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Research Article

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Shintani and Shimura lifts of cusp forms on certain arithmetic groups and their applications

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Abstract: For an odd and squarefree level N, Kohnen proved that there is a canonically defined subspace $S^{\text{new}}_{k+\frac{1}{2}}(N) \subset S_{k+\frac{1}{2}}(N)$, and $S^{\text{new}}_{k+\frac{1}{2}}(N)$ and $S^{\text{new}}_{2k}(N)$ are isomorphic as modules over the Hecke algebra. Later he gave a formula for the product $a_g(m)\overline{a_g(n)}$ of two arbitrary Fourier coefficients of a Hecke eigenform g of half-integral weight and of level $4N$ in terms of certain cycle integrals of the corresponding form f of integral weight. To this end he first constructed Shimura and Shintani lifts, and then combining these lifts with the multiplicity one theorem he deduced the formula in [2, Theorem 3]. In this paper we will prove that there is a Hecke equivariant isomorphism between the spaces $S^{+}_{2k}(p)$ and $\mathbb{S}_{k+\frac{1}{2}}(p)$. We will also construct Shintani and Shimura lifts for these spaces, and prove a result analogous to [2, Theorem 3].

Keywords: Modular forms, Shintani lifts, Shimura lifts

MSC: 11F11, 11F67, 11F37

1 Introduction and statement of results

For a positive integer N, we let

$$\Gamma_0(N) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N}\}$$

be the Hecke subgroup of $\Gamma(1) := SL_2(\mathbb{Z})$, and $\Gamma_0^+(N)$ be the group generated by the Hecke group $\Gamma_0(N)$ and the Fricke involution $W_N := \left(\begin{smallmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{smallmatrix}\right)$. If k is a positive integer, then we write $S_{2k}(N)$ (resp. $S^{+}_{2k}(N)$) for the space of cusp forms of weight $2k$ on $\Gamma_0(N)$ (resp. $\Gamma_0^+(N)$). We also denote by $S_{k+\frac{1}{2}}(N)$ the space of cusp forms of weight $k+\frac{1}{2}$ on $\Gamma_0(4N)$, which have a Fourier expansion $\sum_{n\geq 1} a(n)q^n$ with $a(n) = 0$ unless $(-1)^k n \equiv 0, 1 \pmod{4}$. Let $\mathbb{S}_{k+\frac{1}{2}}(N)$ be the subspace of $S_{k+\frac{1}{2}}(N)$, in which the Fourier expansion of each form is supported only on those $n \in \mathbb{Z}$ for which $(-1)^k n \equiv \square \pmod{4N}$. For each prime divisor p of N we define $S^{\pm,p}_{k+\frac{1}{2}}(N)$ as the subspace of $S_{k+\frac{1}{2}}(N)$ consisting of forms whose n-th Fourier coefficients vanish for $\left(\frac{(-1)^k n}{p}\right) = \mp 1$. In [1, Proposition 4] Kohnen defines, for each prime divisor p of N, a Hermitian involution $w^N_{p,k+\frac{1}{2}}$ on $S_{k+\frac{1}{2}}(N)$ and shows that its $(\pm)$-eigenspace is equal to $S^{\pm,p}_{k+\frac{1}{2}}(N)$. Thus we have an orthogonal decomposition

$$S_{k+\frac{1}{2}}(N) = S^{+,p}_{k+\frac{1}{2}}(N) \bigoplus S^{-,p}_{k+\frac{1}{2}}(N).$$

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We then observe that
$$S^{+,p}_{k+\frac{1}{2}}(p)=\mathbb{S}_{k+\frac{1}{2}}(p).$$
Let $S^{\mathrm{new}}_{2k}(N)$ be the subspace of newforms in $S_{2k}(N)$. We assume that the level N is odd and squarefree. In [1, page 65] Kohnen proved that there is a canonically defined subspace $S^{\mathrm{new}}_{k+\frac{1}{2}}(N)\subset S_{k+\frac{1}{2}}(N)$, and $S^{\mathrm{new}}_{k+\frac{1}{2}}(N)$ and $S^{\mathrm{new}}_{2k}(N)$ are isomorphic as modules over the Hecke algebra. Later in [2, Theorem 3] he gave a formula for the product $a_g(m)\overline{a_g(n)}$ of two arbitrary Fourier coefficients of a Hecke eigenform g of half-integral weight and of level $4N$ in terms of certain cycle integrals of the corresponding form f of integral weight. To this end he first constructed Shimura and Shintani lifts, and then combining these lifts with the multiplicity one theorem [1, Theoerm 2] he deduced the formula in [2, Theorem 3].

In this paper we will prove in Theorem 1.10 that there is a Hecke equivariant isomorphism between the spaces $S^{+}_{2k}(p)$ and $\mathbb{S}_{k+\frac{1}{2}}(p)$. We will also construct Shintani and Shimura lifts for these spaces (see Theorems 1.6 and 1.9), and prove in Theorem 1.11 a result analogous to [2, Theorem 3].

In the following, we introduce Poincaré series, Petersson scalar products, Shintani and Shimura lifts, and develop their properties relavant to our settings. Let $\mathfrak{H}$ denote the complex upper half plane. If f and g are cusp forms of weight $\kappa\in\frac{1}{2}\mathbb{Z}$ on some subgroup Γ of finite index in $\Gamma(1)$ we denote their Petersson product by
$$\langle f,g\rangle=\langle f,g\rangle_\Gamma=\frac{1}{[\Gamma(1):\Gamma]}\int_{\Gamma\backslash\mathfrak{H}} f(z)\overline{g(z)}y^{\kappa-2}dxdy\quad (x=\mathrm{Re}z,\ \ y=\mathrm{Im}z).$$
Let m be a positive integer such that $(-1)^k m\equiv\square \pmod{4p}$. Following [3], for $z\in\mathfrak{H}$, we let $P^{+}_{m}(z)\in\mathbb{S}_{k+\frac{1}{2}}(p)$ be a Poincaré series such that
$$\langle g,P^{+}_{m}(z)\rangle=a_g(m)c(m)\quad\text{for all } g=\sum_{n\ge1}a_g(n)q^n\in\mathbb{S}_{k+\frac{1}{2}}(p),$$
where $c(m)=\frac{\Gamma(k-\frac{1}{2})i_{4p}^{-1}}{(4\pi m)^{k-\frac{1}{2}}}$. Here i_N stands for $[\Gamma(1):\Gamma_0(N)]$. Let α be a positive integer with $(-1)^k\alpha\equiv 0,1 \pmod 4$. As described in [2, (4)], for $z\in\mathfrak{H}$, let $P_{k,p,\alpha}(z)\in S_{k+\frac{1}{2}}(p)$ be the Poincaré series characterized by
$$\langle g,P_{k,p,\alpha}(z)\rangle=a_g(\alpha)c(\alpha)\quad\text{for all } g=\sum_{n\ge1}a_g(n)q^n\in S_{k+\frac{1}{2}}(p).$$
Write
$$P_{k,p,\alpha}=P^{+}_{k,p,\alpha}\oplus P^{-}_{k,p,\alpha}\in\mathbb{S}_{k+\frac{1}{2}}(p)\bigoplus S^{-,p}_{k+\frac{1}{2}}(p).$$

