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Strong and weak convergence of Ishikawa iterations for best proximity pairs

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Abstract: Let A and B be nonempty subsets of a normed linear space X . A mapping $T : A \cup B \rightarrow A \cup B$ is said to be a noncyclic relatively nonexpansive mapping if $T(A) \subseteq A$, $T(B) \subseteq B$ and $\|Tx - Ty\| \leq \|x - y\|$ for all $(x, y) \in A \times B$. A best proximity pair for such a mapping T is a point $(p, q) \in A \times B$ such that $p = Tp$, $q = Tq$ and $d(p, q) = \text{dist}(A, B)$. In this work, we introduce a geometric notion of proximal Opial's condition on a nonempty, closed and convex pair of subsets of strictly convex Banach spaces. By using this geometric notion, we study the strong and weak convergence of the Ishikawa iterative scheme for noncyclic relatively nonexpansive mappings in uniformly convex Banach spaces. We also establish a best proximity pair theorem for noncyclic contraction type mappings in the setting of strictly convex Banach spaces.

Keywords: best proximity pair; uniformly convex Banach space; noncyclic relatively nonexpansive mapping; Ishikawa iteration

MSC: 47H10, 47H09, 46B20

1 Introduction

Let X be a normed linear space. A self-mapping $T : X \rightarrow X$ is said to be *nonexpansive* provided that $\|Tx - Ty\| \leq \|x - y\|$. It is well known that if A is a nonempty, compact and convex subset of a Banach space X , then every nonexpansive mapping of A into itself has a fixed point.

In 1965, Kirk proved that if A is a nonempty, weakly compact and convex subset of a Banach space X with a geometric property, called *normal structure*, then every nonexpansive self-mapping $T : A \rightarrow A$ has a fixed point (Kirk's fixed point theorem [1]).

Now, suppose that (A, B) is a nonempty pair of subsets of a normed linear space X . A mapping $T : A \cup B \rightarrow A \cup B$ is said to be *noncyclic relatively nonexpansive* if T is noncyclic, that is, $T(A) \subseteq A$, $T(B) \subseteq B$, and $\|Tx - Ty\| \leq \|x - y\|$ for all $(x, y) \in A \times B$. Under this weaker assumption over T w.r.t. nonexpansiveness, the existence of the so-called *best proximity pair*, that is, a point $(p, q) \in A \times B$ such that

$$p = Tp, \quad q = Tq \quad \text{and} \quad d(p, q) = \text{dist}(A, B),$$

was first studied in [2] as below.

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Theorem 1.1. (See Theorem 2.2 of [2]) *Let (A, B) be a nonempty, bounded, closed and convex pair of subsets of a uniformly convex Banach space X , and suppose $T : A \cup B \rightarrow A \cup B$ is a noncyclic relatively nonexpansive mapping. Then T has a best proximity pair.*

An interesting observation about Theorem 1.1 is that the mapping T may not be continuous whereas the existence of two fixed points of T which estimates the distance between two sets A and B is guaranteed (see also [3, 4] for more information).

In addition to the existence result of best proximity pairs for noncyclic relatively nonexpansive mappings, the convergence of Krasnoselskii's iteration process for such mappings was discussed as follows.

Theorem 1.2. (Theorem 2.3 of [2]) *Let (A, B) be a nonempty, bounded, closed and convex pair in a uniformly convex Banach space X and $T : A \cup B \rightarrow A \cup B$ be a noncyclic relatively nonexpansive mapping. Let $x_0 \in A_0$ and define $x_{n+1} = \frac{x_n + Tx_n}{2}$. Then $\|x_n - Tx_n\| \rightarrow 0$. Moreover, if $T(A)$ is compact, then $\{x_n\}$ converges to a fixed point of T .*

The current paper is organized as follows: in Section 2, we recall some notions and notations which will be used in our coming discussion. We also introduce a geometric concept of *proximal Opial's condition* on a nonempty, closed and convex pair of subsets of strictly convex Banach spaces. In Section 3, we improve Theorem 1.2 and prove strong and weak convergence theorems for noncyclic relatively nonexpansive mappings in uniformly convex Banach spaces. Finally, in Section 4, we establish a best proximity pair theorem for *noncyclic contraction type mappings* in the setting of strictly convex Banach spaces. We also present some appropriate examples to illustrate our main conclusions.

2 Preliminaries

To describe our results, we need some definitions and notations.

Definition 2.1. *A Banach space X is said to be*

(i) *uniformly convex if there exists a strictly increasing function $\delta : [0, 2] \rightarrow [0, 1]$ such that the following implication holds for all $x, y, p \in X, R > 0$ and $r \in (0, 2R]$:*

$$\begin{cases} \|x - p\| \leq R, \\ \|y - p\| \leq R, \\ \|x - y\| \geq r \end{cases} \Rightarrow \left\| \frac{x+y}{2} - p \right\| \leq (1 - \delta(\frac{r}{R}))R;$$

(ii) *strictly convex if the following implication holds $x, y, p \in X$ and $R > 0$:*

$$\begin{cases} \|x - p\| \leq R, \\ \|y - p\| \leq R, \\ x \neq y \end{cases} \Rightarrow \left\| \frac{x+y}{2} - p \right\| < R.$$

It is well known that Hilbert spaces and l^p spaces ($1 < p < \infty$) are uniformly convex Banach spaces. Also, the Banach space l^1 with the norm

$$\|x\| = \sqrt{\|x\|_1 + \|x\|_2}, \quad \forall x \in l^1,$$

where, $\|\cdot\|_1$ and $\|\cdot\|_2$ are the norms on l^1 and l^2 , respectively is strictly convex but not uniformly convex (see [5] for more details).

The following lemma gives a suitable property for characterization of uniformly convex Banach spaces.

Lemma 2.2. [6] *A Banach space X is uniformly convex if and only if for each fixed number $r > 0$, there exists a continuous strictly increasing function $\varphi : [0, \infty) \rightarrow [0, \infty)$, $\varphi(t) = 0 \Leftrightarrow t = 0$, such that*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\varphi(\|x - y\|),$$

for all $\lambda \in [0, 1]$ and all $x, y \in X$ such that $\|x\| \leq r$ and $\|y\| \leq r$.

We also refer to the following auxiliary lemma.

