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Research Article

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Semilinear problems for the fractional laplacian with a singular nonlinearity

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Abstract: The aim of this paper is to study the solvability of the problem

$$\begin{cases} (-\Delta)^s u = F(x,u) := \lambda \dfrac{f(x)}{u^\gamma} + M u^p \text{ in } \Omega, \\ \qquad u > 0 \text{ in } \Omega, \\ \qquad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N > 2s$, $M \in \{0,1\}$, $0 < s < 1$, $\gamma > 0$, $\lambda > 0$, $p > 1$ and f is a nonnegative function. We distinguish two cases:
- For $M = 0$, we prove the existence of a solution for every $\gamma > 0$ and $\lambda > 0$.
- For $M = 1$, we consider $f \equiv 1$ and we find a threshold Λ such that there exists a solution for every $0 < \lambda < \Lambda$, and there does not for $\lambda > \Lambda$.

Keywords: Fractional Laplacian, Solvability of elliptic equations, Existence and multiplicity

MSC: 49J35, 35A15, 35S15, 47G20, 42B10, 46E35

To Daniela Giachetti, in occasion of her 60th birthday, with our friendship.

1 Motivation and organization of the paper

In this paper we study the existence, regularity and multiplicity of solutions of the following nonlocal problem

$$(\mathcal{P}) = \begin{cases} (-\Delta)^s u = F(x,u) := \lambda \dfrac{f(x)}{u^\gamma} + M u^p \text{ in } \Omega, \\ \qquad u > 0 \text{ in } \Omega, \\ \qquad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N > 2s$, $M \in \{0,1\}$, $0 < s < 1$, $\gamma > 0$, $\lambda > 0$, $p > 1$, and f is a nonnegative function that belongs to some $L^m(\Omega)$, $m \geq 1$. In this context, $(-\Delta)^s$ is the fractional Laplace operator

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defined, up to a normalization factor, by the Riesz potential as

$$-(-\Delta)^s u(x) := P.V. \int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{y^{N+2s}} dy, \quad x \in \mathbb{R}^N$$

when $s \in (0,1)$ is a fixed parameter (see for example [16, 24, 33]).

Such elliptic problems with a singular nonlinearity have a large history in the local case, that is, with a principal part of Laplacian type. The seminal paper by Crandall, Rabinowitz and Tartar [13] is the starting point of a large literature, see for instance [3, 4, 7, 8, 12, 14, 19, 21, 23, 25, 26, 34, 35]. The philosophy to deal with this framework is similar to the one used for problems with a concave or a concave-convex nonlinearity. See for instance the papers [1, 9, 10, 17, 18] and their corresponding references.

In this work we deal with two problems:

$$(P_\gamma) = \begin{cases} (-\Delta)^s u = \lambda \dfrac{f(x)}{u^\gamma} \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

and

$$(D_{\lambda,\gamma,p}) = \begin{cases} (-\Delta)^s u = \dfrac{\lambda}{u^\gamma} + u^p \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Problem (P_γ) is strongly inspired by the following semilinear elliptic problem with a singular nonlinearity (see [7])

$$(\mathcal{L}_1) = \begin{cases} -\Delta u = \dfrac{f(x)}{u^\gamma} \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ in } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, $\gamma > 0$ is a real number and f is a nonnegative function that belongs to some Lebesgue space. The singular problem $(\mathcal{L}_1)$ appears by considering problems with a *convection term* via a change of variable, that is, this problem is related to singular problems with a term depending on $|\nabla v|^2$. See for instance the references [19, 35].

In [7], the authors study existence and regularity results of the solutions to problem $(\mathcal{L}_1)$, depending on γ (splitting in the cases $\gamma = 1$, $\gamma > 1$ and $\gamma < 1$) and on the summability of f. Our aim will be to prove, using similar techniques as in the local case, this kind of results for the nonlocal framework. In particular, to study problem (P_γ), we will work by approximation, that is, analyzing the problems obtained truncating the singular term $\frac{1}{u^\gamma}$ and the datum f, so that the first one becomes non singular at the origin and the second one belongs to $L^\infty(\Omega)$ (see [7, 8] in the local setting). It is worthy to point out that the solvability results obtained for problem (P_γ) when $s = 1$, have been recently extended by D. Giachetti, P. Martinez Aparicio and F. Murat (personal communication), to nonlinearities $f(x,u)$ where f is a function verifying

$$0 \leq f(x,u) \leq g(x)(1 + \frac{1}{u^\gamma}),$$

that is, without the monotonicity assumption. This kind of results can be also extended to the non-local framework. We omit the details.

Regarding problem $(D_{\lambda,\gamma,p})$, the motivation arises from the following semilinear problem, whose nonlinearity combines a singular term and a convex one

$$(\mathcal{L}_2) = \begin{cases} -\Delta u = \dfrac{\lambda}{u^\gamma} + u^p \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ in } \partial\Omega, \end{cases}$$

where γ, p and λ are positive numbers (see, among other papers, [3, 4, 8, 12, 14, 21, 35] for an extensive analysis of this kind of problems). The multiplicity behavior in this case is essentially the same as in *concave-convex* type problems. Firstly, the approach of [9] based on the use of sub/supersolutions method still works in this context (see also [5, 6] where this monotonicity techniques are applied in the nonlocal framework). Then, as in [3], we are also able to show, with some technical variations, the existence of a second solution for the problem $(D_{\lambda,\gamma,p})$ when we add a convex term if $\lambda > 0$ is *small enough.* More precisely, we prove the existence of a first solution for $\gamma > 0$, $p > 1$, up to some threshold on λ, Λ, considering the approximating problems corresponding to $(D_{\lambda,\gamma,p})$, by applying the sub/supersolutions and Sattinger methods.

When we consider the subcritical case, that is, $0 < \gamma < 1$ and $1 < p < 2_s^* - 1 = \frac{N+2s}{N-2s}$, we can show that for $0 < \lambda$ small enough the problem $(D_{\lambda,\gamma,p})$ has indeed two positive energy solutions, obtained by means of variational techniques: the first one by minimization and the second one by the Mountain Pass Lemma (see [2, 20]). As far as we know, the global multiplicity result, that is, the proof of the existence of a second positive solution in the whole range until the threshold Λ, remains open in the nonlocal setting.

The paper is organized as follows: in Section 2, an introduction of the necessary functional framework is presented, as well as the problems that will be treated, and the different type of solutions we will work with. This distinction is related to the assumptions on the parameter of the singularity γ and on the summability of f. Section 3 is devoted to prove the existence and regularity of solution of (P_γ). With this purpose, firstly we need to introduce and analyze the associated approximating problems.

Finally, in Section 4 we deal with problem $(D_{\lambda,\gamma,p})$. The goal of this part of the work is to prove the existence of at least two positive solutions for some range of values of $\lambda > 0$.

2 Introduction and functional setting

Consider the space

$$\mathcal{L}^s := \{u : \mathbb{R}^N \to \mathbb{R} \quad \text{measurable} : \int_{\mathbb{R}^N} \frac{|u(x)|}{(1+|x|^{N+2s})}\, dx < \infty\},$$

endowed with the natural norm

$$\|u\|_{\mathcal{L}^s} := \int_{\mathbb{R}^N} \frac{|u(x)|}{(1+|x|^{N+2s})}\, dx.$$

In this framework, we can calculate $\langle (-\Delta)^s u, \varphi\rangle$ for every $u \in \mathcal{L}^s$ and φ in the Schwartz class. Moreover, we introduce the Sobolev space

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : |\xi|^s \hat{u} \in L^2(\mathbb{R}^N)\},$$

and the space $X_0^s(\Omega)$, defined as

$$X_0^s(\Omega) = \{u \in H^s(\mathbb{R}^N) \text{ with } u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\},$$

endowed with the norm

$$\|u\|_{X_0^s(\Omega)} = \left(\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dx\, dy \right)^{1/2},$$

where $Q = \mathbb{R}^{2N} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$. This space allows us to deal with problems proposed in a bounded domain Ω, as we need. The pair $(X_0^s(\Omega), \|\cdot\|_{X_0^s(\Omega)})$ yields a Hilbert space (see [29, Lemma 7] for more details). Moreover, it can be seen that

$$(-\Delta)^s : X_0^s(\Omega) \to X^{-s}(\Omega)$$

is a continuous operator.

