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Research Article

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On applications of bipartite graph associated with algebraic structures

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Abstract: The latest developments in algebra and graph theory allow us to ask a natural question, what is the application in real world of this graph associated with some mathematical system? Groups can be used to construct new non-associative algebraic structures, loops. Graph theory plays an important role in various fields through edge labeling. In this paper, we shall discuss some applications of bipartite graphs, related with Latin squares of Wilson loops, such as metabolic pathways, chemical reaction networks, routing and wavelength assignment problem, missile guidance, astronomy and x-ray crystallography.

Keywords: Wilson loop, bipartite graph, edge labeling, nucleus

MSC 2010: 05Exx, 05E40, 05E45, 05E15

1 Introduction

Ruth Moufang, German geometer, introduced Quasigroup to associate with non-desarguesian plane significantly. Naturally, this mathematical structure is the generalization of frequently studied algebraic system, group. After the origination, mathematicians discussed it with combinatorial analysis, projective plane, experimental design, algebra, topology, etc. All algebraic nets are the examples of Quasigroups. People worked on different algebraic structures, initiated from magma or groupoid, in the interval 1900 to 1970 and all these developments culminated after the appearance of Moufang loops and Bol loops. Loop theory has not only history of 70 years but also moving in the direction of well-known research areas with modernity.

Let \mathcal{E} be a non-empty set such that with a binary operation \diamond , (\mathcal{E}, \diamond) is a *groupoid* that is $\forall \alpha, \beta \in \mathcal{E}$ we have $\alpha \diamond \beta \in \mathcal{E}$. If the system of equations $p \diamond \alpha = q$ and $\beta \diamond p = q$ have unique solutions for α and β then (\mathcal{E}, \diamond) is known as *Quasigroup*. Furthermore, if there exists a unique identity element $\epsilon \in \mathcal{E}$, then (\mathcal{E}, \diamond) is said to be a *loop*. For each $\alpha \in \mathcal{E}$, the elements $\alpha^\ell, \alpha^r \in \mathcal{E}$ such that $\alpha^\ell \diamond \alpha = \alpha \diamond \alpha^r = \epsilon$ are called *left* and *right inverses* of α respectively. \mathcal{E} is known as *Wilson loop* (WL) if and only if it obeys the Wilson Identity (WI);

$$\alpha \diamond (\alpha \diamond \beta)^r = (\alpha \diamond \gamma)(\alpha \diamond (\beta \diamond \gamma))^r, \forall \alpha, \beta, \gamma \in \mathcal{E}$$

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equivalently;

$$(\beta \diamond \alpha)^\ell \diamond \alpha = ((\gamma \diamond \beta) \diamond \alpha)^\ell \diamond (\gamma \diamond \alpha), \quad \forall \alpha, \beta, \gamma \in \mathcal{E}$$

Any loop \mathcal{E} satisfying $\alpha \diamond (\beta \diamond \alpha) = (\alpha \diamond \beta) \diamond \alpha$ is *flexible loop* $\forall \alpha, \beta \in \mathcal{E}$. Sets $\mathfrak{N}_\ell = \{p \in \mathcal{E}; p \diamond (\alpha \diamond \beta) = (p \diamond \alpha) \diamond \beta \quad \forall \alpha, \beta \in \mathcal{E}\}$, $\mathfrak{N}_\chi = \{p \in \mathcal{E}; \alpha \diamond (p \diamond \beta) = (\alpha \diamond p) \diamond \beta \quad \forall \alpha, \beta \in \mathcal{E}\}$ and $\mathfrak{N}_r = \{p \in \mathcal{E}; (\alpha \diamond \beta) \diamond p = \alpha \diamond (\beta \diamond p) \quad \forall \alpha, \beta \in \mathcal{E}\}$ are said to be *left, middle and right nucleus*, respectively. The set $\mathfrak{N} = \mathfrak{N}_\ell \cap \mathfrak{N}_\chi \cap \mathfrak{N}_r$ consists of all elements that associate with any other two elements and is called the *nucleus* of \mathcal{E} . For Wilson loop we have $\mathfrak{N} = \mathfrak{N}_\ell = \mathfrak{N}_\chi = \mathfrak{N}_r$. \mathcal{E} is *weak inverse property loop* if and only if $(\alpha \diamond \beta) \diamond \gamma = \epsilon$ implies $\alpha \diamond (\beta \diamond \gamma) = \epsilon \quad \forall \alpha, \beta, \gamma \in \mathcal{E}$. \mathcal{E} is called *conjugacy closed loop* if the sets of left and right translations are closed under conjugation.

