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Value distribution of meromorphic solutions of homogeneous and non-homogeneous complex linear differential-difference equations

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Abstract: In this paper, we investigate the value distribution of meromorphic solutions of homogeneous and non-homogeneous complex linear differential-difference equations, and obtain the results on the relations between the order of the solutions and the convergence exponents of the zeros, poles, a -points and small function value points of the solutions, which show the relations in the case of non-homogeneous equations are sharper than the ones in the case of homogeneous equations.

Keywords: Complex linear differential-difference equation, Meromorphic solution, Small function, Order, Convergence exponent

MSC: 30D35, 39B32, 39A10

1 Introduction and main results

In this paper, we use the standard notations and basic results of Nevanlinna's value distribution theory (see [1–3]). In addition, we use the notation $\sigma(f)$ to denote the order of a meromorphic function $f(z)$ in the whole complex plane, and the notations $\lambda(f)$ and $\lambda(\frac{1}{f})$ to denote the convergence exponent of the zeros and the poles of $f(z)$ respectively.

Many scholars applied Nevanlinna theory and its difference analogues to study the properties of meromorphic solutions of complex differential equations and complex difference equations, and obtained fruitful achievement (see [4–8]). Especially, it is an essential respect to study the oscillation property of meromorphic solutions of complex differential equations and complex difference equations.

For complex linear differential equations, Chen, et al. (see [9]) considered the non-homogeneous equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = F(z), \quad (1)$$

where $A_j(z) (j = 0, 1, \dots, k-1)$ and $F(z) (\neq 0)$ are polynomials and obtained that every solution $f(z)$ of (1) satisfies

$$\lambda(f) = \sigma(f). \quad (2)$$

Meanwhile, they pointed out that every solution of the homogeneous linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = 0 \quad (3)$$

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may not satisfy (2). The above facts illustrate that the result on the relation between $\lambda(f)$ and $\sigma(f)$ in the case of the non-homogeneous equation is better than the one in the case of the homogeneous equation.

We note that for homogeneous and non-homogeneous complex linear difference equations, as complex linear differential equations, there is also such important character. Especially, Chen (see [10]) studied the relations between the order and the convergence exponent of entire solutions of homogeneous and non-homogeneous complex linear difference equations, and obtained the following results.

Theorem A ([10]). *Let $h_j(z)(j = 1, 2, \dots, n)$ be polynomials such that $h_n(z) \not\equiv 0, c_i(i = 1, 2, \dots, n)$ be constants which are unequal to each other. Suppose that $f(z)$ is a finite-order transcendental entire solution of the homogeneous linear difference equation*

$$h_n(z)f(z + c_n) + h_{n-1}(z)f(z + c_{n-1}) + \dots + h_1(z)f(z + c_1) = 0. \tag{4}$$

Then:

- (i) $\lambda(f) \geq \sigma(f) - 1$;
- (ii) if $h_n + \dots + h_1 \not\equiv 0$, then $f(z)$ assumes every non-zero finite value a infinitely often, and $\lambda(f - a) = \sigma(f)$.

Theorem B ([10]). *Let $h_j(z)(j = 1, 2, \dots, n)$ and $F(z)$ be polynomials such that $F(z)h_n(z) \not\equiv 0, c_i(i = 1, 2, \dots, n)$ be constants which are unequal to each other. Suppose that $f(z)$ is a finite-order transcendental entire solution of the non-homogeneous linear difference equation*

$$h_n(z)f(z + c_n) + h_{n-1}(z)f(z + c_{n-1}) + \dots + h_1(z)f(z + c_1) = F(z). \tag{5}$$

Then:

- (i) $\lambda(f) = \sigma(f)$;
- (ii) if $\deg F(z) > \max\{\deg h_j(z) : 1 \leq j \leq n\}$, then $f(z)$ assumes every non-zero finite value a infinitely often, and $\lambda(f - a) = \sigma(f)$.

We note that in Theorems A(i) and B(i), the result on the relation between $\lambda(f)$ and $\sigma(f)$ in the case of the non-homogeneous equation is better than the one in the case of the homogeneous equation. Meantime, we note that the value distribution properties of the zeros and a -points of the solutions are distinct in Theorem A(i) and (ii).

