangle$-homomorphism and the $\circ$-homomorphism, respectively. We shall conclude this section by analyzing the connection of these notions with the known notion of p-morphism in modal logic. In Section 5, we shall study the notion of (relative) annihilator-preserving congruence relation. It is well known that in a Boolean algebra or Heyting algebra A, the lattice of congruences of A is isomorphic to the lattice of filters of A. We shall prove that in the class of bounded distributive semilattices the ordered set of all relative annihilator-preserving congruence relations is isomorphic to the lattice of filters. This result may be extended to the variety of bounded distributive lattices.

2 Preliminaries

In this section we shall give some necessary notations and definitions. Let us consider the poset $\langle X, \leq\rangle$. A subset $U \subseteq X$ is said to be *increasing* (*decreasing*) if for all $x \in X$ such that $x \in U$ $(y \in U)$ and $x \leq y$, we have $y \in U$ $(x \in U)$. The set of all subsets of X is denoted by $\mathcal{P}(X)$ and the set of all increasing subsets of X is denoted by $\mathcal{P}_i(X)$. A subset $K \subseteq \mathcal{P}(X)$ is called *dually directed* if for any $U, V \in K$ there exists $W \in K$ such that $W \subseteq U \cap V$. The set complement of subset $Y \subseteq X$ will be denoted by Y^c or $X - Y$. For each $Y \subseteq X$, the increasing (decreasing) set generated by Y is $[Y) = \{x \in X : \exists y \in Y : y \leq x\}$ $((Y] = \{x \in X : \exists y \in Y : x \leq y\})$. If $Y = \{y\}$, then we will write $[y)$ and $(y]$ instead of $[\{y\})$ and $(\{y\}]$, respectively.

Let us recall that a *meet-semilattice with greatest element* is an algebra $\langle A, \wedge, 1\rangle$ of type $(2, 0)$ such that the operation $\wedge$ is idempotent, commutative, associative and $a \wedge 1 = a$ for all $a \in A$. As usual, the binary relation $\leq$ defined by $a \leq b$ if and only if $a \wedge b = a$ is a partial order. In what follows, we will call it *semilattice* instead of meet-semilattice with greatest element. A *bounded* semilattice is an algebra $\langle A, \wedge, 0, 1\rangle$ of type $(2, 0, 0)$ such that $\langle A, \wedge, 1\rangle$ is a semilattice and $a \wedge 0 = 0$ for all $a \in A$.

A *filter* of a semilattice A is a subset $F \subseteq A$ such that $1 \in F$, if $a \leq b$ and $a \in F$, then $b \in F$ and if $a, b \in F$, then $a \wedge b \in F$. The *filter generated by a subset* $H \subseteq A$, in symbols $F(H)$, is the set $F(H) = \{x \in A : \exists\{h_0, ..., h_n\} \subseteq H$ and $h_0 \wedge ... \wedge h_n \leq x\}$. A filter F is said to be *finitely generated* if $F = F(H)$ for some finite non-empty subset H of A. Note that if $H = \{a\}$ then $F(\{a\}) = [a)$. We will denote by $\mathrm{Fi}(A)$ the set of all filters of A. Then, $\mathrm{Fi}(A)$ is a lattice if and only if any pair of elements of A has an upper bound in common.

A proper filter P of A is *irreducible* if for all $F_1, F_2 \in \mathrm{Fi}(A)$ such that $P = F_1 \cap F_2$ we have $P = F_1$ or $P = F_2$. The set of all irreducible filters of A will be denoted by $X(A)$. A subset I of A is called an *order-ideal* of A if I is decreasing and for all $a, b \in I$ there exists an element $c \in I$ such that $a \leq c$ and $b \leq c$. A proper filter F of A is *weakly irreducible* if $I = F^c = \{a \in A : a \notin F\}$ is an order-ideal. We note that in all semilattices, every weakly irreducible filter is an irreducible filter. We will denote by $X_\omega(A)$ and $\mathrm{Id}(A)$ the set of all weakly irreducible filters and proper order-ideals of A, respectively.

Let A be a semilattice and let $F \in \mathrm{Fi}(A)$. We note that F is irreducible if and only if for every $a, b \notin F$ there exists $c \notin F$ and $f \in F$ such that $a \wedge f \leq c$ and $b \wedge f \leq c$. Moreover, F is weakly irreducible if and only if $F_1 \cap F_2 \subseteq F$, implies that $F_1 \subseteq F$ or $F_2 \subseteq F$, for $F_1, F_2 \in \mathrm{Fi}(A)$.

The following result, analogue of the Prime Filter theorem, was proved in [5] for semilattices in general.

Theorem 2.1. *Let A be a semilattice. Let $F \in \mathrm{Fi}(A)$ and $I \in \mathrm{Id}(A)$ such that $F \cap I = \emptyset$. Then there exists $P \in X(A)$ such that $F \subseteq P$ and $P \cap I = \emptyset$.*

A semilattice A is *distributive* if for all $a, b, c \in A$ such that $a \wedge b \leq c$ there exist $a_1, b_1 \in A$ such that $a \leq a_1, b \leq b_1$ and $c = a_1 \wedge b_1$. Recall that a lattice is distributive if and only if it is distributive as a semilattice (see [10] or [7]). We will denote by DS and DS_{01} the class of distributive semilattices and the class of bounded distributive semilattices, respectively.

Now, we will recall the notion of relative annihilator. Let A be a semilattice. For $a, b \in A$, the *annihilator of a relative to b* is the decreasing set

$$\langle a, b\rangle = \{x \in A : x \wedge a \leq b\}.$$

If A is bounded, then

$$a^{\circ} = \langle a, 0\rangle = \{x \in A : x \wedge a = 0\},$$

is the *annihilator* of a. In general the subset $\langle a, b\rangle$ is a decreasing subset, but is not an order-ideal. But, as we will see in Theorem 2.2, in the case of distributive semilattices $\langle a, b\rangle$ is an order-ideal (see [14]). If $X \subseteq A$, then we define the annihilator of X as the set

$$X^{\circ} = \bigcap \left\{x^{\circ} : x \in X\right\}.$$

Then, $a^{\circ\circ} = \bigcap\{x^{\circ} : x \in a^{\circ}\}$, for each $a \in A$.

Theorem 2.2. *Let A be a semilattice. Then the following conditions are equivalents:*
(1) *A is distributive.*
(2) *The set* $\mathrm{Fi}(A)$*, considered as a lattice, is distributive.*
(3) $X(A) = X_{\omega}(A)$.
(4) *Let* $F \in \mathrm{Fi}(A)$ *and* $I \in \mathrm{Id}(A)$ *such that* $F \cap I = \emptyset$*. Then there exists* $P \in X_{\omega}(A)$ *such that* $F \subseteq P$ *and* $P \cap I = \emptyset$.
(5) *The set* $\langle a, b\rangle \in \mathrm{Id}(A)$*, for all* $a, b \in A$.

The equivalence between (1) and (2) was proved by G. Grätzer in [10]. The equivalence between the condition (1) and (4) of Theorem 2.2 was given by J. Varlet in [14] and [13]. This result provides a characterization of distributivity of a semilattice through a separation property and generalizes the Stone's theorem for distributive lattices. Finally, the equivalence of the conditions (1) and (3) was proved by S. Celani in [5].

A filter F of a semilattice A is *maximal* if it is proper and there is no proper filter that is strictly greater. We denote by $X_m(A)$ the set of all maximal filters of a distributive semilattice A. We note that $X_m(A) \subseteq X(A)$.

Remark 2.3. *Let A be a semilattice. If A is distributive, then every maximal filter F is irreducible. Indeed, let $F_1, F_2 \in \mathrm{Fi}(A)$ be such that $F_1 \cap F_2 \subseteq F$. Suppose that $F_1 \nsubseteq F$ and $F_2 \nsubseteq F$. Then there exist $a, b \in A$ such that $a \in F_1 - F$, and $b \in F_2 - F$. As F is maximal, $F \vee [a) = A$ and $F \vee [b) = A$. So, since A is distributive we get*

$$F = F \vee ([a) \cap [b)) = (F \vee [a)) \cap (F \vee [b)) = A,$$

which is a contradiction because F is proper.