Proposition 1.1. *Let α be a positive integer with $(-1)^k\alpha\equiv 0,1 \pmod 4$.*
(i) *If $(-1)^k\alpha\equiv\square \pmod{4p}$, then $P^{+}_{k,p,\alpha}=P^{+}_{\alpha}$.*
(ii) *(ii) If $(-1)^k\alpha\not\equiv\square \pmod{4p}$, then $P^{+}_{k,p,\alpha}$ is identically zero.*
(iii) *If $(-1)^k\alpha\equiv\square \pmod{4p}$ with $p\nmid\alpha$, then $P_{k,p,\alpha}=P^{+}_{\alpha}$.*

The motivation of this paper is as follows. In [3] we have shown that the space $\mathbb{S}_{k+\frac{1}{2}}(p)$ is spanned by the Poincaré series P^{+}_{α}. In [1, 2] Kohnen constructed Shimura lift by making use of the Poincaré series $P_{k,p,\alpha}$ in the space $S_{k+\frac{1}{2}}(p)$. So we expect that these Poincaré series P^{+}_{α} can be used to find certain space of cusp forms of integral weight corresponding to the space $\mathbb{S}_{k+\frac{1}{2}}(p)$ under Shimura and Shintani lifts.

Let $\mathfrak{D}$ be the set of all discriminants, i.e.
$$\mathfrak{D}=\{d\in\mathbb{Z}\mid d\equiv 0,1\pmod 4\}.$$
For $d\in\mathfrak{D}$, we let $\mathcal{Q}_{d,N}$ be the set of all integral binary quadratic forms $Q(x,y)=ax^2+bxy+cy^2=[a,b,c]$ with $N\mid a$ and $b^2-4ac=d$. We observe that $\mathcal{Q}_{d,N}$ is an empty set unless d is congruent to a square modulo $4N$. For $M=\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}\in\Gamma^{+}_{0}(N)$, we set $X'=\alpha X+\beta Y$, $Y'=\gamma X+\delta Y$, and $Q(X,Y)\circ M=Q(X',Y')$. This

defines an action of the group $\Gamma_0^+(N)$ on the set $\mathcal{Q}_{d,N}$. Following [2, page 239], for integers $k \geq 2, N \geq 1$ and $D, D' \in \mathfrak{D}$ with $DD' > 0$ we set

$$f_{k,N}(z; D, D') := \sum_{Q \in \mathcal{Q}_{DD',N}} \frac{\chi_D(Q)}{Q(z,1)^k} \quad (z \in \mathfrak{H}),$$

where $\chi_D(Q)$ is the generalized genus character defined as follows: for $Q = [a, b, c]$,

$$\chi_D(Q) := \begin{cases} 0 & \text{if } (a,b,c,D) > 1 \\ \left(\frac{D}{r}\right) & \text{if } (a,b,c,D) = 1, \text{where } [a,b,c] \text{ represents } r, (r, D) = 1. \end{cases}$$

The series converges absolutely uniformly on compact sets, and $f_{k,N}(z; D, D')$ is a cusp form of weight $2k$ on $\Gamma_0(N)$.

Remark 1.2.
(i) *The series $f_{k,N}(z; D, D')$ is a cusp form of weight $2k$ on $\Gamma_0^+(N)$ since the character χ_D is invariant under the Fricke involution W_N (see [4, Proposition 1 in p.508]).*
(ii) *The series $f_{k,N}(z; D, D')$ is identically zero unless $DD' \equiv \square \pmod{4N}$ and $(-1)^k D > 0$.*

Proposition 1.3. *For cusp forms f and g of weight $\kappa \in \frac{1}{2}\mathbb{Z}$ on $\Gamma_0^+(p)$, we have*

$$\int_{\Gamma_0(p)\backslash\mathfrak{H}} f\bar{g}y^{\kappa-2}dxdy = 2 \int_{\Gamma_0^+(p)\backslash\mathfrak{H}} f\bar{g}y^{\kappa-2}dxdy.$$

If $d \in \mathfrak{D}$ is positive and is not a perfect square, then the group

$$\Gamma_0^+(p)_Q/\{\pm 1\} = \{g \in \Gamma_0^+(p) \mid Q \circ g = Q\}/\{\pm 1\}$$

is infinite cyclic with a distinguished generator g_Q which is explicitly described in [5, Theorem 1.3]. For $Q = [a, b, c] \in \mathcal{Q}_{d,p}$ let S_Q be the oriented semi-circle defined by $a|z|^2 + (\text{Re } z)b + c = 0$, directed counterclockwise if $a > 0$ and clockwise if $a < 0$. For $f \in S_{2k}^+(p)$ and $Q \in \mathcal{Q}_{d,p}$ we define

$$r_Q^+(f) := \int_{C_Q^+} f(z) d_{Q,k}z$$

where $C_Q^+ = C_Q^+(z_0)$ is the directed arc on S_Q from $z_0 \in S_Q$ to $g_Q z_0$, and $d_{Q,k}z = Q(z,1)^{k-1}dz$. Then the value $r_Q^+(f)$ is both independent of $z_0 \in S_Q$ and a class invariant.

For each $f \in S_{2k}^+(p)$ and integers D, m satisfying the following condition

$$(-1)^k m \in \mathfrak{D}, (-1)^k D > 0, m > 0, |D|m \neq \square, \text{ and } |D|m \equiv \square \pmod{4p}, \tag{1}$$

we set

$$r_{k,p}^+(f; D, (-1)^k m) := \sum_{Q \in \mathcal{Q}_{|D|m,p}/\Gamma_0^+(p)} \chi_D(Q) r_Q^+(f).$$

We also define for cusp forms f and g of weight $2k \in 2\mathbb{Z}$ on $\Gamma_0^+(p)$,

$$\langle f, g\rangle_{\Gamma_0^+(p)} = \frac{2}{i_p} \int_{\Gamma_0^+(p)\backslash\mathfrak{H}} f\bar{g}y^{2k-2}dxdy \quad \text{so that} \quad \langle f, g\rangle_{\Gamma_0^+(p)} = \langle f, g\rangle_{\Gamma_0(p)}.$$

Proposition 1.4. *For any $f \in S_{2k}^+(p)$ and integers D, m satisfying* (1) *we have*

$$\begin{aligned}\langle f, f_{k,p}(\ ; D, (-1)^k m)\rangle &= \langle f, f_{k,p}(\ ; D, (-1)^k m)\rangle_{\Gamma_0^+(p)} \\ &= \frac{2}{i_p}\pi \binom{2k-2}{k-1} 2^{-2k+2}(|D|m)^{\frac{1}{2}-k} r_{k,p}^+(f; D, (-1)^k m).\end{aligned}$$

Proposition 1.5. *For $D \in \mathfrak{D}$ with $(-1)^k D > 0$, we let*

$$\varphi_{k,p}(z,\tau) := \sum_{\substack{m\geq 1 \\ (-1)^k m \equiv 0,1 \ (4)}} m^{k-\frac{1}{2}} f_{k,p}(z; D, (-1)^k m) e^{2\pi i m\tau}.$$

Then for fixed z, $\varphi_{k,p}(z,\tau)$ is an element of $S_{k+\frac{1}{2}}(p)$.