In this paper, we continue to consider the oscillation property of meromorphic solutions of complex linear differential equations and complex linear difference equations. Firstly, we consider more general equations than (1) and (3)–(5), namely homogeneous and non-homogeneous complex linear differential-difference equations. Secondly, since the above conclusions are just on entire solutions, we consider meromorphic solutions of the involved equations. Meanwhile, we also consider the value distribution of the poles of meromorphic solutions, and generalize the relative results into the case in which meromorphic solutions assume a small function.

When the coefficients of the complex linear differential-difference equation are zero-order meromorphic functions (including the polynomials and rational functions), we obtain the following Theorem 1.1, Corollary 1.2 and Remark 1.3.

Theorem 1.1. *Let $A_{ij}(z)(i = 0, 1, \dots, n, j = 0, 1, \dots, m)$ and $F(z)$ be zero-order meromorphic functions such that $A_{nm}(z) \not\equiv 0, c_i(i = 0, 1, \dots, n)$ be complex constants which are unequal to each other. Suppose that $f(z)$ is a finite-order meromorphic solution of the complex linear differential-difference equation*

$$\sum_{i=0}^n \sum_{j=0}^m A_{ij}(z)f^{(j)}(z + c_i) = F(z), \tag{6}$$

$\varphi(z)$ is a meromorphic function with $\sigma(\varphi) < \sigma(f)$ such that $\varphi(z)$ is not a solution of (6). Then $\max\{\lambda(\frac{1}{f-\varphi}), \lambda(f - \varphi)\} = \sigma(f)$.

Corollary 1.2. *Let $A_{ij}(z), F(z), c_i$ satisfy the conditions in Theorem 1.1, and suppose that $f(z)$ is a finite-order meromorphic solution of (6). If $a \in \mathbb{C}$ satisfies $F(z) - \sum_{i=0}^n A_{i0}(z)a \not\equiv 0$, then $\max\{\lambda(\frac{1}{f-a}), \lambda(f - a)\} = \sigma(f)$; in particular, if $F(z) \not\equiv 0$, then $\max\{\lambda(\frac{1}{f}), \lambda(f)\} = \sigma(f)$.*

Remark 1.3. From the proof procedure of Theorem 1.1, we see that if the assumptions on the order of meromorphic coefficients $A_{ij}(z)(i = 0, 1, \dots, n, j = 0, 1, \dots, m)$ and $F(z)$ in Theorem 1.1 and Corollary 1.2 are weakened from “zero-order” to “ $\max\{\sigma(A_{ij}), i = 0, 1, \dots, n, j = 0, 1, \dots, m, \sigma(F)\} < \sigma(f)$ ”, the corresponding results still hold.

From Theorem 1.1 and Corollary 1.2, we see that the conclusion “ $\max\{\lambda(\frac{1}{f}), \lambda(f)\} = \sigma(f)$ ” may not hold when $F(z) \equiv 0$, that is, (6) reduces into the corresponding homogeneous equation. Therefore, we consider the corresponding homogeneous equation further and obtain the following Theorem 1.4 and Remark 1.5.

Theorem 1.4. Let $A_j(z)(j = 0, 1, \dots, n), B_j(z)(j = 1, 2, \dots, m), C_j(z)(j = 1, 2, \dots, l)$ be zero-order meromorphic functions such that $A_n(z)B_m(z)C_l(z) \neq 0, b_j(\neq 0)(j = 1, 2, \dots, m), c_j(\neq 0)(j = 1, 2, \dots, l)$ be complex constants which are unequal to each other. Suppose that $f(z)$ is a finite-order meromorphic solution of the homogeneous linear differential-difference equation

$$\sum_{j=0}^n A_j(z)f^{(j)}(z) + \sum_{j=1}^m B_j(z)f^{(j)}(z + b_j) + \sum_{j=1}^l C_j(z)f(z + c_j) = 0. \tag{7}$$

Then:

