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Research Article

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Existence of a common solution for a system of nonlinear integral equations via fixed point methods in b-metric spaces

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Abstract: In this paper we introduce a property and use this property to prove some common fixed point theorems in b-metric space. We also give some fixed point results on b-metric spaces endowed with an arbitrary binary relation which can be regarded as consequences of our main results. As applications, we applying our result to prove the existence of a common solution for the following system of integral equations:

$$x(t) = \int_a^b K_1(t, r, x(r))dr, \qquad x(t) = \int_a^b K_2(t, r, x(r))dr,$$

where $a, b \in \mathbb{R}$ with $a < b$, $x \in C[a, b]$ (the set of continuous real functions defined on $[a, b] \subseteq \mathbb{R}$) and $K_1, K_2 : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are given mappings. Finally, an example is also given in order to illustrate the effectiveness of such result.

Keywords: b-metric spaces, Coincidence points, Common fixed points, Integral equations

MSC: 47H09, 47H10

1 Introduction

The famous Banach contraction mapping principle first appeared in explicit form in Banach's thesis in 1922 where it was used to establish the existence of a solution for an integral equation. Since then, because of its simplicity and usefulness, it has been generalized in several directions over the years. One of the most interesting generalization is an extension of the class of Banach contraction mappings to the class of weak contraction mappings which was first introduced by Alber et al. [1] in the setting of Hilbert spaces. Afterwards, Rhoades [2] considered the class of weak contraction mappings in the setting of metric spaces and proved that the result of Alber et al. [1] is also valid in complete metric spaces. Fixed point theorems and applications for weak contraction mappings have been considered in [3–7] and references therein.

On the other hand, in 1984, Khan et al. [8] introduced the concept of an altering distance function as follows:

Definition 1.1. *The function $\varphi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties hold:*

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(1) *φ is continuous and non-decreasing;*
(2) *$\varphi(t) = 0$ if and only if $t = 0$.*

In 2011, Choudhury et al. [9] generalized the concept of weak contraction mappings by using an altering distance function and proved fixed point theorem for such mappings. Recently, Roshan et al. [10] studied some coincidence point results for four mappings satisfying generalized weak contractive conditions in the framework of partially ordered b-metric spaces (b-metric spaces in the sense of Czerwik [11]).

Inspired by the results of Roshan et al. [10], we introduce some new properties in the setting of b-metric spaces and using such properties, prove some coincidence point and common fixed point theorems for four mappings satisfying generalized weak contractive condition in the setting of b-metric spaces. We also prove some fixed point results in b-metric spaces endowed with a binary relation which generalize and improve the results of Roshan et al. [10] and several fixed point results in metric spaces and b-metric spaces.

Finally, as applications, we show the existence of a common solution for a system of the following integral equations:

$$\begin{cases} x(t) = \int_a^b K_1(t, r, x(r))dr, \\ x(t) = \int_a^b K_2(t, r, x(r))dr, \end{cases} \tag{1}$$

where $a, b \in \mathbb{R}$ with $a < b$, $x \in C[a, b]$ (the set of continuous real value functions defined on $[a, b] \subseteq \mathbb{R}$) and $K_1, K_2 : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are the mappings satisfying some conditions. Furthermore, we give some example to illustrate the existence of a common solution for a system of the integral equations.

2 Preliminaries

In this section, we recollect some essential notations, required definitions and basic results coherent with the literature. Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{R}_+$ and $\mathbb{R}$ the sets of positive integers, non-negative real numbers and real numbers, respectively.

In 1993, Czerwik [11] introduced the concept of a b-metric space as follows:

Definition 2.1 ([11]). *Let X be a nonempty set and $s \geq 1$. Suppose that the mapping $d : X \times X \to \mathbb{R}_+$ satisfies the following conditions: for all $x, y, z, \in X$,*
(BM1) *$d(x, y) = 0$ if and only if $x = y$;*
(BM2) *$d(x, y) = d(y, x)$;*
(BM3) *$d(x, y) \leq s[d(x, z) + d(z, y)]$.*
Then (X, d) is called a b-metric space with coefficient s.

Any metric space is a b-metric space with $s = 1$ and so the class of b-metric spaces is larger than the class of metric spaces. Now, we give some known examples of b-metric spaces as follows:

Example 2.2. *Let $X = \mathbb{R}$ and define a mapping $d : X \times X \to \mathbb{R}_+$ by*

$$d(x, y) = |x - y|^2$$

for all $x, y \in X$. Then (X, d) is a b-metric space with coefficient $s = 2$.

Example 2.3. *The set $l_p(\mathbb{R})$ with $0 < p < 1$, where*

$$l_p(\mathbb{R}) := \Big\{\{x_n\} \subseteq \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\Big\},$$

together with the mapping $d : l_p(\mathbb{R}) \times l_p(\mathbb{R}) \to \mathbb{R}_+$ defined by

$$d(x, y) := (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}}$$

for all $x = \{x_n\}, y = \{y_n\} \in l_p(\mathbb{R})$ is a b-metric space with coefficient $s = 2^{\frac{1}{p}} > 1$. The above result also holds for the general case $l_p(X)$ with $0 < p < 1$, where X is a Banach space.

Example 2.4. *Let p be a given real number in $(0, 1)$. The space*

$$L_p[0,1] := \left\{ x : [0,1] \to \mathbb{R} \;\middle|\; \int_0^1 |x(t)|^p dt < 1 \right\},$$

together with the mapping $d : L_p[0,1] \times L_p[0,1] \to \mathbb{R}_+$ defined by

$$d(x,y) := \Big(\int_0^1 |x(t) - y(t)|^p dt \Big)^{1/p}$$

for all $x, y \in L_p[0,1]$ is a b-metric space with constant $s = 2^{\frac{1}{p}} > 1$.

Next, we give the concepts of b-convergence, b-Cauchy sequence, b-continuity, completeness and closedness in a b-metric space.

Definition 2.5 ([12]). *Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called:*
(1) *b-convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$;*
(2) *a b-Cauchy sequence if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.*

Proposition 2.6 ([12]). *In a b-metric space (X, d), the following assertions hold:*
(1) *a b-convergent sequence has a unique limit;*
(2) *each b-convergent sequence is a b-Cauchy sequence;*
(3) *in general, a b-metric is not continuous.*

From the fact that, in (3) above, we need the following lemma about the b-convergent sequences for our results:

Lemma 2.7 ([13]). *Let (X, d) be a b-metric space with coefficient $s \geq 1$ and let $\{x_n\}$, $\{y_n\}$ be b-convergent to the points $x, y \in X$, respectively. Then we have*

$$\frac{1}{s^2} d(x,y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x,y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for all $z \in X$, we have

$$\frac{1}{s} d(x,z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq s d(x,z).$$

Definition 2.8 ([12]). *A b-metric space (X, d) is said to be b-complete if every b-Cauchy sequence in X b-converges.*

Definition 2.9 ([12]). *Let (X, d) and (X', d') be two b-metric spaces. A function $f : X \to X'$ is said to be b-continuous at a point $x \in X$ if it is b-sequentially continuous at x, that is, whenever $\{x_n\}$ is b-convergent to x, $\{fx_n\}$ is b-convergent to fx.*

Definition 2.10 ([12]). *Let Y be a nonempty subset of a b-metric space (X, d). The closure of Y is denoted by $\overline{Y}$ and it is the set of limits of all b-convergent sequences of points in Y, that is,*

$$\overline{Y} = \{x \in X : \text{there exists a sequence } \{x_n\} \text{ in } Y \text{ so that } \lim_{n\to\infty} x_n = x\}.$$

Definition 2.11 ([12]). *Let (X, d) be a b-metric space. Then a subset $Y \subseteq X$ is said to be closed if, for each sequence $\{x_n\}$ in Y which b-converges to an element x, we have $x \in Y$ (i.e., $\overline{Y} = Y$).*

In 1986, Junck [14] introduced the concept of compatibility in metric space. Now, we give this concept in b-metric space.