Lemma 2.3. Consider a strictly increasing function $\phi : [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$. If a sequence $\{r_n\}$ in $[0, \infty)$ satisfies $\lim_{n \rightarrow \infty} \phi(r_n) = 0$, then $\lim_{n \rightarrow \infty} r_n = 0$.

Another appropriate property of uniformly convex Banach spaces will be explained in the next lemma.

Lemma 2.4. [7] Let (A, B) be a nonempty and closed pair in a uniformly convex Banach space X such that A is convex. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B such that $\lim_{n \rightarrow \infty} \|x_n - y_n\| = \text{dist}(A, B)$ and $\lim_{n \rightarrow \infty} \|z_n - y_n\| = \text{dist}(A, B)$, then we have $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$.

We shall say that a pair (A, B) of subsets of a Banach space X satisfies a property if both A and B satisfy that property. For example, (A, B) is convex if and only if both A and B are convex; $(A, B) \subseteq (C, D) \Leftrightarrow A \subseteq C$, and $B \subseteq D$. We shall also adopt the notation

$$\delta_x(A) = \sup\{d(x, y) : y \in A\} \text{ for all } x \in X,$$

$$\delta(A, B) = \sup\{\delta_x(B) : x \in A\},$$

$$\text{diam}(A) = \delta(A, A).$$

The closed and convex hull of a set A will be denoted by $\overline{\text{conv}}(A)$. Also, $\mathcal{B}(p, r)$ will denote the closed ball in the space X centered at $p \in X$ with radius $r > 0$.

Given (A, B) a pair of nonempty subsets of a Banach space, if $\|x - y\| = \text{dist}(A, B)$, $x \in A$, $y \in B$ then y (x) is called a proximal point of the point x (y). Also, the proximal pair of (A, B) will be denoted by (A_0, B_0) which is given by

$$A_0 = \{x \in A : \|x - y'\| = \text{dist}(A, B) \text{ for some } y' \in B\},$$

$$B_0 = \{y \in B : \|x' - y\| = \text{dist}(A, B) \text{ for some } x' \in A\}.$$

Proximal pairs may be empty but, in particular, if (A, B) is a nonempty, bounded, closed and convex pair in a reflexive Banach space X , then (A_0, B_0) is also nonempty, closed and convex.

For a noncyclic mapping $T : A \cup B \rightarrow A \cup B$ the set of all best proximity pairs of T will be denoted by $\text{Prox}_{A \times B}(T)$.

Suppose A is a nonempty and convex subset of X . A self-mapping $T : A \rightarrow A$ is said to be affine if

$$T(\lambda x + (1 - \lambda)y) = \lambda Tx + (1 - \lambda)Ty,$$

for any $x, y \in A$ and $0 < \lambda < 1$. Also, a mapping $T : A \cup B \rightarrow A \cup B$ is called affine provided that both $T|_A$ and $T|_B$ are affine, where (A, B) is a convex pair in X .

Definition 2.5. A pair (A, B) in a Banach space is said to be proximal if $A = A_0$ and $B = B_0$.

For a nonempty subset A of X a metric projection operator $\mathcal{P}_A : X \rightarrow 2^A$ is defined as

$$\mathcal{P}_A(x) := \{y \in A : \|x - y\| = \text{dist}(\{x\}, A)\},$$

where 2^A denotes the set of all subsets of A . It is well known that if A is a nonempty, closed and convex subset of a reflexive and strictly convex Banach space X , then the metric projection $\mathcal{P}_A$ is single valued from X to A .

Next result will be used in the sequel.

Proposition 2.6. [2, 8] Let (A, B) be a nonempty, bounded, closed and convex pair in a reflexive and strictly convex Banach space X . Define $\mathcal{P} : A_0 \cup B_0 \rightarrow A_0 \cup B_0$ as

$$\mathcal{P}(x) = \begin{cases} \mathcal{P}_{A_0}(x) & \text{if } x \in B_0, \\ \mathcal{P}_{B_0}(x) & \text{if } x \in A_0. \end{cases}$$

Then the following statements hold:

- (i) $\|x - \mathcal{P}x\| = \text{dist}(A, B)$ for any $x \in A_0 \cup B_0$ and $\mathcal{P}(A_0) \subseteq B_0, \mathcal{P}(B_0) \subseteq A_0$.
- (ii) $\mathcal{P}$ is an isometry, that is, $\|\mathcal{P}x - \mathcal{P}y\| = \|x - y\|$ for all $(x, y) \in A_0 \times B_0$.
- (iii) $\mathcal{P}$ is affine.

Definition 2.7. [9] Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. The pair (A, B) is said to have the P -property if and only if

$$\left. \begin{array}{l} d(x_1, y_1) = \text{dist}(A, B) \\ d(x_2, y_2) = \text{dist}(A, B) \end{array} \right\} \Rightarrow d(x_1, x_2) = d(y_1, y_2),$$

whenever $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Next theorem plays an important role in our next conclusions.

Lemma 2.8. [10, 11] Every, nonempty, bounded, closed and convex pair in a uniformly convex Banach space X has the P -property.

We finish this section by introducing a notion of proximal Opial's condition. We recall that a Banach space X is said to satisfy Opial's condition ([12]) if for each sequence $\{x_n\}$ in X with $x_n \xrightarrow{w} x$ we have

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\| \quad \forall y \in X \text{ with } y \neq x.$$

It is well known that finite dimensional Banach spaces, Hilbert spaces and l^p spaces ($1 < p < \infty$) satisfy Opial's condition.

Definition 2.9. Let (A, B) be a nonempty, closed, convex and proximal pair in a strictly convex Banach space X . Then (A, B) is said to satisfy proximal Opial's condition if for every sequence $\{x_n\}$ in A (respectively in B) with $x_n \xrightarrow{w} u \in A$ (respectively $x_n \xrightarrow{w} u \in B$) we have

$$\limsup_{n \rightarrow \infty} \|x_n - \mathcal{P}u\| < \limsup_{n \rightarrow \infty} \|x_n - y\|, \quad \forall y \neq \mathcal{P}u \in B \text{ (respectively } \forall y \neq \mathcal{P}u \in A).$$

We mention that the condition of proximality of the closed and convex pair (A, B) in the above definition is essential even if the considered Banach space X is Hilbert. Let us illustrate this fact with the following example.