In what follows we will use the relation between the norm in the space $X_0^s(\Omega)$ and the L^2 norm of the fractional Laplacian, see [16, Proposition 3.6],

$$\|u\|^2_{X_0^s(\Omega)} = 2C(N,s)^{-1} \|(-\Delta)^{s/2} u\|^2_{L^2(\mathbb{R}^N)} \text{ where } C(N,s) = \frac{4^s \Gamma\left(\frac{N}{2}+s\right)}{-\pi^{\frac{N}{2}} \Gamma(-s)}. \tag{1}$$

Here $C(N,s)$ is a normalization constant, that appears in the definition of $(-\Delta)^s$, see [16, 31], chosen to guarantee that

$$\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s}\widehat{u}(\xi), \quad \xi \in \mathbb{R}^N.$$

We will also need to make use of the classical Sobolev Theorem:

Theorem 2.1. (see for instance [16, Theorem 6.5]) *Let $s \in (0,1)$ and $N > 2s$. There exists a constant $S(N,s)$ such that, for any measurable and compactly supported function $f : \mathbb{R}^N \to \mathbb{R}$, we have*

$$\|f\|^2_{L^{2^*_s}(\mathbb{R}^N)} \le S(N,s) \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|f(x)-f(y)|^2}{|x-y|^{N+2s}}\,dx\,dy,$$

being

$$2^*_s = \frac{2N}{N-2s}$$

the Sobolev critical exponent.

It is easy to check that for u and φ smooth enough, with vanishing condition outside Ω, we have the following duality product,

$$2C(N,s)^{-1}\int\limits_{\mathbb{R}^N} u(-\Delta)^s\varphi\,dx = \int\limits_{Q} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy, \tag{2}$$

what means that $(-\Delta)^s$ is selfadjoint in $X_0^s(\Omega)$.

Before stating the results contained in this work, we need to precise the sense of solutions that we will handle here. In particular, if we consider the general problem

$$(D)\begin{cases} (-\Delta)^s u = F(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

we will distinguish two types of solutions, attending to the regularity of F and u. Indeed,

Definition 2.2. *We say that $u \in X_0^s(\Omega)$ is a positive energy supersolution (respectively subsolution) of problem (D) if $F \in L^{(2^*_s)'}_{loc}(\Omega)$ and*

$$\frac{C(N,s)}{2}\iint\limits_{Q} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy \ge (\le) \int\limits_{\Omega} F(x,u)\varphi(x)\,dx, \tag{3}$$

for every nonnegative $\varphi \in X_0^s(\Omega)$ with compact support contained in Ω.

If u is a supersolution and a subsolution of (D), we say that it is a positive energy solution.

Analogously, when we have less regularity on u, we will make use of a weaker notion of solution. Define first the set

$$\mathcal{T} := \{\phi : \mathbb{R}^N \to \mathbb{R} \text{ measurable s.t. } (-\Delta)^s\phi = \varphi \in L^\infty(\Omega),\ \phi = 0 \text{ on } \mathbb{R}^N \setminus \widetilde{\Omega},\ \widetilde{\Omega} \subset\subset \Omega\}.$$

Notice that every $\phi \in \mathcal{T}$ belongs in particular to $L^\infty(\Omega)$ (see [27]).

Definition 2.3. *We say that $u \in L^1(\Omega)$ is a positive weak supersolution (respectively subsolution) of problem (D) if $F \in L^1_{loc}(\Omega)$, $u = 0$ in $\mathbb{R}^N \setminus \Omega$ and*

$$\int\limits_{\Omega} u(-\Delta)^s\phi\,dx \ge (\le) \int\limits_{\Omega} F(x,u)\phi(x)\,dx, \tag{4}$$

for every nonnegative $\phi \in \mathcal{T}$.

We say that u is a positive weak solution of problem (D) if it is at the same time a supersolution and a subsolution of such a problem.

Notice that, due to the singular term in $(\mathcal{P})$, only by asking $u \in X_0^s(\Omega)$ we cannot expect $F(x,u)$ to belong to $L^{(2_s^*)'}(\Omega)$ in the first case, or $F(x,u) \in L^1(\Omega)$ if $u \in L^1(\Omega)$ in the second one. Thus, the right hand side is not well defined if we test in the natural spaces $X_0^s(\Omega)$ and $\mathcal{T}$ with functions supported in Ω respectively. To deal with this difficulty we restrict our test sets to the functions with compact support.

Remark 2.4. *If $F \in L^{(2_s^*)'}(\Omega)$ in (D), we can extend Definition 2.2, saying that* (3) *holds for every $\varphi \in X_0^s(\Omega)$. This will be the case of the approximating problems,*

$$\begin{cases} (-\Delta)^s u_n = F_n(x,u_n) := \lambda \dfrac{\min(f(x),n)}{\left(u_n+\frac{1}{n}\right)^\gamma} + M u_n^p \ \textit{in}\ \Omega, \\ \qquad u_n > 0 \ \textit{in}\ \Omega, \\ \qquad u_n = 0 \ \textit{in}\ \mathbb{R}^N \setminus \Omega, \end{cases}$$

that we will use to build solutions of $(\mathcal{P})$. Notice that here the first term in the right hand side is no longer singular (it is bounded indeed), so we do not need to restrict our test set to functions compactly supported in Ω.

3 Solvability of the elliptic problem with the singular nonlinearity

Let Ω be a bounded domain in $\mathbb{R}^N$. We consider the following problem

$$(P_\gamma) = \begin{cases} (-\Delta)^s u = \lambda \dfrac{f(x)}{u^\gamma} \ \text{in}\ \Omega, \\ \qquad u > 0 \ \text{in}\ \Omega, \\ \qquad u = 0 \ \text{in}\ \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $N > 2s$, $0 < s < 1$, $\gamma > 0$, $\lambda > 0$ and f is a nonnegative function whose summability conditions will be specified later. By a rescaling argument is sufficient to consider $\lambda = 1$.

3.1 Approximating problems

In order to study the solvability of problem (P_γ), we will analyze the associated approximating problems. Indeed, suppose $f \in L^1(\Omega)$, $f \geq 0$, and for every $n \in \mathbb{N}$, let us define

$$f_n := \min(f(x), n),$$

and consider the problem

$$(P_{n,\gamma}) = \begin{cases} (-\Delta)^s u_n = \dfrac{f_n}{\left(u_n+\frac{1}{n}\right)^\gamma} \ \text{in}\ \Omega, \\ \qquad u_n > 0 \ \text{in}\ \Omega, \\ \qquad u_n = 0 \ \text{in}\ \mathbb{R}^N \setminus \Omega. \end{cases}$$

First, we prove the existence of solution to this problem.

Lemma 3.1. *Problem $(P_{n,\gamma})$ has a nonnegative solution $u_n \in X_0^s(\Omega) \cap L^\infty(\Omega)$.*

Proof. Fix $n \in \mathbb{N}$. Let $v \in L^2(\Omega)$, and define $w = S(v)$ to be the unique solution of the problem

$$\begin{cases} (-\Delta)^s w = \dfrac{f_n}{\left(v^+ + \frac{1}{n}\right)^\gamma} \ \text{in}\ \Omega, \\ \qquad w > 0 \ \text{in}\ \Omega, \\ \qquad w = 0 \ \text{in}\ \mathbb{R}^N \setminus \Omega. \end{cases} \tag{5}$$

Notice that the existence and uniqueness of solution to this problem is given by the Lax-Milgram Theorem, since the right hand side belongs to the dual space $X^{-s}(\Omega)$. Testing now in (5) with w, we get

$$\frac{C(N,s)}{2}\int_Q \frac{(w(x)-w(y))^2}{|x-y|^{N+2s}}\,dx\,dy = \int_\Omega \frac{f_n w}{\left(v^+ + \frac{1}{n}\right)^\gamma} \le n^{\gamma+1}\|w\|_{L^1(\Omega)}, \tag{6}$$

and thus, by the Sobolev embedding given in Theorem 2.1, it follows that

$$\|w\|_{X_0^s(\Omega)} \le Cn^{\gamma+1}, \tag{7}$$

with $C = C(N,s,\Omega)$ independent of v, so that the ball of radius $Cn^{\gamma+1}$ is invariant under S in $X_0^s(\Omega)$. In order to apply the Schauder's Fixed Point Theorem over S to guarantee the existence of a solution of $(P_{n,\gamma})$, apart from the invariance, we need to check the continuity and compactness of S as an operator from $X_0^s(\Omega)$ to $X_0^s(\Omega)$.