Let $w_p := w^p_{p,k+\frac{1}{2}}$ be the Hermitian involution (with respect to τ) on the space $S_{k+\frac{1}{2}}(p)$. We then have a decomposition

$$\varphi_{k,p}(z,\tau) = \varphi^+_{k,p}(z,\tau) \oplus \varphi^-_{k,p}(z,\tau) \in \mathbb{S}_{k+\frac{1}{2}}(p) \bigoplus S^{-,p}_{k+\frac{1}{2}}(p).$$

Theorem 1.6. *Let $D \in \mathfrak{D}$ with $(-1)^k D > 0$ and $D \equiv \square \pmod{4p}$. Define for each $f \in S^+_{2k}(p)$, the D-th Shintani lift*

$$f|\mathcal{S}^*_+ := \frac{i_p}{c^*_{k,D}} \langle f, \varphi^+_{k,p}(\ ,-\bar{\tau})\rangle \in \mathbb{S}_{k+\frac{1}{2}}(p),$$

with $c^*_{k,D} = \frac{i_p(-1)^{[\frac{k}{2}]}3(2\pi)^k}{(k-1)!} c(|D|)$. If we write the Fourier expansion of $g := f|\mathcal{S}^*_+$ as

$$g(\tau) = \sum_{\substack{m\geq 1 \\ (-1)^k m \equiv \square \pmod{4p}}} a_g(m) e^{2\pi i m\tau},$$

then for a positive integer m satisfying $p \nmid m$ and $(-1)^k m \equiv \square \pmod{4p}$ one has

$$a_g(m) = 2r^+_{k,p}(f; D, (-1)^k m).$$

Remark 1.7.

(i) *In the case $p = 1$ and $k = 0$, cycle integrals of the modular invariant j-function were considered in [6] and their generating function was shown to be a mock modular form of weight $1/2$ on $\Gamma_0(4)$.*

(ii) *In [5, 7] cycle integrals of weakly holomorphic modular forms were used to construct modular integrals for certain rational period functions related to indefinite binary quadratic forms.*

We have

$$\varphi^+_{k,p}(z,\tau) = \frac{\varphi_{k,p}(z,\tau) + \varphi_{k,p}(z,\tau)|w_p}{2}.$$

Alternatively, we can find $\varphi^+_{k,p}(z,\tau)$ by means of Poincaré series as follows. Following [2, (3)], for $z, \tau \in \mathfrak{H}$ and $D \in \mathfrak{D}$ with $(-1)^k D > 0$, we let

$$\Omega_{k,p}(z,\tau; D) = i_p c^{-1}_{k,D} \sum_{\substack{m\geq 1 \\ (-1)^k m \equiv 0,1 \ (4)}} m^{k-\frac{1}{2}} \left(\sum_{t|p} \mu(t) \left(\frac{D}{t}\right) t^{k-1} f_{k,\frac{p}{t}}(tz; D, (-1)^k m) \right) e^{2\pi i m\tau}$$

where $c_{k,D} = (-1)^{[\frac{k}{2}]} |D|^{-k+\frac{1}{2}} \pi \binom{2k-2}{k-1} 2^{-3k+2}$ and $\mu(t)$ is the Möbius μ-function. By [2, (3)] or (5) one has

$$\begin{aligned}
\varphi_{k,p}(z,\tau) &= i_p^{-1} c_{k,D} \left(\Omega_{k,p}(z,\tau; D) + p^{k-1} i_p \left(\frac{D}{p}\right) \Omega_{k,1}(pz,\tau; D) \right) \\
&= \frac{(-1)^{[\frac{k}{2}]} 3 \cdot (2\pi)^k}{(k-1)!} \sum_{n\geq 1} n^{k-1} \left(\sum_{\substack{d|n \\ p\nmid d}} \left(\frac{D}{d}\right) \left(\frac{n}{d}\right)^k P_{k,p,n^2|D|/d^2}(\tau) \right) e^{2\pi i n z} \\
&\quad + p^{k-1} \left(\frac{D}{p}\right) \frac{(-1)^{[\frac{k}{2}]} 3 \cdot (2\pi)^k}{(k-1)!} \sum_{n\geq 1} n^{k-1} \left(\sum_{d|n} \left(\frac{D}{d}\right) \left(\frac{n}{d}\right)^k P_{k,1,n^2|D|/d^2}(\tau) \right) e^{2\pi i n p z}.
\end{aligned}$$

This gives rise to the following theorem.

Theorem 1.8. *Let $D \in \mathfrak{D}$ with $(-1)^k D > 0$ and $D \equiv \square \pmod{4p}$. We have*

$$\varphi^+_{k,p}(z,\tau) = \frac{(-1)^{[\frac{k}{2}]}3 \cdot (2\pi)^k}{(k-1)!} \sum_{n \geq 1} n^{k-1} \left(\sum_{\substack{d|n \\ p \nmid d}} \left(\frac{D}{d}\right) \left(\frac{n}{d}\right)^k P^+_{k,p,n^2|D|/d^2}(\tau) \right) e^{2\pi i n z}$$
$$+ p^{k-1} \left(\frac{D}{p}\right) \frac{(-1)^{[\frac{k}{2}]}3 \cdot (2\pi)^k}{(k-1)!} \sum_{n \geq 1} n^{k-1} \left(\sum_{d|n} \left(\frac{D}{d}\right) \left(\frac{n}{d}\right)^k P^+_{k,1,n^2|D|/d^2}(\tau) \right) e^{2\pi i n p z}. \quad (2)$$

Let $D \in \mathfrak{D}$ with $(-1)^k D > 0$ and $D \equiv \square \pmod{4p}$. Now we define for each $g \in \mathbb{S}_{k+\frac{1}{2}}(p)$, the D-th Shimura lift

$$g|\mathcal{S}_+ := \frac{i_p}{c^*_{k,D}} \langle g, \varphi^+_{k,p}(-\bar{z},\tau) \rangle_\tau \in S^+_{2k}(p).$$

It then follows from (2) that

$$g|\mathcal{S}_+ = \frac{i_p}{c^*_{k,D}} \frac{(-1)^{[\frac{k}{2}]}3 \cdot (2\pi)^k}{(k-1)!} \sum_{n \geq 1} n^{k-1} \sum_{\substack{d|n \\ p \nmid d}} \left(\frac{D}{d}\right) \left(\frac{n}{d}\right)^k a_g(n^2|D|/d^2) c(n^2|D|/d^2) e^{2\pi i n z}$$
$$+ \frac{p^{k-1} i_p}{c^*_{k,D}} \left(\frac{D}{p}\right) \frac{(-1)^{[\frac{k}{2}]}3 \cdot (2\pi)^k}{(k-1)!} \sum_{n \geq 1} n^{k-1} \sum_{d|n} \left(\frac{D}{d}\right) \left(\frac{n}{d}\right)^k \langle g, P^+_{k,1,n^2|D|/d^2}(\tau) \rangle_\tau e^{2\pi i n p z}.$$

Theorem 1.9.