Example 2.1. Consider the Hilbert space $X = l^2$ with the canonical basis $\{e_n\}$ and let

$$A = \overline{\text{con}}\{e_{2n} : n \in \mathbb{N}\}, \quad B = \overline{\text{con}}\{2e_1, e_3\}.$$

Then $\text{dist}(A, B) = 1$. Also, the closed and convex pair (A, B) is not proximal. Notice that $e_{2n} \xrightarrow{w} 0$. Let $u = 0$ and $y = \frac{1}{4}e_1 + \frac{7}{8}e_3$. Then $\mathcal{P}u = e_3$ and we have

$$\limsup_{n \rightarrow \infty} \|x_n - \mathcal{P}u\| = \limsup_{n \rightarrow \infty} \|e_{2n} - e_3\| = \sqrt{2} > \sqrt{1 + \frac{1}{16} + \frac{49}{64}} = \limsup_{n \rightarrow \infty} \|e_{2n} - \frac{1}{4}e_1 + \frac{7}{8}e_3\| = \limsup_{n \rightarrow \infty} \|x_n - y\|,$$

that is, (A, B) does not have proximal Opial's condition.

Definition 2.10. [13] A nonempty, closed and convex pair (A, B) in a strictly convex Banach space X is said to have Pythagorean property if for each (x, y) in $A_0 \times B_0$ we have

$$\begin{aligned} \|x - y\|^2 &= \|x - \mathcal{P}x\|^2 + \|\mathcal{P}x - y\|^2, \\ \|x - y\|^2 &= \|x - \mathcal{P}y\|^2 + \|\mathcal{P}y - y\|^2. \end{aligned}$$

Proposition 2.11. *Let (A, B) be a nonempty, closed, convex and proximal pair in a uniformly convex Banach space X which has the Opial property. If (A, B) possesses Pythagorean property, then it satisfies proximal Opial's condition.*

Proof. Let $\{x_n\}$ be a sequence in A such that $x_n \xrightarrow{w} u$ and $y \neq \mathcal{P}u$ in B . By Pythagorean property

$$\|x_n - \mathcal{P}u\|^2 = \|x_n - u\|^2 + \|u - \mathcal{P}u\|^2,$$

$$\|x_n - y\|^2 = \|x_n - \mathcal{P}y\|^2 + \|y - \mathcal{P}y\|^2.$$

Notice that $\|u - \mathcal{P}u\| = \|y - \mathcal{P}y\| = \text{dist}(A, B)$. Since X has the Opial property, $\limsup_{n \rightarrow \infty} \|x_n - u\| < \limsup_{n \rightarrow \infty} \|x_n - \mathcal{P}y\|$ which implies that

$$\limsup_{n \rightarrow \infty} \|x_n - y\|^2 = \limsup_{n \rightarrow \infty} \|x_n - \mathcal{P}y\|^2 + \text{dist}(A, B)^2 > \limsup_{n \rightarrow \infty} \|x_n - u\|^2 + \text{dist}(A, B)^2 = \limsup_{n \rightarrow \infty} \|x_n - \mathcal{P}u\|^2,$$

and the result follows. $\square$

Next corollaries are straightforward consequences of Proposition 2.11.

Corollary 2.12. *Let (A, B) be a nonempty, bounded, closed and convex pair in a Hilbert space $\mathbb{H}$. Then the pair (A_0, B_0) has the proximal Opial's condition.*

Corollary 2.13. *Let (A, B) be a nonempty, bounded, closed and convex pair in l^p ($1 < p < \infty$) spaces such that (A, B) has the Pythagorean property. Then the pair (A_0, B_0) has the proximal Opial's condition.*

3 Convergence results

We begin our results of this section by improving Theorem 1.2 as follows.

Theorem 3.1. *Let (A, B) be a nonempty, bounded, closed and convex pair in a uniformly convex Banach space and $T : A \cup B \rightarrow A \cup B$ be a noncyclic relatively nonexpansive mapping such that $T(A)$ is compact. Let $x_0 \in A_0$ and $y_0 \in B_0$ be a unique proximal point of x_0 . Define*

$$x_{n+1} = \frac{x_n + Tx_n}{2}, \quad y_{n+1} = \frac{y_n + Ty_n}{2}.$$

Then $\|x_n - Tx_n\| \rightarrow 0$, $\|y_n - Ty_n\| \rightarrow 0$ and the sequence $\{(x_n, y_n)\}$ converges to a best proximity pair of T .

Proof. It follows from the proof of Theorem 1.2 that $\|x_n - Tx_n\| \rightarrow 0$ and that the sequence $\{x_n\}$ converges to a fixed point of T in A , say $p \in A$. Besides,

$$\|x_1 - y_1\| = \left\| \frac{x_0 + Tx_0}{2} - \frac{y_0 + Ty_0}{2} \right\| \leq \frac{1}{2} \|x_0 - y_0\| + \frac{1}{2} \|Tx_0 - Ty_0\| \leq \text{dist}(A, B).$$

Equivalently, and by induction we obtain $\|x_n - y_n\| = \text{dist}(A, B)$ for all $n \in \mathbb{N}$. Also for each n , $\|Tx_n - Ty_n\| = \text{dist}(A, B)$ and $\|Tx_n - y_n\| \rightarrow \text{dist}(A, B)$. Using Lemma 2.4, $\|y_n - Ty_n\| \rightarrow 0$. Since B is bounded and X is reflexive, there exists a subsequence $\{y_{n_k}\}$ of the sequence $\{y_n\}$ so that $y_{n_k} \xrightarrow{w} q \in B$. Thus $x_{n_k} - y_{n_k} \xrightarrow{w} p - q$. It now follows from the lower semi-continuity of norm that

$$\|p - q\| \leq \liminf_{k \rightarrow \infty} \|x_{n_k} - y_{n_k}\| = \text{dist}(A, B).$$

By Lemma 2.8 we conclude that $\|x_n - p\| = \|y_n - q\|$ for all $n \in \mathbb{N}$. Thereby, $y_n \rightarrow q$. We finish the proof by showing that q is a fixed point of T in B . Indeed,

$$\text{dist}(A, B) = \|p - q\| \leq \|p - Tq\| = \|Tp - Tq\| \leq \|p - q\| = \text{dist}(A, B).$$

Again by the fact that (A, B) has the P-property, we conclude that $q = Tq$. $\square$

In what follows we prove the strong and weak convergence results of Ishikawa iteration scheme for noncyclic mappings. For more information regarding Ishikawa iteration scheme we refer to [14, 15].