First, we prove the continuity. In order to do this, we want to check that, if we denote $w_k := S(v_k)$ and $w := S(v)$, then

$$\lim_{k\to\infty}\|w_k - w\|_{X_0^s(\Omega)} = 0 \text{ whenever } \lim_{k\to\infty}\|v_k - v\|_{X_0^s(\Omega)} = 0. \tag{8}$$

Notice that from the convergence of v_k in $X_0^s(\Omega)$, by Theorem 2.1 we obtain

$$\begin{aligned} &v_k \to v \text{ in } L^{2_s^*}(\Omega),\\ &v_k \to v \text{ a.e. in } \Omega. \end{aligned} \tag{9}$$

In fact, let $\{v_k\}_{k\in\mathbb{N}}$ be a sequence in $X_0^s(\Omega)$ converging to $v \in X_0^s(\Omega)$. Thus we get

$$\begin{aligned} \frac{C(N,s)}{2}\|w_k - w\|^2_{X_0^s(\Omega)} \le& \int_\Omega \left(\frac{f_n}{\left(v_k^+ + \frac{1}{n}\right)^\gamma} - \frac{f_n}{\left(v^+ + \frac{1}{n}\right)^\gamma}\right)(w_k - w)\,dx \\ \le& \|w_k - w\|_{L^{2_s^*}(\Omega)}\left(\int_\Omega \left(\frac{f_n}{\left(v_k^+ + \frac{1}{n}\right)^\gamma} - \frac{f_n}{\left(v^+ + \frac{1}{n}\right)^\gamma}\right)^{(2_s^*)'} dx\right)^{1/(2_s^*)'}, \end{aligned} \tag{10}$$

where $(2_s^*)' = \frac{2N}{N+2s} < 2_s^*$. Thus, by Hölder's inequality and Theorem 2.1 again, we obtain

$$\|w_k - w\|_{X_0^s(\Omega)} \le C(N,s,n,\Omega)\left(\int_\Omega \left(\frac{f_n}{\left(v_k^+ + \frac{1}{n}\right)^\gamma} - \frac{f_n}{\left(v^+ + \frac{1}{n}\right)^\gamma}\right)^{2_s^*} dx\right)^{1/2_s^*}.$$

Now we observe that both

$$\frac{f_n}{\left(v_k^+ + \frac{1}{n}\right)^\gamma} \le n^{\gamma+1} \quad \text{and} \quad \frac{f_n}{\left(v^+ + \frac{1}{n}\right)^\gamma} \le n^{\gamma+1},$$

and therefore, by the Dominated Convergence Theorem and (9), we conclude that

$$\|w_k - w\|_{X_0^s(\Omega)} \to 0 \text{ as } k \to \infty,$$

and hence, S is continuous from $X_0^s(\Omega)$ to $X_0^s(\Omega)$.

To see that S is compact, we take a sequence $\{v_k\}_{k\in\mathbb{N}}$ such that $\|v_k\|_{X_0^s(\Omega)} \le C$. Therefore, by Rellich-Kondrachov Theorem (see [16, Theorem 7.1]) we conclude that, up to a subsequence,

$$\begin{aligned} &v_k \rightharpoonup v \text{ in } X_0^s(\Omega),\\ &v_k \to v \text{ in } L^r(\Omega),\ 1 \le r < 2_s^*. \end{aligned} \tag{11}$$

Furthermore, since S is continuous,

$$\|S(v_k)\|_{X_0^s(\Omega)} \le C,$$

with C a positive constant independent of k, and hence,

$$\begin{aligned} S(v_k) &\rightharpoonup \tilde{w} \text{ in } X_0^s(\Omega), \\ S(v_k) &\to \tilde{w} \text{ in } L^r(\Omega),\ 1 \le r < 2_s^*. \end{aligned} \tag{12}$$

Because of the continuity of S, necessarily $\tilde{w} = S(v)$. Thus, proceeding as in (10) one can reach

$$\frac{C(N,s)}{2}\|S(v_k) - S(v)\|^2_{X_0^s(\Omega)} \le \|S(v_k) - S(v)\|_{L^2(\Omega)} \left(\int_\Omega \left(\frac{f_n}{\left(v_k^+ + \frac{1}{n}\right)^\gamma} - \frac{f_n}{\left(v^+ + \frac{1}{n}\right)^\gamma} \right)^2 dx \right)^{1/2},$$

and by (12) we conclude

$$\lim_{k\to\infty} \|S(v_k) - S(v)\|_{X_0^s(\Omega)} = 0,$$

and therefore S is compact from $X_0^s(\Omega)$ to $X_0^s(\Omega)$.

Given these conditions on S, Schauder's Fixed Point Theorem provides the existence of $u_n \in X_0^s(\Omega)$ such that $u_n = S(u_n)$, i.e. u_n solves

$$\begin{cases} (-\Delta)^s u_n = \dfrac{f_n}{\left(u_n^+ + \frac{1}{n}\right)^\gamma} \text{ in } \Omega, \\ \qquad u_n = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{13}$$

By the weak maximum principle ([31] and [6, Lemma 2.1]), $u_n \ge 0$, and hence u_n solves $(P_{n,\gamma})$. Finally, since the right hand side of $(P_{n,\gamma})$ belongs to $L^\infty(\Omega)$, by [27] we also get that $u_n \in L^\infty(\Omega)$. □

Moreover, we can prove the following result,

Lemma 3.2. *$\{u_n\}_{n\in\mathbb{N}}$ is an increasing sequence, $u_n > 0$ in Ω, and for every set $\tilde{\Omega} \subset\subset \Omega$ there exists a positive constant $c_{\tilde{\Omega}}$, independent of n, such that*

$$u_n(x) \ge c_{\tilde{\Omega}} > 0, \quad \textit{for every } x \in \Omega \textit{ and every } n \in \mathbb{N}. \tag{14}$$

Proof. Consider the problem satisfied by u_n and u_{n+1} and substracting them we get:

$$\begin{aligned} (-\Delta)^s(u_n - u_{n+1}) &= \frac{f_n}{(u_n + \frac{1}{n})^\gamma} - \frac{f_{n+1}}{(u_{n+1} + \frac{1}{n+1})^\gamma} \\ &\le \frac{f_{n+1}}{(u_n + \frac{1}{n+1})^\gamma} - \frac{f_{n+1}}{(u_{n+1} + \frac{1}{n+1})^\gamma} \\ &= f_{n+1} \frac{(u_{n+1} + \frac{1}{n+1})^\gamma - (u_n + \frac{1}{n+1})^\gamma}{(u_n + \frac{1}{n+1})^\gamma (u_{n+1} + \frac{1}{n+1})^\gamma}. \end{aligned}$$

Now we choose, in the problems satisfied by $u_n - u_{n+1}$, $(u_n - u_{n+1})^+$ as test function. Since

$$\left[(u_{n+1} + \frac{1}{n+1})^\gamma - (u_n + \frac{1}{n+1})^\gamma \right] (u_n - u_{n+1})^+ \le 0,$$

$f_{n+1} \ge 0$, and , by [27],

$$\int_{\mathbb{R}^N} (-\Delta)^s(u_n - u_{n+1})(u_n - u_{n+1})^+ \ge \int_{\mathbb{R}^N} |(-\Delta)^{s/2}(u_n - u_{n+1})^+|^2 \ge 0,$$

we have,

$$0 \le \frac{C(N,s)}{2}\|(u_n - u_{n+1})^+\|^2_{X_0^s(\Omega)} \le 0.$$

Therefore $u_n \le u_{n+1}$.