- (i) $\max\{\lambda(\frac{1}{f}), \lambda(f)\} \geq \sigma(f) - 1$;
- (ii) $\max\{\lambda(\frac{1}{f-a}), \lambda(f - a)\} \geq \sigma(f) - 1$, where $a \in \mathbb{C} \setminus \{0\}$; further, if $A_0(z) + \sum_{j=1}^l C_j(z) \neq 0$, then $\max\{\lambda(\frac{1}{f-a}), \lambda(f - a)\} = \sigma(f)$;
- (iii) if $\varphi(z)$ is a meromorphic function with $\sigma(\varphi) < \sigma(f) - 1$, then $\max\{\lambda(\frac{1}{f-\varphi}), \lambda(f - \varphi)\} \geq \sigma(f) - 1$; if $\varphi(z)$ is a meromorphic function with $\sigma(\varphi) < \sigma(f)$ such that $\varphi(z)$ is not a solution of (7), then $\max\{\lambda(\frac{1}{f-\varphi}), \lambda(f - \varphi)\} = \sigma(f)$.

Remark 1.5. From the proof procedure of Theorem 1.4, we see that if the assumptions on the order of meromorphic coefficients $A_j(z)(j = 0, 1, \dots, n), B_j(z)(j = 1, 2, \dots, m), C_j(z)(j = 1, 2, \dots, l)$ in Theorem 1.4 are weakened from “zero order” to “ $\max\{\sigma(A_j)(j = 0, 1, \dots, n), \sigma(B_j)(j = 1, 2, \dots, m), \sigma(C_j)(j = 1, 2, \dots, l)\} < \sigma(f) - 1$ ”, the corresponding results still hold.

2 Lemmas for proofs of main results

Lemma 2.1 ([3]). Let $n \geq 2, f_i(z)(i = 1, 2, \dots, n)$ be meromorphic functions, $g_i(z)(i = 1, 2, \dots, n)$ be entire functions, and satisfy:

- (i) $\sum_{i=1}^n f_i(z)e^{g_i(z)} \equiv 0$;
- (ii) $g_i(z) - g_s(z)(1 \leq i < s \leq n)$ are not constants;
- (iii) $T(r, f_i) = o\{T(r, e^{g_h - g_s})\}(1 \leq i \leq n, 1 \leq h < s \leq n)$, where $r \notin E, E \subset [1, +\infty)$ is of finite linear measure or finite logarithmic measure. Then $f_i(z) \equiv 0, i = 1, 2, \dots, n$.

Remark 2.2. It is shown in [11] that for any arbitrary complex number $c \neq 0$, the following inequalities

$$(1 + o(1))T(r - |c|, f) \leq T(r, f(z + c)) \leq (1 + o(1))T(r + |c|, f)$$

hold as $r \rightarrow \infty$ for a general meromorphic function $f(z)$. Therefore, it is easy to obtain that $\sigma(f(z + c)) = \sigma(f), \mu(f(z + c)) = \mu(f)$.

3 Proofs of Theorems 1.1 and 1.4

Proof of Theorem 1.1. Suppose that $f(z)$ is a finite-order meromorphic solution of (6) and suppose that $\max\{\lambda(\frac{1}{f-\varphi}), \lambda(f-\varphi)\} < \sigma(f)$ on the contrary. Since $\sigma(\varphi) < \sigma(f)$, then $\max\{\lambda(\frac{1}{f-\varphi}), \lambda(f-\varphi)\} < \sigma(f) = \sigma(f-\varphi)$. Thus, $f(z)$ can be written by Hadamard decomposition theorem as

$$f(z) = \frac{H(z)}{Q(z)}e^{g(z)} + \varphi(z), \tag{8}$$

where $H(z)(\neq 0)$ is the canonical product (or polynomial) formed by zeros of $f(z) - \varphi(z)$, $Q(z)(\neq 0)$ is the canonical product (or polynomial) formed by poles of $f(z) - \varphi(z)$,

$$g(z) = d_k z^k + d_{k-1} z^{k-1} + \dots + d_0, \quad d_k (\neq 0), d_{k-1}, \dots, d_0 \in \mathbf{C} \tag{9}$$

is a polynomial with degree $k(\geq 1)$, and they satisfy

$$\lambda(H) = \sigma(H) = \lambda(f - \varphi) < \sigma(f) = k, \quad \lambda(Q) = \sigma(Q) = \lambda(\frac{1}{f - \varphi}) < \sigma(f) = k. \tag{10}$$