Let (A, B) be a nonempty, closed and convex pair in a strictly convex Banach space X . For $x_1 \in A_0$, put $x'_1 := \mathcal{P}(x_1) \in B_0$. Define the sequence pair $\{(x_n, x'_n)\}$ as follows:

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(y_n), & x'_{n+1} = (1 - \alpha_n)x'_n + \alpha_n T(y'_n); \\ y_n = (1 - \beta_n)x_n + \beta_n T(x_n), & y'_n = (1 - \beta_n)x'_n + \beta_n T(x'_n) \quad n = 1, 2, 3, \dots, \end{cases} \quad (1)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ satisfying one of the following conditions:

(C) $0 < \epsilon \leq \alpha_n(1 - \alpha_n)$ and $\beta_n \rightarrow 0$ as $n \rightarrow \infty$,

(D) $0 < \epsilon \leq \alpha_n \leq 1$ and $0 < \epsilon \leq \beta_n(1 - \beta_n)$.

Lemma 3.2. *Let (A, B) be a nonempty, closed and convex pair in a uniformly convex Banach space X such that either A or B is bounded and let $T : A \cup B \rightarrow A \cup B$ be a noncyclic relatively nonexpansive mapping. Suppose $\{x_n\}$ and $\{x'_n\}$ are given by (1). Then $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists for all $q \in \text{Fix}(T) \cap B_0$ and $\lim_{n \rightarrow \infty} \|x'_n - p\|$ exists for all $p \in \text{Fix}(T) \cap A_0$, where $\text{Fix}(T)$ denotes the set of all fixed points of the mapping T .*

Proof. Notice that (A_0, B_0) is nonempty, closed and convex. For any $q \in \text{Fix}(T) \cap B_0$ we have

$$\begin{aligned} \|x_{n+1} - q\| &= \|\alpha_n T y_n + (1 - \alpha_n)x_n - q\| \\ &\leq \alpha_n \|T y_n - q\| + (1 - \alpha_n) \|x_n - q\| \\ &\leq \alpha_n \|y_n - q\| + (1 - \alpha_n) \|x_n - q\| \\ &\leq \alpha_n \|\beta_n T x_n + (1 - \beta_n)x_n - q\| + (1 - \alpha_n) \|x_n - q\| \\ &\leq \alpha_n \beta_n \|x_n - q\| + \alpha_n (1 - \beta_n) \|x_n - q\| + (1 - \alpha_n) \|x_n - q\| \\ &= \|x_n - q\|. \end{aligned}$$

This implies $\{\|x_n - q\|\}$ is non-increasing and hence $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists. Similarly, we can show that $\lim_{n \rightarrow \infty} \|x'_n - p\|$ exists for all $p \in \text{Fix}(T) \cap A_0$ and hence the lemma. $\square$

Here, we establish the first main result of this section.

Theorem 3.3. *Let (A, B) be a nonempty, bounded, closed and convex pair in a uniformly convex Banach space X and let $T : A \cup B \rightarrow A \cup B$ be a noncyclic relatively nonexpansive mapping. For $x_1 \in A_0$ let $x'_1 \in B_0$ be a unique proximal point of x_1 . Assume $\{x_n\}$ and $\{x'_n\}$ are given by (1) where $\{\alpha_n\}, \{\beta_n\}$ satisfy either (C) or (D). Then*

$$\lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0, \quad \lim_{n \rightarrow \infty} \|x'_n - T x'_n\| = 0.$$

Further the sequence pair $\{(x_n, x'_n)\}$ converges to a best proximity pair of T if $T(A)$ lies in a compact subset.

Proof. By Theorem 1.1, $\text{Fix}(T) \cap B_0$ is nonempty. Let $p \in \text{Fix}(T) \cap B_0$. Then from Lemma 2.2 there exists continuous strictly increasing function $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n T y_n + (1 - \alpha_n)x_n - p\|^2 \\ &= \|\alpha_n(T y_n - p) + (1 - \alpha_n)(x_n - p)\|^2 \\ &\leq \alpha_n \|T y_n - p\|^2 + (1 - \alpha_n) \|x_n - p\|^2 - \alpha_n(1 - \alpha_n) \varphi(\|T y_n - x_n\|) \\ &\leq \alpha_n \|y_n - p\|^2 + (1 - \alpha_n) \|x_n - p\|^2 - \alpha_n(1 - \alpha_n) \varphi(\|T y_n - x_n\|) \\ &= \alpha_n \|\beta_n T x_n + (1 - \beta_n)x_n - p\|^2 + (1 - \alpha_n) \|x_n - p\|^2 - \alpha_n(1 - \alpha_n) \varphi(\|T y_n - x_n\|) \\ &\leq \alpha_n \beta_n \|T x_n - p\|^2 + \alpha_n(1 - \beta_n) \|x_n - p\|^2 - \alpha_n \beta_n(1 - \beta_n) \varphi(\|T x_n - x_n\|) \\ &\quad + (1 - \alpha_n) \|x_n - p\|^2 - \alpha_n(1 - \alpha_n) \varphi(\|T y_n - x_n\|) \\ &\leq \alpha_n \beta_n \|x_n - p\|^2 + \alpha_n(1 - \beta_n) \|x_n - p\|^2 - \alpha_n \beta_n(1 - \beta_n) \varphi(\|T x_n - x_n\|) \end{aligned}$$

$$\begin{aligned}
& + (1 - \alpha_n) \|x_n - p\|^2 - \alpha_n(1 - \alpha_n) \varphi(\|Ty_n - x_n\|) \\
& = \|x_n - p\|^2 - \alpha_n \beta_n (1 - \beta_n) \varphi(\|Tx_n - x_n\|) - \alpha_n(1 - \alpha_n) \varphi(\|Ty_n - x_n\|).
\end{aligned}$$

From the above, we can deduce the following inequalities:

$$\alpha_n(1 - \alpha_n) \varphi(\|Ty_n - x_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2, \quad (2)$$

$$\alpha_n \beta_n (1 - \beta_n) \varphi(\|Tx_n - x_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2. \quad (3)$$

We consider two following cases.

