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Research Article

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Computation of double Hopf points for delay differential equations

DOI 10.1515/math-2015-0076
Received August 6, 2015; accepted November 6, 2015.

Abstract: Relating to the crucial problem of branch switching, the calculation of codimension 2 bifurcation points is one of the major issues in numerical bifurcation analysis. In this paper, we focus on the double Hopf points for delay differential equations and analyze in detail the corresponding eigenspace, which enable us to obtain the finite dimensional defining system of equations of such points, instead of an infinite dimensional one that happens naturally for delay systems. We show that the double Hopf point, together with the corresponding eigenvalues, eigenvectors and the critical values of the bifurcation parameters, is a regular solution of the finite dimensional defining system of equations, and thus can be obtained numerically through applying the classical iterative methods. We show our theoretical findings by a numerical example.

Keywords: double Hopf points, delay differential equations, regular defining system of equations, iterative methods

MSC: 65P30, 37M20

1 Introduction

Delay differential equations (DDEs) have attracted a great deal of attention in the past decades for their excellent performance in modelling the population dynamics, epidemics and many other problems arising in natural science and engineering. Fundamental theories of solutions to DDEs have been well developed, see for instance the monographs by Hale and Lunel [1], Driver [2] and Bellman and Cooke [3]. Among the various solution properties, the periodicity receives many concerns since it arises in numerous model problems and accounts for a lot of phenomena that take place in the real world. After the work of Cooke [4], many works focused on the periodic solutions of delay models in applications, such as neural network or population dynamics. One of the principal mechanisms accounting for the occurence of periodic solutions is the Hopf bifurcation [5]: when the bifurcation parameter crosses a certain critical value, in the phase space a limit cycle surrounding the equilibrium occurs. Mathematically speaking, the Hopf bifurcation occurs when the underlying equation undergoes a pair of purely imaginary eigenvalues. Thus it is of codimension 1 and is relatively easy to analyze. The Hopf point for ODEs can be approximated through numerically solving a simple regular defining system, cf. [6–8]; for DDEs, the similar idea can be applied as well, see [9].

The computation of Hopf points is important in numerical bifurcation analysis, which makes it possible to trace the Hopf point branch using the continuation method [10, 11] when there are two free bifurcation parameters in the underlying system. When two of such Hopf point branches intersect to each other, the so-called double Hopf bifurcation occurs. Recently, many papers have considered the theoretical analysis of the double Hopf bifurcations in DDEs, see for example [12–18]. The double Hopf point is of codimension 2 and thus to accurately calculate such

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a point is harder than the above mentioned Hopf point, however it is of great importance. On the one hand, this is a key issue of branch switching, that is to switch from one Hopf point branch to another. Moreover, this makes it possible to trace the branch of the double Hopf points in a system with more parameters. We note here that for ODEs various techniques for calculating the double Hopf points are already well studied. For example, a bordered matrix technique that can calculate the double Hopf point together with the corresponding eigenvalues and bifurcation parameters simultaneously is discussed in [6] and a defining system based on the eigenspace analysis that can calculate the double Hopf point together with the corresponding eigenvalues, eigenvectors and bifurcation parameters simultaneously is analyzed in [8]. For DDEs a technique based on the characteristic matrix, that can calculate the double Hopf point together with the corresponding eigenvalues and bifurcation parameters simultaneously is applied in the bifurcation analysis software DDE-BIFTOOL[19]. The calculation of normal form coefficients of double Hopf points for DDEs using DDE-BIFTOOL is discussed in [20].

In this paper, based on the eigenspace analysis, we accurately calculate the double Hopf points for the following DDEs

$$\dot{x}(t) = f(x(t), x(t-\tau), \lambda, \mu), \tag{1}$$

where $\lambda,\ \mu \in \mathbb{R}$ are parameters, $\tau > 0$ is a constant delay. In Section 2, for the double Hopf point, we apply the idea used for ODEs to an equivalent abstract ODE form of (1) to give a defining system that is of infinite dimension. In Section 3, we use similar techniques applied in [21, 22] to analyze in detail the eigenspace associated with the purely imaginary eigenvalues. Based on the description of the eigenspace and the defining system obtained in Section 2, we propose a finite dimensional defining system. We also prove that the double Hopf point, together with the corresponding eigenvalues, eigenvectors and the critical bifurcation parameters, is a regular solution of the finite dimensional defining system. Note that the accurate location of the double Hopf point is only the first step of the relating numerical bifurcation analysis and further research include the computations of the critical normal form coefficients and construction of asymptotic of codimension 1 and global bifurcations nearby, where the corresponding eigenfunctions obtained in our algorithm as by-products are required. We present a numerical example to illustrate our theoretical findings in Section 4.

2 Regular defining system for the double Hopf points in Banach space

A simple change of time scale allows us to consider only the case $\tau = 1$ in (1) as follows

$$\dot{x}(t) = f(x(t), x(t-1), \lambda, \mu),\ x \in \mathbb{C}^n, \tag{2}$$

where $\lambda, \mu \in \mathbb{R}$ are bifurcation parameters, $f(x, y, \lambda, \mu)$ is a $C^r\,(r \geq 2)$ smooth function from $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{R} \times \mathbb{R}$ to $\mathbb{C}^n$. The state space of (2), denoted by $C := C([-1, 0], \mathbb{C}^n)$, is the Banach space of continuous mappings from $[-1, 0]$ to $\mathbb{C}^n$ with norm $\|\phi\| = \max_{\theta\in[-1,0]} |\phi(\theta)|$ ($|\cdot|$ is some norm in $\mathbb{C}^n$). Assume that equation (2) has the equilibria $\bar{x}(\lambda, \mu)$ for λ and μ varying in some neighbourhood of (λ^0, μ^0). We then linearize (2) at $\bar{x}(\lambda, \mu)$ to obtain

$$\dot{x}(t) = f_1(\bar{x}(\lambda, \mu), \bar{x}(\lambda, \mu), \lambda, \mu)(x(t) - \bar{x}(\lambda, \mu)) + f_2(\bar{x}(\lambda, \mu), \bar{x}(\lambda, \mu), \lambda, \mu)(x(t-1) - \bar{x}(\lambda, \mu)), \tag{3}$$

where $f_1(x, y, \lambda, \mu) = f_x(x, y, \lambda, \mu)$ and $f_2(x, y, \lambda, \mu) = f_y(x, y, \lambda, \mu)$.