We will recall some topological notions. Let $\langle X, \mathcal{T}\rangle$ be a topological space. The *closure* of a set $Y \subseteq X$ is denoted by $\mathrm{cl}(Y)$. If $Y = \{y\}$, then we will write $\mathrm{cl}(\{y\}) = \mathrm{cl}(y)$. An arbitrary non-empty subset $Y \subseteq X$ is *irreducible* if $Y \subseteq Z \cup W$ for closed subsets Z and W implies $Y \subseteq Z$ or $Y \subseteq W$. Remark that for each $x \in X$ the set $\mathrm{cl}(x)$ is irreducible. We recall that the *specialization order* of X is defined by $x \preceq y$ iff $x \in \mathrm{cl}(y)$. The dual order of $\preceq$ is denoted by $\leq$, i.e., $x \leq y$ if and only if $y \in \mathrm{cl}(x)$. Note that the relation $\preceq$ is reflexive and transitive, but not necessarily antisymmetric. A topological space $\langle X, \mathcal{T}\rangle$ is *sober* if for every irreducible closed set Y of X, there exists a unique $x \in X$ such that $\mathrm{cl}(x) = Y$. If $\langle X, \mathcal{T}\rangle$ is a sober space, then $\preceq$ is an order and is T_0.

Recall that a *DS-space* is a sober topological space $\langle X, \mathcal{T}\rangle$ such that the set of all open and compact subsets $\mathcal{KO}(X)$ of X forms a basis for the topology $\mathcal{T}$ (see [3]). Consider the set $D(X) = \{U \subseteq X : U^c \in \mathcal{KO}(X)\}$. It is clear that $\langle D(X), \cap, X\rangle$ is a distributive semilattice. We note that a DS-space $\langle X, \mathcal{T}\rangle$ is compact if and only if $D(X)$ is a bounded distributive semilattice. We note also that any closed subset in a sober space $\langle X, \mathcal{T}\rangle$ is increasing with respect to the dual specialization order $\leq$ of X. If $\langle X, \mathcal{T}\rangle$ is a DS-space, then the map $H_X : X \to X(D(X))$ defined by $H_X(x) = \{U \in D(X) : x \in U\}$ for each $x \in X$ is bijective, and $x \leq y$ iff $H_X(x) \subseteq H_X(y)$.

Remark 2.4. *By the results given in [3, Theorem 20] and [10] we have that a topological space $\langle X, \mathcal{T}\rangle$ is a DS-space iff*
(1) *the set of all open and compact subsets $\mathcal{KO}(X)$ of $\langle X, \mathcal{T}\rangle$ forms a basis for the topology $\mathcal{T}$,*

(2) $\langle X, \mathcal{T}\rangle$ *is* T_0, *and for each closed subset* Y *and each subset* $L \subseteq \mathcal{KO}(X)$ *dually directed such that* $Y \cap U \neq \emptyset$ *for all* $U \in L$, *we have* $Y \cap \bigcap\{U : U \in L\} \neq \emptyset$.

Let A be a distributive semilattice. Let us consider the set $X(A)$ and the mapping $\varphi : A \to \mathcal{P}(X(A))$ defined by $\varphi(a) = \{P \in X(A) : a \in P\}$. Let $\varphi[A] = \{\varphi(a) : a \in A\}$. Then A is isomorphic to the subalgebra $\varphi[A]$ of $\mathcal{P}(X(A))$, and the pair $\langle X(A), \mathcal{T}_A\rangle$ is a DS-space, where the topology $\mathcal{T}_A$ is generated taking as base of opens the family $\varphi[A]^c = \{X(A) - \varphi(a) = \varphi(a)^c : a \in A\}$. Moreover, $\mathcal{KO}(\langle X(A), \mathcal{T}_A\rangle) = \varphi[A]^c$, i.e., every open and compact subset U of $\langle X(A), \mathcal{T}_A\rangle$ is in the form $U = \varphi(a)^c$ for some $a \in A$. If $A \in \mathsf{DS}_{01}$, then $\langle X(A), \mathcal{T}_A\rangle$ is compact, because $X(A) = \varphi(0)^c$. For more details on the duality between distributive semilattices and DS-spaces see [3, 5] and [6].

For each $F \in \mathrm{Fi}(A)$, we consider the set

$$\Phi(F) = \{P \in X(A) : F \subseteq P\}.$$

Note that $\Phi(F) = \bigcap\{\varphi(a) : a \in F\}$. In [5] it was proved that a subset $U \subseteq X(A)$ is closed in $\langle X(A), \mathcal{T}_A\rangle$ if and only if there exists $F \in \mathrm{Fi}(A)$ such that $U = \Phi(F)^c$.

In [5] (see also [3]) it was shown that there exists a duality between homomorphisms of distributive semilattices and certain binary relations, called *meet-relations*. Let X_1 and X_2 be two sets and let $R \subseteq X_1 \times X_2$ be a binary relation. For each $x \in X_1$, let $R(x) = \{y \in X_2 : (x, y) \in R\}$. Define the mapping $h_R : \mathcal{P}(X_2) \to \mathcal{P}(X_1)$ by

$$h_R(U) = \{x \in X_1 : R(x) \subseteq U\}.$$

It is easy to verify that $h_R(U \cap V) = h_R(U) \cap h_R(V)$ and $h_R(X_2) = X_1$. A *meet-relation* is a subset $R \subseteq X_1 \times X_2$ such that $h_R(U) \in D(X_1)$, and

$$R(x) = \bigcap\{U \in D(X_2) : R(x) \subseteq U\},$$

for all $x \in X_1$. It is not hard to see that if R is a meet-relation, then $(R \circ \leq) \subseteq R$, where $\circ$ denotes the composition of relations. Let A and B be two distributive semilattices. In [5] it was shown that a function $h : A \to B$ is a semilattice homomorphism iff the relation $R_h \subseteq X(B) \times X(A)$ defined by $(P, Q) \in R_h$ iff $h^{-1}(P) \subseteq Q$, is a meet-relation.

3 Relative annihilators in DS_{01}

In this section we will review some results on relative annihilators in bounded distributive semilattices.

Lemma 3.1. *Let* $A \in \mathsf{DS}_{01}$. *Let* $a, b \in A$, $F \in \mathrm{Fi}(A)$, *and* $P \in X(A)$. *Then*
(1) $\langle a, b\rangle \cap F = \emptyset$ *iff there exists* $Q \in X(A)$ *such that* $F \subseteq Q$, $a \in Q$, *and* $b \notin Q$.
(2) $a^\circ \cap F = \emptyset$ *iff there exists* $Q \in X(A)$ *such that* $F \subseteq Q$ *and* $a \in Q$.
(3) $a^\circ \cap P = \emptyset$ *if and only if there exists* $U \in X_m(A)$ *such that* $P \subseteq U$ *and* $a \in U$.
(4) $U \in X_m(A)$ *iff* $\forall a \in A$ ($a \notin U$ *iff* $a^\circ \cap U \neq \emptyset$).
(5) *If* $U \in X_m(A)$, *then* $\forall a \in A$ ($a \notin U$ *iff* $a^{\circ\circ} \cap U = \emptyset$).

Proof. We shall sketch some steps of the proof in order to keep the paper reasonably self-contained.

(1) $\Rightarrow$) Let $F \in \mathrm{Fi}(A)$ be such that $F \cap \langle a, b\rangle = \emptyset$. Let H be the filter generated by $F \cup \{a\}$. We prove that $H \cap (b] = \emptyset$. If there exists $c \in H$ such that $c \leq b$, then there exists $f \in F$ such that $f \wedge a \leq c \leq b$. So, $f \in F \cap \langle a, b\rangle$, which is a contradiction. Thus, by Theorem 2.1 there exists $Q \in X(A)$ such that $F \subseteq Q$, $a \in Q$ and $b \notin Q$.

The direction $\Leftarrow$) is clear.

(2) follows by (1).