Obviously, $f^{(j)}(z)(j = 0, 1, \dots, m)$ are of the following forms

$$f^{(j)}(z) = (\frac{H(z)}{Q(z)}e^{g(z)} + \varphi(z))^{(j)} = \psi_j(z)e^{g(z)} + \varphi^{(j)}(z), \quad j = 0, 1, \dots, m, \tag{11}$$

where $\psi_0(z) = \frac{H(z)}{Q(z)}$, $\psi_j(z)(j = 1, 2, \dots, m)$ are rational expressions formed by $H(z)$, $Q(z)$, $g(z)$ and their derivatives, and we have by (10) that

$$\sigma(\psi_j) < \sigma(f) = k, \quad \sigma(\varphi^{(j)}) < \sigma(f) = k, \quad j = 0, 1, \dots, m. \tag{12}$$

By substituting (8) and (11) into (6), we have

$$\begin{aligned} & \sum_{i=0}^n A_{i0}(z)\psi_0(z+c_i)e^{g(z+c_i)} + \sum_{i=0}^n A_{i1}(z)\psi_1(z+c_i)e^{g(z+c_i)} + \dots \\ & + \sum_{i=0}^n A_{im}(z)\psi_m(z+c_i)e^{g(z+c_i)} = F(z) - \sum_{i=0}^n \sum_{j=0}^m A_{ij}(z)\varphi^{(j)}(z+c_i). \end{aligned} \tag{13}$$

By multiplying $\exp\{-d_k z^k\}$ on both sides of (13), we have

$$\begin{aligned} & \sum_{i=0}^n A_{i0}(z)\psi_0(z+c_i) \exp\{(d_{k-1} + kd_k c_i)z^{k-1} + \dots\} \\ & + \sum_{i=0}^n A_{i1}(z)\psi_1(z+c_i) \exp\{(d_{k-1} + kd_k c_i)z^{k-1} + \dots\} + \dots \\ & + \sum_{i=0}^n A_{im}(z)\psi_m(z+c_i) \exp\{(d_{k-1} + kd_k c_i)z^{k-1} + \dots\} \\ & = (F(z) - \sum_{i=0}^n \sum_{j=0}^m A_{ij}(z)\varphi^{(j)}(z+c_i)) \exp\{-d_k z^k\}. \end{aligned} \tag{14}$$

Since $\varphi(z)$ is not a solution of (6), that is, $F(z) - \sum_{i=0}^n \sum_{j=0}^m A_{ij}(z)\varphi^{(j)}(z+c_i) \neq 0$, we have by (12) and Remark 2.2 that

$$\sigma((F(z) - \sum_{i=0}^n \sum_{j=0}^m A_{ij}(z)\varphi^{(j)}(z+c_i)) \exp\{-d_k z^k\}) = k.$$

By (12) and Remark 2.2 again, we have $\sigma(\psi_j(z+c_i)) < k(i = 0, 1, \dots, n, j = 0, 1, \dots, m)$. By comparing the order of both sides of (14), we deduce a contradiction that the order of the left side is less than k and the order of the right side is equal to k . Hence, $\max\{\lambda(\frac{1}{f-\varphi}), \lambda(f-\varphi)\} = \sigma(f)$.

Therefore, the proof of Theorem 1.1 is complete. □

Proof of Theorem 1.4. Suppose that $f(z)$ is a finite-order meromorphic solution of (7).

(i) If $\sigma(f) \leq 1$, then the result holds obviously. Now, we suppose that $\sigma(f) > 1$ without loss of generality and suppose that $\max\{\lambda(\frac{1}{f}), \lambda(f)\} < \sigma(f) - 1$ on the contrary. Thus, $f(z)$ can be written by Hadamard decomposition theorem as

$$f(z) = \frac{H(z)}{Q(z)}e^{g(z)}, \tag{15}$$

where $H(z)(\neq 0)$ is the canonical product (or polynomial) formed by zeros of $f(z)$, $Q(z)(\neq 0)$ is the canonical product (or polynomial) formed by poles of $f(z)$, $g(z)$ is a polynomial with degree k defined by (9), and they satisfy

$$\begin{aligned} \lambda(H) &= \sigma(H) = \lambda(f) < \sigma(f) - 1 = k - 1, \\ \lambda(Q) &= \sigma(Q) = \lambda(\frac{1}{f}) < \sigma(f) - 1 = k - 1. \end{aligned} \tag{16}$$

By $\max\{\lambda(\frac{1}{f}), \lambda(f)\} < \sigma(f) - 1 = k - 1$, we see that $k \geq 2$.