The above equation can be reformulated as the following retarded functional differential equation

$$\dot{x}(t) = L_{\lambda,\mu}(x_t - \bar{x}(\lambda, \mu)) \tag{4}$$

with $L_{\lambda,\mu}$ the bounded linear operator from C to $\mathbb{C}^n$ defined by $L_{\lambda,\mu}\phi = \int_{-1}^{0} d\eta_{\lambda,\mu}(\theta)\phi(\theta)$ for any $\phi \in C$, where

$$\eta_{\lambda,\mu}(\theta) = \begin{cases} f_1(\bar{x}(\lambda, \mu), \bar{x}(\lambda, \mu), \lambda, \mu) + f_2(\bar{x}(\lambda, \mu), \bar{x}(\lambda, \mu), \lambda, \mu), & \theta = 0, \\ f_2(\bar{x}(\lambda, \mu), \bar{x}(\lambda, \mu), \lambda, \mu), & \theta \in (-1, 0), \\ 0, & \theta = -1 \end{cases}$$

is a matrix-valued function on $[-1, 0]$ of bounded variation and is parameterized by λ and μ.

Denote by $\{T_{\lambda,\mu}(t), t \geqslant 0\}$ the C_0-semigroup generated by the solutions of (3), with the infinitesimal generator $\mathcal{A}_{\lambda,\mu} : C \longrightarrow C$ given by

$$\begin{aligned}&\mathcal{A}_{\lambda,\mu}\phi = \dot{\phi},\\ &D(\mathcal{A}_{\lambda,\mu}) = \{\phi \in C^1([-1,0], \mathbb{C}^n); \dot{\phi}(0) = \textstyle\int_{-1}^{0} d\eta_{\lambda,\mu}(\theta)\phi(\theta)\}.\end{aligned}$$

Thus we have that the spectrum of $\mathcal{A}_{\lambda,\mu}$ coincides with its point spectrum, i.e. $\sigma(\mathcal{A}_{\lambda,\mu}) = \sigma_p(\mathcal{A}_{\lambda,\mu})$ [1], and $z \in \sigma_p(\mathcal{A}_{\lambda,\mu})$ if and only if

$$\det(zI - f_1(\bar{x}(\lambda,\mu), \bar{x}(\lambda,\mu), \lambda, \mu) - f_2(\bar{x}(\lambda,\mu), \bar{x}(\lambda,\mu), \lambda, \mu)e^{-z}) = 0, \tag{5}$$

whose solutions are eigenvalues of (3).

Definition 2.1. *x^0 is called a double Hopf point of (2) at $(\lambda,\mu) = (\lambda^0,\mu^0)$, if*

a) $x^0 = \bar{x}(\lambda^0,\mu^0)$;

b) *the characteristic equation (5) has solutions $z_1^{\pm} = \sigma_1(\lambda,\mu) \pm i w_1(\lambda,\mu)$ and $z_2^{\pm} = \sigma_2(\lambda,\mu) \pm i w_2(\lambda,\mu)$ such that $\sigma_1(\lambda^0,\mu^0) = \sigma_2(\lambda^0,\mu^0) = 0$ but both $w_1(\lambda^0,\mu^0)$ and $w_2(\lambda^0,\mu^0)$ are not zero and $w_1(\lambda^0,\mu^0) \neq w_2(\lambda^0,\mu^0)$;*

c) $\det\begin{pmatrix} \frac{\partial}{\partial\lambda}\sigma_1(\lambda^0,\mu^0) & \frac{\partial}{\partial\mu}\sigma_1(\lambda^0,\mu^0) \\ \frac{\partial}{\partial\lambda}\sigma_2(\lambda^0,\mu^0) & \frac{\partial}{\partial\mu}\sigma_2(\lambda^0,\mu^0)\end{pmatrix} \neq 0$, *that is to say, the map $(\lambda,\mu) \to (\sigma_1(\lambda,\mu), \sigma_2(\lambda,\mu))$ is regular at (λ_0,μ_0);*

d) *the solution z of (5) satisfies $\Re z(\lambda^0,\mu^0) \neq 0$ except $z = z_1^{\pm}$ and $z = z_2^{\pm}$.*

We assume throughout this paper that equation (2) has a double Hopf bifurcation point. We then derive the defining system for the double Hopf point using the eigenspace analysis based techniques similar to the ordinary differential equation case [8]. To this end, we rewrite (2) as the following abstract ODE in C [1]

$$\frac{d}{dt}u = G(u,\lambda,\mu), \tag{6}$$

where $u(t)(\theta) = x_t(\theta)$,

$$G(u,\lambda,\mu)(\theta) = \begin{cases} f(u(0), u(-1), \lambda, \mu), & \theta = 0, \\ \dot{u}(\theta), & \theta \in [-1, 0] \end{cases} \tag{7}$$

and the domain of $G(\cdot,\lambda,\mu)$ is $\{u \in C^1 : \dot{u}(0) = f(u(0), u(-1), \lambda, \mu)\}$. Note that this reformulation does not change its eigenvalues. Thus, $\pm i w_1^0 := \pm i w_1(\lambda^0,\mu^0)$ and $\pm i w_2^0 = \pm i w_2(\lambda^0,\mu^0)$ are eigenvalues of $G_u^0 = G_u(u^0,\lambda^0,\mu^0)$, where $u^0 = x^0$. That is to say, there exist $c_j^R, c_j^I \in C$ and $d_j^R, d_j^I \in C^*$, $j = 1, 2$, such that

$$(G_u^0 - i w_j^0 I)(c_j^R + i c_j^I) = 0,\ (d_j^R + i d_j^I)(G_u^0 + i w_j^0 I) = 0, \quad j = 1, 2,$$

where $C^* = C([0,1], \mathbb{C}^{n*})$ is the formal adjoint space of C with $\mathbb{C}^{n*}$ being the n-dimensional space of row vectors. The formal adjoint bilinear form on $C^* \times C$ is then defined by [1]

$$(\psi,\phi) = \psi(0)\phi(0) - \int_{-1}^{0}\int_{0}^{\theta} \psi(\xi - \theta) d\eta_{\lambda,\mu}(\theta)\phi(\xi) d\xi. \tag{8}$$