We prove (3). Suppose that $a^\circ \cap P = \emptyset$. Then there exists $Q \in X(A)$ such that $P \subseteq Q$ and $a \in Q$. Consider the family $\mathcal{Z} = \{F \in \mathrm{Fi}(A) - \{A\} \mid P \subseteq F$ and $a \in F\}$. It is clear that $\mathcal{Z} \neq \emptyset$, since $Q \in \mathcal{Z}$. Due to Zorn's lemma there is a maximal element in $\mathcal{Z}$, since every chain of elements of $\mathcal{Z}$, ordered by inclusion, has supremum in $\mathcal{Z}$. Let

U be such an element. Clearly, U is proper. We prove that U is a maximal filter. Let $b \notin U$. We prove that there exists $c \in U$ such that $c \wedge b = 0$. If $b^\circ \cap U = \emptyset$, then the filter $F = F(U \cup \{b\})$ is proper and $F \in \mathcal{Z}$, which is a contradiction because U is a maximal element in $\mathcal{Z}$. Therefore there exists $c \in U$ such that $c \wedge b = 0$, i.e., U is a maximal filter.

Conversely. Let $P \in X(A)$ and $U \in X_m(A)$. Suppose that $P \subseteq U$ and $a \in U$. If $a^\circ \cap P \neq \emptyset$, then there exists $p \in P$ such that $a \wedge p = 0$. So, $p \in U$, and consequently we have that $0 = a \wedge p \in U$, which is impossible. Thus, $a^\circ \cap P = \emptyset$.

(4) Assume that $U \in X_m(A)$. Let $a \notin U$. Since U is a maximal filter, $F(U \cup \{a\}) = A$. So, there exists $p \in U$ such that $p \wedge a = 0$, i.e. $p \in a^\circ \cap U$.

If $a^\circ \cap U \neq \emptyset$ and $a \in U$, then there exists $p \in U$ such that $p \wedge a = 0$, which is a contradiction.

Conversely. Let Q be a filter such that $U \subset Q$. Then there exists $a \in Q$ and $a \notin U$. As $a \notin U$, $a^\circ \cap U \neq \emptyset$. So there exists $p \in U$ such that $p \wedge a = 0$. As $U \subset Q$, $p \wedge a = 0 \in Q$. Thus, $Q = A$, and this implies that U is a maximal filter.

(5) Assume that $U \in X_m(A)$. If $a \notin U$ but $a^{\circ\circ} \cap U \neq \emptyset$, there exists $b \in a^{\circ\circ} = \bigcap\{z^\circ : z \in a^\circ\}$ and $b \in U$. As U is maximal, $a^\circ \cap U \neq \emptyset$, i.e., there exists $u \in U$ such that $u \in a^\circ$. So, $b \in u^\circ$, i.e., $b \wedge u = 0 \in U$, which is impossible. Thus, $a^{\circ\circ} \cap U = \emptyset$.

Assume that $a^{\circ\circ} \cap U = \emptyset$. As $a \in a^{\circ\circ}$, we get that $a \notin U$. □

Recall that a bounded distributive semilattice A is *normal* if each irreducible filter P is contained in a unique maximal filter [13]. It is clear that a bounded distributive lattice is normal iff it is normal as a bounded distributive semilattice (see [8]). While studying normal bounded distributive semilattices Pawar and Lokhande [13] have presented several characterizations of normal bounded distributive semilattices. The following result is proved in [13], but we give here a different proof for completeness. This result we shall need in Proposition 4.7.

Proposition 3.2 ([13]). *Let $A \in \mathsf{DS}_{01}$. Then the following conditions are equivalent:*

(1) *A is normal.*

(2) *For every $P \in X(A)$, and for every $a, b \in A$ with $a \wedge b = 0$, $a^\circ \cap P \neq \emptyset$ or $b^\circ \cap P \neq \emptyset$.*

Proof. (1) ⇒ (2) Let $P \in X(A)$ and $a, b \in A$ with $a \wedge b = 0$. If $a^\circ \cap P = \emptyset$ and $b^\circ \cap P = \emptyset$, then by Lemma 3.1, there are $U_1, U_2 \in X_m(A)$ such that $P \subseteq U_1$, and $P \subseteq U_2$ and $a \in U_1$, $b \in U_2$. As A is normal, $U_1 = U_2$. Thus, $a \wedge b = 0 \in U_1$, which is a contradiction. Thus $a^\circ \cap P \neq \emptyset$ or $b^\circ \cap P \neq \emptyset$.

(2) ⇒ (1) Let $P \in X(A)$ and suppose that $P \subseteq U_1$, and $P \subseteq U_2$, where $U_1, U_2 \in X_m(A)$. We prove that $U_1 = U_2$. If $U_1 \nsubseteq U_2$, there exists $a \in U_1$ such that $a \notin U_2$. Since U_2 is maximal, there exists $b \in U_2$ such that $a \wedge b = 0$. By hyphotesis, $a^\circ \cap P \neq \emptyset$ or $b^\circ \cap P \neq \emptyset$. If $a^\circ \cap P \neq \emptyset$, then there exists $p \in P$ such that $a \wedge p = 0$, and as $a \in U_1$, we get that $a \wedge p = 0 \in U_1$, which is impossible. If $b^\circ \cap P \neq \emptyset$ we arrive also to a contradiction. Thus, $U_1 \subseteq U_2$, and as U_1 is maximal, we have that $U_1 = U_2$. □

4 Relative annihilator-preserving semilattice homomorphisms

Let $A, B \in \mathsf{DS}_{01}$. Recall that a function $h : A \to B$ is a bounded semilattice homomorphism if $h(1) = 1$, $h(0) = 0$, and $h(a \wedge b) = h(a) \wedge h(b)$, for every $a, b \in A$. We note that if $I \in \mathrm{Id}(A)$, then $(h(I)] \in \mathrm{Id}(B)$. Indeed, for each $a, b \in I$ and $c \in B$ such that $c \leq h(a)$ and $c \leq h(b)$, there exists $d \in I$ such that $a \leq d$ and $b \leq d$. Since h is monotonic, $h(a) \leq h(d)$ and $h(b) \leq h(d)$. Thus, $c \leq h(d)$, and consequently $c \in (h(I)]$.

Definition 4.1. *A* relative annihilator-preserving semilattice homomorphism*, or $\langle\,\rangle$-homomorphism, is a bounded semilattice homomorphism $h : A \to B$ such that*

$$(h(\langle a, b\rangle)] = \langle h(a), h(b)\rangle\,,$$

for all $a, b \in A$.

An annihilator-preserving semilattice homomorphism, *or* $\circ$*-homomorphism, is a bounded semilattice homomorphism* $h : A \to B$ *such that* $(h(a^\circ)] = h(a)^\circ$*, for all* $a \in A$.

Clearly any $\langle\,\rangle$-homomorphism is also a $\circ$-homomorphism.

Remark 4.2. *If* $h : A \to B$ *is a bounded semilattice homomorphism, then the inclusion* $(h(\langle a, b\rangle)] \subseteq \langle h(a), h(b)\rangle$ *is valid for all* $a, b \in A$. *Indeed, if* $c \leq h(d)$ *with* $d \wedge a \leq b$*, then as h is a meet-homomorphism,* $h(d \wedge a) = h(d) \wedge h(a) \leq h(b)$. *So,* $c \wedge h(a) \leq h(b)$*, i.e.,* $c \in \langle h(a), h(b)\rangle$.

The set of all $\langle\,\rangle$-homomorphisms from A to B contains the identity function and it is closed under composition, as we see in the lemma below. Thus the class of bounded meet-semilattices, taken as objects, and the class of $\langle\,\rangle$-homomorphisms, taken as arrows, form a category.

Lemma 4.3. *The composition of* $\langle\,\rangle$*-homomorphisms (*$\circ$*-homomorphisms) is a* $\langle\,\rangle$*-homomorphism (*$\circ$*-homomorphism).*

Proof. Let $A_1, A_2, A_3 \in \mathsf{DS}_{01}$. Let $f : A_1 \to A_2$ and $g : A_2 \to A_3$ be two $\langle\,\rangle$-homomorphisms. We show that $g \circ f$ is a $\langle\,\rangle$-homomorphism. We need to prove that $((g \circ f)[\langle a, b\rangle]] = \langle g(f(a)), g(f(b))\rangle$ for every $a, b \in A_1$. Let $I = f(\langle a, b\rangle)$.In order to prove this identity, it is enough to show that

$$(g[I]] = (g[(I]]], \tag{1}$$

because $(g[I]] = (g[(I]]] = (g(\langle f(a), f(b)\rangle)] = \langle g(f(a)), g(f(b))\rangle$. Let us prove (1). Since $g[I] \subseteq g[(I]]$, we get that $(g[I]] \subseteq (g[(I]]]$.