CASE I:

Suppose α_n and β_n satisfy (C). From (2) we conclude that

$$\sum_{n=1}^m \alpha_n(1 - \alpha_n) \varphi(\|Ty_n - x_n\|) \leq \|x_1 - p\|^2 - \|x_{m+1} - p\|^2.$$

Letting $m \rightarrow \infty$, we obtain $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) \varphi(\|Ty_n - x_n\|) < \infty$. In view of the fact that $\alpha_n(1 - \alpha_n) \geq \epsilon$, it results that $\lim_n \varphi(\|Ty_n - x_n\|) = 0$. Therefore, $\|Ty_n - x_n\| \rightarrow 0$. Since $\mathcal{P}$ is affine and isometry and $\mathcal{P}T = T\mathcal{P}$ on $A_0 \cup B_0$,

$$\begin{aligned}
\|Tx_n - \mathcal{P}x_n\| & \leq \|Tx_n - \mathcal{P}Ty_n\| + \|\mathcal{P}Ty_n - \mathcal{P}x_n\| \\
& = \|Tx_n - T(\mathcal{P}y_n)\| + \|Ty_n - x_n\| \\
& \leq \|x_n - \mathcal{P}y_n\| + \|Ty_n - x_n\| \\
& \leq \|x_n - \mathcal{P}\{(1 - \beta_n)x_n + \beta_n Tx_n\}\| + \|Ty_n - x_n\| \\
& = \|x_n - \mathcal{P}x_n + \beta_n(\mathcal{P}x_n - \mathcal{P}Tx_n)\| + \|Ty_n - x_n\| \\
& \leq \|x_n - \mathcal{P}x_n\| + \beta_n \|\mathcal{P}x_n - \mathcal{P}Tx_n\| + \|Ty_n - x_n\| \\
& = \|x_n - \mathcal{P}x_n\| + \beta_n \|x_n - Tx_n\| + \|Ty_n - x_n\|.
\end{aligned}$$

Thus,

$$\|Tx_n - \mathcal{P}x_n\| \leq \|x_n - \mathcal{P}x_n\| + \beta_n \|x_n - Tx_n\| + \|Ty_n - x_n\|. \quad (4)$$

If $n \rightarrow \infty$ in above relation, we obtain

$$\lim_n \|Tx_n - \mathcal{P}x_n\| \leq \lim_n \|x_n - \mathcal{P}x_n\| = \text{dist}(A, B).$$

Lemma 2.4 implies that $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| \rightarrow 0$.

CASE II:

Suppose the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy (D). By (3) we deduce that

$$\sum_{n=1}^m \alpha_n \beta_n (1 - \beta_n) \varphi(\|Tx_n - x_n\|) \leq \|x_1 - p\|^2 - \|x_{m+1} - p\|^2.$$

As $m \rightarrow \infty$, we get $\sum_{n=1}^{\infty} \alpha_n \beta_n (1 - \beta_n) \varphi(\|Tx_n - x_n\|) < \infty$. On account of the facts that $\alpha_n \beta_n (1 - \beta_n) \geq \epsilon^2$, one can be sure that $\limsup_{n \rightarrow \infty} \varphi(\|Tx_n - x_n\|) = 0$ which means $\limsup_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$.

Therefore, $\|x_n - Tx_n\| \rightarrow 0$ in both the cases.

Now, suppose that $T(A)$ lies in a compact subset. Then $\{Tx_n\}$ has a convergent subsequence $\{Tx_{n_k}\}$, converging to some point $u \in A_0$. Since $\lim_{n \rightarrow \infty} \|Tx_{n_k} - x_{n_k}\| = 0$, $x_{n_k} \rightarrow u$. Besides, from $T(\mathcal{P}u) = \mathcal{P}(Tu)$ we have

$$\|Tx_{n_k} - \mathcal{P}(Tu)\| = \|Tx_{n_k} - T(\mathcal{P}u)\| \leq \|x_{n_k} - \mathcal{P}u\| \rightarrow \text{dist}(A, B),$$

which deduces that $Tx_{n_k} \rightarrow Tu$. Thus $Tu = u$ and so $T(\mathcal{P}u) = \mathcal{P}u$. Therefore by Lemma 3.2, $\lim_n \|x_n - \mathcal{P}u\|$ exists and

$$\lim_n \|x_n - \mathcal{P}u\| = \lim_k \|x_{n_k} - \mathcal{P}u\| = \|u - \mathcal{P}u\| = \text{dist}(A, B),$$

which implies $x_n \rightarrow u \in \text{Fix}(T) \cap A$. Besides, by the assumption $\|x_1 - x'_1\| = \text{dist}(A, B)$. We now have

$$\begin{aligned} \|x_2 - x'_2\| &= \|(1 - \alpha_1)x_1 + \alpha_1 T(y_1) - ((1 - \alpha_1)x'_1 + \alpha_1 Ty'_1)\| \\ &\leq (1 - \alpha_1)\|x_1 - x'_1\| + \alpha_1\|Ty_1 - Ty'_1\| \\ &\leq (1 - \alpha_1)\|x_1 - x'_1\| + \alpha_1\|y_1 - y'_1\| \\ &= (1 - \alpha_1)\|x_1 - x'_1\| + \alpha_1\|(1 - \beta_1)x_1 + \beta_1 T(x_1) - ((1 - \beta_1)x'_1 + \beta_1 Tx'_1)\| \\ &\leq (1 - \alpha_1)\|x_1 - x'_1\| + \alpha_1\|x_1 - x'_1\| \\ &= \|x_1 - x'_1\| \\ &= \text{dist}(A, B). \end{aligned}$$

Continuing this process and by induction we obtain $\|x_n - x'_n\| = \text{dist}(A, B)$ for each $n \in \mathbb{N}$. By a similar argument of aforesaid discussion we conclude that

$$\|x'_n - Tx'_n\| \rightarrow 0 \quad \text{and} \quad x'_n \rightarrow v \in \text{Fix}(T) \cap B.$$