Note here C^* is not the true dual space for C and thus the above bilinear form is not a pairing between C and its true dual. For details, we refer the reader to [1], see also [23]. By the relation between x_t and u, we only need to calculate the double Hopf points for (6), which can be determined by the following defining system in Banach space [8].

$$H(\xi) = \begin{pmatrix} G(u,\lambda,\mu) \\ G_u(u,\lambda,\mu)\alpha_j + w_j\beta_j \\ (L_j,\alpha_j) - 1 \\ G_u(u,\lambda,\mu)\beta_j - w_j\alpha_j \\ (L_j,\beta_j) \end{pmatrix} = 0, \ j = 1, 2 \tag{9}$$

where $\xi = (u, \alpha_1, \lambda, \beta_1, w_1, \alpha_2, \mu, \beta_2, w_2)^T \in Y = C \times C \times \mathbb{R} \times C \times \mathbb{R} \times C \times \mathbb{R} \times C \times \mathbb{R}$, H is a smooth map from Y to itself, L_j, $j = 1, 2$ are given nonzero functions in C^* to normalize α_j and theoretically should be chosen a priori not orthogonal in the sense of (8) to the eigenspace of G_u^0 corresponding to $\pm i w_j^0$; while in implementation, these conditions will be satisfied for almost any functions in C^*.

Theorem 2.2 ([8]). *Assume*

$$\det \begin{pmatrix} d_1^R A_\lambda c_1^R + d_1^I A_\lambda c_1^I & d_1^R A_\mu c_1^R + d_1^I A_\mu c_1^I \\ d_2^R A_\lambda c_2^R + d_2^I A_\lambda c_2^I & d_2^R A_\mu c_2^R + d_2^I A_\mu c_2^I \end{pmatrix} \neq 0, \tag{10}$$

where

$$A_\lambda = G_{uu}^0 V_\lambda + G_{u\lambda}^0; V_\lambda = -(G_u^0)^{-1} G_\lambda^0,$$
$$A_\mu = G_{uu}^0 V_\mu + G_{u\mu}^0; V_\mu = -(G_u^0)^{-1} G_\mu^0.$$

Then $H(\xi)$ is regular at $\xi^0 = (x^0, c_1^R, \lambda^0, c_1^I, w_1^0, c_2^R, \mu^0, c_2^I, w_2^0)^T$.

In fact, the regularity condition (10) used in Theorem 2.2 is exactly the explicit expression of the transversality condition (c) in Definition 2.1. To see this, let $\varphi(\lambda, \mu)$ be the eigenfunction of $G_u(u(\lambda, \mu), \lambda, \mu)$ associated with the eigenvalue $\alpha(\lambda, \mu)$, i.e., $G_u(u(\lambda, \mu), \lambda, \mu)\varphi(\lambda, \mu) - \alpha(\lambda, \mu)\varphi(\lambda, \mu) = 0$. Differentiating this equation with respect to λ and evaluating at (λ^0, μ^0) gives

$$G_u^0 \varphi_\lambda^0 + (G_{uu}^0 u_\lambda^0 + G_{u\lambda}^0)\varphi^0 - \alpha_\lambda^0 \varphi^0 - \alpha^0 \varphi_\lambda^0 = 0, \tag{11}$$

where the superscript "^{0}" represents the function's value evaluated at (λ^0, μ^0). In addition, we consider (λ, μ) such that $G(u(\lambda, \mu), \lambda, \mu) = 0$, which shows by a derivative with respect to λ and evaluating at (λ^0, μ^0) that $G_u^0 u_\lambda^0 + G_\lambda^0 = 0$. Hence $u_\lambda^0 = -(G_u^0)^{-1} G_\lambda^0$. Inserting this result into (11), and multiplying from the left $d_j^R - i d_j^I$, we obtain

$$(d_j^R - i d_j^I) A_\lambda (c_j^R + i c_j^I) - (d_j^R - i d_j^I)(\sigma_\lambda^0 + i w_\lambda^0)(c_j^R + i c_j^I) = 0.$$

Taking the real part of the above equation and noting the orthogonality of eigenfunctions we arrive at

$$\sigma_\lambda^0 = d_j^R A_\lambda c_1^R + d_1^I A_\lambda c_1^I.$$

Similar techniques show the other three equalities. Consequently, the assumption (10) holds naturally if x^0 is a double Hopf point of (2) at $(\lambda, \mu) = (\lambda^0, \mu^0)$.

3 Simplification of the defining system

Solving the defining system (9) would give the desired double Hopf bifurcation points, together with its eigenvalues, eigenfunctions and the critical values of bifurcation parameters. However, solving such an infinite dimensional system directly is exhausted and expensive to use. Here come some reasons. A large amount of data storage would be required and the discretization error would be introduced during the necessary discretization applied to the infinite dimensional space Y. In addition, the special form of the function $G(u, \lambda, \mu)$ is therefore difficult to deal with.