To prove the other inclusion, let $x \in (g[(I]]]$. Hence, there is $z \in g[(I]]$ such that $x \leq z$. Let $y \in (f(\langle a, b\rangle)]$ be such that $z = g(y)$. Then $x \leq g(y)$, and as $y \in (f(\langle a, b\rangle)]$, there is $d \in \langle a, b\rangle$ such that $y \leq f(d)$. So, $x \leq g(f(d))$ and since $d \in \langle a, b\rangle$, we obtain $x \in (g[f(\langle a, b\rangle)]]$.

It is easy to see that the composition of $\circ$-homomorphisms is a $\circ$-homomorphism. □

Let $A, B \in \mathsf{DS}_{01}$. Let $h : A \to B$ be a semilattice homomorphism. It is clear that $h^{-1}[P] \in \mathrm{Fi}(A)$, for every $P \in X(B)$, but in general $h^{-1}[P] \notin X(A)$. Now, we prove that when P is maximal, then $h^{-1}[P]$ is also maximal, and thus it is irreducible.

Lemma 4.4. *Let* $A, B \in \mathsf{DS}_{01}$. *Let* $h : A \to B$ *be a* $\circ$*-homomorphism. Then* $h^{-1}[P] \in X_m(A)$*, for all* $P \in X_m(B)$.

Proof. Let $P \in X_m(B)$. As h is a semilattice homomorphism, $h^{-1}[P]$ is a filter of A. Since $h(0) = 0 \notin P$, we get that $h^{-1}[P]$ is proper. Let $a \notin h^{-1}[P]$. Then $h(a) \notin P$, and as P is maximal, $h(a)^\circ \cap P \neq \emptyset$. So, $(h(a^\circ)] \cap P \neq \emptyset$. Then there exists $b \in A$ such that $b \in a^\circ$, and $h(b) \in P$. As $b \in h^{-1}[P]$, we get that $a^\circ \cap h^{-1}[P] \neq \emptyset$. It now follows that $h^{-1}[P]$ is maximal. □

Now we will characterize the $\langle\,\rangle$-homomorphisms.

Theorem 4.5. *Let* $A, B \in \mathsf{DS}_{01}$. *Let* $h : A \to B$ *be a bounded semilattice homomorphism. Then the following conditions are equivalent:*

(1) *h is a* $\langle\,\rangle$*-homomorphism.*

(2) *For all* $P \in X(B)$*, and for all* $Q \in X(A)$*, if* $h^{-1}[P] \subseteq Q$*, then there exists* $D \in X(B)$ *such that* $P \subseteq D$ *and* $Q = h^{-1}[D]$.

Proof. (1) $\Rightarrow$ (2) Let $P \in X(B)$ and $Q \in X(A)$ be such that $h^{-1}[P] \subseteq Q$. Let us consider the filter $F = F(P \cup h(Q))$ and the order-ideal $(h[Q^c]] = \{b \in B : \exists d \notin Q\ (b \leq h(d))\}$. We prove that

$$F(P \cup h[Q]) \cap \left(h\left[Q^c\right]\right] = \emptyset.$$

If we assume the opposite, then there exist $p \in P$, $q \in Q$ and $d \notin Q$ such that $p \wedge h(q) \leq h(d)$, i.e., $p \in \langle h(q), h(d)\rangle$. As h is a $\langle\ \rangle$-homomorphism, $\langle h(q), h(d)\rangle = (h(\langle q, d\rangle))]$. Then there exists $w \in \langle q, d\rangle$ such that $p \leq h(w)$. So, $h(w) \in P$, and as $h^{-1}[P] \subseteq Q$, we get that $w \in Q$. Since $w \wedge q \leq d$, we have that $d \in Q$, which is a contradiction. So, by Theorem 2.1, there exists $D \in X(B)$ such that $P \subseteq D$ and $Q = h^{-1}[D]$.

(2) $\Rightarrow$ (1) Let $a, b \in A$. We prove that $\langle h(a), h(b)\rangle \subseteq (h(\langle a, b\rangle))]$. Let $c \in \langle h(a), h(b)\rangle$. If $c \notin (h(\langle a, b\rangle))]$, then by Theorem 2.1 there exists $P \in X(B)$ such that $c \in P$ and $(h(\langle a, b\rangle))] \cap P = \emptyset$. So, $\langle a, b\rangle \cap h^{-1}[P] = \emptyset$. As $h^{-1}[P]$ is a filter and $\langle a, b\rangle$ is an order-ideal, by Lemma 3.1 there exists $Q \in X(A)$ such that $h^{-1}[P] \subseteq Q$, and $a \in Q$ and $b \notin Q$. Then there exists $D \in X(B)$ such that $P \subseteq D$ and $Q = h^{-1}[D]$. But this implies that $h(a) \in D$, and as $c \wedge h(a) \leq h(b)$, we get that $h(b) \in D$, i.e., $b \in h^{-1}[D] = Q$, which is impossible. Therefore, $c \in (h(\langle a, b\rangle))]$. □

Now, we characterize the $\circ$-homomorphisms.

Theorem 4.6. *Let $A, B \in \mathsf{DS}_{01}$. Let $h : A \to B$ be a bounded semilattice homomorphism. Then the following conditions are equivalent:*
(1) *h is a $\circ$-homomorphism.*
(2) *For all $P \in X(B)$, and for all $Q \in X(A)$, if $h^{-1}[P] \subseteq Q$, then there exists $D \in X(B)$ such that $P \subseteq D$ and $Q \subseteq h^{-1}[D]$.*
(3) *$X_m(A) \cap \left[h^{-1}[P]\right) \subseteq h^{-1}[X_m(B) \cap [P)]$, for all $P \in X(B)$.*

Proof. (1) $\Rightarrow$ (2) Let $P \in X(B)$ and $Q \in X(A)$ be such that $h^{-1}[P] \subseteq Q$. Let us consider the filter $F = F(P \cup h[Q])$. We note that $0 \notin F$, because otherwise, we have elements $p \in P$ and $q \in Q$ such that $p \wedge h(q) = 0$. So, $p \in h(q)^\circ = (h(q^\circ))]$. Then there exists $d \in q^\circ$ such that $p \leq h(d)$. It follows that, $d \in h^{-1}[P] \subseteq Q$, which is a contradiction, because $q^\circ \cap Q = \emptyset$. Thus F is a proper filter of B. By Theorem 2.1, there exists $D \in X(B)$ such that $P \subseteq D$ and $Q \subseteq h^{-1}[D]$.

(2) $\Rightarrow$ (3) Let $D \in X_m(A)$. Assume that $h^{-1}[P] \subseteq D$. Then by hypothesis, there exists $Q \in X(B)$ such that $P \subseteq Q$ and $D \subseteq h^{-1}[Q]$. As h is a semilattice homomorphism, $h^{-1}[P] \subseteq h^{-1}[Q]$, and since D is maximal, $D = h^{-1}[Q]$. Thus, $h^{-1}[Q] \in h^{-1}[X_m(B) \cap [P)]$.