As worked by Goodaire and Robinson [1, Theorem 1], a loop \mathcal{E} is a Wilson loop iff it is weak inverse property loop [2, p. 295][3, p. 132] and conjugacy closed loop [4, p. 843]. Originally Wilson loop is introduced by E. L. Wilson in [5, Theorem 5] where it is also given that a Moufang loop [6, p. 42][7, p. 194] is Wilson loop if and only if $\alpha^2 \in \mathfrak{N} \quad \forall \alpha \in \mathcal{E}$. The developments of loop theory remained eclipsed under the fast moving research horizon of the theory of groups. After the completion of the list of simple groups, the research environment is getting more suitability for the structures of non-associative models like loops and Quasigroups. In the literature of loop theory, the groups are being used to derive new families of loops.

In the recent time researchers are using computers rapidly for mostly used applications and the second approach is graph theory. We can understand many real world applications by associating with several graphs. Graph theory is the extensively used branch of mathematics. In 1735, Koinser bridge's problem gave the origin of graph theory and later on researchers did work on Eulerian graph, complete graph and bipartite graph comprehensively. After Leonhard Euler's work, Cauchy and L'Huilier played an important role to initiate a new branch, topology, of mathematics tremendously. Arthur Cayley was first mathematician who used trees for chemical composition in theoretical chemistry. Sylvester used term "graph" first time in his work and Frank Harary wrote an eminent book on graph theory in 1969 to connect mathematicians, biologists, computer experts, chemists, engineers and social scientists see Figure 1.

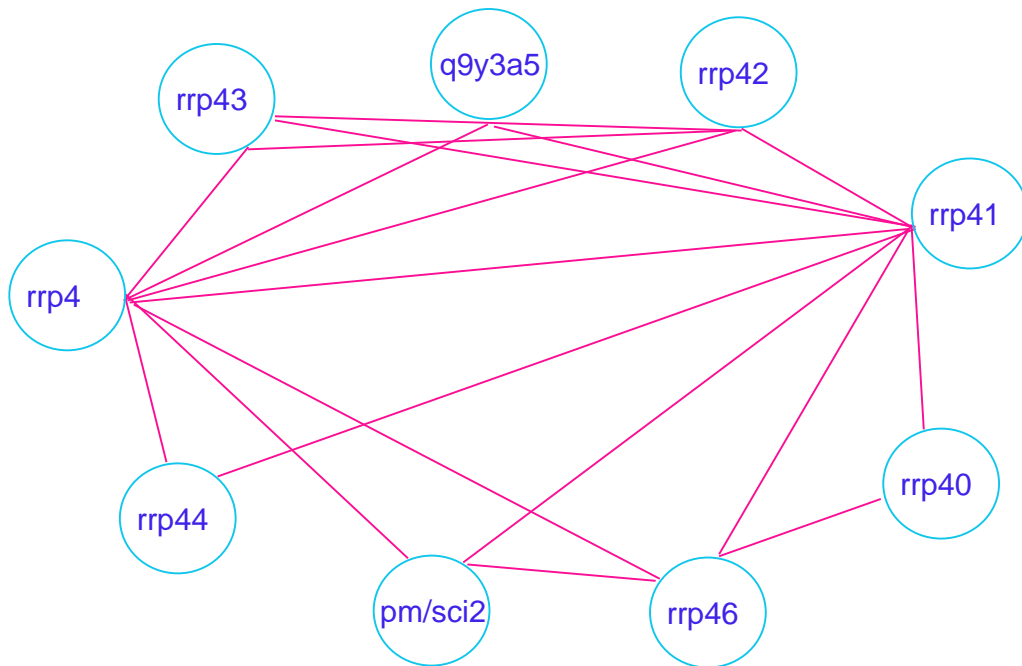


Figure 1: A module of a protein interaction graph.

Graph $\Gamma = (\Sigma, Y)$ is known as a *simple graph* if it does not contain loops and multiple edges where Σ and Y are respectively sets of vertices and edges of Γ . A simple graph $\Gamma = (\Sigma, Y)$ is said to be *complete* if there is an edge between any pair of distinct vertices. Secondly, $\Gamma = (\Sigma_1, \Sigma_2, E)$ is *bipartite* (or 2-mode network or bigraph) if $\forall e \in Y$ has one end in Σ_1 and the other in Σ_2 where the sets Σ_1, Σ_2 are disjoint [8, p. 2][9, p. 225]. Equivalently, Γ is bipartite if it does not contain any odd length cycle. For instance, $K_{n,n}$ is a complete bipartite graph with cardinality of both Σ_1, Σ_2 is n . Graph, $\Gamma = (\Sigma_1, \Sigma_2, Y)$ is *balanced bipartite graph* if $|\Sigma_1| = |\Sigma_2|$. Bipartite graphs can be used broadly to consider bioentities, signal transduction, gene regulation, evolutionary relationships, metabolic pathways, gene expression etc. as vertices and their correlation as edges within a network.