On the other hand, from Lemma 3.1 we know that u_1 belongs to $L^\infty(\Omega)$, that is, $\|u_1\|_{L^\infty(\Omega)} \le C$, and thus,

$$(-\Delta)^s u_1 = \frac{f_1}{(u_1+1)^\gamma} \ge \frac{f_1}{(\|u_1\|_{L^\infty(\Omega)}+1)^\gamma} \ge \frac{f_1}{(C+1)^\gamma}.$$

Once again, since $\dfrac{f_1}{(C+1)^\gamma}$ is not identically zero, a further use of the Strong Maximum Principle ([31]) gives us that $u_1 > 0$ in Ω and hence, since $\{u_n\}$ is an increasing sequence, u_n verifies (14), for every $n \ge 1$. □

Corollary 3.3. *The solution u_n to problem $(P_{n,\gamma})$ is unique.*

Proof. Let us consider $v_n \ne u_n$ a solution of $(P_{n,\gamma})$. Taking $(u_n - v_n)^+$ ($(v_n - u_n)^+$ resp.) as a test function in $(P_{n,\gamma})$, we conclude $v_n \le u_n$ ($u_n \le v_n$ resp.), and the uniqueness follows. See [10] for more details and general results. □

Now, the goal is passing to the limit in the sequence $\{u_n\}_{n\in\mathbb{N}}$ to achieve a solution of (P_γ). With this purpose, we must distinguish three cases, attending to the value of the power γ.

3.2 Case $\gamma \le 1$

Lemma 3.4. *Let u_n be the solution of the problem $(P_{n,\gamma})$. Then,*
- *if $\gamma = 1$ and $f \in L^1(\Omega)$, or*
- *if $\gamma < 1$ and f belongs to $L^m(\Omega)$ with $m = \frac{2N}{N+2s+\gamma(N-2s)} = \left(\frac{2_s^*}{1-\gamma}\right)' > 1$,*

u_n is uniformly bounded in $X_0^s(\Omega)$.

Proof. Let us first consider the case $\gamma = 1$. Taking u_n as a test function in $(P_{n,\gamma})$, just by noticing that $\dfrac{u_n}{u_n+\frac{1}{n}} \le 1$, one gets,

$$\frac{C(N,s)}{2}\|u_n\|^2_{X_0^s(\Omega)} = \int_\Omega \frac{f_n u_n}{u_n+\frac{1}{n}} \le \int_\Omega f < +\infty,$$

that is, $\|u_n\|_{X_0^s(\Omega)} \le C$, with C independent of n.

In the case $\gamma < 1$ taking again u_n as a test function in $(P_{n,\gamma})$, by Theorem 2.1 and Hölder inequality, we get

$$\frac{C(N,s)}{2}\|u_n\|^2_{X_0^s(\Omega)} \le \int_\Omega f u_n^{1-\gamma} \le \|f\|_{L^m(\Omega)} \left(\int_\Omega u_n^{2_s^*}\right)^{\frac{1}{m'}} \le S(N,s)\|f\|_{L^m(\Omega)} \|u_n\|^{\frac{2_s^*}{m'}}_{X_0^s(\Omega)}. \tag{15}$$

Since $\frac{2}{2_s^*} > \frac{1}{m'}$, we get an uniform estimate of u_n in the space $X_0^s(\Omega)$, as desired. □

Theorem 3.5. *Let $f \in L^{(2_s^*)'}(\Omega)$ and $\gamma \le 1$. Then there exists an energy solution $u \in X_0^s(\Omega)$ of problem (P_γ).*

Proof. First of all, notice that

$$(2_s^*)' = \frac{2N}{N+2s} \ge \frac{2N}{N+2s+\gamma(N-2s)} = \left(\frac{2_s^*}{1-\gamma}\right)' > 1.$$

Therefore, for both cases $\gamma = 1$ and $\gamma < 1$, since, by Lemma 3.4, u_n is bounded in the Hilbert space $X_0^s(\Omega)$, u_n weakly converges to u in $X_0^s(\Omega)$. Therefore we are able to pass to the limit in the left hand side of $(P_{n,\gamma})$, that is,

$$\lim_{n\to\infty} \int_Q \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy = \int_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy,\ \varphi \in X_0^s(\Omega).$$

In the right hand side we observe that, for every $\varphi \in H^s(\mathbb{R}^N)$ with $\mathrm{supp}(\varphi) = \omega \subset\subset \Omega$, since $f \in L^{(2_s^*)'}(\Omega)$, we have

$$0 \le \left|\frac{f_n\varphi}{\left(u_n+\frac{1}{n}\right)^\gamma}\right| \le \frac{|\varphi||f|}{c_\omega^\gamma} \in L^1(\Omega).$$

Therefore, by the Dominated Convergence Theorem we get

$$\lim_{n\to\infty}\int_\Omega \frac{f_n\varphi}{\left(u_n+\frac{1}{n}\right)^\gamma} = \int_\Omega \frac{f\varphi}{u^\gamma}.$$ □

3.3 Case $\gamma > 1$

Lemma 3.6. *Let $f \in L^1(\Omega)$ and let u_n be the solution of problem $(P_{n,\gamma})$ for $\gamma > 1$. Then, $u_n^{\frac{\gamma+1}{2}}$ is uniformly bounded in $X_0^s(\Omega)$.*

Proof. Let be $T > 1$. We consider, for $\beta > 1$, the convex function $\Phi_\beta : [0,+\infty) \to [0,+\infty)$ defined as

$$\Phi_\beta(r) := \begin{cases} r^\beta & \text{if } 0 \le r < T, \\ \beta T^{\beta-1} r - (\beta-1)T^\beta & if\ r \ge T > 1. \end{cases} \tag{16}$$

Let us take $\beta = \frac{\gamma+1}{2} > 1$ and we call $\Phi(r) := \Phi_{\frac{\gamma+1}{2}}(r)$. Since $\Phi(r)$ is a Lipschitz (with constant $L_{\Phi(r)} = \frac{\gamma+1}{2}T^{\frac{\gamma-1}{2}}$), then $\Phi(r)$ and $\Phi(r)\Phi'(r)$ belong to $X_0^s(\Omega)$. Using [27, Proposition 2.4], we have

$$(-\Delta)^s\Phi(u_n) \le \Phi'(u_n)(-\Delta)^s u_n. \tag{17}$$

Therefore

$$\int_{\mathbb{R}^N} \Phi(u_n)(-\Delta)^s\Phi(u_n) \le \int_{\mathbb{R}^N} \Phi'(u_n)\Phi(u_n)(-\Delta)^s u_n = \int_\Omega \frac{f}{\left(u_n+\frac{1}{n}\right)^\gamma}\Phi'(u_n)\Phi(u_n). \tag{18}$$

Since $\Phi'(u_n)\Phi(u_n) \le \frac{\gamma+1}{2}u_n^\gamma$, from (18) it follows that

$$\frac{C(N,s)}{2}\|\Phi(u_n)\|^2_{X_0^s(\Omega)} \le \frac{(\gamma+1)}{2}\|f\|_{L^1(\Omega)} \le C,$$

where $C > 0$ is independent of n. Letting $T \to +\infty$ we conclude. □

Theorem 3.7. *Let $f \in L^1(\Omega)$ and $\gamma > 1$. Then there exists a weak solution u of problem (P_γ). Moreover, $u^{\frac{\gamma+1}{2}} \in X_0^s(\Omega)$.*