(3) $\Rightarrow$ (1) Let $a \in A$. We prove that $h(a)^\circ \subseteq (h(a^\circ))]$. Suppose the contrary. Then there exists $b \in h(a)^\circ$ and $b \notin (h(a^\circ))]$. Then there exists $P \in X(B)$ such that $b \in P$ and $(h(a^\circ))] \cap P = \emptyset$. Then,

$$h(a)^\circ \cap P \neq \emptyset \text{ and } h(a^\circ) \cap P = \emptyset.$$

So, $a^\circ \cap h^{-1}[P] = \emptyset$. By Lemma 3.1 there exists $Q \in X_m(A)$ such that $h^{-1}[P] \subseteq Q$ and $a \in Q$. So, taking into account the hypothesis we get that

$$Q \in X_m(A) \cap \left[h^{-1}[P]\right) \subseteq h^{-1}[X_m(B) \cap [P)].$$

Then there exists $D \in X_m(B)$ such that $P \subseteq D$ and $Q = h^{-1}[D]$. As $a \in Q = h^{-1}[D]$, $h(a) \in D$. So, $h(a)^\circ \cap D = \emptyset$. On the other hand, as $h(a)^\circ \cap P \neq \emptyset$ and $P \subseteq D$, $h(a)^\circ \cap D \neq \emptyset$, which is a contradiction. Therefore, $h(a)^\circ \subseteq (h(a^\circ))]$.

Let $b \in (h(a^\circ))]$. Then there exists $c \in A$ such that $c \wedge a = 0$ and $b \leq h(c)$. As h is a semilattice homomorphism, $h(b \wedge c) = h(c) \wedge h(a) = h(0) = 0$, i.e., $h(c) \in h(a)^\circ$, and since $h(a)^\circ$ is decreasing we get that $b \in h(a)^\circ$. Thus, $(h(a^\circ))] \subseteq h(a)^\circ$. □

Now we see that in the case of bounded distributive normal semilattices it is possible to give another characterization of the $\circ$-homomorphisms.

Proposition 4.7. *Let $A, B \in \mathsf{DS}_{01}$. Suppose that B is normal. Let $h : A \to B$ be a bounded semilattice homomorphism. Then h is a $\circ$-homomorphism if and only if*
(1) *For all $P \in X(B)$, and for all $Q_1, Q_2 \in X_m(A)$, if $h^{-1}[P] \subseteq Q_1 \cap Q_2$, then $Q_1 = Q_2$.*
(2) *$h^{-1}[P] \in X_m(A)$, for each $P \in X_m(B)$.*

Proof. $\Rightarrow$) Suppose that h is a $\circ$-homomorphism.

(1) Let $P \in X(A)$ be such that $h^{-1}[P] \subseteq Q_1 \cap Q_2$, where $Q_1, Q_2 \in X_m(A)$. Suppose that there exists $a \in Q_1 - Q_2$. Since Q_2 is maximal, there exists $b \in Q_2$ such that $a \wedge b = 0$. Then

$$h(a) \wedge h(b) = h(a \wedge b) = h(0) = 0.$$

As B is normal, by Proposition 3.2, we get that

$$h(a)^\circ \cap P \neq \emptyset \text{ or } h(b)^\circ \cap P \neq \emptyset,$$

and since h is a $\circ$-homomorphism,

$$\big(h(a^\circ)\big] \cap P \neq \emptyset \text{ or } \big(h(b^\circ)\big] \cap P \neq \emptyset.$$

If $(h(a^\circ)] \cap P \neq \emptyset$, then there exists $x \in a^\circ$ such that $h(x) \in P$. Then $x \in h^{-1}[P] \subseteq Q_1 \cap Q_2$. So, $x \in Q_1$, and as $a \in Q_1$, we get $a \wedge x = 0 \in Q_1$, which is a contradiction. If we take the case $(h(b^\circ)] \cap P \neq \emptyset$ we get also a contradiction. Thus, $Q_1 \subseteq Q_2$, and consequently, $Q_1 = Q_2$.

(2) is the conclusion of Lemma 4.4.

$\Leftarrow$) Let $a \in A$. We prove the inclusion $h(a)^\circ \subseteq (h(a^\circ)]$. Suppose that there exists $x \in A$ such that $x \in h(a)^\circ$, but $x \notin (h(a^\circ)]$. Then there exists a prime filter P of B such that $h(a)^\circ \cap P \neq \emptyset$, and $(h(a^\circ)] \cap P = \emptyset$. As B is normal, there exists a unique $Q \in X_m(B)$ such that $P \subseteq Q$. Then $h(a) \notin Q$. So, $a \notin h^{-1}[Q]$, and as $h^{-1}[Q]$ is a maximal filter, $a^\circ \cap h^{-1}[Q] \neq \emptyset$, i.e. $(h(a^\circ)] \cap Q \neq \emptyset$. As $(h(a^\circ)] \cap P = \emptyset$, we get that $a^\circ \cap h^{-1}[P] = \emptyset$. So there exists a maximal filter M such that $h^{-1}[P] \subseteq M$ and $a \in M$. As $h^{-1}[P] \subseteq h^{-1}[Q] \cap M$, and $h^{-1}[Q]$ and M are maximal filters of A, we get $h^{-1}[Q] = M$, which is a contradiction, because $a \notin h^{-1}[Q]$ and $a \in M$.

The inclusion $(h(a^\circ)] \subseteq h(a)^\circ$ has already been proved in the direction (3) $\Rightarrow$ (1) in the proof of Theorem 12. □

We define the dual meet-relations of $\langle\,\rangle$-homomorphisms and $\circ$-homomorphisms. Let $h : A \to B$ be a semilattice homomorphism. Recall that the relation $R_h \subseteq X(B) \times X(A)$ defined by $(P, Q) \in R_h$ iff $h^{-1}[P] \subseteq Q$, is a meet-relation.

Definition 4.8. *Let X_1 and X_2 be two DS-spaces. Let $R \subseteq X_1 \times X_2$ be a meet-relation. We shall say that R is a* $\langle\,\rangle$-relation *if $\forall x \in X_1 \forall y \in X_2$ $((x, y) \in R$, implies that $\exists z \in X_1 (x \leq z$ & $R(z) = [y))$. We shall say that R is a* $\circ$-relation *if $\forall x \in X_1 \forall y \in X_2$ $((x, y) \in R$, implies that $\exists z \in X_1 (x \leq z$ & $R(z) \subseteq [y))$.*

It is clear that every $\langle\,\rangle$-relation is also a $\circ$-relation. We note if $R \subseteq X_1 \times X_2$ is a meet-relation, then the function $h_R : D(X_2) \to D(X_1)$ is a semilattice homomorphism. From Theorem 4.5 and Theorem 4.6 we have as a consequence the following result.

Proposition 4.9. *Let $A, B \in \mathsf{DS}_{01}$. Let $h : A \to B$ be a bounded semilattice homomorphism. Then*
(1) *h is a $\langle\,\rangle$-homomorphism iff the meet relation R_h is a $\langle\,\rangle$-relation.*
(2) *h is a $\circ$-homomorphism iff the meet relation R_h is a $\circ$-relation.*

Supplement: some applications

We recall that an implicative semilattice is a meet-semilattice A with an additional binary operation $\to: A \to A$ such that for all $a, b, c \in A$ we have $a \wedge c \leq b$ iff $c \leq a \to b$. For two implicative semilattices A and B, we recall that a map $h : A \to B$ is an implicative semilattice homomorphism if h is a meet-homomorphism and $h(a \to b) = h(a) \to h(b)$, for each $a, b \in A$. If A and B are bounded implicative semilattices, then a implicative semilattice homomorphism is *bounded* if $h(0) = 0$. A Heyting algebra is a bounded distributive lattice with an additional operation $\to$ such that it is also an implicative semilattice. If A and B are Heyting algebras, a Heyting homomorphism $h : A \to B$ is a bounded lattice homomorphism and an implicative semilattice homomorphism. If A

and B are pseudocomplemented distributive lattices, a pseudocomplemented homomorphism $h : A \to B$ is bounded lattice homomorphism such that $h(a^*) = h(a)^*$, for each $a \in A$.

We recall that a *p-morphism* between two ordered sets $\langle X_1, \leq_1 \rangle$ and $\langle X_2, \leq_2 \rangle$ is a map $f : X_1 \to X_2$ satisfying the conditions: (1) if $x \leq_1 y$ then $f(x) \leq_2 f(y)$, and (2) if $f(x) \leq_2 z$ then there exists $y \in X_1$ such that $x \leq_1 y$ and $f(y) = z$ (see [2]). A *weak p-morphism* is between two ordered sets $\langle X_1, \leq_1 \rangle$ and $\langle X_2, \leq_2 \rangle$ is a map $f : X_1 \to X_2$ such that $f(x) \leq_2 f(y)$, when $x \leq_1 y$, and if $f(x) \leq_2 z$ then there exists $y \in X_1$ such that $x \leq_1 y$ and $z \leq f(y)$. Given a p-morphism or a weak p-morphism $f : X \to Y$ we consider the relation $R_f \subseteq X_1 \times X_2$ defined by

$$(x, z) \in R_f \text{ iff } f(x) \leq_2 z.$$

It is well known that a bounded lattice homomorphism is a Heyting homomorphism iff its dual is a p-morphism (see [2]).