Now biologists can understand more about yeast-two-hybrid [10, p. 246], protein-protein interactions (PPIs) for particular organisms [11, p. 822][12, p. 4570][13, p. 4880][14, p. 212][15, p. 624]. Microarrays and RNA-seq [16, p. 57][17, p. 201] with the help of bipartite graphs. Graph theory is a companionable and prolific tool to handle chemical reaction networks (CRNs) [18, p. 2309]. Absolutely, it has become an important structure to study in different fields specially computer science and chemistry.

In the modern world, it seems impossible to discuss properties of classical random graphs associated with the models of real-world complex networks. Instead of classical random graphs bipartite graphs can be used to overcome this difficulty [19, p. 800]. Bipartite graphs are very expedient to decode the code words in advance coding theory and Query Log Analysis, Personnel Assignment Problem, Optimal Assignment Problem. A factor graph (subclass of bipartite graph) and belief network are very closed to each other. They give us probabilistic decoding of low-density parity-check and turbo codes in [20, p. 143]. Inspired by [21, p. 332] for projective geometry, taking into account the fact that every Levi graph is the bipartite graph, we are able to model the incidences between points and lines in a configuration.

Document/Word Graphs are the bipartite graphs where (say) Σ_1 and Σ_2 respectively consists of documents and words, $e = (v_1, v_2) \in Y$ represents word v_2 is in the document v_1 . *Edge labeling* of a simple graph $\Gamma = (\Sigma, Y)$ is a mapping, $\Theta : Y \rightarrow \clubsuit$, from Y to \clubsuit , set of integers or symbols. And with this Θ the graph Γ is called edge-labeled graph. For an healthier understanding of graph labeling, its consequences and algebraic properties see [22-27]. Without any restriction, algebraic operation, we can assign a Wilson Latin square to a complete bipartite graph through edge labeling. In Figure 2, we label an element $(-1, 1)$ as an edge with respect to any two arbitrary vertices A and B so $K_{4,4}$ is desired bipartite graph for table 1 with Figure 2.

Table 1: Wilson loop of order 4.

(1,0)	(1,1)	(-1,0)	(-1,1)
(1,1)	(1,0)	(-1,1)	(-1,0)
(-1,0)	(-1,1)	(1,1)	(1,0)
(-1,1)	(-1,0)	(1,0)	(1,1)



Figure 2: Edge $(-1, 1)$.

A *path* from u to v in the simple graph Γ is a sequence of edges $(\zeta_0, \zeta_1), (\zeta_1, \zeta_2), (\zeta_2, \zeta_3), \dots, (\zeta_{m-1}, \zeta_m)$ in Γ , where m is a nonnegative integer, and $\zeta_0 = u$ and $\zeta_m = v$. It can be denoted by $\zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{m-1}, \zeta_m$ and has length m . In case of directed graphs, we say a path is *increasing* if the sequence of its edge labels is non-decreasing. *Good edge-labeling* is an edge-labeling in which for any two distinct vertices u, v we have at most one increasing (u, v) -path. Subcubic $\{C_3, K_{2,3}\}$ -free graphs, planar graphs of girth at least 6, C_3 -free outerplanar graphs, forests are the examples of graphs which admit the good edge-labeling and help us to

overcome RWA (Routing and Wavelength Assignment) problem for UPP-DAG [25, 28-30]. Graph labeling plays a vital role in a number of applications like data base management, communication network addressing, circuit design, x-ray crystallography, astronomy, radar and missile guidance. For further information see [31-33].

2 Main results

Let Ψ_1 and Ψ_2 be respectively groups under multiplication and addition. Moreover Ψ_2 is abelian group. The function $\flat : \Psi_1 \times \Psi_1 \rightarrow \Psi_2$ with;

$$(1, F)\flat = (F, 1)\flat = 0, \quad \forall F \in \Psi_1,$$

is called a *factor set*. Binary operation \diamond on $\Psi_1 \times \Psi_2$ can be defined, with the help of \flat , as follows;

$$(F_1, v_1) \diamond (F_2, v_2) = (F_1 F_2, v_1 + v_2 + (F_1, F_2)\flat) \quad \forall F_1, F_2 \in \Psi_1 \text{ and } \forall v_1, v_2 \in \Psi_2.$$

Clearly the resulting groupoid is a loop denoted by (Ψ_1, Ψ_2, \flat) with neutral element $(1, 0)$. Note that $(F, v)^{-1} = (F^{-1}, -v - (F, F^{-1})\flat)$ is the inverse of (F, v) in (Ψ_1, Ψ_2, \flat) . The following theorem provides construction of the Wilson loops.