(i) *Our Shimura lift is Hecke equivariant, i.e. for each $g \in \mathbb{S}_{k+\frac{1}{2}}(p)$ and a prime $r(\neq p)$, we have*

$$(g|\mathcal{S}_+)|T(r) = (g|T(r^2))|\mathcal{S}_+.$$

(ii) *Our Shintani lift is Hecke equivariant, i.e. for each $f \in S^+_{2k}(p)$ and a prime $r(\neq p)$, we have*

$$(f|\mathcal{S}^*_+)|T(r^2) = (f|T(r))|\mathcal{S}^*_+.$$

Theorem 1.10. *Let p be an odd prime. There exists a Hecke equivariant isomorphism*

$$\acute{\mathbf{L}}(= \acute{\mathbf{L}}_p) : S^+_{2k}(p) \to \mathbb{S}_{k+\frac{1}{2}}(p).$$

Theorem 1.11. *Let $D \in \mathfrak{D}$ with $(-1)^k D > 0$ and $D \equiv \square \pmod{4p}$. Let p be an odd prime and $\acute{\mathbf{L}}(= \acute{\mathbf{L}}_p) : S^+_{2k}(p) \to \mathbb{S}_{k+\frac{1}{2}}(p)$ be the Hecke equivariant isomorphism discussed in Theorem 1.10 and let $\{f_1, \ldots, f_t\}$ be a basis for $S^+_{2k}(p)$ consisting of normalized Hecke eigenforms. For each $j \in \{1, \ldots, t\}$ we put $g_j := \acute{\mathbf{L}}(f_j)$. Then the following assertions are true.*

(i) *Let $\mathcal{S}_+$ denote the D-th Shimura lift. Then*

$$g_j|\mathcal{S}_+ = c_j(|D|) f_j$$

where $c_j(|D|)$ denotes the Fourier coefficient of $q^{|D|}$ in g_j.

(ii) *For each positive integer n, the Fourier coefficients $c_j(n)$ of q^n in g_j satisfy*

$$\overline{c_j(n)} = c \cdot c_j(n)$$

for some nonzero constant c.

(iii) *Let $\mathcal{S}^*_+$ denote the D-th Shintani lift. Then*

$$f_j|\mathcal{S}^*_+ = \overline{c_j(|D|)} \langle f_j, f_j \rangle \langle g_j, g_j \rangle^{-1} g_j.$$

(iv) *For each positive integer m such that $(-1)^k m \equiv \square \pmod{4p}$ and $p \nmid m$, we have*

$$r^+_{k,p}(f_j, D, (-1)^k m) = \frac{1}{2} \overline{c_j(|D|)} c_j(m) \langle f_j, f_j \rangle \langle g_j, g_j \rangle^{-1}.$$

(v) *For each* $f \in S_{2k}^{+}(p)$ *and* $g \in \mathbb{S}_{k+\frac{1}{2}}(p)$, *one has*

$$\langle g|\mathcal{S}_{+}, f\rangle = \langle g, f|\mathcal{S}_{+}^{*}\rangle,$$

i.e. $\mathcal{S}_{+}$ *and* $\mathcal{S}_{+}^{*}$ *are adjoint.*

Remark 1.12.

(i) *Let* $\mathcal{S}_{k,p,D}^{*}$ *denote the Kohnen's Shintani lift [2, (8)] and* f *be a newform in* $S_{2k}^{+}(p)$. *Then our Shintani lift* $f|\mathcal{S}_{+}^{*}$ *is the projection of Kohnen's Shintani lift* $f|\mathcal{S}_{k,p,D}^{*}$ *on* $\mathbb{S}_{k+\frac{1}{2}}(p)$, *up to multiplication. But they are different for oldforms.*

(ii) *Let* $\mathcal{S}_{k,p,D}$ *denote the Kohnen's Shimura lift [2, (6)] and g be a newform in* $S_{k+\frac{1}{2}}(p)$. *Our Shimura lift* $g|\mathcal{S}_{+}$ *and Kohnen's Shimura lift* $g|\mathcal{S}_{k,p,D}$ *are the same up to multiplication. But they are different for oldforms.*

(iii) *Kohnen's results [1] do not give a Hecke equivariant isomorphism between the spaces* $S_{2k}^{+}(p)$ *and* $\mathbb{S}_{k+\frac{1}{2}}(p)$ *while we give it in Theorem 1.10.*

(iv) *The results in Theorem 1.11 hold for all Hecke eigenforms in* $S_{2k}^{+}(p)$ *or* $\mathbb{S}_{k+\frac{1}{2}}(p)$. *But when we consider the spaces* $S_{2k}^{+}(p)$ *and* $\mathbb{S}_{k+\frac{1}{2}}(p)$, *Kohnen's results hold only for newforms.*

Throughout this paper, given a modular form f we denote by $a_f(n)$, the Fourier coefficient of q^n in f. This paper is organized as follows. The proofs of Propositions 1.1, 1.3, 1.4 and 1.5 are given in Section 2. In Section 3 the proofs of Theorems 1.6 and 1.9 are given. In Section 4 and Section 5 we prove Theorems 1.10 and 1.11, respectively.

2 Proofs of Propositions 1.1, 1.3, 1.4 and 1.5

Proof of Proposition 1.1. We observe that for $f \in \mathbb{S}_{k+\frac{1}{2}}(p)$ and $g \oplus h \in S_{k+\frac{1}{2}}(p) = \mathbb{S}_{k+\frac{1}{2}}(p) \bigoplus S_{k+\frac{1}{2}}^{-,p}(p)$,

$$\langle f, g+h\rangle = \langle f, g\rangle.$$

Write $P_{k,p,\alpha}^{+}(\tau) = \sum_{(-1)^k l \equiv \square \ (4p)} a_{P_{k,p,\alpha}^{+}}(l) e^{2\pi i l\tau}$. We then have for each positive integer l with $(-1)^k l \equiv \square$ (mod $4p$),

$$a_{P_l^{+}}(\alpha)c(\alpha) = \langle P_l^{+}, P_{k,p,\alpha}\rangle = \langle P_l^{+}, P_{k,p,\alpha}^{+}\rangle = \overline{\langle P_{k,p,\alpha}^{+}, P_l^{+}\rangle} = c(l)\overline{a_{P_{k,p,\alpha}^{+}}(l)},$$

so that

$$a_{P_{k,p,\alpha}^{+}}(l) = \frac{c(\alpha)}{c(l)}\overline{a_{P_l^{+}}(\alpha)}, \tag{3}$$

which is zero unless $(-1)^k\alpha \equiv \square$ (mod $4p$). This proves the second assertion. To prove the first assertion we observe that

$$a_{P_l^{+}}(\alpha)c(\alpha) = \langle P_l^{+}, P_\alpha^{+}\rangle = \overline{\langle P_\alpha^{+}, P_l^{+}\rangle} = \overline{a_{P_\alpha^{+}}(l)}c(l). \tag{4}$$