On the other hand,

$$\|u - v\| = \lim_{n \rightarrow \infty} \|x_n - x'_n\| = \text{dist}(A, B),$$

which deduces that $(u, v) \in \text{Prox}_{A \times B}(T)$. □

Theorem 3.4. *Let (A, B) be a nonempty, bounded, closed and convex pair in a uniformly convex Banach space X with (A_0, B_0) satisfying proximal Opial's condition and $T : A \cup B \rightarrow A \cup B$ be a noncyclic relatively nonexpansive mapping. Then $I - T$ is demi-closed at zero, i.e., for each sequence pair $\{(x_n, x'_n)\}$ in (A_0, B_0) with $\|x_n - x'_n\| = \text{dist}(A, B)$ for each $n \in \mathbb{N}$ if $\{(x_n, x'_n)\}$ converges weakly to (x, y) and $\{(Tx_n - x_n, Tx'_n - x'_n)\}$ converges to $(0, 0)$ then $(x, y) \in \text{Prox}_{A \times B}(T)$.*

Proof. Let $\{x_n\}$ weakly converge to x in A_0 and $x_n - Tx_n \rightarrow 0$. Since $\mathcal{P}Tx = T(\mathcal{P}x)$ for all $x \in A_0$,

$$\begin{aligned} \|x_n - \mathcal{P}Tx\| &\leq \|x_n - Tx_n\| + \|Tx_n - \mathcal{P}Tx\| \\ &= \|x_n - Tx_n\| + \|Tx_n - T(\mathcal{P}x)\| \\ &\leq \|x_n - Tx_n\| + \|x_n - \mathcal{P}x\|. \end{aligned}$$

Thereby,

$$\limsup_{n \rightarrow \infty} \|x_n - \mathcal{P}Tx\| \leq \limsup_{n \rightarrow \infty} \|x_n - \mathcal{P}x\|.$$

Due to the fact that (A_0, B_0) has the proximal Opial's condition, we must have $Tx = x$. Equivalently, we can prove that $Ty = y$. Moreover,

$$\|x - y\| \leq \liminf_{n \rightarrow \infty} \|x_n - x'_n\| = \text{dist}(A, B),$$

that is, $(x, y) \in \text{Prox}_{A \times B}(T)$. □

Next theorem guarantees the weak convergence of the iterative sequence defined in (1) to finding best proximity pairs of noncyclic relatively nonexpansive mappings in uniformly convex Banach spaces.

Theorem 3.5. *Let (A, B) be a nonempty, bounded, closed and convex pair in a uniformly convex Banach space X with (A_0, B_0) satisfying proximal Opial's condition and let $T : A \cup B \rightarrow A \cup B$ be a noncyclic relatively nonexpansive mapping. Let $x_1 \in A_0$ and $x'_1 \in B_0$ be a unique proximal point of x_1 . Suppose $\{x_n\}$ and $\{x'_n\}$ are given by (1) where $\{\alpha_n\}, \{\beta_n\}$ satisfy either condition (C) or (D). Then $\{(x_n, x'_n)\}$ converges weakly to a best proximity pair of T .*

Proof. We assert that $\{x_n\}$ weakly converges to a member of $\text{Fix}(T) \cap A_0$. Let $\{x_{n_k}\}$ and $\{x_{m_k}\}$ be subsequences of $\{x_n\}$ converging weakly to u and v respectively, such that $u \neq v$. Since $\|x_{n_k} - Tx_{n_k}\|$ and $\|x_{m_k} - Tx_{m_k}\|$ converge to 0, by Theorem 3.4 $Tu = u$ and $Tv = v$. From Lemma 3.2, $\lim_{n \rightarrow \infty} \|x_n - \mathcal{P}u\|$ and $\lim_{n \rightarrow \infty} \|x_n - \mathcal{P}v\|$ exist. It now follows from the proximal Opial's condition that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|x_n - \mathcal{P}u\| &= \limsup_{n \rightarrow \infty} \|x_{n_k} - \mathcal{P}u\| \\ &< \limsup_{n \rightarrow \infty} \|x_{n_k} - \mathcal{P}v\| \\ &= \limsup_{n \rightarrow \infty} \|x_{m_k} - \mathcal{P}v\| \\ &< \limsup_{n \rightarrow \infty} \|x_{m_k} - \mathcal{P}u\| \\ &= \limsup_{n \rightarrow \infty} \|x_n - \mathcal{P}u\|, \end{aligned}$$

which is a contradiction. Thus $u = v$ and so $\{x_n\}$ converges weakly to u in A_0 . Similarly we can show that $\{x'_n\}$ converges weakly to a member u' in $\text{Fix}(T) \cap B_0$. By Theorem 3.3, $\|x_n - x'_n\| = \text{dist}(A, B)$ for each $n \in \mathbb{N}$ and that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ and $\lim_{n \rightarrow \infty} \|x'_n - Tx'_n\| = 0$. Therefore, from Theorem 3.4, $(u, u') \in \text{Prox}_{A \times B}(T)$ and this completes the proof of theorem. $\square$

4 Noncyclic contraction type mappings

We begin our main results of this section with the following well known fixed point theorem which is the main idea for coming discussions.

Theorem 4.1. *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a continuous self-mapping such that*

$$d(Tx, T^2x) \leq \alpha d(x, Tx), \quad \forall x \in X,$$

where $\alpha \in [0, 1)$, then T has a fixed point.

In fact, the condition on T ensures that $\{T^n(x)\}$ is a Cauchy sequence for each $x \in X$, and continuity does the rest.

Here, we state the following existence theorems of best proximity pairs for two different classes of noncyclic mappings.

Theorem 4.2. [16] *Let (A, B) be a nonempty weakly compact convex pair in a strictly convex Banach space X . Assume that $T : A \cup B \rightarrow A \cup B$ is a noncyclic contraction mapping, that is, T is noncyclic on $A \cup B$ and*

$$\|Tx - Ty\| \leq r\|x - y\| + (1 - r)\text{dist}(A, B),$$

for some $r \in [0, 1)$ and for all $(x, y) \in A \times B$. Then $\text{Prox}_{A \times B}(T)$ is nonempty.

Let (A, B) be a nonempty pair in a normed linear space X . For a noncyclic mapping $T : A \cup B \rightarrow A \cup B$ and $(x, y) \in A \times B$, we set

$$\mathcal{O}(x, \infty) := \{x, Tx, T^2x, \dots\}, \quad \mathcal{O}(y, \infty) := \{y, Ty, T^2y, \dots\}.$$

We note that $(\mathcal{O}(x, \infty), \mathcal{O}(y, \infty)) \subseteq (A, B)$.