We then aim to reduce the defining system (9) to an equivalent finite dimensional form that can be solved with low cost. To this end, we need to analyze in detail the eigenspace of G_u^0 associated with its purely imaginary eigenvalues. Noting that G_u^0 has the same eigen structure as the infinitesimal generator $\mathcal{A}_0 := \mathcal{A}_{\lambda^0, \mu^0}$, we therefore consider only the eigen structure of $\mathcal{A}_0$.

For any given nonzero complex vectors $c_j^1, c_j^2 \in \mathbb{C}^n$, we denote $\phi_j^R(\theta) = i(c_j^1 e^{i w_j^0 \theta} + c_j^2 e^{-i w_j^0 \theta})$, $\phi_j^I(\theta) = c_j^1 e^{i w_j^0 \theta} - c_j^2 e^{-i w_j^0 \theta}$. For any given nonzero complex vectors $d_j^1, d_j^2 \in \mathbb{C}^{n*}$, we denote $\psi_j^R(s) = d_j^1 e^{i w_j^0 s} - d_j^2 e^{-i w_j^0 s}$ and $\psi_j^I(s) = i(d_j^1 e^{i w_j^0 s} + d_j^2 e^{-i w_j^0 s})$. Let $\phi_j(\theta) = \phi_j^R(\theta) + i\phi_j^I(\theta)$ and $\psi_j(s) = \psi_j^R(s) + i\psi_j^I(s)$, $j = 1, 2$. Besides, we indicate again by the superscript "^{0}" the function's value evaluated at $(x^0, x^0, \lambda^0, \mu^0)$, for example $f_j^0 = f_j(x^0, x^0, \lambda^0, \mu^0)$ for $j = 1, 2$. We then have the following lemma regarding the eigenspace of $\mathcal{A}_0$ associated with its purely imaginary eigenvalues.

Lemma 3.1. *For* $j = 1, 2$,

1) $\phi_j(\theta)$ *are the eigenfunctions of* $\mathcal{A}_0$ *associated with the purely imaginary eigenvalues* iw_j^0, *if and only if*
   (I) $(f_1^0 + e^{-iw_j^0} f_2^0 - iw_j^0 I_n)c_j^1 = 0$,
   (II) $(f_1^0 + e^{iw_j^0} f_2^0 + iw_j^0 I_n)c_j^2 = 0$.
2) $\psi_j(s)$ *are the eigenfunctions of* $\mathcal{A}_0^*$ *associated with the purely imaginary eigenvalues* $-iw_j^0$, *if and only if*
   (III) $d_j^1(f_1^0 + e^{iw_j^0} f_2^0 + iw_j^0 I_n) = 0$,
   (IV) $d_j^2(f_1^0 + e^{-iw_j^0} f_2^0 - iw_j^0 I_n) = 0$.

Proof. Note that $\phi_j(\theta)$ are the eigenfunctions of $\mathcal{A}_0$ associated with iw_j^0, i.e. $\mathcal{A}_0\phi_j(\theta) = iw_j^0\phi_j(\theta)$. Solving this equation using the definition of $\mathcal{A}_0$ and techniques applied in [21] gives assertion 1). Note that the adjoint operator $\mathcal{A}_0^*$ of $\mathcal{A}_0$ is an operator from C^* to itself defined as

$$\begin{aligned} &\mathcal{A}_0^*\psi = -\dot{\psi}, \\ &D(\mathcal{A}_0^*) = \{\psi \in C^1([0,1], \mathbb{C}^{n*}) : -\dot{\psi}(0) = \textstyle\int_{-1}^0 \psi(-\theta)d\eta_{\lambda^0,\mu^0}(\theta)\}. \end{aligned}$$

Thus, assertion 2) is obtained similarly. □

We remark that c_j^1, c_j^2 and d_j^1, d_j^2 cannot be uniquely determined by relations (I)–(IV) described in Lemma 3.1. Thus we can further require

$$(\psi_j, \phi_j) = -4id_j^2(I_n + f_2^0 e^{-iw_j^0})c_j^1 = 1, \text{ for } j = 1, 2, \tag{12}$$

$$(\psi_j, \phi_k) = 4d_j^2(-iI_n + f_2^0 \frac{e^{-iw_k^0} - e^{-iw_j^0}}{w_k^0 - w_j^0})c_k^1 = 0, \quad \text{for } j, k = 1, 2 \text{ but } j \neq k, \tag{13}$$

$$(\bar{\psi}_j, \bar{\phi}_j) = 4id_j^1(I_n + f_2^0 e^{iw_j^0})c_j^2 = 1, \text{ for } j = 1, 2, \tag{14}$$

and

$$(\bar{\psi}_j, \bar{\phi}_k) = 4d_j^1(iI_n + f_2^0 \frac{e^{iw_k^0} - e^{iw_j^0}}{w_k^0 - w_j^0})c_k^2 = 0, \text{ for } j, k = 1, 2 \text{ but } j \neq k, \tag{15}$$

where $\bar{\psi}_j = \psi_j^R - i\psi_j^I, \bar{\phi}_j = \phi_j^R - i\phi_j^I$, for $j = 1, 2$.

We remark as well that, different from the ODE case, the eigenfunctions of $\mathcal{A}_0$ associated with the purely imaginary eigenvalues are not constant vectors.