Combining (3) and (4) we get that $a_{P_{k,p,\alpha}^{+}}(l) = a_{P_\alpha^{+}}(l)$, from which the first assertion is immediate. Now we prove the third assertion. For each positive integer α and β with $(-1)^k\alpha \equiv \square$ (mod $4p$) and $(-1)^k\beta \equiv \square$ (mod $4p$), we consider

$$\begin{aligned}
\langle P_{k,p,\alpha}^{-}, P_{k,p,\beta}^{-}\rangle &= \langle P_{k,p,\alpha} - P_\alpha^{+}, P_{k,p,\beta} - P_\beta^{+}\rangle \\
&= c(\beta)a_{P_{k,p,\alpha}}(\beta) - c(\alpha)\overline{a_{P_\beta^{+}}(\alpha)} \\
&= c(\beta)a_{P_{k,p,\alpha}}(\beta) - c(\beta)a_{P_\alpha^{+}}(\beta) \quad \text{by (4)} \\
&= c(\beta)a_{P_{k,p,\alpha}^{-}}(\beta).
\end{aligned}$$

Thus

$$\langle P_{k,p,\alpha}, P_{k,p,\beta}^{-}\rangle = \langle P_\alpha^{+} + P_{k,p,\alpha}^{-}, P_{k,p,\beta}^{-}\rangle = \langle P_{k,p,\alpha}^{-}, P_{k,p,\beta}^{-}\rangle$$

$$= c(\beta)a_{P^-_{k,p,\alpha}}(\beta) = \overline{c(\alpha)a_{P^-_{k,p,\beta}}(\alpha)}$$
$$= 0 \quad \text{since } p \nmid \alpha \text{ and } \left(\tfrac{(-1)^k\alpha}{p}\right) = 1.$$

This implies that $a_{P^-_{k,p,\alpha}}(\beta) = 0$ for arbitrary positive integer β with $(-1)^k\beta \equiv \square \pmod{4p}$, so that $P_{k,p,\alpha} = P^+_\alpha$. $\square$

Proof of Proposition 1.3. Let F^+_p be a fundamental domain of $\Gamma^+_0(p)$. Then $F^+_p \dot{\bigcup} W_p F^+_p$ is a fundamental domain of $\Gamma_0(p)$. Thus we see that

$$\begin{aligned}\int\limits_{\Gamma_0(p)\backslash\mathfrak{H}} f\bar{g}y^{\kappa-2}dxdy &= \int\limits_{F^+_p} f\bar{g}y^{\kappa-2}dxdy + \int\limits_{W_pF^+_p} f\bar{g}y^{\kappa-2}dxdy \\ &= \int\limits_{F^+_p} f\bar{g}y^{\kappa-2}dxdy + \int\limits_{F^+_p} f|_\kappa W_p\, \overline{g|_\kappa W_p}\, y^{\kappa-2}dxdy \\ &= 2\int\limits_{F^+_p} f\bar{g}y^{\kappa-2}dxdy \\ &= 2\int\limits_{\Gamma^+_0(p)\backslash\mathfrak{H}} f\bar{g}y^{\kappa-2}dxdy,\end{aligned}$$

as desired. $\square$

Proof of Proposition 1.4. For a $\Gamma^+_0(p)$-class $\mathfrak{U}$ of forms $Q \in \mathcal{Q}_{|D|m,p}$, we define

$$f_{k,p}(z;\mathfrak{U}) = \sum_{Q\in\mathfrak{U}} Q(z,1)^{-k}.$$

We then have $f_{k,p}(z;\mathfrak{U}) \in S^+_{2k}(p)$ and

$$\begin{aligned} i_p\langle f, f_{k,p}(\ ;\mathfrak{U})\rangle_{\Gamma^+_0(p)} &= 2\int\limits_{\Gamma^+_0(p)\backslash\mathfrak{H}} \sum_{\gamma\in\Gamma_Q\backslash\Gamma^+_0(p)} (Q\circ\gamma)(\bar{z},1)^{-k} f(z)y^{2k-2}dxdy \\ &= 2\int\limits_{\Gamma_Q\backslash\mathfrak{H}} Q(\bar{z},1)^{-k} f(z)y^{2k-2}dxdy,\end{aligned}$$

where Q is any element of $\mathfrak{U}$ and Γ_Q is the stabilizer of Q in $\Gamma^+_0(p)$. By similar argument in [2, Proposition 7], we obtain

$$\langle f, f_{k,p}(\ ;\mathfrak{U})\rangle_{\Gamma^+_0(p)} = \frac{2}{i_p}(|D|m)^{\frac{1}{2}-k}\int_0^\pi \sin^{2k-2}\theta d\theta \int\limits_{c^+_Q} f(z)d_{Q,k}z$$

and hence

$$\langle f, f_{k,p}(\ ;D,(-1)^k m)\rangle_{\Gamma^+_0(p)} = \frac{2}{i_p}\pi\binom{2k-2}{k-1}2^{-2k+2}(|D|m)^{\frac{1}{2}-k} r^+_{k,p}(f;D,(-1)^k m). \qquad \square$$

Proof of Proposition 1.5. We observe that

$$\begin{aligned}\Omega_{k,p}(z,\tau;D) = {}& i_p c^{-1}_{k,D} \sum_{\substack{m\geq 1\\ (-1)^k m\equiv 0,1\ (4)}} m^{k-\frac{1}{2}} f_{k,p}(z;D,(-1)^k m)e^{2\pi i m\tau} \\ & - i_p c^{-1}_{k,D} \sum_{\substack{m\geq 1\\ (-1)^k m\equiv 0,1\ (4)}} m^{k-\frac{1}{2}} \left(\frac{D}{p}\right) p^{k-1} f_{k,1}(pz;D,(-1)^k m)e^{2\pi i m\tau} \in S_{k+\frac{1}{2}}(p).\end{aligned}$$

Since

$$\Omega_{k,1}(z,\tau;D) = c_{k,D}^{-1} \sum_{\substack{m\geq 1 \\ (-1)^k m\equiv 0,1\ (4)}} m^{k-\frac{1}{2}} f_{k,1}(z;D,(-1)^k m)e^{2\pi i m\tau}$$

belongs to $S_{k+\frac{1}{2}}(1) \subseteq S_{k+\frac{1}{2}}(p)$ for fixed τ, one has

$$\Omega_{k,1}(pz,\tau;D) = c_{k,D}^{-1} \sum_{\substack{m\geq 1 \\ (-1)^k m\equiv 0,1\ (4)}} m^{k-\frac{1}{2}} f_{k,1}(pz;D,(-1)^k m)e^{2\pi i m\tau} \in S_{k+\frac{1}{2}}(p).$$