Definition 2.12. *Let (X,d) be a b-metric space with coefficient $s \geq 1$ and $f,g : X \to X$ be two mappings. The pair (f,g) is said to be compatible if*

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$. If $s = 1$, then it becomes compatible in the sense of Junck [14].

In [15], Junck gave the following concept:

Definition 2.13 ([15]). *Let f and g be two self mappings on a nonempty set X. The pair (f,g) is said to be weakly compatible if f and g commute at their coincidence point (i.e., $fgx = gfx$ whenever $fx = gx$).*

The weak compatibility leads to the compatibility but its converse need not be true.

In 2014, Hussain et al. [16] introduced the concept of α-completeness in metric spaces. Inspired by this concept, we give this concept in b-metric spaces as follows:

Definition 2.14. *Let (X,d) be a b-metric space with coefficient $s \geq 1$ and let $\alpha : X \times X \to [0,\infty)$ be a given mapping. The b-metric space X is said to be α-complete if every Cauchy sequence $\{x_n\} \subseteq X$ with*

$$\alpha(x_n, x_{n+1}) \geq 1$$

for all $n \in \mathbb{N}$ converges in X.

In 2014, Sintunavarat [17] (see also [6]) introduced the useful concept of transitivity for mappings as follows:

Definition 2.15. *Let X be a nonempty set. The mapping $\alpha : X \times X \to [0,\infty)$ is said to be transitive if, for $x,y,z \in X$, we have*

$$\alpha(x,y) \geq 1, \quad \alpha(y,z) \geq 1 \implies \alpha(x,z) \geq 1.$$

3 Main results

In this section, we introduce some new properties and establish coincidence point and common fixed point theorems by using these concepts.

Definition 3.1. *Let X be a nonempty set, $\alpha : X \times X \to [0,\infty)$ and $f,g : X \to X$ be three mappings. The ordered pair (f,g) is said to be:*

(1) *α-weakly increasing if $\alpha(fx, gfx) \geq 1$ and $\alpha(gx, fgx) \geq 1$ for all $x \in X$;*
(2) *partially α-weakly increasing if $\alpha(fx, gfx) \geq 1$ for all $x \in X$.*

Throughout this paper, for a self mapping f on a nonempty set X and a point $x \in X$, we use the following notation:

$$f^{-1}(x) := \{u \in X : fu = x\}.$$

Definition 3.2. *Let X be a nonempty set, $\alpha : X \times X \to [0,\infty)$ and $f,g,h : X \to X$ be four mappings such that $f(X) \subseteq h(X)$ and $g(X) \subseteq h(X)$. The ordered pair (f,g) is said to be:*

(1) *α-weakly increasing with respect to h if, for all $x \in X$, we have $\alpha(fx, gy) \geq 1$ for all $y \in h^{-1}(fx)$ and $\alpha(gx, fy) \geq 1$ for all $y \in h^{-1}(gx)$;*

(2) *partially α-weakly increasing with respect to h if* $\alpha(fx, gy) \geq 1$ *for all* $y \in h^{-1}(fx)$.

Remark 3.3. *From Definition 3.2, we have the following assertions:*
(1) *If* $h = I_X$ *(: the identity mapping on* X*), then Definition 3.2 reduces to Definition 3.1;*
(2) *If* $g = f$*, then we say that* f *is α-weakly increasing with respect to* h *(partially α-weakly increasing with respect to* h*). Also, if* $h = I_X$*, then we say that* f *is α-weakly increasing (partially α-weakly increasing).*

Definition 3.4. *Let* (X, d) *be a b-metric space,* $\alpha : X \times X \to [0, \infty)$ *and* $f, g : X \to X$ *be three mappings. The pair* (f, g) *is said to be α-compatible if*

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$$

whenever $\{x_n\}$ *is a sequence in* X *such that*

$$\alpha(x_n, x_{n+1}) \geq 1$$

for all $n \in \mathbb{N}$ *and*

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$.

Definition 3.5. *Let* (X, d) *be a b-metric space,* $\alpha : X \times X \to [0, \infty)$ *and* $f : X \to X$ *be two mappings. We say that* f *is α-continuous at a point* $x \in X$ *if, for each sequence* $\{x_n\}$ *in* X *with* $x_n \to x$ *as* $n \to \infty$ *and* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$*, we have*

$$\lim_{n\to\infty} fx_n = f\Big(\lim_{n\to\infty} x_n\Big).$$

Let (X, d) be a b-metric space with coefficient $s \geq 1$ and $f, g, R, S : X \to X$ be four mappings. Throughout this paper, unless otherwise stated, for all $x, y \in X$, let

$$M_s(x, y) := \max\left\{d(Sx, Ry), d(Sx, fx), d(Ry, gy), \frac{d(Sx, gy) + d(Ry, fx)}{2s}\right\}.$$

Now, we give a coincidence point result in this paper.

Theorem 3.6. *Let* (X, d) *be a b-metric space with coefficient* $s \geq 1$*,* $\alpha : X \times X \to [0, \infty)$ *and* $f, g, R, S : X \to X$ *be five mappings such that* $f(X) \subseteq R(X)$ *and* $g(X) \subseteq S(X)$*. Suppose that, for all* $x, y \in X$*, we have*

$$\alpha(Sx, Ry) \geq 1 \ \text{ or } \ \alpha(Ry, Sx) \geq 1 \Longrightarrow \psi(s^3 d(fx, gy)) \leq \psi(M_s(x, y)) - \varphi(M_s(x, y)), \tag{2}$$

where $\psi, \varphi : [0, \infty) \to [0, \infty)$ *are altering distance functions. If the following conditions hold:*
(1) (X, d) *is α-complete;*
(2) f, g, R *and* S *are α-continuous;*
(3) *the pairs* (f, S) *and* (g, R) *are α-compatible;*
(4) *the pairs* (f, g) *and* (g, f) *are partially α-weakly increasing with respect to* R *and* S*, respectively;*
(5) α *is a transitive mapping,*
then the pair (f, S) *and* (g, R) *have a coincidence point* $z \in X$*. Moreover, if* $\alpha(Rz, Sz) \geq 1$ *or* $\alpha(Sz, Rz) \geq 1$*, then* z *is a coincidence point of* f, g, R *and* S.