Then, based on Lemma 3.1 and the discussion above, we introduce the following defining system of equations for the double Hopf points of (2)

$$H(v) = \begin{pmatrix} f(x, x, \lambda, \mu), \\ (f_1(x, x, \lambda, \mu) + e^{-iw_1} f_2(x, x, \lambda, \mu) - iw_1 I_n)\alpha_1 \\ 4il_1^2(I_n + e^{-iw_1} f_2(x, x, \lambda, \mu))\alpha_1 + 1 \\ (f_1(x, x, \lambda, \mu) + e^{iw_1} f_2(x, x, \lambda, \mu) + iw_1 I_n)\alpha_2 \\ 4il_1^1(I_n + e^{iw_1} f_2(x, x, \lambda, \mu))\alpha_2 - 1 \\ (f_1(x, x, \lambda, \mu) + e^{-iw_2} f_2(x, x, \lambda, \mu) - iw_2 I_n)\beta_1 \\ 4il_2^2(I_n + e^{-iw_2} f_2(x, x, \lambda, \mu))\beta_1 + 1 \\ (f_1(x, x, \lambda, \mu) + e^{iw_2} f_2(x, x, \lambda, \mu) + iw_2 I_n)\beta_2 \\ 4il_2^1(I_n + e^{iw_2} f_2(x, x, \lambda, \mu))\beta_2 - 1 \end{pmatrix} = 0, \tag{16}$$

where $v = (x, \alpha_1, \alpha_2, \lambda, w_1, \beta_1, \beta_2, \mu, w_2)^T \in V = \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{R} \times \mathbb{R}$, $H : V \to V$. In addition, in view of the expressions of the eigenfunctions of $\mathcal{A}_0^*$, l_j^1, l_j^2, $j = 1, 2$ should be chosen a priori such that (12) and (14) are nonzero if where d_j^1 and d_j^2 were replaced by l_j^1 and l_j^2, respectively; while in implementation these conditions can again be satisfied by almost any nonzero row vectors only if l_1^j and l_2^j are linearly independent for $j = 1, 2$. We note here that for ease of calculation and analysis, we use different normalization equations from those applied in (9). Besides, L_j in (9) can but not necessarily be the real part of the functions $2l_j^1 e^{iw_j^0 s}$ or $2l_j^2 e^{-iw_j^0 s}$ for $s \in [0, 1]$.

Theorem 3.2. *Assume*

$$d_0 = \det \begin{pmatrix} d_1^2 A_\lambda^1 c_1^1 - d_1^1 B_\lambda^1 c_1^2 & d_1^2 A_\mu^1 c_1^1 - d_1^1 B_\mu^1 c_1^2 \\ d_2^2 A_\lambda^2 c_2^1 - d_2^1 B_\lambda^2 c_2^2 & d_2^2 A_\mu^2 c_2^1 - d_2^1 B_\mu^2 c_2^2 \end{pmatrix} \neq 0, \tag{17}$$

where for $j = 1, 2$

$$\begin{aligned} A_\lambda^j &= [(f_{11}^0 + f_{12}^0) + e^{-iw_j^0}(f_{21}^0 + f_{22}^0)]v_\lambda + (f_{1\lambda}^0 + e^{-iw_j^0} f_{2\lambda}^0), \\ A_\mu^j &= [(f_{11}^0 + f_{12}^0) + e^{-iw_j^0}(f_{21}^0 + f_{22}^0)]v_\mu + (f_{1\mu}^0 + e^{-iw_j^0} f_{2\mu}^0), \\ B_\lambda^j &= [(f_{11}^0 + f_{12}^0) + e^{iw_j^0}(f_{21}^0 + f_{22}^0)]v_\lambda + (f_{1\lambda}^0 + e^{iw_j^0} f_{2\lambda}^0), \\ B_\mu^j &= [(f_{11}^0 + f_{12}^0) + e^{iw_j^0}(f_{21}^0 + f_{22}^0)]v_\mu + (f_{1\mu}^0 + e^{iw_j^0} f_{2\mu}^0), \\ v_\lambda &= -(f_1^0 + f_2^0)^{-1} f_\lambda^0, v_\mu = -(f_1^0 + f_2^0)^{-1} f_\mu^0. \end{aligned}$$

Then the defining system (16) is regular at its zero $v^0 = (x^0, c_1^1, c_1^2, \lambda^0, w_1^0, c_2^1, c_2^2, \mu^0, w_2^0)^T$.

Proof. Obviously, we only need to prove that the Jacobi matrix H_v^0 of H at v^0 is one to one if (17) holds.

We first prove that the map H_v^0 is injective. Let $y = (y_1, y_2, y_3, b_1, b_2, y_4, y_5, b_3, b_4)^T \in V$. We only need to prove that $H_v^0 y = 0$ has only trivial solution. In fact, the equation $H_v^0 y = 0$ reads the following nine equations:

$$(f_1^0 + f_2^0)y_1 + f_\lambda^0 b_1 + f_\mu^0 b_3 = 0, \tag{18}$$

$$\begin{aligned} &\left[f_{11}^0 + f_{12}^0 + e^{-iw_1^0}(f_{21}^0 + f_{22}^0)\right]c_1^1 y_1 + (f_1^0 + e^{-iw_1^0} f_2^0 - iw_1^0 I_n)y_2 + (f_{1\lambda}^0 + e^{-iw_1^0} f_{2\lambda}^0)c_1^1 b_1 \\ &-i(e^{-iw_1^0} f_2^0 + I_n)c_1^1 b_2 + (f_{1\mu}^0 + e^{-iw_1^0} f_{2\mu}^0)c_1^1 b_3 = 0, \end{aligned} \tag{19}$$

$$\begin{aligned} &4ie^{-iw_1^0} l_1^2 (f_{21}^0 + f_{22}^0)c_1^1 y_1 + 4il_1^2(I_n + e^{-iw_1^0} f_2^0)y_2 + 4ie^{-iw_1^0} l_1^2 f_{2\lambda}^0 c_1^1 b_1 + 4e^{-iw_1^0} l_1^2 f_2^0 c_1^1 b_2 \\ &+4ie^{-iw_1^0} l_1^2 f_{2\mu}^0 c_1^1 b_3 = 0, \end{aligned} \tag{20}$$

$$\begin{aligned} &\left[f_{11}^0 + f_{12}^0 + e^{iw_1^0}(f_{21}^0 + f_{22}^0)\right]c_1^2 y_1 + (f_1^0 + e^{iw_1^0} f_2^0 + iw_1^0 I_n)y_3 + (f_{1\lambda}^0 + e^{iw_1^0} f_{2\lambda}^0)c_1^2 b_1 \\ &+i(e^{iw_1^0} f_2^0 + I_n)c_1^2 b_2 + (f_{1\mu}^0 + e^{iw_1^0} f_{2\mu}^0)c_1^2 b_3 = 0, \end{aligned} \tag{21}$$