Let A and B be two bounded distributive lattices. Recall that if $h : A \to B$ is a semilattice homomorphism, then the relation $R_h \subseteq X(B) \times X(A)$ by $(P, Q) \in R_h$ iff $h^{-1}(P) \subseteq Q$, is a meet-relation. If h is a lattice homomorphism, then we can define a function $f_h : X(B) \to X(A)$ as $f_h(P) = h^{-1}(P)$, for each $P \in X(B)$.

In an implicative semilattice or in a Heyting algebra A we have that $\langle a, b \rangle = (a \to b]$, for each $a, b \in A$. So, by Proposition 4.5 and Proposition 4.9 we get the following result:

Corollary 4.10. *Let A and B be two bounded distributive lattices. Let $h : A \to B$ be a lattice homomorphism.*
(1) *If A and B are Heyting algebras, then h is a Heyting homomorphism iff h is a $\langle\ \rangle$-homomorphism iff the meet-relation $R_h \subseteq X(B) \times X(A)$ is a $\langle\ \rangle$-relation iff the function $f_h : X(B) \to X(A)$ is a p-morphism.*
(2) *If A and B are pseudocomplemented distributive lattices. Then h is a pseudocomplemented homomorphism iff h a $\circ$-homomorphism iff the meet relation $R_h \subseteq X(B) \times X(A)$ is a $\circ$-relation iff $f_h : X(B) \to X(A)$ is a weak p-morphism.*

For implicative semilattice the situation is a little different, because the dual of implicative homomorphism is not a function. In [6] it was shown that the dual of implicative semilattice homomorphism is precisely a $\langle\ \rangle$-relation (in [6] the $\langle\ \rangle$-relations are called *functional meet-relations*). By Proposition 4.5, and Proposition 4.9 we get the following result first proved in [6].

Corollary 4.11. *Let A and B be two implicative semilattices. Let $h : A \to B$ be a semilattice homomorphism. Then h is an implicative homomorphism iff h a $\langle\ \rangle$-homomorphism iff the meet-relation $R_h \subseteq X(B) \times X(A)$ is a $\langle\ \rangle$-relation.*

5 Relative annihilator-preserving congruence relations

Let $A \in \mathsf{DS}_{01}$ and let θ be a congruence of A. We will write $(a, b) \in \theta$ or $a \equiv_\theta b$. The equivalence class of an element $a \in A$ is denoted by $|a|_\theta = \{b \in A \mid a \equiv_\theta b\}$, or directly by $|a|$. The *canonical* or *natural map* with respect to θ is the function $q_\theta : A \to A/\theta$ defined by $q_\theta(a) = |a|_\theta$. For a subset $S \subseteq A$, we will write $|S|_\theta = \{|a|_\theta \mid a \in S\}$.

Remark 5.1. *Let $A \in \mathsf{DS}$ and let θ be a congruence of A. We note that in general the quotient A/θ may be not a distributive semilattice, because the class of* DS *is not a variety.*

The notion of $\langle\ \rangle$-homomorphism leads naturally to a notion of congruence that preserves relative annihilators. First, we establish the relevant properties of the kernel of a $\langle\ \rangle$-homomorphism.

Proposition 5.2. *Let $A_1, A_2 \in \mathsf{DS}_{01}$ and $h : A_1 \to A_2$ a $\langle\ \rangle$-homomorphism. Then $\ker h = \{(a, b) : h(a) = h(b)\}$ is a congruence of A_1, and for every $(a, b), (c, d) \in \ker h$,*
(1) $\forall x \in \langle a, c \rangle\ \exists y \in \langle b, d \rangle : (x, y) \in \ker h$,
(2) $\forall x \in \langle b, d \rangle\ \exists y \in \langle a, c \rangle : (x, y) \in \ker h$.

Proof. It is well known that ker$h$ is a semilattice congruence of A_1. Suppose that $(a,b),(c,d) \in$ ker$h$. So, $h(a) = h(b)$ and $h(c) = h(d)$. To prove (1) let $x \in \langle a,c\rangle$. Then $h(x) \in h[\langle a,c\rangle] \subseteq (h[\langle a,c\rangle]]$. Since h is a $\langle\ \rangle$-homomorphism, $h(x) \in \langle h(a),h(c)\rangle$. Therefore, $h(x) \in \langle h(b),h(d)\rangle = (h[\langle b,d\rangle]]$. Since $\langle b,d\rangle$ is an order-ideal, there must be $y \in \langle b,d\rangle$ such that $h(x) \leq h(y)$. Then $h(x) = h(x \wedge y)$ and $x \wedge y \in \langle b,d\rangle$. So, $(x, x\wedge y) \in$ ker$h$. So, (1) holds. (2) follows from (1) and the symmetry of θ. □

Proposition 5.2 suggests the following definition.

Definition 5.3. *Let $A \in \mathsf{DS}_{01}$ and let θ be a semilattice congruence of A. We say that θ is a* relative annihilator-preserving congruence *or* RA-congruence *of A, if*

***(RA)** for every $(a,b),(c,d) \in \theta$, the following condition holds*

$$\forall x \in \langle a,c\rangle\ \exists y \in \langle b,d\rangle : (x,y) \in \theta. \tag{2}$$

This condition together with the symmetry of θ implies

$$\forall x \in \langle b,d\rangle\ \exists y \in \langle a,c\rangle : (x,y) \in \theta. \tag{3}$$

Let θ be a congruence of $A \in \mathsf{DS}_{01}$. To indicate that the pairs (a,b) and (c,d) satisfies the condition (2) of Definition 5.3 we will use the following notation:

$$(\langle a,b\rangle, \langle c,d\rangle) \in \tilde{\theta}, \text{ or } \langle a,c\rangle \equiv_{\tilde{\theta}} \langle b,d\rangle .$$

Thus, a semilattice congruence θ is a RA-congruence if for all $a,b \in A$, $\langle a,c\rangle \equiv_{\tilde{\theta}} \langle b,d\rangle$, whenever $a \equiv_\theta b$ and $c \equiv_\theta d$. Since $a^\circ = \langle a,0\rangle$, we write $a^\circ \equiv_{\tilde{\theta}} b^\circ$, when $a \equiv_\theta b$. A semilattice congruence θ is an *annihilator-preserving congruence* if $a^\circ \equiv_{\tilde{\theta}} b^\circ$, when $a \equiv_\theta b$.

We denote by $\mathrm{Con}_{RA}A$ the set of all RA-congruences of A, and by $\mathrm{Con}_A A$ the set of all annihilator-preserving congruence of A. We note that $\mathrm{Con}_{RA}A \subseteq \mathrm{Con}_A A$.

Let F be a filter of $A \in \mathsf{DS}_{01}$. It is well known that the relation

$$\theta(F) = \left\{(a,b) \in A^2 : \exists f \in F\ (a \wedge f = b \wedge f)\right\}$$

is a congruence. On the other hand, if θ is a semilattice congruence of A, then the class $|1|_\theta$ is a filter. In general, $F = |1|_{\theta(F)}$ and $\theta(|1|_\theta) \subseteq \theta$. If A is an implicative semilattice (see [7]), then is valid the other inclusion, i.e., $\theta \subseteq \theta(|1|_\theta)$.

Now we prove that if a semilattice congruence θ is a *RA*-congruence, then $\theta = \theta(|1|_\theta)$. As a consequence of this fact we have that there exists a bijective correspondence between the sets $\mathrm{Con}_{RA}A$ and $\mathrm{Fi}(A)$.