Theorem 1. *Let $\flat : \Psi_1 \times \Psi_1 \rightarrow \Psi_2$ be a factor set. Then (Ψ_1, Ψ_2, \flat) is Wilson loop if and only if*

$$\begin{aligned} & (F_1, F_3)\flat + (F_1, F_2)\flat + (F_1 F_2, (F_1 F_2)^{-1})\flat + (F_1 F_3, (F_1 \cdot F_2 F_3)^{-1})\flat \\ & = (F_2, F_3)\flat + (F_1, F_2 F_3)\flat + (F_1 \cdot F_2 F_3, (F_1 \cdot F_2 F_3)^{-1})\flat + (F_1, (F_1 F_2)^{-1})\flat. \end{aligned}$$

Proof. Let (Ψ_1, Ψ_2, \flat) is the Wilson loop so it satisfies the identity

$$(F_1, v_1) \diamond ((F_1, v_1) \diamond (F_2, v_2))^r = ((F_1, v_1) \diamond (F_3, v_3)) \diamond ((F_1, v_1) \diamond ((F_2, v_2) \diamond (F_3, v_3)))^r$$

for all $(F_1, v_1), (F_2, v_2), (F_3, v_3) \in (\Psi_1, \Psi_2, \flat)$.

Now

$$\begin{aligned} (F_1, v_1) \diamond ((F_1, v_1) \diamond (F_2, v_2))^r &= (F_1, v_1) \diamond ((F_1 F_2, v_1 + v_2 + (F_1, F_2)\flat))^r \\ &= (F_1, v_1) \diamond ((F_1 F_2)^{-1}, -v_1 - v_2 - (F_1, F_2)\flat - (F_1 F_2, (F_1 F_2)^{-1})\flat) \\ &= (F_1, v_1) \diamond (F_2^{-1} F_1^{-1}, -v_1 - v_2 - (F_1, F_2)\flat - (F_1 F_2, F_2^{-1} F_1^{-1})\flat) \\ &= (F_1 \cdot F_2^{-1} F_1^{-1}, v_1 - v_1 - v_2 - (F_1, F_2)\flat - (F_1 F_2, F_2^{-1} F_1^{-1})\flat + (F_1, F_2^{-1} F_1^{-1})\flat) \\ &= (F_1 \cdot F_2^{-1} F_1^{-1}, -v_2 - (F_1, F_2)\flat - (F_1 F_2, F_2^{-1} F_1^{-1})\flat + (F_1, F_2^{-1} F_1^{-1})\flat) \end{aligned}$$

and

$$\begin{aligned} & ((F_1, v_1) \diamond (F_3, v_3)) \diamond ((F_1, v_1) \diamond ((F_2, v_2) \diamond (F_3, v_3)))^r \\ &= (F_1 F_3, v_1 + v_3 + (F_1, F_3)\flat) \diamond ((F_1, v_1) \diamond (F_2 F_3, v_2 + v_3 + (F_2, F_3)\flat))^r \\ &= (F_1 F_3, v_1 + v_3 + (F_1, F_3)\flat) \diamond ((F_1 \cdot F_2 F_3, v_1 + v_2 + v_3 + (F_2, F_3)\flat + (F_1, F_2 F_3)\flat))^r \\ &= (F_1 F_3, v_1 + v_3 + (F_1, F_3)\flat) \diamond ((F_1 \cdot F_2 F_3)^{-1}, -v_1 - v_2 - v_3 - (F_2, F_3)\flat \\ & \quad - (F_1, F_2 F_3)\flat - (F_1 \cdot F_2 F_3, (F_1 \cdot F_2 F_3)^{-1})\flat) \\ &= (F_1 F_3, v_1 + v_3 + (F_1, F_3)\flat) \diamond ((F_3^{-1} F_2^{-1} \cdot F_1^{-1}, -v_1 - v_2 - v_3 - (F_2, F_3)\flat - (F_1, F_2 F_3)\flat \\ & \quad - (F_1 \cdot F_2 F_3, F_3^{-1} F_2^{-1} \cdot F_1^{-1})\flat) \\ &= ((F_1 F_3)(F_3^{-1} F_2^{-1} \cdot F_1^{-1}), v_1 + v_3 + (F_1, F_3)\flat - v_1 - v_2 - v_3 - (F_2, F_3)\flat - (F_1, F_2 F_3)\flat \\ & \quad - (F_1 \cdot F_2 F_3, F_3^{-1} F_2^{-1} \cdot F_1^{-1})\flat) \\ &= (F_1 \cdot F_2^{-1} F_1^{-1}, (F_1, F_3)\flat - (F_2, F_3)\flat - (F_1, F_2 F_3)\flat - (F_1 \cdot F_2 F_3, F_3^{-1} F_2^{-1} \cdot F_1^{-1})\flat \\ & \quad - v_2 + (F_1 F_3, F_3^{-1} F_2^{-1} \cdot F_1^{-1})\flat) \end{aligned}$$