Proof. Let x_0 be an arbitrary point of X. Choose $x_1 \in X$ such that $fx_0 = Rx_1$ and $x_2 \in X$ such that $gx_1 = Sx_2$. Now, we can construct a sequence $\{z_n\}$ defined by

$$z_{2n+1} := Rx_{2n+1} = fx_{2n}, \quad z_{2n+2} := Sx_{2n+2} = gx_{2n+1}$$

for all $n \in \mathbb{N} \cup \{0\}$. Since $x_1 \in R^{-1}(fx_0)$, $x_2 \in S^{-1}(gx_1)$ and the pair (f, g) and (g, f) are partially α-weakly increasing with respect to R and S, respectively, we have

$$\alpha(z_1, z_2) = \alpha(fx_0, gx_1) \geq 1, \quad \alpha(z_2, z_3) = \alpha(Sx_2, fx_2) \geq 1.$$

Repeating this process, we obtain

$$\alpha(z_n, z_{n+1}) \geq 1 \tag{3}$$

for all $n \in \mathbb{N} \cup \{0\}$. Furthermore, by the transitive property of α, we have

$$\alpha(z_{2n}, z_{2n+2}) \geq 1 \tag{4}$$

and

$$\alpha(z_{2n+1}, z_{2n+3}) \geq 1 \tag{5}$$

for all $n \in \mathbb{N}$.

Now, we will complete the proof in three steps:

Step I. We prove that $\lim\limits_{k\to\infty} d(z_k, z_{k+1}) = 0$. For all $k \in \mathbb{N} \cup \{0\}$, we define $d_k := d(z_k, z_{k+1})$. We assume that $d_{k_0} = 0$ for some $k_0 \in \mathbb{N} \cup \{0\}$, which implies that $z_{k_0} = z_{k_0+1}$. If $k_0 = 2n$ such that $n \in \mathbb{N} \cup \{0\}$, then $z_{2n} = z_{2n+1}$. Next, we show that $z_{2n+1} = z_{2n+2}$. Since $\alpha(Sx_{2n}, Rx_{2n+1}) = \alpha(z_{2n}, z_{2n+1}) \geq 1$, we have

$$\begin{aligned}\psi(s^3 d(z_{2n+1}, z_{2n+2})) &= \psi(s^3 d(fx_{2n}, gx_{2n+1}))\\ &\leq \psi(M_s(x_{2n}, x_{2n+1})) - \varphi(M_s(x_{2n}, x_{2n+1})),\end{aligned} \tag{6}$$

where

$$\begin{aligned}&M_s(x_{2n}, x_{2n+1})\\ &= \max\left\{d(Sx_{2n}, Rx_{2n+1}), d(Sx_{2n}, fx_{2n}), d(Rx_{2n+1}, gx_{2n+1}), \frac{d(Sx_{2n}, gx_{2n+1}), d(Rx_{2n+1}, fx_{2n})}{2s}\right\}\\ &= \max\left\{d(z_{2n}, z_{2n+1}), d(z_{2n}, z_{2n+1}), d(z_{2n+1}, z_{2n+2}), \frac{d(z_{2n}, z_{2n+2}) + d(z_{2n+1}, z_{2n+1})}{2s}\right\}\\ &= \max\left\{0, 0, d(z_{2n+1}, z_{2n+2}), \frac{d(z_{2n}, z_{2n+2})}{2s}\right\}\\ &\leq \max\left\{0, 0, d(z_{2n+1}, z_{2n+2}), \frac{0 + d(z_{2n+1}, z_{2n+2})}{2}\right\}\\ &= d(z_{2n+1}, z_{2n+2}).\end{aligned}$$

Therefore, from (6), we have

$$\begin{aligned}\psi(s^3 d(z_{2n+1}, z_{2n+2})) &\leq \psi(d(z_{2n+1}, z_{2n+2})) - \varphi\Big(\max\left\{d(z_{2n+1}, z_{2n+2}), \frac{d(z_{2n}, z_{2n+2})}{2s}\right\}\Big)\\ &\leq \psi(s^3 d(z_{2n+1}, z_{2n+2})) - \varphi\Big(\max\left\{d(z_{2n+1}, z_{2n+2}), \frac{d(z_{2n}, z_{2n+2})}{2s}\right\}\Big).\end{aligned}$$

This implies that

$$\varphi\Big(\max\left\{d(z_{2n+1}, z_{2n+2}), \frac{d(z_{2n}, z_{2n+2})}{2s}\right\}\Big) = 0$$

and so

$$d(z_{2n+1}, z_{2n+2}) = 0.$$

Therefore, we have $z_{2n+1} = z_{2n+2}$. Similarly, if $k_0 = 2n + 1$ such that $n \in \mathbb{N} \cup \{0\}$, then $z_{2n+1} = z_{2n+2}$ gives $z_{2n+2} = z_{2n+3}$. Consequently, the sequence $\{z_n\}$ becomes constant for $k \geq k_0$ and hence $\lim_{k\to\infty} d(z_k, z_{k+1}) = 0$. This completes this step.

Therefore, we will suppose that

$$d_k = d(z_k, z_{k+1}) > 0 \tag{7}$$

for all $k \in \mathbb{N} \cup \{0\}$. Next, we show that

$$d(z_{k+1}, z_{k+2}) \leq d(z_k, z_{k+1}) \tag{8}$$

for all $k \in \mathbb{N} \cup \{0\}$. Assume that

$$d(z_{k+1}, z_{k+2}) > d(z_k, z_{k+1})$$

for some $k \in \mathbb{N} \cup \{0\}$. If $k = 2n$ such that $n \in \mathbb{N} \cup \{0\}$, then we have

$$d(z_{2n+1}, z_{2n+2}) > d(z_{2n}, z_{2n+1}).$$

Since

$$\alpha(Sx_{2n}, Rx_{2n+1}) \geq 1,$$

using (2), we obtain

$$\begin{aligned}\psi(s^3 d(z_{2n+1}, z_{2n+2})) &= \psi(s^3 d(fx_{2n}, gx_{2n+1})) \\ &\leq \psi(M_s(x_{2n}, x_{2n+1})) - \varphi(M_s(x_{2n}, x_{2n+1})),\end{aligned} \tag{9}$$

where

$$\begin{aligned}&M_s(x_{2n}, x_{2n+1}) \\ &= \max\left\{d(Sx_{2n}, Rx_{2n+1}), d(Sx_{2n}, fx_{2n}), d(Rx_{2n+1}, gx_{2n+1}), \frac{d(Sx_{2n}, gx_{2n+1}) + d(Rx_{2n+1}, fx_{2n})}{2s}\right\} \\ &= \max\left\{d(z_{2n}, z_{2n+1}), d(z_{2n}, z_{2n+1}), d(z_{2n+1}, z_{2n+2}), \frac{d(z_{2n}, z_{2n+2}) + d(z_{2n+1}, z_{2n+1})}{2s}\right\} \\ &\leq \max\left\{d(z_{2n}, z_{2n+1}), d(z_{2n+1}, z_{2n+2}), \frac{d(z_{2n}, z_{2n+1}) + d(z_{2n+1}, z_{2n+2})}{2}\right\} \\ &= d(z_{2n+1}, z_{2n+2}).\end{aligned}$$