$$\begin{aligned} &4ie^{iw_1^0} l_1^1 (f_{21}^0 + f_{22}^0)c_1^2 y_1 + 4il_1^1(I_n + e^{iw_1^0} f_2^0)y_3 + 4il_1^1 e^{iw_1^0} f_{2\lambda}^0 c_1^2 b_1 - 4l_1^1 e^{iw_1^0} f_2^0 c_1^2 b_2 \\ &+4il_1^1 e^{iw_1^0} f_{2\mu}^0 c_1^2 b_3 = 0, \end{aligned} \tag{22}$$

$$\begin{aligned} &\left[f_{11}^0 + f_{12}^0 + e^{-iw_2^0}(f_{21}^0 + f_{22}^0)\right]c_2^1 y_1 + (f_{1\lambda}^0 + e^{-iw_2^0} f_{2\lambda}^0)c_2^1 b_1 + (f_1^0 + e^{-iw_2^0} f_2^0 \\ &-iw_2^0 I_n)y_4 + (f_{1\mu}^0 + e^{-iw_2^0} f_{2\mu}^0)c_2^1 b_3 - i(I_n + e^{-iw_2^0} f_2^0)c_2^1 b_4 = 0, \end{aligned} \tag{23}$$

$$\begin{aligned} &4ie^{-iw_2^0} l_2^2 (f_{21}^0 + f_{22}^0)c_2^1 y_1 + 4ie^{-iw_2^0} l_2^2 f_{2\lambda}^0 c_2^1 b_1 + 4il_2^2(I_n + e^{-iw_2^0} f_2^0)y_4 \\ &+4ie^{-iw_2^0} l_2^2 f_{2\mu} c_2^1 b_3 + 4e^{-iw_2^0} l_2^2 f_2^0 c_2^1 b_4 = 0, \end{aligned} \tag{24}$$

$$\begin{aligned} &\left[f_{11}^0 + f_{12}^0 + e^{iw_2^0}(f_{21}^0 + f_{22}^0)\right]c_2^2 y_1 + (f_{1\lambda}^0 + e^{iw_2^0} f_{2\lambda}^0)c_2^2 b_1 + (f_1^0 + e^{iw_2^0} f_2^0 + iw_2^0 I_n)y_5 \\ &+(f_{1\mu}^0 + e^{iw_2^0} f_{2\mu}^0)c_2^2 b_3 + i(I_n + e^{iw_2^0} f_2^0)c_2^2 b_4 = 0, \end{aligned} \tag{25}$$

and

$$\begin{aligned} &4ie^{iw_2^0} l_2^1 (f_{21}^0 + f_{22}^0)c_2^2 + 4ie^{iw_2^0} l_2^1 f_{2\lambda}^0 c_2^2 b_1 + 4il_2^1(I_n + e^{iw_2^0} f_2^0)y_5 \\ &+4ie^{iw_2^0} l_2^1 f_{2\mu}^0 c_2^2 b_3 - 4e^{iw_2^0} l_2^1 f_2^0 c_2^2 b_4 = 0. \end{aligned} \tag{26}$$

Since zero is not an eigenvalue of $\mathcal{A}_0$, we have that $\det(f_1^0 + f_2^0) \neq 0$ and thus $f_1^0 + f_2^0$ is invertible. We then obtain from (18)

$$y_1 = -(f_1^0 + f_2^0)^{-1} f_\lambda^0 b_1 - (f_1^0 + f_2^0)^{-1} f_\mu^0 b_3 = \nu_\lambda b_1 + \nu_\mu b_3.$$

Inserting this expression into equations (19), (21), (23) and (25) yields

$$A_\lambda^1 c_1^1 b_1 + A_\mu^1 c_1^1 b_3 - i(I_n + e^{-iw_1^0} f_2^0) c_1^1 b_2 + (f_1^0 + e^{-iw_1^0} f_2^0 - iw_1^0 I_n) y_2 = 0, \tag{27}$$

$$B_\lambda^1 c_1^2 b_1 + B_\mu^1 c_1^2 b_3 + i(I_n + e^{iw_1^0} f_2^0) c_1^2 b_2 + (f_1^0 + e^{iw_1^0} f_2^0 + iw_1^0 I_n) y_3 = 0, \tag{28}$$

$$A_\lambda^2 c_2^1 b_1 + A_\mu^2 c_2^1 b_3 - i(I_n + e^{-iw_2^0} f_2^0) c_2^1 b_4 + (f_1^0 + e^{-iw_2^0} f_2^0 - iw_2^0 I_n) y_4 = 0, \tag{29}$$

and

$$B_\lambda^2 c_2^2 b_1 + B_\mu^2 c_2^2 b_3 + i(I_n + e^{iw_2^0} f_2^0) c_2^2 b_4 + (f_1^0 + e^{iw_2^0} f_2^0 + iw_2^0 I_n) y_5 = 0. \tag{30}$$

Multiplying (27) from left on both sides by d_1^2, we obtain by Lemma 3.1

$$d_1^2 A_\lambda^1 c_1^1 b_1 + d_1^2 A_\mu^1 c_1^1 b_3 + \frac{1}{4} b_2 = 0. \tag{31}$$

Similarly, multiplying (28), (29) and (30) from left on both sides by d_1^1, d_2^2 and d_2^1 respectively gives

$$d_1^1 B_\lambda^1 c_1^2 b_1 + d_1^1 B_\mu^1 c_1^2 b_3 + \frac{1}{4} b_2 = 0, \tag{32}$$

$$d_2^2 A_\lambda^2 c_2^1 b_1 + d_2^2 A_\mu^2 c_2^1 b_3 + \frac{1}{4} b_4 = 0, \tag{33}$$

and

$$d_2^1 B_\lambda^2 c_2^2 b_1 + d_2^1 B_\mu^2 c_2^2 b_3 + \frac{1}{4} b_4 = 0. \tag{34}$$