Using both results in the above Wilson identity

$$\begin{aligned} & (F_1, F_3)\flat + (F_1, F_2)\flat + (F_1F_2, (F_1F_2)^{-1})\flat + (F_1F_3, (F_1 \cdot F_2F_3)^{-1})\flat \\ & = (F_2, F_3)\flat + (F_1, F_2F_3)\flat + (F_1 \cdot F_2F_3, (F_1 \cdot F_2F_3)^{-1})\flat + (F_1, (F_1F_2)^{-1})\flat \end{aligned} \quad (1)$$

Which is the required identity. The converse is easy to verify. \square

2.1 Wilson factor set

A factor set $\flat : \Psi_1 \times \Psi_1 \rightarrow \Psi_2$ with equation (1) is called *Wilson factor-set*. If $|\Psi_1| = 2^s$ where s is the whole number then equation (1) reduces to

$$\begin{aligned} & (F_1, F_3)\flat + (F_1, F_2)\flat + (F_1F_2, F_1F_2)\flat + (F_1F_3, F_1 \cdot F_2F_3)\flat \\ & = (F_2, F_3)\flat + (F_1, F_2F_3)\flat + (F_1 \cdot F_2F_3, F_1 \cdot F_2F_3)\flat + (F_1, F_1F_2)\flat. \end{aligned} \quad (2)$$

This Wilson-factor set is very helpful in construction of Wilson loops by the following manner.

Proposition 1. *Let Ψ_2 be an additive abelian group with cardinality k , positive integer greater than 1, and $0 \neq p \in \Psi_2$. Let $\Psi_1 = \{1, \sqsupset\}$ be the multiplicative group where $\sqsupset = \cos\pi + i\sin\pi$. We define function $\flat : \Psi_1 \times \Psi_1 \rightarrow \Psi_2$ by*

$$(F_1, F_2)\flat = \begin{cases} p, & \text{if } (F_1, F_2) = (\sqsupset, \sqsupset); \\ 0, & \text{if } (F_1, F_2) = (1, 1), (1, \sqsupset), (\sqsupset, 1). \end{cases}$$

Then (Ψ_1, Ψ_2, \flat) is a flexible, non-associative Wilson loop with nucleus $\aleph = (1, v) \forall v \in \Psi_2$ and $\forall F_1, F_2 \in \Psi_1$.

Let $\Psi_1 = \{1, \sqsupset\}$, multiplicative group, and $\Psi_2 = \{0, 1, 2, 3, \dots, n-1\}$, additive abelian group of modulo n , table 2 shows a pattern of Wilson loops of even orders.

Proposition 2. *Let Ψ_2 be an additive abelian group with $|\Psi_2| > 2$, and order of p is greater than 2 where $0 \neq p \in \Psi_2$. Let $\Psi_1 = \{1, \wp_1, \wp_2, \wp_3\}$ be the Klein group. Define $\flat : \Psi_1 \times \Psi_1 \rightarrow \Psi_2$ by*

$$(F_1, F_2)\flat = \begin{cases} p, & \text{if } (F_1, F_2) = (\wp_1, \wp_3), (\wp_3, \wp_2), (\wp_2, \wp_1); \\ -p, & \text{if } (F_1, F_2) = (\wp_1, \wp_2), (\wp_2, \wp_3), (\wp_3, \wp_1); \\ 0, & \text{otherwise.} \end{cases}$$

Then (Ψ_1, Ψ_2, \flat) is a non-flexible (implies non-associative) Wilson loop with nucleus $\aleph = (1, v) \forall v \in \Psi_2$ and $\forall F_1, F_2 \in \Psi_1$.

Proof. Following table shows that function \flat is obviously Wilson-factor set.