We recall that if A is a distributive semilattice, then the mapping $\varphi : A \to \mathcal{P}(X(A))$ is defined by $\varphi(a) = \{P \in X(A) : a \in P\}$.

Lemma 5.4. *Let $A \in \mathsf{DS}_{01}$. Let $F \in \mathrm{Fi}(A)$. Then*

$$\theta(F) = \left\{(a,b) \in A^2 : \varphi(a) \cap \Phi(F) = \varphi(b) \cap \Phi(F)\right\},$$

where $\Phi(F) = \{P \in X(A) : F \subseteq P\}$.

Proof. It is easy to see $\theta(F) \subseteq \left\{(a,b) \in A^2 : \varphi(a) \cap \Phi(F) = \varphi(b) \cap \Phi(F)\right\}$. Assume that $\varphi(a) \cap \Phi(F) = \varphi(b) \cap \Phi(F)$. As $\Phi(F) = \bigcap\{\varphi(f) : f \in F\}$, we have that

$$\varphi(a) \cap \varphi(b)^c \cap \bigcap\{\varphi(f) : f \in F\} = \emptyset.$$

So,

$$\varphi(b)^c \subseteq \varphi(a)^c \cup \bigcup\left\{\varphi(f)^c : f \in F\right\}.$$

As $\varphi(b)^c$ is a compact subset of the DS-space $\langle X(A), \mathcal{T}_A\rangle$, there exist $f_1, \ldots, f_n \in F$ such that

$$\varphi(b)^c \subseteq \varphi(a)^c \cup \varphi(f_1)^c \cup \ldots \cup \varphi(f_n)^c,$$

i.e.,

$$\varphi(b) \supseteq \varphi(a) \cap \varphi(f_1) \cap \ldots \cap \varphi(f_n) = \varphi(a) \cap \varphi(f) = \varphi(a \wedge f),$$

where $f = f_1 \wedge \ldots \wedge f_n$. Then $a \wedge f \leq b$. Similarly we can to prove that there exists $f' \in F$ such that $b \wedge f' \leq a$. Let $h = f \wedge f' \in F$. Then it is easy to see that $a \wedge h = b \wedge h$, i.e., $(a, b) \in \theta(F)$. □

Let $A \in \mathsf{DS}_{01}$. For each order-ideal I of A consider the set

$$\alpha(I) = \bigcup \{\varphi(a) : a \in I\}.$$

It is clear that $\alpha(I)$ is an open subset of the DS-space $\langle X(A), \mathcal{T}_A\rangle$.

Lemma 5.5. *Let $A \in \mathsf{DS}_{01}$. Let $F \in \mathrm{Fi}(A)$. Then for every $(a, b), (c, d) \in \theta(F)$, the following conditions are equivalent,*
(1) $\langle a, c\rangle \equiv_{\widetilde{\theta(F)}} \langle b, d\rangle$,
(2) $\alpha(\langle a, c\rangle) \cap \Phi(F) = \alpha(\langle b, d\rangle) \cap \Phi(F)$.

Proof. Assume (1). First we prove that $\alpha(\langle a, c\rangle) \cap \Phi(F) \subseteq \alpha(\langle b, d\rangle) \cap \Phi(F)$. Suppose $P \in \alpha(\langle a, c\rangle) \cap \Phi(F)$. Let $d \in \langle a, c\rangle$ be such that $P \in \varphi(d)$. By (1), there is $e \in \langle b, d\rangle$ such that $(d, e) \in \theta(F)$. Thus, $\varphi(d) \cap \Phi(F) = \varphi(e) \cap \Phi(F)$. Since $P \in \varphi(d) \cap \Phi(F)$, we get that $e \in P$, and therefore $P \in \varphi(e) \subseteq \bigcup \{\varphi(x) : x \in \langle b, d\rangle\} = \alpha(\langle b, d\rangle)$. Hence, $P \in \alpha(\langle b, d\rangle) \cap \Phi(F)$. The other inclusion is proved similarly.

Now, assume (2). To prove condition **(RA)** on Definition 5.3, let $d \in \langle a, c\rangle$. Then,

$$\begin{aligned}\varphi(d) \cap \Phi(F) &\subseteq \bigcup \{\varphi(d) \cap \Phi(F) : d \in \langle a, c\rangle\} = \alpha(\langle a, c\rangle) \cap \Phi(F)\\ &= \alpha(\langle b, d\rangle) \cap \Phi(F) \subseteq \bigcup \{\varphi(e) : e \in \langle b, d\rangle\}.\end{aligned}$$

We note that $\varphi(d) \cap \Phi(F)$ is a closed subset of $\langle X(A), \mathcal{T}_A\rangle$. Moreover, as $\langle b, d\rangle$ is an order-ideal, we have that the family $\{\varphi(e)^c : e \in \langle b, d\rangle\}$ is dually directed. Since $\varphi(d) \cap \Phi(F) \cap \bigcap \{\varphi(e)^c : e \in \langle b, d\rangle\} = \emptyset$, by Remark 2.4, we have that there exists $e \in \langle b, d\rangle$ such that $\varphi(d) \cap \Phi(F) \subseteq \varphi(e)$. Thus, $\varphi(d) \cap \Phi(F) \subseteq \varphi(e) \cap \Phi(F)$, and consequently $\varphi(d) \cap \Phi(F) = \varphi(d) \cap \Phi(F) \cap \varphi(e) = \varphi(d \wedge e) \cap \Phi(F)$. Let $y = d \wedge e$. Consequently $(d, y) \in \theta(F)$. Moreover, since $y \leq e$ and $e \in \langle b, d\rangle$, we obtain that $y \in \langle b, d\rangle$. □

Lemma 5.6. *Let $A \in \mathsf{DS}_{01}$. Let $F \in \mathrm{Fi}(A)$. Then $\theta(F) \in \mathrm{Con}_{RA} A$.*

Proof. Let $(a, b), (c, d) \in \theta(F)$. Then by Lemma 5.4 $\varphi(a) \cap \Phi(F) = \varphi(b) \cap \Phi(F)$ and $\varphi(c) \cap \Phi(F) = \varphi(d) \cap \Phi(F)$. By Lemma 5.5, it is sufficient to prove that $\alpha(\langle a, c\rangle) \cap \Phi(F) = \alpha(\langle b, d\rangle) \cap \Phi(F)$.

Let $P \in \alpha(\langle a, c\rangle) \cap \Phi(F)$. Suppose that $P \notin \alpha(\langle b, d\rangle)$, i.e., $\langle b, d\rangle \cap P = \emptyset$. By Lemma 3.1 there exists $Q \in X(A)$ such that $P \subseteq Q$, $b \in Q$ and $d \notin Q$. So, $Q \in \varphi(b) \cap \Phi(F) = \varphi(a) \cap \Phi(F)$, i.e., $a \in Q$.

As $P \in \alpha(\langle a, c\rangle)$, there exists $x \in A$ such that $a \wedge x \leq c$ and $x \in P$. So $a \wedge x \in Q$, and consequently $c \in Q$. So, $Q \in \varphi(c) \cap \Phi(F) = \varphi(d) \cap \Phi(F)$, i.e., $d \in Q$ which is a contradiction. Thus, $\alpha(\langle a, c\rangle) \cap \Phi(F) \subseteq \alpha(\langle b, d\rangle) \cap \Phi(F)$. The other inclusion is proved similarly. □

Theorem 5.7. *Let $A \in \mathsf{DS}_{01}$. Then there exists a bijective correspondence between the sets* $\mathrm{Con}_{RA} A$ *and* $\mathrm{Fi}(A)$. *Thus,* $\mathrm{Con}_{RA} A$ *is a bounded distributive lattice.*

Proof. Let $\theta \in \mathrm{Con}_{RA} A$. We prove that $\theta = \theta(|1|_\theta)$. It is clear that $|1|_\theta \in \mathrm{Fi}(A)$ and that the inclusion $\theta(|1|_\theta) \subseteq \theta$ is always true. We prove that $\theta \subseteq \theta(|1|_\theta)$. By Lemma 5.4 it suffices to prove that

$$\theta \subseteq \left\{(a, b) \in A^2 : \varphi(a) \cap \Phi(|1|_\theta) = \varphi(b) \cap \Phi(|1|_\theta)\right\}.$$

Let $(a, b) \in \theta$. Let $P \in \varphi(a) \cap \Phi(|1|_\theta)$. Since $\theta \in \mathrm{Con}_{RA} A$, we get that

$$(\langle a, a\rangle, \langle a, b\rangle) = (A, \langle a, b\rangle) \in \tilde{\theta}.$$

As $1 \in A$, there exists $c \in \langle a, b\rangle$ such that $(1, c) \in \theta$. So, $c \wedge a \leq b$ and $c \in |1|_\theta \subseteq P$. Thus, $c \wedge a \in P$, and consequently $b \in P$. Therefore $\varphi(a) \cap \Phi(|1|_\theta) \subseteq \varphi(b)$. Similarly we can prove that $\varphi(b) \cap \Phi(|1|_\theta) \subseteq \varphi(a)$. Then $\theta \subseteq \theta(|1|_\theta)$.