Obviously, $f^{(j)}(z)(j = 0, 1, \dots, \max\{n, m\})$ are of the following forms

$$f^{(j)}(z) = (\frac{H(z)}{Q(z)}e^{g(z)})^{(j)} = \psi_j(z)e^{g(z)}, \quad j = 0, 1, \dots, \max\{n, m\}, \tag{17}$$

where $\psi_0(z) = \frac{H(z)}{Q(z)}$, $\psi_j(z)(j = 1, 2, \dots, \max\{n, m\})$ are rational expressions formed by $H(z)$, $Q(z)$, $g(z)$ and their derivatives, and we have by (16) that

$$\sigma(\psi_j) < \sigma(f) - 1 = k - 1, \quad j = 0, 1, \dots, \max\{n, m\}. \tag{18}$$

By substituting (15) and (17) into (7), we have

$$\sum_{j=0}^n A_j(z)\psi_j(z)e^{g(z)} + \sum_{j=1}^m B_j(z)\psi_j(z + b_j)e^{g(z+b_j)} + \sum_{j=1}^l C_j(z)\psi_0(z + c_j)e^{g(z+c_j)} = 0. \tag{19}$$

Since $k \geq 2$ and $b_j(\neq 0)(j = 1, 2, \dots, m), c_j(\neq 0)(j = 1, 2, \dots, l)$ are unequal to each other, we have by (9) that

$$\begin{cases} g(z + b_i) - g(z + b_s) = kd_k(b_i - b_s)z^{k-1} + \dots, & 1 \leq i < s \leq m, \\ g(z + c_i) - g(z + c_s) = kd_k(c_i - c_s)z^{k-1} + \dots, & 1 \leq i < s \leq l, \\ g(z + b_i) - g(z + c_s) = kd_k(b_i - c_s)z^{k-1} + \dots, & 1 \leq i \leq m, 1 \leq s \leq l, \\ g(z + b_i) - g(z) = kd_k b_i z^{k-1} + \dots, & 1 \leq i \leq m, \\ g(z + c_s) - g(z) = kd_k c_s z^{k-1} + \dots, & 1 \leq s \leq l \end{cases} \tag{20}$$

are all polynomials with degree $k - 1(\geq 1)$. By (18) and Remark 2.2, we have $\sigma(\psi_j(z + b_j)) < k - 1(j = 1, 2, \dots, m)$ and $\sigma(\psi_0(z + c_j)) < k - 1(j = 1, 2, \dots, l)$. Thus, by (18) and (20), we have

$$\begin{cases} T(r, \sum_{j=0}^n A_j(z)\psi_j(z)) = o\{T(r, \exp\{g(z + b_i) - g(z + b_s)\})\}, & 1 \leq i < s \leq m, \\ T(r, \sum_{j=0}^n A_j(z)\psi_j(z)) = o\{T(r, \exp\{g(z + c_i) - g(z + c_s)\})\}, & 1 \leq i < s \leq l, \\ T(r, \sum_{j=0}^n A_j(z)\psi_j(z)) = o\{T(r, \exp\{g(z + b_i) - g(z + c_s)\})\}, & 1 \leq i \leq m, 1 \leq s \leq l, \\ T(r, \sum_{j=0}^n A_j(z)\psi_j(z)) = o\{T(r, \exp\{g(z + b_i) - g(z)\})\}, & 1 \leq i \leq m, \\ T(r, \sum_{j=0}^n A_j(z)\psi_j(z)) = o\{T(r, \exp\{g(z + c_s) - g(z)\})\}, & 1 \leq s \leq l, \end{cases} \tag{21}$$