\flat	1	\wp_1	\wp_2	\wp_3
1	0	0	0	0
\wp_1	0	0	$-p$	p
\wp_2	0	p	0	$-p$
\wp_3	0	$-p$	p	0

To show that (Ψ_1, Ψ_2, \flat) is Wilson loop we verify equation (2). Since \flat is factor set, there is nothing to prove when $F_3 = 1$;

$$(F_1, F_2)\flat + (F_1F_2, F_1F_2)\flat + (F_1, F_1F_2)\flat = (F_1, F_2)\flat + (F_1F_2, F_1F_2)\flat + (F_1, F_1F_2)\flat.$$

When $F_2 = 1$;

Table 2: Wilson loop of order $2n$.

\diamond	(1,0)	(1,1)	(1,2)	(1,3)	...	(1,n-2)	(1,n-1)	($\bar{1}$,0)	($\bar{1}$,1)	($\bar{1}$,2)	($\bar{1}$,3)	...	($\bar{1}$,n-2)	($\bar{1}$,n-1)
(1,0)	(1,0)	(1,1)	(1,2)	(1,3)	...	(1,n-2)	(1,n-1)	($\bar{1}$,0)	($\bar{1}$,1)	($\bar{1}$,2)	($\bar{1}$,3)	...	($\bar{1}$,n-2)	($\bar{1}$,n-1)
(1,1)	(1,1)	(1,2)	(1,3)	(1,4)	...	(1,n-1)	(1,0)	($\bar{1}$,1)	($\bar{1}$,2)	($\bar{1}$,3)	($\bar{1}$,4)	...	($\bar{1}$,n-1)	($\bar{1}$,0)
(1,2)	(1,2)	(1,3)	(1,4)	(1,5)	...	(1,0)	(1,1)	($\bar{1}$,2)	($\bar{1}$,3)	($\bar{1}$,4)	($\bar{1}$,5)	...	($\bar{1}$,0)	($\bar{1}$,1)
(1,3)	(1,3)	(1,4)	(1,5)	(1,6)	...	(1,1)	(1,2)	($\bar{1}$,3)	($\bar{1}$,4)	($\bar{1}$,5)	($\bar{1}$,6)	...	($\bar{1}$,1)	($\bar{1}$,2)
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(1,n-2)	(1,n-2)	(1,n-1)	(1,0)	(1,1)	...	(1,n-4)	(1,n-3)	($\bar{1}$,n-2)	($\bar{1}$,n-1)	($\bar{1}$,0)	($\bar{1}$,1)	...	($\bar{1}$,n-4)	($\bar{1}$,n-3)
(1,n-1)	(1,n-1)	(1,0)	(1,1)	(1,2)	...	(1,n-3)	(1,n-2)	($\bar{1}$,n-1)	($\bar{1}$,0)	($\bar{1}$,1)	($\bar{1}$,2)	...	($\bar{1}$,n-3)	($\bar{1}$,n-2)
($\bar{1}$,0)	($\bar{1}$,0)	($\bar{1}$,1)	($\bar{1}$,2)	($\bar{1}$,3)	...	($\bar{1}$,n-2)	($\bar{1}$,n-1)	(1,n-1)	(1,0)	(1,1)	(1,2)	...	(1,n-3)	(1,n-2)
($\bar{1}$,1)	($\bar{1}$,1)	($\bar{1}$,2)	($\bar{1}$,3)	($\bar{1}$,4)	...	($\bar{1}$,n-1)	($\bar{1}$,0)	(1,0)	(1,1)	(1,2)	(1,3)	...	(1,n-2)	(1,n-1)
($\bar{1}$,2)	($\bar{1}$,2)	($\bar{1}$,3)	($\bar{1}$,4)	($\bar{1}$,5)	...	($\bar{1}$,0)	($\bar{1}$,1)	(1,1)	(1,2)	(1,3)	(1,4)	...	(1,n-1)	(1,0)
($\bar{1}$,3)	($\bar{1}$,3)	($\bar{1}$,4)	($\bar{1}$,5)	($\bar{1}$,6)	...	($\bar{1}$,1)	($\bar{1}$,2)	(1,2)	(1,3)	(1,4)	(1,5)	...	(1,0)	(1,1)
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($\bar{1}$,n-2)	($\bar{1}$,n-2)	($\bar{1}$,n-1)	($\bar{1}$,0)	($\bar{1}$,1)	...	($\bar{1}$,n-4)	($\bar{1}$,n-3)	(1,n-3)	(1,n-2)	(1,n-1)	(1,0)	...	(1,n-5)	(1,n-4)
($\bar{1}$,n-1)	($\bar{1}$,n-1)	($\bar{1}$,0)	($\bar{1}$,1)	($\bar{1}$,2)	...	($\bar{1}$,n-3)	($\bar{1}$,n-2)	(1,n-2)	(1,n-1)	(1,0)	(1,1)	...	(1,n-4)	(1,n-3)

$$(F_1, F_3)^b + (F_1, F_1)^b + (F_1 F_3, F_1 F_3)^b = (F_1, F_3)^b + (F_1 F_3, F_1 F_3)^b + (F_1, F_1)^b.$$

Similarly it can be proved for $F_1 = 1$.