Since ψ is nondecreasing and

$$M_s(x_{2n}, x_{2n+1}) \leq d(z_{2n+1}, z_{2n+2}),$$

we have

$$\psi(M_s(x_{2n}, x_{2n+1})) \leq \psi(d(z_{2n+1}, z_{2n+2})).$$

Now, the inequality (9) implies that

$$\psi(s^3 d(z_{2n+1}, z_{2n+2})) \leq \psi(d(z_{2n+1}, z_{2n+2})) - \varphi(M_s(x_{2n}, x_{2n+1})),$$

which is possible only $M_s(x_{2n}, x_{2n+1}) = 0$, that is, $d(z_{2n+1}, z_{2n+2}) = 0$, which contradicts (7). Hence we have

$$d(z_{2n+1}, z_{2n+2}) \leq d(z_{2n}, z_{2n+1}).$$

Therefore, (8) is proved for $k = 2n$. Also, we have

$$d(z_{2n}, z_{2n+1}) = M_s(x_{2n}, x_{2n+1}).$$

Similarly, we can shown that

$$d(z_{2n+2}, z_{2n+3}) \leq d(z_{2n+1}, z_{2n+2}) = M_s(x_{2n+1}, x_{2n+2}) \tag{10}$$

for all $n \in \mathbb{N} \cup \{0\}$. Hence the inequality (8) holds and then $\{d(z_k, z_{k+1})\}$ is a nonincreasing sequence of nonnegative real numbers. Since $\{d(z_k, z_{k+1})\}$ is bounded below, there exists $r \geq 0$ such that

$$\lim_{k\to\infty} d(z_k, z_{k+1}) = r \tag{11}$$

and then

$$\lim_{k\to\infty} M_s(x_k, x_{k+1}) = r. \tag{12}$$

Furthermore, we get

$$\psi(s^3 d(z_{k+1}, z_{k+2})) \leq \psi(d(z_k, z_{k+1})) - \varphi(M_s(x_k, x_{k+1})) \tag{13}$$

for all $k \in \mathbb{N} \cup \{0\}$. Letting $k \to \infty$ in (13), using (11), (12) and the continuity of ψ, φ, we have

$$\psi(s^3 r) \leq \psi(r) - \varphi(r).$$

This implies that $\varphi(r) = 0$. From the property of φ, we have $r = 0$ and so

$$\lim_{k\to\infty} d(z_k, z_{k+1}) = 0. \tag{14}$$

Step II. We now claim that $\{z_n\}$ is a b-Cauchy sequence in X. That is, for any $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $d(z_m, z_n) < \epsilon$, for all $m, n \geq k$. Assume that there exists $\epsilon > 0$ for which we can find subsequences $\{z_{m(k)}\}$ and $\{z_{n(k)}\}$ of $\{z_n\}$ such that $n(k) > m(k) \geq k$ and

(a) $m(k) = 2t$ and $n(k) = 2t' + 1$, where $t, t' \in \mathbb{N}$;

(b)

$$d(z_{m(k)}, z_{n(k)}) \geq \epsilon; \tag{15}$$

(c) $n(k)$ is the smallest number such that the condition (b) holds, i.e.,

$$d(z_{m(k)}, z_{n(k)-1}) < \epsilon. \tag{16}$$

By the triangle inequality, (15) and (16), we obtain

$$\begin{aligned}\epsilon &\leq d(z_{m(k)}, z_{n(k)})\\ &\leq s[d(z_{m(k)}, z_{n(k)-1}) + d(z_{n(k)-1}, z_{n(k)})]\\ &< s\epsilon + d(z_{n(k)-1}, z_{n(k)}).\end{aligned} \tag{17}$$

Taking limit supremum as $k \to \infty$ in (17) and using (14), we have

$$\epsilon \leq \limsup_{k\to\infty} d(z_{m(k)}, z_{n(k)}) \leq s\epsilon. \tag{18}$$

From the triangle inequality, we have

$$d(z_{m(k)}, z_{n(k)}) \leq s[d(z_{m(k)}, z_{n(k)+1}) + d(z_{n(k)+1}, z_{n(k)})] \tag{19}$$

and

$$d(z_{m(k)}, z_{n(k)+1}) \leq s[d(z_{m(k)}, z_{n(k)}) + d(z_{n(k)}, z_{n(k)+1})]. \tag{20}$$

Taking limit supremum as $k \to \infty$ in (19) and (20), from (14) and (18), it follows that

$$\epsilon \leq s\left(\limsup_{k\to\infty} d(z_{m(k)}, z_{n(k)+1})\right)$$

and

$$\limsup_{k\to\infty} d(z_{m(k)}, z_{n(k)+1}) \leq s^2\epsilon,$$

This implies that

$$\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d(z_{m(k)}, z_{n(k)+1}) \leq s^2\epsilon. \tag{21}$$

Again, using above process, we get

$$\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d(z_{n(k)}, z_{m(k)+1}) \leq s^2\epsilon. \tag{22}$$

Finally, we obtain that

$$d(z_{m(k)}, z_{n(k)+1}) \leq s[d(z_{m(k)}, z_{m(k)+1}) + d(z_{m(k)+1}, z_{n(k)+1})]. \tag{23}$$

Taking limit supremum as $k \to \infty$ in (23), from (14) and (21), we obtain that

$$\frac{\epsilon}{s^2} \leq \limsup_{k\to\infty} d(z_{m(k)+1}, z_{n(k)+1}). \tag{24}$$

By similar method, we have

$$\limsup_{k\to\infty} d(z_{m(k)+1}, z_{n(k)+1}) \leq s^3\epsilon. \tag{25}$$

From (24) and (25) implies that

$$\frac{\epsilon}{s^2} \leq \limsup_{k\to\infty} d(z_{m(k)+1}, z_{n(k)+1}) \leq s^3\epsilon. \tag{26}$$

Since α is transitive, we have

$$\alpha(z_{m(k)+1}, z_{n(k)+1}) \geq 1.$$

From (2), it follows that

$$\begin{aligned}\psi(s^3 d(z_{m(k)+1}, z_{n(k)+1})) &= \psi(s^3 d(fx_{m(k)}, gx_{n(k)})) \\ &\leq \psi(M_s(x_{m(k)}, x_{n(k)})) - \varphi(M_s(x_{m(k)}, x_{n(k)})),\end{aligned} \tag{27}$$

where

$$\begin{aligned}M_s(x_{m(k)}, x_{n(k)}) &= \max\Big\{d(Sx_{m(k)}, Rx_{n(k)}), d(Sx_{m(k)}, fx_{m(k)}), d(Rx_{n(k)}, gx_{n(k)}), \\ &\qquad \frac{d(Sx_{m(k)}, gx_{n(k)}) + d(Rx_{n(k)}, fx_{m(k)})}{2s}\Big\} \\ &= \max\Big\{d(z_{m(k)}, z_{n(k)}), d(z_{m(k)}, z_{m(k)+1}), d(z_{n(k)}, z_{n(k)+1}), \\ &\qquad \frac{d(z_{m(k)}, z_{n(k)+1}) + d(z_{n(k)}, z_{m(k)+1})}{2s}\Big\}.\end{aligned}$$

Taking limit supremum as $k \to \infty$ in the above equation and using (14), (18), (21), (23) and (22), we have

$$\epsilon = \max\left\{\epsilon, \frac{\frac{\epsilon}{s} + \frac{\epsilon}{s}}{2s}\right\} \leq \limsup_{k\to\infty} M_s(x_{m(k)}, x_{n(k)}) \leq \max\left\{s\epsilon, \frac{s^2\epsilon + s^2\epsilon}{2s}\right\} = s\epsilon.$$

Also, we can show that

$$\epsilon = \max\left\{\epsilon, \frac{\frac{\epsilon}{s} + \frac{\epsilon}{s}}{2s}\right\} \leq \liminf_{k\to\infty} M_s(x_{m(k)}, x_{n(k)}) \leq \max\left\{s\epsilon, \frac{s^2\epsilon + s^2\epsilon}{2s}\right\} = s\epsilon.$$

Taking limit supremum as $k \to \infty$ in (27), we have

$$\begin{aligned}\psi(s\epsilon) &= \psi\Big(s\Big(s^2\frac{\epsilon}{s^2}\Big)\Big) \\ &\leq \psi\Big(s^3 \limsup_{k\to\infty} d(z_{m(k)+1}, z_{n(k)+1})\Big) \\ &\leq \psi\Big(\limsup_{k\to\infty} M_s(x_{m(k)}, x_{n(k)})\Big) - \varphi\Big(\liminf_{k\to\infty} M_s(x_{m(k)}, x_{n(k)})\Big) \\ &\leq \psi(s\epsilon) - \varphi(\epsilon).\end{aligned} \tag{28}$$

This implies that $\varphi(\epsilon) \leq 0$ and so $\epsilon = 0$, which is a contradiction. Therefore, $\{z_n\}$ is a b-Cauchy sequence.