Thus we obtain that

$$\begin{aligned}\varphi_{k,p}(z,\tau) &= \sum_{\substack{m\geq 1 \\ (-1)^k m\equiv 0,1\ (4)}} m^{k-\frac{1}{2}} f_{k,p}(z;D,(-1)^k m)e^{2\pi i m\tau} \\ &= i_p^{-1} c_{k,D}\left(\Omega_{k,p}(z,\tau;D) + p^{k-1} i_p \left(\frac{D}{p}\right)\Omega_{k,1}(pz,\tau;D)\right) \in S_{k+\frac{1}{2}}(p), \qquad (5)\end{aligned}$$

as desired. □

3 Proofs of Theorems 1.6 and 1.9

Proof of Theorem 1.6. We note that for each positive integer m such that $(-1)^k m \equiv \square \pmod{4p}$ and $p \nmid m$,

$$a_{\varphi_{k,p}^+(z,\tau)}(m) = m^{k-\frac{1}{2}} f_{k,p}(z;D,(-1)^k m).$$

Thus we have

$$\begin{aligned} g(\tau) = f|\mathcal{S}_+^* &= \frac{i_p}{c_{k,D}^*}\langle f, \varphi_{k,p}^+(\ ,-\bar{\tau})\rangle \\ &= \frac{i_p}{c_{k,D}^*}\langle f, \sum_{\substack{m\geq 1,\ p\nmid m \\ (-1)^k m\equiv \square\ (4p)}} m^{k-\frac{1}{2}} f_{k,p}(\ ;D,(-1)^k m)e^{-2\pi i m\bar{\tau}}\rangle + \sum_{\substack{m\geq 1,\ p| m \\ (-1)^k m\equiv \square\ (4p)}} a_g(m)e^{2\pi i m\tau} \\ &= \frac{i_p}{c_{k,D}^*}\sum_{\substack{m\geq 1,\ p\nmid m \\ (-1)^k m\equiv \square\ (4p)}} m^{k-\frac{1}{2}}\langle f, f_{k,p}(\ ;D,(-1)^k m)\rangle e^{2\pi i m\tau} + \sum_{\substack{m\geq 1,\ p| m \\ (-1)^k m\equiv \square\ (4p)}} a_g(m)e^{2\pi i m\tau} \\ &= 2\sum_{\substack{m\geq 1,\ p\nmid m \\ (-1)^k m\equiv \square\ (4p)}} r_{k,p}^+(f;D,(-1)^k m)e^{2\pi i m\tau} + \sum_{\substack{m\geq 1,\ p| m \\ (-1)^k m\equiv \square\ (4p)}} a_g(m)e^{2\pi i m\tau} \quad \text{by Proposition 1.4,}\end{aligned}$$

from which the assertion is immediate. □

Proof of Theorem 1.9. First we need a lemma.

Lemma 3.1. *One has*

$$(\varphi_{k,p}|T(r^2))^+ = \varphi_{k,p}^+|T(r^2) \qquad (6)$$

and

$$(\varphi_{k,p}|T(r))^+ = \varphi_{k,p}^+|T(r). \qquad (7)$$

Proof. We observe that

$$(\varphi_{k,p} = \varphi_{k,p}^+ \oplus \varphi_{k,p}^-)|T(r^2) = \varphi_{k,p}^+|T(r^2) \oplus \varphi_{k,p}^-|T(r^2) \in \mathbb{S}_{k+\frac{1}{2}}(p) \bigoplus S_{k+\frac{1}{2}}^{-,p}(p),$$

from which (6) is immediate. By a similar reasoning one obtains (7). □

For the proof of the first assertion of Theorem 1.9 we observe that for $l \in \{1, p\}$ and each $g \in S_{k+\frac{1}{2}}(l)$,

$$\begin{aligned}\langle g, \Omega_{k,l}(-\bar{z},\ , D)|T(r^2)\rangle &= \langle g|T(r^2), \Omega_{k,l}(-\bar{z},\ , D)\rangle \\ &= (\langle g, \Omega_{k,l}(-\bar{z},\ , D)\rangle)|T(r) \quad \text{by [2, p.241]} \\ &= \langle g, \Omega_{k,l}(-\bar{z},\ , D)|T(r)\rangle.\end{aligned}$$

Thus one obtains

$$\Omega_{k,l}(-\bar{z}, \tau, D)|T(r^2) = \Omega_{k,l}(-\bar{z}, \tau, D)|T(r) \tag{8}$$

where the operator $T(r^2)$ (resp. $T(r)$) acts on a function with variable τ (resp. z). In particular,

$$\begin{aligned}\Omega_{k,1}(-p\bar{z}, \tau, D)|T(r^2) &= \Omega_{k,1}(-\bar{z}, \tau, D)|T(r^2)|V(p) = \Omega_{k,1}(-\bar{z}, \tau, D)|T(r)|V(p) \\ &= \Omega_{k,1}(-\bar{z}, \tau, D)|V(p)|T(r) \\ &= \Omega_{k,1}(-p\bar{z}, \tau, D)|T(r)\end{aligned} \tag{9}$$

where the operator $V(p)$ acts on a function with variable z. Consequently, for $g \in \mathbb{S}_{k+\frac{1}{2}}(p)$,

$$\begin{aligned}(g|\mathcal{S}_+)|T(r) &= (\langle g, \varphi^+_{k,p}(-\bar{z}, \tau)\rangle_\tau)|T(r) = (\langle g, \varphi_{k,p}(-\bar{z}, \tau)\rangle_\tau)|T(r) \\ &= \langle g, i_p^{-1} c_{k,D}\Omega_{k,p}(-\bar{z}, \tau, D)\rangle|T(r) \\ &\quad + \langle g, p^{k-1} c_{k,D} \left(\frac{D}{p}\right) \Omega_{k,1}(-p\bar{z}, \tau, D)\rangle|T(r) \quad \text{by (5)} \\ &= \langle g, i_p^{-1} c_{k,D}\Omega_{k,p}(-\bar{z}, \tau, D)|T(r)\rangle \\ &\quad + \langle g, p^{k-1} c_{k,D} \left(\frac{D}{p}\right) \Omega_{k,1}(-p\bar{z}, \tau, D)|T(r)\rangle \\ &= \langle g, i_p^{-1} c_{k,D}\Omega_{k,p}(-\bar{z}, \tau, D)|T(r^2)\rangle \\ &\quad + \langle g, p^{k-1} c_{k,D} \left(\frac{D}{p}\right) \Omega_{k,1}(-p\bar{z}, \tau, D)|T(r^2)\rangle \quad \text{by (8) and (9)} \\ &= \langle g|T(r^2), \varphi_{k,p}(-\bar{z}, \tau)\rangle = \langle g|T(r^2), \varphi^+_{k,p}(-\bar{z}, \tau)\rangle \\ &= (g|T(r^2))|\mathcal{S}_+.\end{aligned}$$

Next we will prove the second assertion of Theorem 1.9. For $f \in S_{2k}(l)$, let $f|\mathcal{S}^*$ denote the D-th Shintani lifting of f, defined in [2, p.240]. We then remark from [2, Theorem 2] that

$$\langle f, \Omega_{k,l}(\ , -\bar{\tau}, D)\rangle = f|\mathcal{S}^* \quad \text{for all } f \in S_{2k}(l) \tag{10}$$

and

$$(f|T(r))|\mathcal{S}^* = (f|\mathcal{S}^*)|T(r^2) \quad \text{for all prime } r (\neq l). \tag{11}$$