Proof. Consider the sequence $\{u_n\}_{n\in\mathbb{N}}$ of solutions to the problem $(P_{n,\gamma})$. By Lemma 3.2 we know that this sequence is increasing, and thus we can define $u := \lim_{n\to\infty} u_n$. Hence, due to the weak lower semicontinuity of the norm, by Lemma 3.6 we obtain

$$\|u^{\frac{\gamma+1}{2}}\|_{X_0^s(\Omega)} \le \liminf_{n\to\infty}\|u_n^{\frac{\gamma+1}{2}}\|_{X_0^s(\Omega)} \le C,$$

with C a positive constant independent of n, i.e., $u^{\frac{\gamma+1}{2}} \in X_0^s(\Omega)$. Moreover, by the Sobolev embedding this implies $u^{\frac{\gamma+1}{2}} \in L^{2_s^*}(\Omega)$ and, since $\frac{\gamma+1}{2}2_s^* > 1$, in particular $u \in L^1(\Omega)$. Thus, for every $\phi \in \mathcal{T}$,

$$\lim_{n\to\infty}\int_\Omega u_n(-\Delta)^s\phi\,dx = \int_\Omega u(-\Delta)^s\phi\,dx < +\infty.$$

Since $\mathrm{supp}(\phi) =: \omega \subset\subset \Omega$ whether $\phi \in \mathcal{T}$, by Lemma 3.2 we have

$$0 \le \left|\frac{f_n\phi}{(u_n+\frac{1}{n})^\gamma}\right| \le \frac{|f||\phi|}{c_\omega^\gamma} \in L^1(\Omega).$$

Therefore, by the Dominated Convergence Theorem we can pass to the limit in the right hand side of the weak formulation of $(P_{n,\gamma})$, concluding the existence of a weak solution of problem (P_γ). □

3.4 Regularity of solutions of problem (P_γ)

Proposition 3.8. *Let $f \in L^{(2_s^*)'}(\Omega)$, $\gamma \leq 1$ and u be the solution of (P_γ) provided by Theorem 3.5. Then $u \in L^{(\gamma+1)2_s^*}(\Omega)$.*

Proof. We begin proving the case $\gamma = 1$. Following the idea given in [27, Theorem 3.10] we consider $\Phi_\beta(r)$, the convex function defined in (16) as a test function in $(P_{n,\gamma})$, for $\beta = 2$. Doing the same as in Lemma 3.6, since $\Phi_2'(u_n)\Phi_2(u_n) \leq 2u_n^3$, we get that

$$\int_{\mathbb{R}^N} \Phi_2(u_n)(-\Delta)^s \Phi_2(u_n) \leq \int_{\mathbb{R}^N} \Phi_2'(u_n)\Phi_2(u_n)(-\Delta)^s u_n = \int_\Omega \frac{f}{u_n}\Phi_2'(u_n)\Phi_2(u_n) \leq 2\int_\Omega f u_n^2. \tag{19}$$

The integral in the left hand side of (19) can be estimated, by Theorem 2.1, in the following way

$$\begin{aligned} \int_{\mathbb{R}^N} \left|(-\Delta)^{s/2}\Phi_2(u_n)\right| &= \|\Phi_2(u_n)\|^2_{X_0^s(\Omega)} \geq \frac{1}{S(N,s)}\|\Phi_2(u_n)\|^2_{L^{2_s^*}(\Omega)} \\ &= \frac{1}{S(N,s)}\left[\int_{\{u_n<T\}} u_n^{2\cdot 2_s^*} + \int_{\{u_n\geq T\}} (2Tu_n - T^2)^{2_s^*}\right]^{\frac{2}{2_s^*}} \\ &\geq \frac{1}{S(N,s)}\left[\int_{\{u_n<T\}} u_n^{2\cdot 2_s^*} + \int_{\{u_n\geq T\}} T^{2\cdot 2_s^*}\right]^{\frac{2}{2_s^*}} \\ &\geq \frac{1}{S(N,s)}\left[\int_{\{u_n<T\}} u_n^{2\cdot 2_s^*} + \operatorname{meas}\{u_n \geq T\}\right]^{\frac{2}{2_s^*}}. \end{aligned} \tag{20}$$

Since $u_n \in L^\infty(\Omega)$ then

$$\lim_{T\to+\infty} (\operatorname{meas}\{u_n \geq T\}) = 0.$$

Thus, from (19) and (20), we find that

$$\|u_n^2\|^2_{L^{2_s^*}(\Omega)} \leq C(N,s)\int_\Omega f u_n^2 \leq C(N,s)\|f\|_{L^{(2_s^*)'}(\Omega)}\|u_n^2\|_{L^{2_s^*}(\Omega)}.$$

That is, by Fatou's Lemma we conclude that u belongs to $L^{2\cdot 2_s^*}(\Omega)$.

Consider now the case $\gamma < 1$. In this case we take $\Phi_\beta(r)$, defined in (16), with $\beta = \gamma + 1$, and thus, repeating the previous argument, one gets

$$\begin{aligned} \|u_n^{\gamma+1}\|^2_{L^{2_s^*}(\Omega)} &\leq C(N,s)\int_\Omega f u_n^{2(\gamma+1)-1-\gamma} \leq C(N,s)\|f\|_{L^{(2_s^*)'}(\Omega)}\|u_n^{\gamma+1}\|_{L^{2_s^*}(\Omega)} \\ &= C(N,s)\|f\|_{L^{(2_s^*)'}(\Omega)}\|u_n^{\gamma+1}\|_{L^{2_s^*}(\Omega)}. \end{aligned}$$

Therefore the previous inequality and Fatou's Lemma imply that $u \in L^{(\gamma+1)2_s^*}(\Omega)$, as desired. □

Remark 3.9. *Observe that the exponent of summability $(\gamma + 1)2_s^*$ coincides when $s = 1$ with the one given in [7, Lemmas 3.3 and 5.5] in the local case.*

The summability of the solution obtained in the previous proposition could be improved requesting more regularity to the function f. In order to prove this result, we will adapt to the nonlocal framework the ideas given in [4, Lemma 1], to obtain the following:

Lemma 3.10. *Let $\lambda > 0$ and $\gamma > 0$. If $w \in X_0^s(\Omega)$, $w > 0$ in Ω, satisfies*

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2} w (-\Delta)^{s/2} \phi \, dx \le \lambda \int_{\Omega} \frac{\phi}{w^\gamma} \, dx \qquad \text{for every } \phi \in X_0^s(\Omega),\ \phi \ge 0 \tag{21}$$

then there exists a constant $C > 0$, independent of w, such that

$$\|w\|_{L^\infty(\Omega)} \le C \lambda^{\frac{1}{\gamma+1}}.$$

Proof. Let us first consider the case $\lambda = 1$. For $k \ge 1$, define the function

$$G_k(\sigma) := \sigma - \max\{-k, \min\{k, \sigma\}\}.$$

Thus, testing with $G_k(w)$ in (21), by [27, Proposition 2.7 ii)], we get that

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2} G_k(w)|^2 \le \int_{A_k} \frac{G_k(w)}{w^\gamma} \le \int_{A_k} G_k(w),$$

where

$$A_k := \{x \in \Omega : \ w(x) > k\}.$$

Since, from the previous inequality and Theorem 2.1, we have that

$$\|G_k(w)\|_{L^{2_s^*}(\Omega)} \le S(N,s) |A_k|^{\frac{N+2s}{2N}},$$

using Stampacchia's method ([32, Lemma 4.1]), we conclude that there exists a positive constant $C = C(N,s)$ such that $\|w\|_{L^\infty(\Omega)} \le C$.

The general case, $\lambda > 0$, follows easily using that the fractional Laplacian is a linear operator. In fact, if we consider the modified function

$$\widetilde{w} = \left(\frac{1}{\lambda}\right)^{\frac{1}{\gamma+1}} w \in X_0^s(\Omega),$$

for every $\varphi \in X_0^s(\Omega)$, we get that

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2} \widetilde{w} (-\Delta)^{s/2} \varphi = \left(\frac{1}{\lambda}\right)^{\frac{1}{\gamma+1}} \int_{\mathbb{R}^N} (-\Delta)^{s/2} w (-\Delta)^{s/2} \varphi \le \left(\frac{1}{\lambda}\right)^{\frac{1}{\gamma+1}} \int_{\Omega} \frac{\lambda \varphi}{w^\gamma} = \int_{\Omega} \frac{\varphi}{\widetilde{w}^\gamma}.$$

That is, $\widetilde{w}$ is an energy solution of the Dirichlet problem (P) with $\lambda = 1$. Therefore, for the calculations done before for the case $\lambda = 1$, there exists $C > 0$ such that $\|\widetilde{w}\|_{L^\infty(\Omega)} \le C$. That is, $\|w\|_{L^\infty(\Omega)} \le C\lambda^{\frac{1}{\gamma+1}}$. □

Using the previous Lemma we get the next result.