and for $j = 1, 2, \dots, m$, we have

$$\begin{cases} T(r, B_j(z)\psi_j(z + b_j)) = o\{T(r, \exp\{g(z + b_i) - g(z + b_s)\})\}, & 1 \leq i < s \leq m, \\ T(r, B_j(z)\psi_j(z + b_j)) = o\{T(r, \exp\{g(z + c_i) - g(z + c_s)\})\}, & 1 \leq i < s \leq l, \\ T(r, B_j(z)\psi_j(z + b_j)) = o\{T(r, \exp\{g(z + b_i) - g(z + c_s)\})\}, & 1 \leq i \leq m, 1 \leq s \leq l, \\ T(r, B_j(z)\psi_j(z + b_j)) = o\{T(r, \exp\{g(z + b_i) - g(z)\})\}, & 1 \leq i \leq m, \\ T(r, B_j(z)\psi_j(z + b_j)) = o\{T(r, \exp\{g(z + c_s) - g(z)\})\}, & 1 \leq s \leq l, \end{cases} \tag{22}$$

for $j = 1, 2, \dots, l$, we have

$$\begin{cases} T(r, C_j(z)\psi_0(z + c_j)) = o\{T(r, \exp\{g(z + b_i) - g(z + b_s)\})\}, & 1 \leq i < s \leq m, \\ T(r, C_j(z)\psi_0(z + c_j)) = o\{T(r, \exp\{g(z + c_i) - g(z + c_s)\})\}, & 1 \leq i < s \leq l, \\ T(r, C_j(z)\psi_0(z + c_j)) = o\{T(r, \exp\{g(z + b_i) - g(z + c_s)\})\}, & 1 \leq i \leq m, 1 \leq s \leq l, \\ T(r, C_j(z)\psi_0(z + c_j)) = o\{T(r, \exp\{g(z + b_i) - g(z)\})\}, & 1 \leq i \leq m, \\ T(r, C_j(z)\psi_0(z + c_j)) = o\{T(r, \exp\{g(z + c_s) - g(z)\})\}, & 1 \leq s \leq l. \end{cases} \tag{23}$$

By combining (20)–(23), we see that Lemma 2.1 holds for (19), that is,

$$\begin{cases} \sum_{j=0}^n A_j(z)\varphi_j(z) \equiv 0, \\ B_j(z)\varphi_j(z + b_j) \equiv 0, \quad j = 1, 2, \dots, m, \\ C_j(z)\psi_0(z + c_j) \equiv 0, \quad j = 1, 2, \dots, l. \end{cases} \tag{24}$$

Since $B_m(z)C_l(z) \neq 0$, (24) can not hold. Hence, $\max\{\lambda(\frac{1}{f}), \lambda(f)\} \geq \sigma(f) - 1$.

(ii) Here, we use the similar reasoning method as the one of (i). Suppose that $\sigma(f) > 1$ without loss of generality and suppose that $\max\{\lambda(\frac{1}{f-a}), \lambda(f - a)\} < \sigma(f) - 1$ on the contrary. Thus, $f(z)$ can be written by Hadamard decomposition theorem as

$$f(z) = \frac{H(z)}{Q(z)}e^{g(z)} + a, \tag{25}$$

where $H(z) (\neq 0)$ is the canonical product (or polynomial) formed by zeros of $f(z) - a$, $Q(z) (\neq 0)$ is the canonical product (or polynomial) formed by poles of $f(z) - a$, $g(z)$ is a polynomial with degree $k (\geq 2)$ defined by (9), and they satisfy

$$\begin{aligned} \lambda(H) = \sigma(H) = \lambda(f - a) < \sigma(f) - 1 = k - 1, \\ \lambda(Q) = \sigma(Q) = \lambda(\frac{1}{f-a}) < \sigma(f) - 1 = k - 1. \end{aligned} \tag{26}$$