Substracting (32) from (31) gives

$$\left(d_1^2 A_\lambda^1 c_1^1 - d_1^1 B_\lambda^1 c_1^2\right) b_1 + \left(d_1^2 A_\mu^1 c_1^1 - d_1^1 B_\mu^1 c_1^2\right) b_3 = 0. \tag{35}$$

Similarly, substracting (34) from (33) yields

$$\left(d_2^2 A_\lambda^2 c_2^1 - d_2^1 B_\lambda^2 c_2^2\right) b_1 + \left(d_2^2 A_\mu^2 c_2^1 - d_2^1 B_\mu^2 c_2^2\right) b_3 = 0. \tag{36}$$

Noting (17), we obtain from (32) and (33) $b_1 = b_3 = 0$. Inserting them into equations (31) and (33) gives $b_2 = 0$ and $b_4 = 0$, respectively.

Inserting $b_1 = b_3 = 0$ into equation (18) gives $y_1 = 0$. Inserting $b_1 = b_2 = b_3 = y_1 = 0$ into equations (27) and (28) gives respectively

$$(f_1^0 + e^{-iw_1^0} f_2^0 - iw_1^0 I_n) y_2 = 0 \tag{37}$$

and

$$(f_1^0 + e^{iw_1^0} f_2^0 + iw_1^0 I_n) y_3 = 0. \tag{38}$$

Noting that w_1^0 is a simple eigenvalue of $\mathcal{A}_0$, we have $\mathrm{rank}(f_1^0 + e^{-iw_1^0} f_2^0 - iw_1^0 I_n) = \mathrm{rank}(f_1^0 + e^{iw_1^0} f_2^0 + iw_1^0 I_n) = n - 1$. Therefore, from Lemma 3.1 we obtain $y_2 = k_1 c_1^1$, $y_3 = k_2 c_1^2$, where k_1, k_2 are constants to be determined. Inserting $b_1 = b_2 = b_3 = y_1 = 0$ and $y_2 = k_1 c_1^1$ into equation (20), we get

$$4il_1^2 (I_n + e^{-iw_1^0} f_2^0) k_1 c_1^1 = 0. \tag{39}$$

From the defining system (16) we know that $4il_1^2(I_n + e^{-iw_1^0} f_2^0) c_1^1 = -1$, thus we get $k_1 = 0$ and consequently $y_2 = 0$. Analogously, inserting $b_1 = b_2 = b_3 = y_1 = 0$ and $y_3 = k_2 c_1^2$ into equation (22) shows that

$$4il_1^1 (I_n + e^{iw_1^0} f_2^0) k_2 c_1^2 = 0,$$

which together with the defining system (16) gives $y_3 = 0$.

Similarly, we obtain $y_4 = y_5 = 0$ based on (24),(29),(26) and (30). In a word, we have now $y = 0$. Hence, H_v^0 is injective. In a similar fashion, we can prove that H_v^0 is surjective. Which ends the proof. □

Remark 3.3. *In fact, the regularity conditions (17) in Theorem 3.2 and (10) in Theorem 2.2 are equivalent, and thus they are equivalent to the regularity condition c) in Definition 2.1. To see this, one only need to simply calculate each entry of the matrix appeared in condition (10) in Theorem 2.2 using the techniques applied in Theorem 5 in [22]. As a consequence, the assumption (17) holds naturally again if x^0 is a double Hopf point of (2) at $(\lambda, \mu) = (\lambda^0, \mu^0)$.*

Noting that system (16) is of finite dimension, Theorem 3.2 shows that (16) can be solved by classical iterative methods. Since (16) is still of high dimension, we propose therefore using the Newton downhill method as a solver to get a widely convergent iteration, i.e.

$$v_{k+1} = v_k - \delta[J_H(v_k)]^{-1} H(v_k),\ k = 0, 1, 2, ... \tag{40}$$

where $J_H(\cdot)$ is the Jacobian matrix, $0 < \delta \leq 1$ is the downhill factor to be determined such that $\|H(v_{k+1})\| < \|H(v_k)\|$ with $\|\cdot\|$ some vector norm.

4 Numerical example

We illustrate our theoretical findings by a numerical example. To this end, we consider the following class-B laser model with incoherent delayed feedback [24]

$$\begin{cases} \dot{x}_1(t) = W - x_1(t) - x_1(t)(x_2(t) + \gamma x_2(t-T)), \\ \dot{x}_2(t) = r(x_1(t) - 1)x_2(t), \end{cases} \tag{41}$$

where $x_1(t)$ and $x_2(t)$ denote the population density and the photon density respectively, W, r, γ are parameters satisfying $W > 1$, $0 < \gamma < 1$, $r >> 1$, and $T > 0$ is time delay. For the physical meanings of these parameters, we refer the reader to [24].

It is easy to check that system (41) has two equilibria, $(W, 0)$ and $(1, \frac{W-1}{\gamma+1})$. At the positive equilibrium $(1, \frac{W-1}{\gamma+1})$ system (41) undergoes the double Hopf bifurcation for certain critical values of bifurcation parameters T and γ, see [14] for detail. As an illustration of our method, we will show how to apply the techniques developed in this paper to calculate numerically such bifurcation points.