Proposition 3.11. *Let us consider $0 < \gamma$ and let u be a solution of (P_γ) provided by Theorems 3.5 and Theorem 3.7. If $f \in L^m(\Omega)$ with $m > \frac{N}{2s}$, then $u \in L^\infty(\Omega)$.*

Proof. Following the same ideas done in the proof of Lemma 3.10, applying Hölder inequality and the Stampacchia's method for the solutions u_n of the truncated problems $(P_{n,\lambda})$. That is, one proves that there exists $C = C(N, s, \|f\|_{L^m(\Omega)}, |\Omega|) > 0$ such that

$$\|u_n\|_{L^\infty(\Omega)} \le C,$$

so, recalling that $u := \lim_{n\to\infty} u_n$ we conclude. □

4 Some remarks on solvability and multiplicity of solutions for the elliptic problem when adding a convex term

In this section, we consider the previous Dirichlet problem, but adding a convex term of the form u^p, with $p > 1$.

$$(D_{\lambda,\gamma,p}) = \begin{cases} (-\Delta)^s u = \dfrac{\lambda}{u^\gamma} + u^p \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

4.1 First solution

We will prove the following

Theorem 4.1. *Assume $\gamma > 0$ and $p > 1$. Then, there exists $0 < \Lambda < \infty$ such that, for every $0 < \lambda < \Lambda$, there exists a positive solution u to the problem $(D_{\lambda,\gamma,p})$ in the following sense:*
- *if $0 < \gamma \le 1$, then $u \in X_0^s(\Omega) \cap L^\infty(\Omega)$ is an energy solution;*
- *if $\gamma > 1$, then $u \in L^\infty(\Omega)$ is a weak solution, satisfying $u^{\frac{\gamma+1}{2}} \in X_0^s(\Omega)$.*

Proof. Step 1:
Consider the following approximated problems:

$$(D_{n,\lambda,\gamma,p}) = \begin{cases} (-\Delta)^s u_n = \dfrac{\lambda}{\left(u_n^+ + \frac{1}{n}\right)^\gamma} + (u_n^+)^p \text{ in } \Omega, \\ u_n = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

First, we prove that there exists a solution of this problem, by applying the Sattinger method.

Step 2: We construct a subsolution.
Consider the solution $\underline{u_n} \in X_0^s(\Omega)$ to the problem

$$\begin{cases} (-\Delta)^s \underline{u_n} = \dfrac{\lambda}{\left(\underline{u_n}^+ + \frac{1}{n}\right)^\gamma} \text{ in } \Omega, \\ \underline{u_n} = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{22}$$

that is constructed applying Schauder's fixed point theorem, in the same way as the solution of problem (5). By the maximum principle, $\underline{u_n} \ge 0$, and thus

$$\frac{\lambda}{\left(\underline{u_n}^+ + \frac{1}{n}\right)^\gamma} \le \frac{\lambda}{\left(\underline{u_n} + \frac{1}{n}\right)^\gamma} + \underline{u_n}^p,$$

that is, $\underline{u_n}$ is a subsolution of $(D_{n,\lambda,\gamma,p})$.

Step 3: We find a supersolution.
Let $t > \lambda$, that will be chosen later, and let $0 \le \overline{u_n}$ be the energy solution to the problem

$$\begin{cases} (-\Delta)^s \overline{u_n} = \dfrac{t}{\left(\overline{u_n}^+ + \frac{1}{n}\right)^\gamma} \text{ in } \Omega, \\ \overline{u_n} = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{23}$$

By Lemma 3.10, we know that there exists $C_0 > 0$ such that $\|\overline{u_n}\|_{L^\infty(\Omega)} \le C_0 t^{\frac{1}{\gamma+1}}$. In order to have that $\overline{u_n}$ is a supersolution of $(D_{n,\lambda,\gamma,p})$, we need to prove that we can choose t large enough so that

$$\frac{t}{\left(\overline{u_n} + \frac{1}{n}\right)^\gamma} \ge \frac{\lambda}{\left(\overline{u_n} + \frac{1}{n}\right)^\gamma} + \overline{u_n}^p. \tag{24}$$

But this is equivalent to

$$t \geq \lambda + \overline{u_n}^p \left(\overline{u_n} + \frac{1}{n}\right)^\gamma ,$$

what holds if t satisfies

$$t \geq \lambda + (C_0 t^{\frac{1}{\gamma+1}})^p \left(C_0 t^{\frac{1}{\gamma+1}} + \frac{1}{n}\right)^\gamma .$$

Notice first that, since $\dfrac{\gamma + p}{\gamma + 1} > 1$, for λ small enough one can find $t > 0$ satisfying

$$t \geq \lambda + 2^\gamma C_0^{p+\gamma} t^{\frac{\gamma+p}{\gamma+1}}.$$

Hence, if

$$\frac{1}{n} \leq C_0 t^{\frac{1}{\gamma+1}},$$

(24) holds. Thus, we have proved the existence of $\Lambda_0 > 0$ such that for $0 < \lambda < \Lambda_0$, $\overline{u_n}$ is a supersolution of problem $(D_{n,\lambda,\gamma,p})$.

Step 4: We prove that $\underline{u_n} \leq \overline{u_n}$.

We procced as in the proof of Lemma 3.2, that is, we consider the problem satisfied by $\underline{u_n} - \overline{u_n}$ and we use $(\underline{u_n} - \overline{u_n})^+$ as test function. Therefore, since $t > \lambda$ and $\dfrac{1}{\left(r + \frac{1}{n}\right)^\gamma}$ is decreasing for $r > 0$, we get that

$$\begin{aligned}
\int_{\mathbb{R}^N} (\underline{u_n} - \overline{u_n})^+ (-\Delta)^s (\underline{u_n} - \overline{u_n}) &= \int_\Omega \left(\frac{\lambda}{\left(\underline{u_n} + \frac{1}{n}\right)^\gamma} - \frac{t}{\left(\overline{u_n} + \frac{1}{n}\right)^\gamma} \right) (\underline{u_n} - \overline{u_n})^+ \\
&= \int_\Omega \left(\frac{\lambda}{\left(\underline{u_n} + \frac{1}{n}\right)^\gamma} - \frac{\lambda}{\left(\overline{u_n} + \frac{1}{n}\right)^\gamma} \right) (\underline{u_n} - \overline{u_n})^+ \\
&\quad + \int_\Omega \left(\frac{\lambda}{\left(\overline{u_n} + \frac{1}{n}\right)^\gamma} - \frac{t}{\left(\overline{u_n} + \frac{1}{n}\right)^\gamma} \right) (\underline{u_n} - \overline{u_n})^+ \\
&\leq 0.
\end{aligned} \tag{25}$$

Thus, using that

$$\int_{\mathbb{R}^N} (\underline{u_n} - \overline{u_n})^+ (-\Delta)^s (\underline{u_n} - \overline{u_n}) \geq \int_{\mathbb{R}^N} |(-\Delta)^{s/2} (\underline{u_n} - \overline{u_n})^+|^2 \geq 0,$$

it follows that $\underline{u_n} \leq \overline{u_n}$.

Step 5: Sattinger method.