Obviously, $f^{(j)}(z) (j = 0, 1, \dots, \max\{n, m\})$ are of the following forms

$$f^{(j)}(z) = (\frac{H(z)}{Q(z)}e^{g(z)} + a)^{(j)} = \psi_j(z)e^{g(z)}, \quad j = 1, 2, \dots, \max\{n, m\}, \tag{27}$$

where $\psi_j(z) (j = 1, 2, \dots, \max\{n, m\})$ are rational expressions formed by $H(z), Q(z), g(z)$ and their derivatives, and we have by (26) that

$$\sigma(\psi_j) < \sigma(f) - 1 = k - 1, \quad j = 0, 1, \dots, \max\{n, m\}, \tag{28}$$

where $\psi_0(z) = \frac{H(z)}{Q(z)}$. By substituting (25) and (27) into (7), we have

$$\begin{aligned} & \sum_{j=0}^n A_j(z)\psi_j(z)e^{g(z)} + \sum_{j=1}^m B_j(z)\psi_j(z + b_j)e^{g(z+b_j)} \\ & + \sum_{j=1}^l C_j(z)\psi_0(z + c_j)e^{g(z+c_j)} + a(A_0(z) + \sum_{j=1}^l C_j(z))e^0 = 0. \end{aligned} \tag{29}$$

By using the similar reasoning method as the one dealing with (19) in (i), we see that Lemma 2.1 holds for (29), that is, (24) and $A_0(z) + \sum_{j=1}^l C_j(z) \equiv 0$ hold, which is a contradiction. Hence, $\max\{\lambda(\frac{1}{f-a}), \lambda(f - a)\} \geq \sigma(f) - 1$.

Further, if $A_0(z) + \sum_{j=1}^l C_j(z) \neq 0$, then (7) can be rewritten as

$$\sum_{j=0}^n A_j(z)h^{(j)}(z) + \sum_{j=1}^m B_j(z)h^{(j)}(z + b_j) + \sum_{j=1}^l C_j(z)h(z + c_j) = -a(A_0(z) + \sum_{j=1}^l C_j(z)),$$

where $h(z) = f(z) - a$. By Corollary 1.2, we have $\max\{\lambda(\frac{1}{h}), \lambda(h)\} = \sigma(h)$, that is, $\max\{\lambda(\frac{1}{f-a}), \lambda(f-a)\} = \sigma(f)$.

(iii) By using the similar reasoning method as the ones of (i) and (ii), we have $\max\{\lambda(\frac{1}{f-\varphi}), \lambda(f-\varphi)\} \geq \sigma(f) - 1$ when $\sigma(\varphi) < \sigma(f) - 1$. In the meantime, we have by Theorem 1.1 that $\max\{\lambda(\frac{1}{f-\varphi}), \lambda(f-\varphi)\} = \sigma(f)$, when $\sigma(\varphi) < \sigma(f)$ and $\varphi(z)$ is not a solution of (7).

Therefore, the proof of Theorem 1.4 is complete. \square

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References

- [1] Hayman W. K., Meromorphic functions, Clarendon Press, Oxford, 1964.
- [2] Laine I., Nevanlinna theory and complex differential equations, Walter de Gruyter, Berlin, 1993.
- [3] Yang C. C., Yi H. X., Uniqueness theory of meromorphic functions, Klumer Academic Publishers Group, Dordrecht, 2003.
- [4] Ablowitz M., Halburd R. G., Herbst B., On the extension of Painlevé property to difference equations, Nonlinearity, 2000, 13(3), 889-905.
- [5] Chen W. J., Xu J. F., Growth of meromorphic solutions of higher-order linear differential equations, Electron. J. Qual. Theory Differ. Equ., 2009, 1, 1-13.
- [6] Chen Z. X., Growth and zeros of meromorphic solution of some linear difference equations, J. Math. Anal. Appl., 2011, 373, 235-241.
- [7] Bank S., Laine I., On the zeros of meromorphic solution of second order linear differential equations, Comment. Math. Helv., 1983, 58, 656-677.
- [8] Luo L. Q., Zheng X. M., The complex oscillation of a second order linear differential equation with meromorphic coefficients of $[p, q] - \varphi$ order, Journal of Jiangxi Normal University(Natural Science), 2016, 40(4), 331-337. (in Chinese).
- [9] Gao S. A., Chen Z. X., Chen T. W., Complex oscillation theory of linear differential equations, Huazhong University of Science and Technology Press, Wuhan, 1998. (in Chinese).
- [10] Chen Z. X., Zeros of entire solutions to complex linear difference equations, Acta Math. Sci., 2012, 32B(3), 1141-1148.
- [11] Goldberg A. A., Ostrovskii I. V., The distribution of values of meromorphic functions, Nauka, Moscow, 1970. (in Russian).