Step III. We will show that f, g, R and S have a coincidence point. From Step II, we show that $\{z_n\}$ is a b-Cauchy sequence in X. Since the inequality (3) holds, by the α-completeness of b-metric space X, there exists $z \in X$ such that

$$\lim_{n\to\infty} d(z_n, z) = 0$$

and so

$$\lim_{n\to\infty} d(z_{2n+1}, z) = \lim_{n\to\infty} d(Rx_{2n+1}, z) = \lim_{n\to\infty} d(fx_{2n}, z) = 0 \tag{29}$$

and

$$\lim_{n\to\infty} d(z_{2n+2}, z) = \lim_{n\to\infty} d(Sx_{2n+2}, z) = \lim_{n\to\infty} d(gx_{2n+1}, z) = 0. \tag{30}$$

From (29) and (30), we have $fx_{2n} \to z$ and $Sx_{2n} \to z$ as $n \to \infty$. Since (f, S) is α-compatible, by (4), we have

$$\lim_{n\to\infty} d(Sfx_{2n}, fSx_{2n}) = 0. \tag{31}$$

By (4), the α-continuity of S, f and Lemma 2.7, we obtain

$$\lim_{n\to\infty} d(Sfx_{2n}, Sz) = 0 = \lim_{n\to\infty} d(fSx_{2n}, fz). \tag{32}$$

By the triangle inequality, we have

$$\begin{aligned} d(Sz, fz) &\le s[d(Sz, Sfx_{2n}) + d(Sfx_{2n}, fz)] \\ &\le sd(Sz, Sfx_{2n}) + s^2[d(Sfx_{2n}, fSx_{2n}) + d(fSx_{2n}, fz)] \end{aligned} \tag{33}$$

for all $n \in \mathbb{N} \cup \{0\}$. Taking limit as $n \to \infty$ in the above inequality and using (23) and (31), we obtain

$$d(Sz, fz) \le 0.$$

This implies that $d(Sz, fz) = 0$ and so $fz = Sz$, that is, z is a coincidence point of f and S. Similarly, we can prove that z is also a coincidence point of g and R.

Finally, we prove that z is a coincidence point of f, g, R and S provide that

$$\alpha(Rz, Sz) \ge 1 \quad \text{or} \quad \alpha(Sz, Rz) \ge 1. \tag{34}$$

From (2) and (34), we have

$$\psi(s^3 d(fz, gz)) \le \psi(M_s(z, z)) - \varphi(M_s(z, z)), \tag{35}$$

where

$$\begin{aligned} M_s(z, z) &= \max\left\{d(Sz, Rz), d(Sz, fz), d(Rz, gz), \frac{d(Sz, gz) + d(Rz, fz)}{2s}\right\} \\ &= d(Sz, Rz) \\ &= d(fz, gz). \end{aligned}$$

Therefore, (35) implies

$$\psi(s^3 d(fz, gz)) \le \psi\big(d(fz, gz)\big) - \varphi\big(d(fz, gz)\big).$$

Now we obtain that

$$\psi\big(d(fz, gz)\big) = 0.$$

This implies that $d(fz, gz) = 0$ and so

$$fz = gz = Rz = Sz.$$

Hence z is a coincidence point of f, g, R and S. This completes the proof. □

Next, we give an example to illustrate Theorem 3.6.

Example 3.7. *Let $X = \mathbb{R}$ and b-metric $d : X \times X \to [0, \infty)$ be given by $d(x, y) = |x - y|^2$ for all $x, y \in X$. Define mapping $f, g, R, S : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by*

$$fx = \begin{cases} \sinh^{-1} x, & x \in [0, \infty); \\ -0.2, & x \in (-\infty, 0), \end{cases} \tag{36}$$

$$gx = \begin{cases} \sinh^{-1}\left(\frac{x}{2}\right), & x \in [0, \infty); \\ -0.2, & x \in (-\infty, 0), \end{cases} \tag{37}$$

$$Rx = \begin{cases} \sinh 6x, & x \in [0, \infty); \\ -0.2, & x \in (-\infty, 0), \end{cases} \tag{38}$$

$$Sx = \begin{cases} \sinh 12x, & x \in [0, \infty); \\ -0.2, & x \in (-\infty, 0), \end{cases} \tag{39}$$

and

$$\alpha(x, y) = \begin{cases} 1, & x \ge y \text{ and } x, y \in [0, \infty); \\ 0, & \text{otherwise}. \end{cases} \tag{40}$$

It is easy to see that (X, d) is an α-complete b-metric space with coefficient $s = 2$. Also, we can see that f, g, R and S are α-continuous.

To prove that (f, g) is partially α-weakly increasing with respect to R. Let $x, y \in X$ be such that $y \in R^{-1}(fx)$, that is, $Ry = fx$. By the definition of f and R, we divide two cases, that is,

$$x, y \in [0, \infty) \quad or \quad x, y \in (-\infty, 0).$$

If $x, y \in [0, \infty)$, then we have

$$\sinh 6y = \sinh^{-1} x$$

and so

$$y = \frac{\sinh^{-1}(\sinh^{-1} x)}{6}.$$

Since $\sinh x \geq \sinh^{-1} x$ for all $x \geq 0$, we have $12x \geq \sinh^{-1}(\sinh^{-1} x)$ or

$$fx = \sinh^{-1} x \geq \sinh^{-1}\left(\frac{1}{12}\sinh^{-1}(\sinh^{-1} x)\right) = \sinh^{-1}\left(\frac{1}{2}y\right) = gy.$$

If $x, y \in (-\infty, 0)$, then $fx = gy$ and so $\alpha(fx, gy) = 1$. Hence (f, g) is partially α-weakly increasing with respect to R.

To prove that (g, f) is partially α-weakly increasing with respect to S. Let $x, y \in X$ be such that $y \in S^{-1}(gx)$, that is, $Sy = gx$. By the definition of g and S, we divide two cases, that is,

$$x, y \in [0, \infty) \quad or \quad x, y \in (-\infty, 0).$$

If $x, y \in [0, \infty)$, then we have

$$\sinh 12y = \sinh^{-1} \frac{x}{2}$$

and so

$$y = \frac{\sinh^{-1}(\sinh^{-1} \frac{x}{2})}{12}.$$

Since $\sinh x \geq \sinh^{-1} x$ for all $x \geq 0$, we have

$$\frac{x}{2} \geq \sinh^{-1}\left(\sinh^{-1}\frac{x}{2}\right) \geq \frac{\sinh^{-1}(\sinh^{-1}\frac{x}{2})}{12}$$

and so

$$gx = \sinh^{-1}\frac{x}{2} \geq \sinh^{-1}\left(\frac{1}{12}\sinh^{-1}\left(\sinh^{-1}\frac{x}{2}\right)\right) = \sinh^{-1} y = fy.$$

If $x, y \in (-\infty, 0)$, then $gx = fy$ and so $\alpha(gx, fy) = 1$. Hence (g, f) is partially α-weakly increasing with respect to R.