As indicated by [14], in this experiment, we fix $W = 1.5$ and $r = 1000$ and take γ and T as parameters. Then, after a change of time scale $t = Ts$ and using t to represent the time again, we reformulate (41) as

$$\dot{x}(t) = f(x(t), x(t-1), T, \gamma) \tag{42}$$

with $x(t) = (x_1(t), x_2(t))^T$ and

$$f(x(t), x(t-1), T, \gamma) = \begin{pmatrix} T(1.5 - x_1(t) - x_1(t)(x_2(t) + \gamma x_2(t-1))) \\ 1000T(x_1(t) - 1)x_2(t) \end{pmatrix}.$$

For any given $x \in \mathbb{R}^2$, the derivatives of f with respect to the first and the second variables at x are respectively given by

$$f_1(x, x, T, \gamma) = \begin{pmatrix} -T - Tx_2 - T\gamma x_2 & -Tx_1 \\ 1000Tx_2 & 1000T(x_1 - 1) \end{pmatrix}$$

and

$$f_2(x, x, T, \gamma) = \begin{pmatrix} 0 & -T\gamma x_1 \\ 0 & 0 \end{pmatrix}.$$

Inserting the above notations into (16) gives the defining system of equations for the double Hopf points of (41).

We then choose $l_1^1 = l_1^2 = (1, 0)$ and $l_2^1 = l_2^2 = (0, 1)$ and apply the Newton downhill method (40) to the resulting defining system. We list the results in Table 1 and 2 with the initial values, the corresponding zeros of the defining system and the number of iterations required. We found from Table 1 and 2 that the method has the ability to find each double Hopf point of the underlying time delayed model.

Table 1. Initial values applied to the Newton downhill method and the corresponding convergent results, as well as the number of iterations required to converge.

# iter	initial value	zero
22	1.0000	$1.0000 + 0.0000i$
	0.2000	$0.2538 - 0.0000i$
	$1.0000 - 1.0000i$	$0.0193 + 0.0042i$
	$1.0000 - 1.0000i$	$0.0478 - 0.2192i$
	$1.0000 - 1.0000i$	$0.0193 - 0.0042i$
	$1.0000 - 1.0000i$	$0.0478 + 0.2192i$
	1.0000	$1.1327 + 0.0000i$
	17.0000	$25.2691 - 0.0000i$
	$1.0000 - 1.0000i$	$-0.0027 + 0.0000i$
	$1.0000 - 1.0000i$	$0 + 0.2500i$
	$1.0000 - 1.0000i$	$-0.0027 + 0.0000i$
	$1.0000 - 1.0000i$	$0 - 0.2500i$
	0.7000	$0.9701 + 0.0000i$
	5.0000	$3.1248 + 0.0000i$
# iter	initial value	zero
23	1.0000	$1.0000 - 0.0000i$
	0.5000	$0.3095 - 0.0000i$
	$1.0000 - 1.0000i$	$0.0316 + 0.0100i$
	$1.0000 - 1.0000i$	$0.1395 - 0.4392i$
	$1.0000 - 1.0000i$	$0.0316 - 0.0100i$
	$1.0000 - 1.0000i$	$0.1395 + 0.4392i$
	1.0000	$0.8534 + 0.0000i$
	25.0000	$19.0260 + 0.0000i$
	$1.0000 - 1.0000i$	$-0.0088 + 0.0000i$
	$1.0000 - 1.0000i$	$0 + 0.2500i$
	$1.0000 - 1.0000i$	$-0.0088 + 0.0000i$
	$1.0000 - 1.0000i$	$0 - 0.2500i$
	0.4000	$0.6154 + 0.0000i$
	16.0000	$9.3385 + 0.0000i$

5 Conclusion

We presented in this paper an alternative method for calculating the double Hopf points of DDEs. Through investigating in detail the eigenspace associated with the distinct purely imaginary eigenvalues, we proposed a finite dimensional defining system that is proved to be regular at the double Hopf point and enable us to calculate such points and the corresponding eigenvalues, eigenfunctions, critical bifurcation parameters simultaneously. The analysis is performed for the case of DDEs with one constant delay, while the method can be straightforwardly extended to more general cases, for example the case of multiple-delay, where one only needs to analyze the eigenspace associated with the purely imaginary eigenvalues and insert the corresponding relations into (16) correctly to find a regular defining system of equations that can be solved iteratively.

Acknowledgement: The research was supported by NSFC grants 11201061, 11271065, 11371171, 11271157, CPSF-2012M520657 and the Science and Technology Development Planning of Jilin Province 20140520058JH.

Table 2. Initial values applied to the Newton downhill method and the corresponding convergent results, as well as the number of iterations required to converge (continued).

# iter	initial value	zero
20	1.0000	$1.0000 + 0.0000i$
	0.4000	$0.4070 - 0.0000i$
	$0.0441 - 0.0334i$	$0.0506 + 0.0344i$
	$0.6378 + 0.8410i$	$0.6297 - 0.9266i$
	$0.0441 + 0.0334i$	$0.0506 - 0.0344i$
	$0.6378 - 0.8410i$	$0.6297 + 0.9266i$
	0.8000	$0.8647 - 0.0000i$
	20.0000	$19.2160 - 0.0000i$
	-0.0111	$-0.0110 + 0.0000i$
	$0 - 0.2500i$	$0 + 0.2500i$
	-0.0111	$-0.0110 + 0.0000i$
	$0 + 0.2500i$	$0 - 0.2500i$
	0.2400	$0.2286 + 0.0000i$
	15.0000	$15.4164 - 0.0000i$
# iter	**initial value**	**zero**
10	1.0000	$1.0000 - 0.0000i$
	0.4000	$0.4231 - 0.0000i$
	$0.0300 - 0.0100i$	$0.0441 + 0.0334i$
	$0.1500 + 0.4000i$	$0.6378 - 0.8410i$
	$0.0300 + 0.0100i$	$0.0441 - 0.0334i$
	$0.1500 - 0.4000i$	$0.6378 + 0.8410i$
	1.2000	$1.1528 + 0.0000i$
	27.0000	$25.5807 + 0.0000i$
	-0.0100	$-0.0111 + 0.0000i$
	$0 - 0.2500i$	$0 + 0.2500i$
	-0.0100	$-0.0111 + 0.0000i$
	$0 + 0.2500i$	$0 - 0.2500i$
	0.3500	$0.1816 + 0.0000i$
	20.0000	$21.6165 + 0.0000i$

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