Let $F \in \mathrm{Fi}(A)$. By Lemma 5.6 $\theta(F) \in \mathrm{Con}_{RA}A$. As identity $F = |1|_{\theta(F)}$ is always true, we have that every filter is the kernel of a congruence from $\mathrm{Con}_{RA}A$. As for each $F \in \mathrm{Fi}(A)$, we get $F = |1|_{\theta(F)}$, and for each $\theta \in \mathrm{Con}_{RA}A$, we have that $\theta = \theta(|1|_\theta)$, we have as consequence that the maps

$$F \to \theta(F) \text{ and } \theta \to |1|_\theta$$

show that there is a bijective correspondence between the sets $\mathrm{Con}_{RA}A$ and $\mathrm{Fi}(A)$. Moreover, since $\theta_1 \subseteq \theta_2$ iff $|1|_{\theta_1} \subseteq |1|_{\theta_2}$, we have that the ordered sets $(\mathrm{Con}_{RA}A, \subseteq)$ and $(\mathrm{Fi}(A), \subseteq)$ are isomorphic. Since $(\mathrm{Fi}(A), \subseteq)$ is a bounded distributive lattice, because A is distributive, we get that $(\mathrm{Con}_{RA}A, \subseteq)$ is also a bounded distributive lattice. □

Remark 5.8. *Let $A \in \mathsf{DS}_{01}$. As A/θ is not a distributive semilattice, in general $\langle |a|_\theta, |b|_\theta\rangle$ may be not an order-ideal of A/θ.*

Theorem 5.9. *Let $A \in \mathsf{DS}_{01}$. Then the following conditions are equivalent:*
(1) *$\theta \in \mathrm{Con}_{RA}A$,*
(2) *$\langle |a|_\theta, |b|_\theta\rangle = (|\langle a, b\rangle|_\theta]$, for all $a, b \in A$.*

Proof. (1) $\Rightarrow$ (2) Let $\theta \in \mathrm{Con}_{RA}A$. Let $a, b \in A$. Suppose that $|c|_\theta \in (|\langle a, b\rangle|_\theta]$. Then there exists $d \in \langle a, b\rangle$ such that $|c|_\theta \leq |d|_\theta$. As $d \wedge a \leq b$ and θ is a semilattice congruence,

$$|c|_\theta \wedge |a|_\theta \leq |d|_\theta \wedge |a|_\theta = |d \wedge a|_\theta \leq |b|_\theta .$$

Then $|c|_\theta \in \langle |a|_\theta, |b|_\theta\rangle$.

Assume that $|c|_\theta \in \langle |a|_\theta, |b|_\theta\rangle$, i.e., $|c \wedge a|_\theta \leq |b|_\theta$. Suppose that $|c|_\theta \notin (|\langle a, b\rangle|_\theta]$. It is not hard to see that $(|\langle a, b\rangle|_\theta]$ is an order-ideal of A/θ. Then by Theorem 2.1 there exists an irreducible filter P of A/θ such that $(|\langle a, b\rangle|_\theta] \cap P = \emptyset$ and $|c|_\theta \in P$. So, $|\langle a, b\rangle|_\theta \cap P = \emptyset$ and $|c|_\theta \in P$. Then $\langle a, b\rangle \cap q_\theta^{-1}[P] = \emptyset$ and $c \in q_\theta^{-1}[P]$. As $q_\theta^{-1}[P]$ is a filter of A, we have by Lemma 3.1 that there exists $Q \in X(A)$ such that $q_\theta^{-1}[P] \subseteq Q$, $a \in Q$, and $b \notin Q$. Let H be the filter generated by $P \cup q_\theta[Q]$. It is easy to see that $(q_\theta[Q^c]]$ is an order-ideal such that $H \cap (q_\theta[Q^c]] = \emptyset$. By Theorem 2.1 there exists an irreducible filter D of A/θ such that $P \subseteq D$, $q_\theta[Q] \subseteq D$ and $D \cap q_\theta[Q^c] = \emptyset$. Then, $P \subseteq D$ and $Q = q_\theta^{-1}[D]$. As $a \in Q$, we get that $|a|_\theta \in D$, and since $c \in q_\theta^{-1}[P]$, we have that $|c|_\theta \in D$. Then $|c \wedge a|_\theta \in D$, and so $|b|_\theta \in D$. Thus, $b \in q_\theta^{-1}[D] = Q$, which is impossible. Therefore, $|c|_\theta \notin |\langle a, b\rangle|_\theta$.

(2) $\Rightarrow$ (1) Let $(a, b), (c, d) \in \theta$. We need to prove that if $x \in \langle a, c\rangle$, then there exists $y \in \langle b, d\rangle$ such that $(x, y) \in \theta$. Let $x \in \langle a, c\rangle$. Then

$$|x|_\theta \in |\langle a, c\rangle|_\theta \subseteq (|\langle a, c\rangle|_\theta] = \langle |a|_\theta, |c|_\theta\rangle = \langle |b|_\theta, |d|_\theta\rangle = (|\langle b, d\rangle|_\theta] .$$

So, there exists $x' \in \langle b, d\rangle$ such that $|x|_\theta \leq |x'|_\theta$. Since $|x|_\theta = |x \wedge x'|_\theta$ and $x \wedge x' \in \langle b, d\rangle$, we have that there exists $y = x \wedge x' \in \langle b, d\rangle$ such that $(x, y) \in \theta$. □

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References

[1] Bezhanishvili G, and Jansana, R., Esakia Style Duality for Implicative Semilattices, Applied Categorical Structures, 2013, 21 (2), 181-208

[2] Chagrov, A., and Zakharyaschev M., Modal Logic, Oxford University Press, 1997

[3] Celani S. A and Calomino I. M., Some remarks on distributive semilattices, Comment. Math. Univ. Caroline, 2013, 54 (3), 407-428

[4] Celani S. A., Remarks on annihilators preserving congruence relations, Mathematica Slovaca, 2012, 62 (3), 689-698

[5] Celani S. A., Topological representation of distributive semilattices, Scientiae Math. Japonicae online, 2003, (8), 41-51

[6] Celani, S. A., Representation of Hilbert algebras and implicative semilattices, Central European Journal of Mathematics, 2003, 4, 561-572

[7] Chajda I., Halaš R. and Kühr J., Semilattice Structures. Heldermann Verlag, Research and Exposition in Mathematics, 30, 2007

[8] Cornish, W. H., Normal lattices, Australian Mathematical Society. Journal. Pure Mathematics and Statistics, 1972, 14, 200-215

[9] Cornish W., Annulets and α-ideals in a distributive lattice, J. Austral Math. Soc., 1973, 15, 70-77

[10] Grätzer G., General Lattice Theory. Birkhäuser Verlag, 1998

[11] Janowitz M. F., Annihilator preserving congruence relations of lattices, Algebra Universalis,1975, 5 (1), 391–394

[12] Mandelker M., Relative annihilators in lattices, Duke Math. J., 1970, 37, 377–386

[13] Pawar S. P. and Lokhande A. D., Normal semilattices, Indian J. Pure and Appl. Math., 1998, 29, (12), 1245-1249

[14] Varlet J. C., Relative annihilators in semilattices, Bull. Austral. Math. Soc., 1973, 9, 169-185