Thus for all $f \in S_{2k}(l)$, we have that

$$\begin{aligned}\langle f, \Omega_{k,l}(\ , -\bar{\tau}, D)|T(r)\rangle &= \langle f|T(r), \Omega_{k,l}(\ , -\bar{\tau}, D)\rangle \\ &= \langle f, \Omega_{k,l}(\ , -\bar{\tau}, D)\rangle|T(r^2) \quad \text{by (10) and (11)} \\ &= \langle f, \Omega_{k,l}(\ , -\bar{\tau}, D)|T(r^2)\rangle,\end{aligned}$$

which implies that

$$\Omega_{k,l}(z, -\bar{\tau}, D)|T(r) = \Omega_{k,l}(z, -\bar{\tau}, D)|T(r^2).$$

In particular,

$$\begin{aligned}\Omega_{k,1}(pz, -\bar{\tau}, D)|T(r) &= \Omega_{k,1}(z, -\bar{\tau}, D)|V(p)|T(r) = \Omega_{k,1}(z, -\bar{\tau}, D)|T(r)|V(p) \\ &= \Omega_{k,1}(pz, -\bar{\tau}, D)|T(r^2).\end{aligned}$$

Hence we obtain that for all $f \in S_{2k}^{+}(p)$,

$$\begin{aligned}\frac{c_{k,D}^{*}}{i_p}(f|\mathcal{S}_{+}^{*})|T(r^2) &= \langle f, \varphi_{k,p}^{+}(\ ,-\bar{\tau})\rangle|T(r^2) = \langle f, \varphi_{k,p}^{+}(\ ,-\bar{\tau})|T(r^2)\rangle \\ &= \langle f, (\varphi_{k,p}(\ ,-\bar{\tau})|T(r^2))^{+}\rangle \quad \text{by (6)} \\ &= \langle f, (\varphi_{k,p}(\ ,-\bar{\tau})|T(r))^{+}\rangle \\ &= \langle f, \varphi_{k,p}^{+}(\ ,-\bar{\tau})|T(r)\rangle \quad \text{by (7)} \\ &= \langle f|T(r), \varphi_{k,p}^{+}(\ ,-\bar{\tau})\rangle \\ &= \frac{c_{k,D}^{*}}{i_p}(f|T(r))|\mathcal{S}_{+}^{*}. \end{aligned}$$

□

4 Proof of Theorem 1.10

Following [1, p.64 lines 7-9], let $S_{k+\frac{1}{2}}^{\text{old}}(p)$ be the set of oldforms in $S_{k+\frac{1}{2}}(p)$. Let $\mathbb{S}_{k+\frac{1}{2}}^{\text{old}}(p) := S_{k+\frac{1}{2}}^{\text{old}}(p) \cap \mathbb{S}_{k+\frac{1}{2}}(p)$.

Lemma 4.1. *Define a map*

$$\Phi : S_{k+\frac{1}{2}}(1) \to \mathbb{S}_{k+\frac{1}{2}}^{\text{old}}(p)$$

by $\Phi(f) = f + f|w_p$. Then Φ is a Hecke equivariant isomorphism.

Proof. According to [1, p.65 line 4], the Hermitian involution w_p leaves the space of oldforms stable. Let $f \in S_{k+\frac{1}{2}}(1)$. It then follows from [1, p.66 line 5] that $f|w_p$ is an oldform, too. Therefore we have an inclusion

$$\{f + f|w_p \mid f \in S_{k+\frac{1}{2}}(1)\} \subseteq S_{k+\frac{1}{2}}^{\text{old}}(p) \cap \mathbb{S}_{k+\frac{1}{2}}(p) = \mathbb{S}_{k+\frac{1}{2}}^{\text{old}}(p).$$

While [8, Theorem 5.11 in p.2612] says

$$\mathbb{S}_{k+\frac{1}{2}}(p) = \mathbb{S}_{k+\frac{1}{2}}^{\text{new}}(p) \bigoplus \{f + f|w_p \mid f \in S_{k+\frac{1}{2}}(1)\},$$

in [1] it is shown that

$$\mathbb{S}_{k+\frac{1}{2}}(p) = \mathbb{S}_{k+\frac{1}{2}}^{\text{new}}(p) \bigoplus \mathbb{S}_{k+\frac{1}{2}}^{\text{old}}(p).$$

Thus we have

$$\mathbb{S}_{k+\frac{1}{2}}^{\text{old}}(p) = \{f + f|w_p \mid f \in S_{k+\frac{1}{2}}(1)\}. \tag{12}$$

It follows from (12) that the map Φ is surjective. Next we will show that Φ is injective. Indeed if we assume $\Phi(f) = f + f|w_p = 0$ for $f = \sum_{n\geq 1} a(n)q^n \in S_{k+\frac{1}{2}}(1)$, then we have

$$\begin{aligned} 0 &= f + f|w_p \\ &= \sum_{n\geq 1} a(n)q^n + \sum_{n\geq 1}\left(\left(\frac{(-1)^k n}{p}\right)a(n) + p^k a(n/p^2)\right)q^n \quad \text{by [1, (44)].}\end{aligned}$$

Comparing the coefficients of $q^{p^2 m}$ we obtain that $a(p^2 m) = -p^k a(m)$ for all positive integers m, which means that $f|U(p^2) = -p^k f$. But it follows from [1, Theorem 2] that

$$S_{k+\frac{1}{2}}(p) = S_{k+\frac{1}{2}}(1) \bigoplus S_{k+\frac{1}{2}}(1)|U(p^2) \bigoplus S_{k+\frac{1}{2}}^{\text{new}}(p),$$

from which we deduce that $f = 0$. This shows that Φ is injective. Finally we will show that Φ is Hecke equivariant. By [1, lines 2-3 in p.66] we have

$$f|w_p = p^{1-k}(-f|U(p^2) + f|T_{k+\frac{1}{2},1}(p^2).$$

Thus for any prime $r(\neq p)$, one has

$$\begin{aligned}\Phi(f)|T(r^2) &= (f + f|w_p)|T(r^2) = f|T(r^2) + (f|w_p)|T(r^2)\\ &= f|T(r^2) + (f|T(r^2))|w_p \quad \text{since } U(p^2) \text{ and } T(p^2) \text{ commute with } T(r^2)\\ &= \Phi(f|T(r^2)).\end{aligned}$$

□

Let $\Phi_1 : S_{2k}(1) \to S_{k+\frac{1}{2}}(1)$ be the Kohnen-Shimura isomorphism which is Hecke equivariant, and we let $\Phi_2 : S_{2k}(1) \to S_{2k}^{+\text{old}}(p)$ be a Hecke equivariant isomorphism defined by $\Phi_2(f) = f + f|W_p$. These facts combined with Lemma 4.1 show that the map $\varphi := \Phi \circ \Phi_1 \circ \Phi_2^{-1}$ defines a Hecke equivariant isomorphism from $S_{2k}^{+\text{old}}(p)$ to $\mathbb{S}_{k+\frac{1}{2}}^{\text{old}}(p)$. Meanwhile, according to [1, page 71] there exists a Hecke equivariant isomorphism ψ from $S_{2k}^{+\text{new}}(p)$ to $\mathbb{S}_{k+\frac{1}{2}}^{\text{new}}(p)$. Now we see that the map

$$\acute{\mathbf{L}} : S_{2k}^{+}(p)(= S_{2k}^{+\text{old}}(p) \bigoplus S_{2k}^{+\text{new}}(p)) \to \mathbb{S}_{k+\frac{1}{2}}(p)$$

defined by

$$\acute{\mathbf{L}}(f \oplus g) = \varphi(f) \oplus \psi(g)$$

is a Hecke equivariant isomorphism.