Furthermore, $f(X) = g(X) = S(X) = R(X) = \{-0.2\} \cup [0, \infty)$. Next, we will show that (f, S) is α-compatible. Suppose that $\{x_n\}$ is a sequence in X such that

$$\alpha(x_n, x_{n+1}) \geq 1$$

for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} d(t, fx_n) = \lim_{n\to\infty} d(t, Sx_n) = 0$$

for some $t \in X$. Therefore, we have $x_n \in [0, \infty)$ and $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$ and so

$$\lim_{n\to\infty} |\sinh^{-1} x_n - t| = \lim_{n\to\infty} |\sinh 12x_n - t| = 0.$$

The continuity of hyperbolic sine and inverse hyperbolic sine functions imply that

$$\lim_{n\to\infty} |x_n - \sinh t| = \lim_{n\to\infty} \left|x_n - \frac{\sinh^{-1} t}{12}\right| = 0.$$

By the uniqueness of the limit, we get $\sinh t = \frac{\sinh^{-1} t}{12}$. *But we have*

$$\sinh t = \frac{\sinh^{-1} t}{12} \iff t = 0.$$

Thus we have $t = 0$. *Therefore*

$$\lim_{n\to\infty} d(fSx_n, Sfx_n) = \lim_{n\to\infty} |fSx_n - Sfx_n|^2 = 0.$$

Similarly, we can prove that (g, R) *s also* α*-compatible. Define two functions* $\psi, \varphi : [0,\infty) \to [0,\infty)$ *by* $\psi(t) = bt$ *and* $\varphi(t) = (b-1)t$ *for all* $t \in [0,\infty)$, *where* $b \in (1, 18)$.

Next, we show that the condition (2) holds with the functions ψ *and* φ. *Assume that* $\alpha(Sx, Ry) \geq 1$ *or* $\alpha(Ry, Sx) \geq 1$. *From the definition of* S *and* R, *we conclude that* $x, y \in [0,\infty)$. *Using the Mean Value Theorem simultaneously for the hyperbolic sine function and inverse hyperbolic sine functions, we have*

$$\begin{aligned}
\psi(s^3 d(fx, gy)) &= 8b|fx - gy|^2 \\
&= 8b\left|\sinh^{-1} x - \sinh^{-1} \frac{y}{2}\right|^2 \\
&\leq 8b\left|x - \frac{y}{2}\right|^2 \\
&= 8b\frac{|12x - 6y|^2}{144} \\
&\leq \frac{b}{18}|\sinh 12x - \sinh 6y|^2 \\
&\leq |Sx - Ry|^2 \\
&= d(Sx, Ry) \\
&\leq M_s(x, y) \\
&\leq \psi(M_s(x, y)) - \varphi(M_s(x, y)).
\end{aligned}$$

This implies that (2) holds. Therefore, all the conditions of Theorem 3.6 are satisfied. Then we can conclude that f, g, R *and* S *have coincidence point, that is, a point 0.*

Theorem 3.8. *Let* (X, d) *be a b-metric space with coefficient* $s \geq 1$, $\alpha : X \times X \to [0,\infty)$ *and* $f, g, R, S : X \to X$ *be five mappings such that* $f(X) \subseteq R(X)$ *and* $g(X) \subseteq S(X)$ *and* $R(X)$ *and* $S(X)$ *be b-closed subset of* X. *Suppose that, for all* $x, y \in X$, *we have*

$$\alpha(Sx, Ry) \geq 1 \ \text{ or } \ \alpha(Ry, Sx) \geq 1 \Longrightarrow \psi(s^3 d(fx, gy)) \leq \psi(M_s(x, y)) - \varphi(M_s(x, y)), \tag{41}$$

where $\psi, \varphi : [0,\infty) \to [0,\infty)$ *are altering distance functions. If the following conditions hold:*

(1) (X, d) *is* α*-complete;*

(2) X *is* α*-regular, i.e., if* $\{x_n\}$ *is sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$ *and* $x_n \to x \in X$ *as* $n \to \infty$, *then* $\alpha(x_n, x) \geq 1$ *for all* $n \in \mathbb{N}$*;*

(3) *the pairs* (f, S) *and* (g, R) *are weakly compatible;*

(4) *the pairs* (f, g) *and* (g, f) *are partially* α*-weakly increasing with respect to* R *and* S*, respectively;*

(5) α *is a transitive mapping,*

then the pair (f, S) *and* (g, R) *have a coincidence point* $z \in X$. *Moreover, if* $\alpha(Rz, Sz) \geq 1$ *or* $\alpha(Sz, Rz) \geq 1$, *then* z *is a coincidence point of* f, g, R *and* S.

Proof. First, we prove the following in Theorem 3.6, that is, there exists $z \in X$ such that

$$\lim_{n\to\infty} d(z_n, z) = 0. \tag{42}$$

Since $R(X)$ is b-closed and $\{z_{2n+1}\} \subseteq R(X)$, it follows that $z \in R(X)$. Hence there exists $u \in X$ such that $z = Ru$ and

$$\lim_{n\to\infty} d(z_{2n+1}, Ru) = \lim_{n\to\infty} d(Rx_{2n+1}, Ru) = 0. \tag{43}$$

Similarly, there exists $v \in X$ such that $z = Ru = Sv$ and

$$\lim_{n\to\infty} d(z_{2n}, Sv) = \lim_{n\to\infty} d(Sx_{2n}, Sv) = 0. \tag{44}$$

Next, we prove that v is a coincidence point of f and S. Since $Rx_{2n+1} \to z = Sv$ as $n \to \infty$, it follows from the α-regularity of X that $\alpha(Rx_{2n+1}, Sv) \geq 1$. Therefore, from (41), we have

$$\psi(s^3 d(fv, gx_{2n+1})) \leq \psi(M_s(v, x_{2n+1})) - \varphi(M_s(v, x_{2n+1})), \tag{45}$$

where

$$M_s(v, x_{2n+1}) = \max\left\{d(Sv, Rx_{2n+1}), d(Sv, fv), d(Rx_{2n+1}, gx_{2n+1}), \frac{d(Sv, gx_{2n+1}) + d(Rx_{2n+1}, fv)}{2s}\right\}.$$

Taking limit as $n \to \infty$ in (45) and using Lemma 2.7, we obtain

$$\psi(sd(fv, z)) = \psi\left(s^3 \frac{1}{s^2} d(fv, z)\right) \leq \psi(d(fv, z)) - \varphi(d(fv, z)),$$

which implies that $fv = z = Sv$. Since f and S are weakly compatible, we have

$$fz = fSv = Sfv = Sz.$$

Thus z is a coincidence point of f and S. Similarly, it can be shown that z is a coincidence point of pair (g, R). The rest of the proof follows from similar arguments as in Theorem 3.6. This completes the proof. □

Corollary 3.9. *Let (X, d) be a b-metric space with coefficient $s \geq 1$, $\alpha : X \times X \to [0, \infty)$ and $f, g : X \to X$ be three mappings. Suppose that, for all $x, y \in X$, we have*

$$\alpha(x, y) \geq 1 \;\; or \;\; \alpha(y, x) \geq 1 \Longrightarrow \psi(s^3 d(fx, gy)) \leq \psi(N_s(x, y)) - \varphi(N_s(x, y)), \tag{46}$$

where

$$N_s(x, y) = \max\left\{d(x, y), d(x, fx), d(y, gy), \frac{d(x, gy) + d(y, fx)}{2s}\right\}$$

and $\psi, \varphi : [0, \infty) \to [0, \infty)$ are altering distance functions. If the following conditions hold:

(1) *(X, d) is α-complete;*

(2) *f and g are α-continuous (or X is α-regular);*

(3) *the pairs (f, g) and (g, f) are partially α-weakly increasing;*

(4) *α is a transitive mapping,*

then the pair (f, g) have a common fixed point in X.