When $F_3 = \wp_1$;

$$\begin{aligned} & (F_1, \wp_1)^b + (F_1, F_2)^b + (F_1 F_2, F_1 F_2)^b + (F_1 \wp_1, F_1 F_2 \cdot \wp_1)^b \\ &= (F_2, \wp_1)^b + (F_1, F_2 \wp_1)^b + (F_1 \cdot F_2 \wp_1, F_1 \cdot F_2 \wp_1)^b + (F_1, F_1 F_2)^b. \end{aligned}$$

Putting $F_1 = \wp_2, F_2 = \wp_3$ in the last identity, we have

$$\begin{aligned} & (F_1, \wp_1)^b + (F_1, F_2)^b + (F_1 F_2, F_1 F_2)^b + (F_1 \wp_1, F_1 F_2 \cdot \wp_1)^b \\ &= (\wp_2, \wp_1)^b + (\wp_2, \wp_3)^b + (\wp_2 \wp_3, \wp_2 \wp_3)^b + (\wp_2 \wp_1, \wp_2 \cdot \wp_3 \wp_1)^b \\ &= p - p + 0 + (\wp_2 \wp_1, \wp_2 \cdot \wp_3 \wp_1)^b \\ &= (\wp_2 \wp_1, \wp_2 \cdot \wp_3 \wp_1)^b \\ &= (\wp_3, 1)^b \end{aligned}$$

$$\begin{aligned} & (F_2, \wp_1)^b + (F_1, F_2 \wp_1)^b + (F_1 \cdot F_2 \wp_1, F_1 \cdot F_2 \wp_1)^b + (F_1, F_1 F_2)^b \\ &= (\wp_3, \wp_1)^b + (\wp_2, \wp_3 \wp_1)^b + (\wp_2 \cdot \wp_3 \wp_1, \wp_2 \cdot \wp_3 \wp_1)^b + (\wp_2, \wp_2 \wp_3)^b \\ &= -p + (\wp_2, \wp_3 \wp_1)^b + (\wp_2, \wp_2 \wp_3)^b \\ &= -p + (\wp_2, \wp_2)^b + (\wp_2, \wp_1)^b \end{aligned}$$

Similarly we can check other cases when $F_3 = \wp_1$. By using same procedure for F_2, F_1 we can verify (2). Thus (Ψ_1, Ψ_2, b) is Wilson loop. (Ψ_1, Ψ_2, b) is non-commutative, non-associative Wilson loop. As let $\forall v \in \Psi_2$ and $0 \neq p, p \neq -p$

$$((\wp_1, v) \diamond (\wp_2, v)) \diamond (\wp_1, v) = (\wp_1 \wp_2, v + v + (\wp_1, \wp_2)^b) \diamond (\wp_1, v)$$

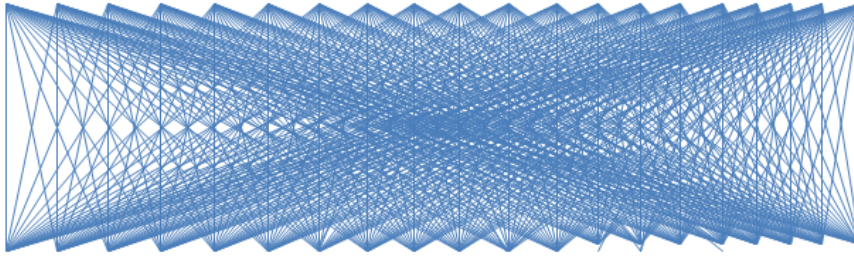


Figure 3: Complete bipartite graph $K_{20,20}$.