Consider the function

$$g(r) = \frac{\lambda}{\left(r + \frac{1}{n}\right)^\gamma} + r^p + n^{\gamma+1} \lambda \gamma r, \quad r \in [0, +\infty). \tag{26}$$

It can be easily checked that g is an increasing function. Thus, consider $u_{n,1} \in X_0^s(\Omega)$ the solution of the problem

$$\begin{cases} (-\Delta)^s u_{n,1} + n^{\gamma+1} \lambda \gamma u_{n,1} = g(\underline{u_n}) \text{ in } \Omega, \\ u_{n,1} > 0 \text{ in } \Omega, \\ u_{n,1} = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{27}$$

Since the added term in the left hand side is linear, it can be seen ([6, Lemma 2.1]) that the new operator satisfies a comparison principle. Hence, using that

$$(-\Delta)^s \underline{u_n} + n^{\gamma+1} \lambda \gamma \underline{u_n} \leq g(\underline{u_n}) \text{ in } \Omega,$$

we conclude that $\underline{u_n} \leq u_{n,1}$. Likewise, it can be proved that $u_{n,1} \leq \overline{u_n}$. Consider now for every $k \in \mathbb{N}$ the iterated problems

$$\begin{cases} (-\Delta)^s u_{n,k+1} + n^{\gamma+1} \lambda \gamma u_{n,k+1} = g(u_{n,k}) \text{ in } \Omega, \\ u_{n,k+1} > 0 \text{ in } \Omega, \\ u_{n,k+1} = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{28}$$

Then, since g is increasing, the solutions of these problems satisfy

$$\underline{u_n} \leq u_{n,1} \leq \cdots \leq u_{n,k} \leq u_{n,k+1} \leq \overline{u_n}.$$

Thus, we can define the pointwise limit $u_n := \lim_{k\to\infty} u_{n,k}$. Moreover, using $u_{n,k+1}$ as a test function in (28), by Lemma 3.10, we get that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u_{n,k+1}|^2 + n^{\gamma+1}\lambda \int_\Omega \gamma u_{n,k+1}^2 \\
&\qquad = \lambda \int_\Omega \frac{u_{n,k+1}}{\left(u_{n,k} + \frac{1}{n}\right)^\gamma} + \int_\Omega u_{n,k}^p u_{n,k+1} + n^{\gamma+1}\lambda \int_\Omega \gamma u_{n,k} u_{n,k+1} \\
&\qquad \leq \lambda \int_\Omega n^\gamma \overline{u_n} + \int_\Omega \overline{u_n}^{p+1} + n^{\gamma+1}\lambda \int_\Omega \gamma \overline{u_n}^2 \leq C_n,
\end{aligned}$$

where C_n is a constant dependent on n but independent of k. Hence, up to a subsequence, we can conclude that $u_{n,k} \rightharpoonup u_n$ in $X_0^s(\Omega)$. Thus, u_n is an energy solution of problem $(D_{n,\lambda,\gamma,p})$.

Furthermore, by Lemma 3.2, since the sequence of subsolutions $\{\underline{u_n}\}_{n\in\mathbb{N}}$ is increasing with respect to n, there exists a constant $c_{\tilde{\Omega}} > 0$, independent of n, such that

$$u_n \geq \underline{u_n} \geq \underline{u_1} \geq c_{\tilde{\Omega}} > 0, \text{ for every } x \in \tilde{\Omega} \text{ and every } n. \tag{29}$$

Remark 4.2. *Note that, by construction, the solution u_n of problem $(D_{n,\lambda,\gamma,p})$ is a minimal solution, that is, if $\widetilde{u_n}$ is another solution of $(D_{n,\lambda,\gamma,p})$ then $u_n \leq \widetilde{u_n}$.*

Step 6: Regularity.

The idea now is passing to the limit in the sequence $\{u_n\}$ in order to get a solution to problem $(D_{\lambda,\gamma,p})$. Consider first the case $\gamma \leq 1$. Using $u_n \in X_0^s(\Omega)$ as a test function in the problem $(D_{n,\lambda,\gamma,p})$, one gets

$$\begin{aligned}
\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u_n|^2 &= \lambda \int_\Omega \frac{u_n}{\left(u_n + \frac{1}{n}\right)^\gamma} + \int_\Omega u_n^{p+1} \leq \lambda \int_\Omega u_n^{1-\gamma} + \int_\Omega u_n^{p+1} \\
&\leq C_1 \|\overline{u_n}\|_{L^\infty(\Omega)}^{1-\gamma} + C_2 \|\overline{u_n}\|_{L^\infty(\Omega)}^{p+1} \leq \tilde{C},
\end{aligned}$$

where, by Lemma 3.10, $\tilde{C}$ is a constant independent of n. Therefore, u_n is uniformly bounded in $X_0^s(\Omega)$ and, up to a subsequence, $u_n \rightharpoonup u$ in $X_0^s(\Omega)$. Moreover, for every $\varphi \in X_0^s(\Omega)$ with $\operatorname{supp}(\varphi) \subset \tilde{\Omega} \subset\subset \Omega$, by (29), we have that

$$0 \leq \frac{\lambda\varphi}{\left(u_n + \frac{1}{n}\right)^\gamma} \leq \frac{\lambda\varphi}{c_{\tilde{\Omega}}^\gamma} \in L^1(\Omega).$$

Thus, we can pass to the limit in the energy formulation of the approximated problems $(D_{n,\lambda,\gamma,p})$ to conclude that u is an energy solution of $(D_{\lambda,\gamma,p})$.

Consider now $\gamma > 1$. We proceed as in the proof of Lemma 3.6, considering the Lipschitz convex function $\Phi_{\frac{\gamma+1}{2}}(r)$ for $r > 0$, defined in (16). Therefore, by Lemma 3.10, we have that

$$\begin{aligned}
\frac{C(N,s)}{2} \|\Phi_{\frac{\gamma+1}{2}}(u_n)\|_{X_0^s(\Omega)}^2 &\leq \int_{\mathbb{R}^N} \Phi'_{\frac{\gamma+1}{2}}(u_n)\Phi_{\frac{\gamma+1}{2}}(u_n)(-\Delta)^s u_n \\
&= \int_\Omega \frac{\lambda}{(u_n + \frac{1}{n})^\gamma} \Phi'_{\frac{\gamma+1}{2}}(u_n)\Phi_{\frac{\gamma+1}{2}}(u_n) + \int_\Omega u_n^p \Phi'_{\frac{\gamma+1}{2}}(u_n)\Phi_{\frac{\gamma+1}{2}}(u_n) \\
&\leq \frac{(\gamma+1)\lambda}{2}|\Omega| + \frac{\gamma+1}{2}\int_\Omega u_n^{\gamma+p} \leq C,
\end{aligned}$$

where C is a constant independent of n. Letting $T \to +\infty$ in the definition of $\Phi_{\frac{\gamma+1}{2}}$, we conclude that $u_n^{\frac{\gamma+1}{2}}$ is uniformly bounded in $X_0^s(\Omega)$, and thus, by the Rellich-Kondrachov Theorem (see [16, Theorem 7.1]), there exists

$u^{\frac{\gamma+1}{2}} \in X_0^s(\Omega)$ such that

$$u_n^{\frac{\gamma+1}{2}} \to u^{\frac{\gamma+1}{2}} \text{ in } L^r(\Omega),\ \forall\, 1 \le r < 2_s^*,$$
$$u_n^{\frac{\gamma+1}{2}} \to u^{\frac{\gamma+1}{2}} \text{ a.e. in } \Omega.$$

Notice that this is the meaning of u satisfying the boundary data. In particular, from here we deduce that $u_n \to u$ a.e. in Ω, and from Lemma 3.10, that $u \in L^\infty(\Omega)$. Hence, by this convergence and (29), we can pass to the limit in the weak formulation of $(D_{n,\mu,\gamma,p})$ to get

$$\int_{\mathbb{R}^N} u(-\Delta)^s\phi\, dx = \mu \int_\Omega \frac{\phi}{u^\gamma}\, dx + \int_\Omega u^p\phi\, dx,$$

for every $\phi \in \mathcal{T}$, that is, to conclude that $u \in L^\infty(\Omega)$ is a weak solution of $(D_{n,\mu,\gamma,p})$.