Proof. If R and S are the identity mappings in Theorem 3.6 and Theorem 3.8, then we have this result. □

4 Fixed point results in b-metric spaces endowed with a binary relation

In this section, we give some fixed point results in b-metric spaces endowed with a binary relation, which can be regarded as consequences of the results presented in the previous section.

The following notions and definitions are needed.

Definition 4.1. *Let (X, d) be a b-metric space endowed with a binary relation $\mathcal{R}$ in X. The b-metric space X is said to be $\mathcal{R}$-complete if every Cauchy sequence $\{x_n\}$ in X with $x_n \mathcal{R} x_{n+1}$ for all $n \in \mathbb{N}$ converges in X.*

Definition 4.2. *Let (X, d) be a b-metric space endowed with a binary relation $\mathcal{R}$ in X and $f : X \to X$ be a mapping. We say that f is $\mathcal{R}$-continuous if, for any sequence $\{x_n\}$ in X with $x_n \to x \in X$ as $n \to \infty$ and $x_n \mathcal{R} x_{n+1}$ for all $n \in \mathbb{N}$, we have*

$$\lim_{n\to\infty} fx_n = f\left(\lim_{n\to\infty} x_n\right).$$

Definition 4.3. *Let (X, d) be a b-metric space endowed with a binary relation $\mathcal{R}$ in X and $f, g : X \to X$ be two mappings. The pair (f, g) is said to be $\mathcal{R}$-compatible if*

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$x_n \mathcal{R} x_{n+1}$$

for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$.

Definition 4.4. *Let X be a nonempty set, $\mathcal{R}$ be a binary relation in X and $f, g, h : X \to X$ be three mappings such that $f(X) \subseteq h(X)$ and $g(X) \subseteq h(X)$. The ordered pair (f, g) is said to be:*
(1) *$\mathcal{R}$-weakly increasing with respect to h if, for each $x \in X$, we have $(fx)\mathcal{R}(gy)$ for all $y \in h^{-1}(fx)$ and $(gx)\mathcal{R}(fy)$ for all $y \in h^{-1}(gx)$;*
(2) *partially $\mathcal{R}$-weakly increasing with respect to h if $(fx)\mathcal{R}(gy)$ for all $y \in h^{-1}(fx)$.*

Definition 4.5. *Let X be a nonempty set. A relation $\mathcal{R}$ on X is said to be transitive if, for $x, y, z \in X$,*

$$x\mathcal{R}y \quad \text{and} \quad y\mathcal{R}z \implies x\mathcal{R}z.$$

Theorem 4.6. *Let (X, d) be a b-metric space with coefficient $s \geq 1$ endowed with a binary relation $\mathcal{R}$ in X and $f, g, R, S : X \to X$ be four mappings such that $f(X) \subseteq R(X)$ and $g(X) \subseteq S(X)$. Suppose that, for all $x, y \in X$, we have*

$$(Sx)\mathcal{R}(Ry) \geq 1 \ \text{ or } \ (Ry)\mathcal{R}(Sx) \geq 1 \implies \psi(s^3 d(fx, gy)) \leq \psi(M_s(x, y)) - \varphi(M_s(x, y)), \tag{47}$$

where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are altering distance functions. If the following conditions hold:
(1) *(X, d) is $\mathcal{R}$-complete;*
(2) *f, g, R and S are $\mathcal{R}$-continuous;*
(3) *the pairs (f, S) and (g, R) are $\mathcal{R}$-compatible;*
(4) *the pairs (f, g) and (g, f) are partially $\mathcal{R}$-weakly increasing with respect to R and S, respectively;*
(5) *$\mathcal{R}$ is a transitive relation,*

then the pair (f, S) and (g, R) have a coincidence point $z \in X$. Moreover, if $(Rz)\mathcal{R}(Sz)$ or $(Sz)\mathcal{R}(Rz)$, then z is a coincidence point of f, g, R and S.

Proof. Consider a mapping $\alpha : X \times X \to [0, \infty)$ defined by

$$\alpha(x, y) = \begin{cases} 1, & x\mathcal{R}y; \\ 0, & \text{otherwise}. \end{cases} \tag{48}$$

It is easy to see that all the conditions in Theorem 3.6 hold and so the conclusion of this theorem follows from Theorem 3.6. □

Theorem 4.7. *Let (X, d) be a b-metric space with coefficient $s \geq 1$ endowed with a binary relation $\mathcal{R}$ in X and $f, g, R, S : X \to X$ be four mappings such that $f(X) \subseteq R(X)$, $g(X) \subseteq S(X)$ and $R(X)$, $S(X)$ be b-closed subset of X. Suppose that, for all $x, y \in X$, we have*

$$(Sx)\mathcal{R}(Ry) \geq 1 \ \text{ or } \ (Ry)\mathcal{R}(Sx) \geq 1 \implies \psi(s^3 d(fx, gy)) \leq \psi(M_s(x, y)) - \varphi(M_s(x, y)), \tag{49}$$

where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are altering distance functions. If the following conditions hold:
(1) *(X, d) is $\mathcal{R}$-complete;*
(2) *X is $\mathcal{R}$-regular, i.e., if $\{x_n\}$ is a sequence in X such that $x_n\mathcal{R}x_{n+1}$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then $x_n\mathcal{R}x$ for all $n \in \mathbb{N}$;*

(3) *the pairs* (f, S) *and* (g, R) *are weakly compatible;*
(4) *the pairs* (f, g) *and* (g, f) *are partially* $\mathcal{R}$*-weakly increasing with respect to* R *and* S*, respectively;*
(5) $\mathcal{R}$ *is a transitive relation,*
then the pair (f, S) *and* (g, R) *have a coincidence point* $z \in X$. *Moreover, if* $(Rz)\mathcal{R}(Sz)$ *or* $(Sz)\mathcal{R}(Rz)$*, then* z *is a coincidence point of* f, g, R *and* S.

Proof. The results follow from Theorem 3.8 by considering a mapping $\alpha : X \times X \to [0, \infty)$ given by (48). □

Remark 4.8. *Since* $\preceq$ *is a transitive relation in partially ordered b-metric spaces* $(X, \preceq, d)$*, Theorems 4.6 and 4.7 are a generalization and an improvement of Theorems 2.1 and 2.2 of Roshan et al. [10].*

5 Existence of a common solution for a system of integral equations

Consider the following system of nonlinear integral equations:

$$\begin{cases} x(t) = \int_a^b K_1(t, r, x(r))dr, \\ x(t) = \int_a^b K_2(t, r, x(r))dr, \end{cases} \tag{50}$$

where $a, b \in \mathbb{R}$ with $a < b$, $x \in C[a, b]$ (the set of continuous real value functions defined on $[a, b] \subseteq \mathbb{R}$) and $K_1, K_2 : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are mappings. The purpose of this section is to present the existence theorem for a solution of the system (50) by using the result in previous section. We furnish an illustrative example to demonstrate the validity of the hypotheses and degree of utility of our results.