5 Proof of Theorem 1.11

(i) Utilizing Theorems 1.9 and 1.10 we get that for each prime $r(\neq p)$,

$$(g_j|\mathcal{S}_+)|T(r) = (g_j|T(r^2))|\mathcal{S}_+ = \acute{\mathbf{L}}(f_j|T(r))|\mathcal{S}_+ = \lambda_j(r)\acute{\mathbf{L}}(f_j)|\mathcal{S}_+ = \lambda_j(r)g_j|\mathcal{S}_+,$$

for some $\lambda_j(r) \in \mathbb{C}$. By multiplicity one theorem we have

$$g_j|\mathcal{S}_+ = cf_j \quad \text{for some } c \in \mathbb{C}. \tag{13}$$

Comparing Fourier cofficient of q on both sides of (13) we obtain $c = c_j(|D|)$, which proves the assertion.

(ii) We note that for each prime $r(\neq p)$,

$$\acute{\mathbf{L}}^{-1}(g_j|T(r^2)) = \acute{\mathbf{L}}^{-1}(g_j)|T(r) = f_j|T(r) = \lambda_j(r)f_j = \acute{\mathbf{L}}^{-1}(\lambda_j(r)g_j),$$

so that

$$g_j|T(r^2) = \lambda_j(r)g_j. \tag{14}$$

Thus we have

$$\overline{g_j(-\bar{\tau})}|T(r^2) = \overline{(g_j|T(r^2))(-\bar{\tau})} = \lambda_j(r)\overline{g_j(-\bar{\tau})}$$

and therefore

$$\acute{\mathbf{L}}^{-1}(\overline{g_j(-\bar{\tau})})|T(r) = \acute{\mathbf{L}}^{-1}(\overline{g_j(-\bar{\tau})}|T(r^2)) = \lambda_j(r)\acute{\mathbf{L}}^{-1}(\overline{g_j(-\bar{\tau})}).$$

This implies that

$$\acute{\mathbf{L}}^{-1}(\overline{g_j(-\bar{\tau})}) = cf_j = c\acute{\mathbf{L}}^{-1}(g_j) \quad \text{for some nonzero } c \in \mathbb{C}.$$

Hence we have

$$\overline{g_j(-\bar{\tau})} = cg_j(\tau)$$

and therefore

$$\overline{c_j(n)} = c \cdot c_j(n).$$

(iii) We note that $g_1, \ldots, g_t$ are mutually orthogonal with respect to Petersson scalar product. Write

$$\varphi_{k,p}^{+}(z,\tau) = \sum_{j=1}^{t} h_j(z)g_j(\tau).$$

We then have

$$\varphi^{+}_{k,p}(-\bar{z},\tau)=\sum_{j=1}^{t}h_j(-\bar{z})g_j(\tau),$$

so that

$$g_j|\mathcal{S}_+=\langle g_j,\varphi^{+}_{k,p}(-\bar{z},\tau)\rangle_\tau=\overline{h_j(-\bar{z})}\langle g_j,g_j\rangle.$$

Meanwhile by the assertion (i), one has $g_j|\mathcal{S}_+=c_j(|D|)f_j$. Thus we come up with

$$h_j(z)=\frac{\overline{c_j(|D|)}}{\langle g_j,g_j\rangle}\overline{f_j(-\bar{z})}=\frac{\overline{c_j(|D|)}}{\langle g_j,g_j\rangle}f_j(z),$$

since f_j has real Fourier coefficients. Hence we get that

$$\varphi^{+}_{k,p}(z,\tau)=\sum_{j=1}^{t}\frac{\overline{c_j(|D|)}}{\langle g_j,g_j\rangle}f_j(z)g_j(\tau),$$

and therefore

$$\begin{aligned}\frac{c^{*}_{k,D}}{i_p}f_j|\mathcal{S}^{*}_{+}&=\langle f_j(z),\varphi^{+}_{k,p}(z,-\bar{\tau})\rangle_z=c_j(|D|)\frac{\langle f_j,f_j\rangle}{\langle g_j,g_j\rangle}\overline{g_j(-\bar{\tau})}\\&=\overline{c_j(|D|)}\frac{\langle f_j,f_j\rangle}{\langle g_j,g_j\rangle}g_j\quad\text{by the assertion (ii).}\end{aligned}$$

(iv) Recalling from Theorem 1.6 that

$$F_j:=f_j|\mathcal{S}^{*}_{+}=\sum_{\substack{m\geq 1\ \ p\nmid m\\ (-1)^k m\equiv\square\ (4p)}}2r^{+}_{k,p}(f_j;D,(-1)^k m)e^{2\pi i m\tau}+\sum_{\substack{m\geq 1,\ \ p|m\\ (-1)^k m\equiv\square\ (4p)}}a_{F_j}(m)e^{2\pi i m\tau},$$

we see that the assertion (iv) is immediate from (iii). (v) First we claim that for each $i,j\in\{1,\ldots,t\}$, one has

$$\langle g_i|\mathcal{S}_+,f_j\rangle=\langle g_i,f_j|\mathcal{S}^{*}_{+}\rangle.$$

Indeed we obtain that

$$\langle g_i|\mathcal{S}_+,f_j\rangle=\langle c_j(|D|)f_i,f_j\rangle=c_j(|D|)\delta_{ij}\langle f_i,f_j\rangle$$

and by the assertion (iii)

$$\langle g_i,f_j|\mathcal{S}^{*}_{+}\rangle=\langle g_i,g_j\rangle c_j(|D|)\frac{\langle f_j,f_j\rangle}{\langle g_j,g_j\rangle}=c_j(|D|)\delta_{ij}\langle f_i,f_j\rangle.$$

More generally if we write $g=\sum_{i=1}^{t}a_i g_i$ and $f=\sum_{j=1}^{t}b_j f_j$, then

$$\begin{aligned}\langle g|\mathcal{S}_+,f\rangle&=\langle\sum_{i=1}^{t}a_i g_i|\mathcal{S}_+,\sum_{j=1}^{t}b_j f_j\rangle\\&=\sum_{i,j}a_i\overline{b_j}\langle g_i|\mathcal{S}_+,f_j\rangle\\&=\sum_{i,j}a_i\overline{b_j}\langle g_i,f_j|\mathcal{S}^{*}_{+}\rangle\quad\text{by the claim}\\&=\langle g,f|\mathcal{S}^{*}_{+}\rangle.\end{aligned}$$

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