Definition 4.3. [17] *Let (A, B) be a nonempty pair of subsets of a normed linear space X . A mapping $T : A \cup B \rightarrow A \cup B$ is said to be a generalized noncyclic relatively nonexpansive mapping provided that T is noncyclic on $A \cup B$ and*

$$\|Tx - Ty\| = \|x - y\|, \quad \text{for all } (x, y) \in A \times B \quad \text{with} \quad \|x - y\| = \text{dist}(A, B),$$

and

$$\|Tx - Ty\| \leq \alpha\|x - y\| + (1 - \alpha) \min\{\delta_x(\mathcal{O}(Ty, \infty)), \delta_y(\mathcal{O}(Tx, \infty))\},$$

for some $\alpha \in [0, 1]$ and for all $(x, y) \in A \times B$ with $\|x - y\| > \text{dist}(A, B)$.

Theorem 4.4. (see Theorem 3.2 of [17]) *Let (A, B) be a nonempty, bounded, closed and convex pair in a uniformly convex Banach space X . Suppose that $T : A \cup B \rightarrow A \cup B$ is a generalized noncyclic relatively nonexpansive mapping. Then T has a best proximity pair.*

In this section, we establish a best proximity pair result under some sufficient conditions. For this purpose, we introduce a new class of noncyclic contraction mappings as below.

Definition 4.5. *Let (A, B) be nonempty pair of subsets of a metric space (X, d) . A mapping $T : A \cup B \rightarrow A \cup B$ is said to be a noncyclic contraction type mapping if there exists $\alpha \in [0, 1)$ such that*

$$d(Tx, T^2y) \leq \alpha d(x, Ty) + (1 - \alpha)\text{dist}(A, B),$$

for all $(x, y) \in A \times B$.

It is clear that every noncyclic contraction mapping is noncyclic contraction type mapping in the sense of Definition 4.5. The next example shows that the reverse implication does not hold.

Example 4.1. Let X be the real Banach space l^2 renormed according to

$$\|x\| = \max\{\|x\|_2, \sqrt{2}\|x\|_\infty\},$$

where, $\|x\|_\infty$ denotes the l^∞ -norm and $\|x\|_2$ the l^2 norm. Let $\{e_n\}$ be the canonical basis of l^2 . Note that this norm is equivalent to $\|\cdot\|_2$ and so, $(X, \|\cdot\|)$ is a reflexive Banach space. Consider

$$A = \{x := e_1 + e_2\} \quad \text{and} \quad B = \{y = (y_n) : y_3 = 1, \|y\| \leq \sqrt{2}\}.$$

Set $u := e_1 + e_3, v := e_2 + e_3$ and $w := e_3 + e_4$. Then u, v, w are three distinct elements in B and we have $\|x - u\| = \|x - v\| = \sqrt{2}$. Also, for each $y = (y_1, y_2, 1, y_4, \dots) \in B$ we have $\|y\|_2 \leq \sqrt{2}$ which implies that $\sum_{i \neq 3} |y_i|^2 \leq 1$ and since $\|y\|_\infty \leq 1$, we conclude that $|y_i| \leq 1$ for each $i \in \mathbb{N}$. Thus, for all $y \in B$ we have $\|x - y\| \geq \sqrt{2}$ which deduces that $\text{dist}(A, B) = \sqrt{2}$. Let $T : A \cup B \rightarrow A \cup B$ be a mapping defined as follows

$$Tx = x, \text{ and for each } y \in B, Ty = \begin{cases} v & \text{if } y \neq u, \\ w & \text{if } y = u. \end{cases}$$

Then T is noncyclic and for each $\alpha \in [0, 1)$ and $y \in B$ we have $T^2y = v$ and so,

$$\|Tx - T^2y\| = \sqrt{2} = \alpha\sqrt{2} + (1 - \alpha)\sqrt{2} \leq \alpha\|x - Ty\| + (1 - \alpha)\text{dist}(A, B),$$

that is, T is noncyclic contraction type mapping. We also note that if $y = u$, then

$$\|Tx - Ty\| = \|(e_1 + e_2) - (e_3 + e_4)\| = 2 > \sqrt{2} = \|x - y\|.$$

Therefore, T is not a generalized noncyclic relatively nonexpansive mapping.

Next lemma will be used in our main result of this section.

Lemma 4.6. [18] *Let (K_1, K_2) be a nonempty pair of subsets of a Banach space X . Then*

$$\delta(K_1, K_2) = \delta(\overline{\text{conv}}(K_1), \overline{\text{conv}}(K_2)).$$

The following theorem guarantees the existence of best proximity pairs for noncyclic contraction type mappings which are affine.

Theorem 4.7. (compare with Theorems 4.2, 4.4) *Let (A, B) be a nonempty, weakly compact and convex pair of subsets of a strictly convex Banach space X , and let $T : A \cup B \rightarrow A \cup B$ be a noncyclic contraction type mapping. Assume that $T|_A, T|_B$ are affine. Then T has a best proximity pair.*

Proof. Assume that $\mathfrak{F}$ denotes the collection of all nonempty, closed and convex pair $(E, F) \subseteq (A, B)$ such that T is noncyclic on $E \cup F$. Then $(A, B) \in \mathfrak{F} \neq \emptyset$. By using Zorn's Lemma we get an element say (K_1, K_2) which is minimal with respect to being nonempty, closed, convex and T -invariant. Note that $(\overline{\text{conv}}(T(K_1)), \overline{\text{conv}}(T(K_2))) \subseteq (K_1, K_2)$ is a nonempty, bounded, closed and convex pair in X . Further,

$$T(\overline{\text{conv}}(T(K_1))) \subseteq T(K_1) \subseteq \overline{\text{conv}}(T(K_1)),$$

and also,

$$T(\overline{\text{conv}}(T(K_2))) \subseteq \overline{\text{conv}}(T(K_2)),$$

that is, T is noncyclic on $\overline{\text{conv}}(T(K_1)) \cup \overline{\text{conv}}(T(K_2))$. It now follows from the minimality of (K_1, K_2) that

$$\overline{\text{conv}}(T(K_1)) = K_1, \quad \overline{\text{conv}}(T(K_2)) = K_2.$$