Now, we prove the following result:

Theorem 5.1. *Consider the system of integral equations* (50). *Suppose that the following conditions hold:*
(1) $K_1, K_2 : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ *are continuous;*
(2) *for all* $x \in X$ *and* $t, r \in [a, b]$*, we have*

$$K_1(t, r, x(r)) \leq K_2\Big(t, r, \int_a^b K_1(r, w, x(w))dw\Big)$$

and

$$K_2(t, r, x(r)) \leq K_1\Big(t, r, \int_a^b K_2(r, w, x(w))dw\Big);$$

(3) *there exists* $p > 1$ *such that for all* $x, y \in X$ *and* $r, t \in [a, b]$ *with* $x(v) \leq y(v)$ *for all* $v \in [a, b]$*, we have*

$$|K_1(t, r, x(r)) - K_2(t, r, y(r))| \leq \xi(t, r)(\Omega(|x(r) - y(r)|^p)),$$

where $\xi : [a, b] \times [a, b] \to [0, \infty)$ *is a continuous function satisfying*

$$\sup_{t\in[a,b]} \Big(\int_a^b \xi(t, r)^p dr\Big) < \frac{1}{2^{3p^2-3p}(b-a)^{p-1}}$$

and $\Omega : [0, \infty) \to [0, \infty)$ *is continuous nondecreasing and satisfying the following conditions:*
(Ω_1) $\Omega(h) = 0$ *if and only if* $h = 0$*;*
(Ω_2) $\Omega(h) < h$ *for all* $h > 0$.
Then the system of nonlinear integral equations (50) *has a common solution.*

Proof. Let $X = C[a,b]$ and let $f, g : X \to X$ be two mappings which are defined by

$$(fx)(t) = \int_a^b K_1(t,r,x(r))dr, \quad (gx)(t) = \int_a^b K_2(t,r,x(r))dr$$

for all $x \in X$ and $t \in [a,b]$. Then the existence of a solution of (50) is equivalent to the existence of a common fixed point of f and g. Clearly, X with the b-metric given by

$$d(u,v) = \sup_{t \in [a,b]} |u(t) - v(t)|^p$$

for all $u, v \in X$ is a complete b-metric space. Define a mapping $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, \; x(t) \le y(t) \text{ for all } t \in [a,b]; \\ 0, \text{ otherwise.} \end{cases}$$

It is easy to observe that X is α-regular. From the condition (2), the ordered pairs (f,g) and (g,f) are partially α-weakly increasing.

Next, we will show that the condition (47) holds. To prove this fact, we first choose $q \in \mathbb{R}$ such that $q := 1 - \frac{1}{p} > 1$. Now, let $x, y \in X$ be such that $\alpha(x,y) \ge 1$. From condition (3) and the Hölder inequality, for each $t \in [a,b]$ we have

$$\begin{aligned}
&(2^{3p-3}|(fx)(t) - (gy)(t)|)^p \\
&\le 2^{3p^2-3p} \left(\int_a^b |K_1(t,r,x(r)) - K_2(t,r,y(r))|dr \right)^p \\
&\le 2^{3p^2-3p} \left[\left(\int_a^b 1^q dr \right)^{\frac{1}{q}} \left(\int_a^b |K_1(t,r,x(r)) - K_2(t,r,y(r))|^p dr \right)^{\frac{1}{p}} \right]^p \\
&\le 2^{3p^2-3p}(b-a)^{\frac{p}{q}} \left(\int_a^b \xi(t,r)^p (\Omega(|x(r) - y(r)|^p))^p dr \right) \\
&\le 2^{3p^2-3p}(b-a)^{\frac{p}{q}} \left(\int_a^b \xi(t,r)^p (\Omega(d(x,y)))^p dr \right) \\
&\le 2^{3p^2-3p}(b-a)^{\frac{p}{q}} \left(\int_a^b \xi(t,r)^p (\Omega(M_s(x,y)))^p dr \right) \\
&= 2^{3p^2-3p}(b-a)^{p-1} \left(\int_a^b \xi(t,r)^p dr \right) (\Omega(M_s(x,y)))^p \\
&< (\Omega(M_s(x,y)))^p \\
&= M_s(x,y)^p - (M_s(x,y)^p - (\Omega(M_s(x,y)))^p).
\end{aligned}$$

In case of $\alpha(y,x) \ge 1$, we also get

$$(2^{2p-2}|(fx)(t) - (gy)(t)|)^p \le M_s(x,y)^p - (M_s(x,y)^p - (\Omega(M_s(x,y)))^p).$$

Hence, we have

$$(s^3 d(fx,gy))^p = \left(s^3 \sup_{t \in [a,b]} |(fx)(t) - (gy)(t)| \right)^p \le M_s(x,y)^p - [M_s(x,y)^p - (\Omega(M_s(x,y)))^p]$$

for all $x, y \in X$, where $\psi, \varphi : [0,\infty) \to [0,\infty)$ are the functions defined by $\psi(t) = t^p$ and $\varphi(t) = t^p - (\Omega(t))^p$ for all $t \in [0,\infty)$. Thus, by the conditions (Ω_1) and (Ω_2), we can prove that φ is the altering distance function.

Also, it is easy to see that ψ is the altering distance. Therefore, the condition (47) holds. By using Corollary 3.9, there exists $\hat{x} \in X$ which is a common fixed point of f and g, that is, $\hat{x}$ is a common solution for the system (50). This completes the proof. □

Finally, we provide example to illustrate the Theorem 5.1 presented herein.

Example 5.2. *The system of nonlinear integral equations*

$$\begin{cases} x(t) = \int_1^5 K_1(t, r, x(r))\, dr, \\ x(t) = \int_1^5 K_2(t, r, x(r))\, dr, \end{cases} \tag{51}$$

where $x \in C[1, 5]$ *and* $K_1, K_2 : [1, 5] \times [1, 5] \times \mathbb{R} \to \mathbb{R}$ *are defined by*

$$K_1(u, v, w) = K_2(u, v, w) = e^u + v$$

for all $u, v \in [1, 5]$ *and* $w \in \mathbb{R}$, *has a common solution.*

Proof. It is easy to see that the system of nonlinear integral equations (51) becomes the system of nonlinear integral equations (50) with $a = 1$ and $b = 5$.

Now, we will show that all conditions in Theorem 5.1 are satisfied. It is easy to see that K_1, K_2 are continuous. Next, we claim that the condition (2) in Theorem 5.1 holds. Assume that $x \in C[1, 5]$ and $t, r \in [1, 5]$. Then we have

$$K_1(t, r, x(r)) = e^t + r = K_2\Big(t, r, \int_1^5 K_1(r, w, x(w))dw\Big)$$

and

$$K_2(t, r, x(r)) = e^t + r = K_1\Big(t, r, \int_1^5 K_2(r, w, x(w))dw\Big).$$

Hence, the condition (2) in Theorem 5.1 is proved. Also, we can easily prove that the condition (3) in Theorem 5.1 is true. Therefore, all the conditions of Theorem 5.1 hold and then system of nonlinear integral equations (51) has a common solution. This completes the proof. □

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