Step 7: We prove that $\Lambda < +\infty$.
Let us define

$$\Lambda := \sup\{\lambda > 0 \text{ such that problem } (D_{\lambda,\gamma,p}) \text{ has a solution}\}. \tag{30}$$

Following the ideas of [8, Remark 2.2] we will prove the nonexistence for large λ, that is, we obtain that $\Lambda < +\infty$. Let $\Omega' \subset\subset \Omega$ and consider the eigenvalue problem

$$\begin{cases} (-\Delta)^s\varphi_1 = \lambda_1\varphi_1 \text{ in } \Omega', \\ \varphi_1 > 0 \text{ in } \Omega', \\ \varphi_1 = 0 \text{ in } \mathbb{R}^N \setminus \Omega'. \end{cases} \tag{31}$$

By [5, Proposition 2.2] and [30, Proposition 9] we know that $0 \le \varphi_1 \in X_0^s(\Omega') \cap L^\infty(\Omega')$, and hence it can be used as a test function in $(D_{\lambda,\gamma,p})$. Moreover, since

$$(-\Delta)^2\varphi_1(x) \le 0 = \lambda_1\varphi_1(x),\ x \in \Omega \setminus \Omega',$$

it follows that

$$(-\Delta)^s\varphi_1 \le \lambda_1\varphi_1 \text{ in weak sense in } \Omega.$$

Let u be a solution of $(D_{\lambda,\gamma,p})$. Then, by testing $(D_{\lambda,\gamma,p})$ with φ_1 and applying the Young's inequality, one gets the following,

$$\lambda\int_\Omega \frac{\varphi_1}{u^\gamma} + \int_\Omega u^p\varphi_1 \le \lambda_1\int_\Omega u\varphi_1 \le \frac{1}{p}\int_\Omega u^p\varphi_1 + \frac{\lambda_1^{p'}}{p'}\int_\Omega \varphi_1,$$

that is,

$$\int_\Omega \left(\frac{\lambda}{u^\gamma} + \frac{p-1}{p}u^p - \frac{\lambda_1^{p'}}{p'}\right)\varphi_1 \le 0. \tag{32}$$

But it can be seen that there exists

$$C = \left(\frac{p-1}{\gamma}\right)^{\frac{\gamma}{p+\gamma}} + \frac{p-1}{p}\left(\frac{\gamma}{p-1}\right)^{\frac{p}{p+\gamma}} > 0,$$

such that $\dfrac{\lambda}{u^\gamma} + \frac{p-1}{p}u^p \ge C\lambda^{\frac{p}{p+\gamma}}$, and hence, (32) implies that

$$\int_\Omega \left(C\lambda^{\frac{p}{p+\gamma}} - \frac{\lambda_1^{p'}}{p'}\right)\varphi_1 \le 0,$$

which is impossible for λ large enough.

Step 8: There exists at least a solution of $(D_{\lambda,\gamma,p})$ *for every* $0 < \lambda < \Lambda$.
In Step 5 and Step 6 we have proved the existence of solutions for $0 < \lambda < \Lambda_0$, where Λ_0 was small enough so that

we could construct the supersolution of Step 3. The purpose now is to prove that indeed we can find a solution for every $0 < \lambda < \Lambda$, where Λ was defined in (30).

Take $0 < \lambda < \Lambda$. Given the definition of Λ, we can find $\bar{\lambda}$ as close as we want to Λ so that problem $(D_{\bar{\lambda},\gamma,p})$ has a solution $u_{\bar{\lambda}}$. In particular, taking $\lambda < \bar{\lambda} < \Lambda$, it is easy to check that $u_{\bar{\lambda}}$ is a supersolution of $(D_{n,\lambda,\gamma,p})$. Proceeding as in Step 4, one can prove that $\underline{u_n} \le u_{\bar{\lambda}}$, where $\underline{u_n}$ is the subsolution of $(D_{n,\lambda,\gamma,p})$ constructed at Step 2. Therefore, repeating Step 5 and Step 6 using the new supersolution $u_{\bar{\lambda}}$, the result follows. □

4.2 About existence of second solution

For the truncated problems $(D_{n,\lambda,\gamma,p})$ it is easy to check that in fact there exists at least a second solution for every $\lambda \in (0, \Lambda_n)$, where Λ_n is the critical parameter for the existence of minimal solution for every $n \in \mathbb{N}$ and $p < 2_s^* - 1$. Notice that $\{\Lambda_n\} \downarrow \Lambda$ as $n \to \infty$.

More precisely, by means of variational techniques, one can find two different energy solutions, the first one by minimization, and the second one by the Mountain Pass Lemma (see [2, 20]). In fact it would be natural to consider the energy functional $\mathcal{J}_n : X_0^s(\Omega) \to \mathbb{R}^N$ defined as

$$\mathcal{J}_n(u) = \frac{C(N,s)}{4}\|u\|^2_{X_0^s(\Omega)} - \frac{\lambda}{1-\gamma}\int\limits_\Omega \left(u^+ + \frac{1}{n}\right)^{1-\gamma} - \frac{1}{p+1}\int\limits_\Omega (u^+)^{p+1}. \tag{33}$$

To find the critical points of $\mathcal{J}_n$, it can be used an adaptation of the ideas developed in [1] (see [5] for the details in the non local case). Indeed, let us define the class

$$\mathcal{C}_s(\Omega) := \left\{ w \in \mathcal{C}^0(\overline{\Omega}) : \|w\|_{\mathcal{C}^s(\Omega)} := \left\| \frac{w}{\delta^s} \right\|_{L^\infty(\Omega)} < +\infty \right\},$$

where $\delta^s(x)$ =dist$(x, \partial\Omega)$. Notice that, by the Hopf Lemma (see [22, Lemma 1.2]), $\mathcal{C}_s(\Omega)$ is a natural space where one can *separate* the minimal solutions (see [15]).

More precisely, let $u_{n,1}$ and $u_{n,2}$ be the minimal solutions of $(D_{n,\lambda_1,\gamma,p})$ and $(D_{n,\lambda_2,\gamma,p})$, with $0 < \lambda_1 < \lambda_2$. Thus, $w := u_{n,2} - u_{n,1} \ge 0$. The idea now is to apply a Hopf lemma on w but, since we cannot assure that $(-\Delta)^s w \ge 0$ in Ω, we will apply again the idea of adding a linear term in the problem (see Step 5 in the proof Theorem 4.1). That is, considering the increasing function g given in (26), it follows that

$$\begin{cases} (-\Delta)^s w + n^{\gamma+1}\lambda_1\gamma w \ge g(u_{n,2}) - g(u_{n,1}) \ge 0 \text{ in } \Omega, \\ \qquad w \ge 0 \text{ in } \Omega, \\ \qquad w = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

Therefore, since we are under the hypotheses of [22, Lemma 1.2], we conclude that

$$u_{n,2}(x) - u_{n,1}(x) \ge C\delta^s(x), \quad x \in \Omega.$$

Moreover, by Theorem 4.1, we know that the solutions to the truncated problems are bounded, and thus the right hand side of $(D_{n,\lambda,\gamma,p})$ is bounded as well. Hence, by [28, Proposition 1.1] we obtain $u_{n,2} - u_{n,1} \in \mathcal{C}^s(\overline{\Omega}) \subseteq \mathcal{C}_s(\Omega)$, and therefore the minimal solutions are *separated* in $\mathcal{C}_s(\Omega)$.

It is worth to remark that this strategy cannot be followed to get the second solution of $(D_{\lambda,\gamma,p})$ for every $0 < \lambda < \Lambda$ because, due to the singular term, we cannot apply any bootstrapping result to reach a Hölder regularity up to boundary.

Using the ideas in [11], one can prove that there exists a local minimum of the problem $(D_{n,\lambda,\gamma,p})$ in the $\mathcal{C}_s$-topology, and then that every minimum of $\mathcal{J}_n$ in $\mathcal{C}_s(\Omega)$ is indeed a minimum in $X_0^s(\Omega)$, separating the minimal solutions of this problem for different λ. Finally, following the ideas of [1] (see also [5]), it can be made a translation of the truncated functional and to use the Mountain Pass Lemma, checking that the translated functional satisfies the suitable geometry and compactness properties to prove the existence of a non-trivial critical point.

Despite the fact that we cannot apply this strategy for the non truncated problem, for $p < 2_s^* - 1$ and λ small enough a second solution can be still obtained using the Mountain Pass Theorem (see for example Section 3 of [6], where these ideas are developed for a nonlocal concave-convex problem). The analysis of multiplicity of solutions for the whole range $0 < \lambda < \Lambda$, as far as we know, remains as an open problem.

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