We now assert that

$$\overline{\text{conv}}(T^2(K_1)) = K_1 \quad \text{and} \quad \overline{\text{conv}}(T^2(K_2)) = K_2.$$

Since $T(K_1) \subseteq K_1$, we have $T^2(K_1) \subseteq T(K_1)$ and so, $\overline{\text{conv}}(T^2(K_1)) \subseteq \overline{\text{conv}}(T(K_1)) = K_1$. Then $T[\overline{\text{conv}}(T^2(K_1))] \subseteq T[\overline{\text{conv}}(T(K_1))]$. We show that $T[\overline{\text{conv}}(T(K_1))] \subseteq \overline{\text{conv}}(T^2(K_1))$. Suppose that $u \in T[\overline{\text{conv}}(T(K_1))]$. Then there exists an element $u' \in \overline{\text{conv}}(T(K_1))$ such that $u = Tu'$. By the fact that $u' \in \overline{\text{conv}}(T(K_1))$, we have $u' = \sum_{i=1}^n \alpha_i Tx_i$, where $0 \leq \alpha_i \leq 1$ and $\sum_{i=1}^n \alpha_i = 1$ and $x_i \in K_1$ for $i = 1, 2, \dots, n$. Since $T|_A$ is affine, we obtain

$$u = Tu' = T\left(\sum_{i=1}^n \alpha_i Tx_i\right) = \sum_{i=1}^n \alpha_i T(Tx_i) = \sum_{i=1}^n \alpha_i T^2x_i,$$

which implies that $u \in \overline{\text{conv}}(T^2(K_1))$. Therefore, $T[\overline{\text{conv}}(T^2(K_1))] \subseteq \overline{\text{conv}}(T^2(K_1))$. By the similar manner, we have $T[\overline{\text{conv}}(T^2(K_2))] \subseteq \overline{\text{conv}}(T^2(K_2))$, that is, T is noncyclic on $\overline{\text{conv}}(T^2(K_1)) \cup \overline{\text{conv}}(T^2(K_2))$. Minimality of (K_1, K_2) implies that $\overline{\text{conv}}(T^2(K_1)) = K_1$ and $\overline{\text{conv}}(T^2(K_2)) = K_2$. Let $x \in K_1$. Then for each $y \in K_2$ we have

$$\|Tx - T^2y\| \leq \alpha\|x - Ty\| + (1 - \alpha)\text{dist}(A, B) \leq \alpha\delta_x(K_2) + (1 - \alpha)\text{dist}(A, B).$$

Thus $T^2y \in \mathcal{B}(Tx; \alpha\delta_x(K_2) + (1 - \alpha)\text{dist}(A, B))$ for each $y \in K_2$ and so, $T^2(K_2) \subseteq \mathcal{B}(Tx; \alpha\delta_x(K_2) + (1 - \alpha)\text{dist}(A, B))$. Therefore,

$$K_2 = \overline{\text{conv}}(T^2(K_2)) \subseteq \mathcal{B}(Tx; \alpha\delta_x(K_2) + (1 - \alpha)\text{dist}(A, B)).$$

Hence, $\|y - Tx\| \leq \alpha\delta_x(K_2) + (1 - \alpha)\text{dist}(A, B)$ for each $y \in K_2$ and

$$\delta_{Tx}(K_2) \leq \alpha\delta_x(K_2) + (1 - \alpha)\text{dist}(A, B) \leq \alpha\delta(K_1, K_2) + (1 - \alpha)\text{dist}(A, B), \quad \forall x \in K_1.$$

This implies that

$$\delta(T(K_1), K_2) = \sup_{x \in K_1} \delta_{Tx}(K_2) \leq \alpha\delta(K_1, K_2) + (1 - \alpha)\text{dist}(A, B).$$

It now follows from Lemma 4.6 that

$$\delta(K_1, K_2) = \delta(\overline{\text{conv}}(T(K_1)), \overline{\text{conv}}(K_2)) = \delta(T(K_1), K_2) \leq \alpha\delta(K_1, K_2) + (1 - \alpha)\text{dist}(A, B).$$

Therefore,

$$\delta(K_1, K_2) = \text{dist}(A, B) (= \text{dist}(K_1, K_2)).$$

Strictly convexity of the Banach space X yields that both K_1 and K_2 must be singleton and so $\text{Prox}_{A \times B}(T) \neq \emptyset$. $\square$

Next fixed point result is concluded from Theorem 4.7.

Corollary 4.8. *Let (A, B) be a nonempty, weakly compact and convex pair of subsets of a strictly convex Banach space X , and let $T : A \cup B \rightarrow A \cup B$ be a noncyclic mapping and suppose there exists $\alpha \in [0, 1)$ such that*

$$\|Tx - T^2y\| \leq \alpha\|x - Ty\|,$$

for each $(x, y) \in A \times B$. If $T|_A, T|_B$ are affine, then $A \cap B$ is nonempty and T has a fixed point in $A \cap B$.

Example 4.2. Consider the real Banach space l_2 with the canonical basis $\{e_n\}$. Let

$$A = \{x_1 e_1 : 0 \leq x_1 \leq 1\} \quad \text{and} \quad B = \{x_1 e_1 + x_2 e_2 : 0 \leq x_1, x_2 \leq 1\}.$$

Define $T : A \cup B \rightarrow A \cup B$ with $T(x_1 e_1) = 0$ and $T(x_1 e_1 + x_2 e_2) = x_2 e_1$. Then T is noncyclic on $A \cup B$ and it is easy to see that $T|_A$ and $T|_B$ are affine. Let $\mathbf{x} := x_1 e_1$ and $\mathbf{y} := y_1 e_1 + y_2 e_2$. Thus for each $\alpha \in [0, 1)$ we have

$$\|T\mathbf{x} - T^2\mathbf{y}\| = 0 \leq \alpha\|\mathbf{x} - T\mathbf{y}\|.$$

That is, T is a noncyclic contraction type mapping. Now, by Corollary 4.8, T has a fixed point. Note that existence of fixed point for T cannot be obtained from Theorem 4.1. Indeed, if $\mathbf{z} := ze_1 + ze_2$, then

$$\|T\mathbf{z} - T^2\mathbf{z}\| = \|ze_1 - 0\| = z \geq \|\mathbf{z} - T\mathbf{z}\|.$$

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