$$\begin{aligned}
 &= (\wp_3, 2v - p) \diamond (\wp_1, v) \\
 &= (\wp_3 \wp_1, 2v - p + v + (\wp_3, \wp_1)b) \\
 &= (\wp_2, 3v - 2p)
 \end{aligned} \tag{3}$$

$$\begin{aligned}
 (\wp_1, v) \diamond ((\wp_2, v) \diamond (\wp_1, v)) &= (\wp_1, v) \diamond (\wp_2 \wp_1, v + v + (\wp_2, \wp_1)b) \\
 &= (\wp_1, v) \diamond (\wp_3, 2v + p) \\
 &= (\wp_2, 3v + p + (\wp_1, \wp_3)b) \\
 &= (\wp_2, 3v + 2p)
 \end{aligned} \tag{4}$$

from (3) and (4)

$$((\wp_1, v) \diamond (\wp_2, v)) \diamond (\wp_1, v) \neq (\wp_1, v) \diamond ((\wp_2, v) \diamond (\wp_1, v)).$$

It implies that (Ψ_1, Ψ_2, b) is not flexible and $(\wp_1, v), (\wp_2, v)$ are not in \aleph . Similarly $((\wp_1, v) \diamond (\wp_3, v)) \diamond (\wp_1, v) \neq ((\wp_1, v) \diamond (\wp_3, v)) \diamond (\wp_1, v)$ gives (\wp_3, v) also not in \aleph . Finally $\forall F_1, F_2 \in \Psi_1$ and $\forall v_2, v_3 \in \Psi_2$

$$((1, v) \diamond (F_1, v_2)) \diamond (F_2, v_3) = (F_1 F_2, v + v_2 + v_3 + (F_1, F_2)b) = (1, v) \diamond ((F_1, v_2) \diamond (F_2, v_3))$$

shows that $(1, v) \in \aleph$ represents star graph through above mentioned edge labeling. □

Example: If $\Psi_1 = \{1, \wp_1, \wp_2, \wp_3\}$ and $\Psi_2 = \{0, 1, 2, 3, 4\}$, with modulo 5, then $K_{20,20}$ is the associated graph see Figure 3. Let $\Psi_1 = \{1, \wp_1, \wp_2, \wp_3\}$, Klein four group, and $\Psi_2 = \{0, 1, 2, 3, \dots, n - 1\}$, additive abelian group of modulo n , table 3 also shows a pattern of Wilson loops.

We can recapitulate all the above discussion in the table 4.

3 Conclusion

This article deals with the application of graph theory in the pure mathematics. In particular the aim is to discover those algebraic structures and quasigroups which are closely associated with bipartite graphs. We have shown that graph labeling is a powerful tool to understand algebraic object namely the Wilson loop. The field is quite open in the sense, one can discover more connections between these two areas.

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Table 4: Complete bipartite graphs associated with loops.

Multiplicative Group	Additive group	Loop	p	$-p$	Bipartite graph	Star graph
$\Psi_1 = \{1, \sqsupset\}$	$\Psi_2 = \{0, 1\}$	(Ψ_1, Ψ_2, b)	1	1	$K_{4,4}$	$K_{1,2}$
$\Psi_1 = \{1, \sqsupset\}$	$\Psi_2 = \{0, 1, 2\}$	(Ψ_1, Ψ_2, b)	2	1	$K_{6,6}$	$K_{1,3}$
$\Psi_1 = \{1, \sqsupset\}$	$\Psi_2 = \{0, 1, 2, 3\}$	(Ψ_1, Ψ_2, b)	3	1	$K_{8,8}$	$K_{1,4}$
$\Psi_1 = \{1, \sqsupset\}$	$\Psi_2 = \{0, 1, 2, 3, \dots, n-1\}$	(Ψ_1, Ψ_2, b)	$n-1$	1	$K_{2n,2n}$	$K_{1,n}$
$\Psi_1 = \{1, \wp_1, \wp_2, \wp_3\}$	$\Psi_2 = \{0, 1, 2\}$	(Ψ_1, Ψ_2, b)	2	1	$K_{12,12}$	$K_{1,3}$
$\Psi_1 = \{1, \wp_1, \wp_2, \wp_3\}$	$\Psi_2 = \{0, 1, 2, 3\}$	(Ψ_1, Ψ_2, b)	3	1	$K_{16,16}$	$K_{1,4}$
$\Psi_1 = \{1, \wp_1, \wp_2, \wp_3\}$	$\Psi_2 = \{0, 1, 2, 3, 4\}$	(Ψ_1, Ψ_2, b)	4	1	$K_{20,20}$	$K_{1,5}$
$\Psi_1 = \{1, \wp_1, \wp_2, \wp_3\}$	$\Psi_2 = \{0, 1, 2, 3, \dots, n-1\}$	(Ψ_1, Ψ_2, b)	$n-1$	1	$K_{4n,4n}$	$K_{1,n}$

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