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The 2-pebbling property of squares of paths and Graham's conjecture

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Abstract: A pebbling move on a graph G consists of taking two pebbles off one vertex and placing one pebble on an adjacent vertex. The pebbling number of a connected graph G , denoted by $f(G)$, is the least n such that any distribution of n pebbles on G allows one pebble to be moved to any specified vertex by a sequence of pebbling moves. In this paper, we determine the 2-pebbling property of squares of paths and Graham's conjecture on P_{2n}^2 .

Keywords: square of paths, pebbling number, pebbling move

MSC 2010: 05C70

1 Introduction

Throughout this paper, G denotes a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. Let p pebbles be distributed onto the vertices of a graph G . A pebbling move consists of removing two pebbles from one vertex and then placing one pebble at an adjacent vertex. The pebbling number of a connected graph G , denoted by $f(G)$, is the least n such that any distribution of n pebbles on G allows one pebble to be moved to any specified vertex by a sequence of pebbling moves.

If each vertex (except v) has at most one pebble, then no pebble can be moved to v . Also, if u is of distance d from v and at most $2^d - 1$ pebbles are placed on u (and none elsewhere), then no pebble can be moved from u to v . So it is clear that $f(G) \geq \max\{|V(G)|, 2^D\}$, where $|V(G)|$ is the number of vertices of G , and D is the diameter of G . Furthermore, $f(K_n) = n$ and $f(P_n) = 2^{n-1}$ in [1], where K_n denotes a complete graph with n vertices and P_n denotes a path with n vertices.

Pebbling of graphs was first introduced by Chung [1]. In [2], Pachter et al. obtained the pebbling number of P_n^2 and showed that the most graphs have the 2-pebbling property. Y. S. Ye et al. gave that the pebbling number of C_n^2 [3, 4]. There are many known results about pebbling numbers [5-9]. Motivated by these results, we first show that squares of paths have the 2-pebbling property and then establish Graham's conjecture holds for P_{2n}^2 . We now introduce definitions and lemmas, which will be used in the subsequent proofs.

Definition 1.1 Let G be a connected graph. For $u, v \in V(G)$, we denote by $d_G(u, v)$ the distance between u and v in G . The k -th power of G , denoted by G^k , is the graph obtained from G by adding the edge uv to G whenever $2 \leq d_G(u, v) \leq k$ in G . That is, $E(G^k) = \{uv : 1 \leq d_G(u, v) \leq k\}$.

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Obviously, G^k is the complete graph whenever k is at least the diameter of G . In particular, let P_n be a path with n vertices, written $\langle v_1, v_2, \dots, v_n \rangle$, i.e., $P_n = \langle v_1, v_2, \dots, v_n \rangle$. P_n^2 is obtained from P_n by joining $v_i v_{i+2}$ ($i = 1, 2, \dots, n-2$).

Definition 1.2 We call a graph G satisfies the 2-pebbling property if two pebbles can be moved to any specified vertex when the total starting number of pebbles is $2f(G) - q + 1$, where q is the number of vertices with at least one pebble.

Lemma 1.1 [2] $f(P_{2k}^2) = 2^k, f(P_{2k+1}^2) = 2^k + 1$.

Lemma 1.2 [5] Let $P_n = \langle v_1, v_2, \dots, v_n \rangle$. If $p(v_1) + 2p(v_2) + \dots + 2^{i-1}p(v_i) + \dots + 2^{n-2}p(v_{n-1}) \geq 2^{n-1}$, then one pebble can be moved to v_n from v_1 .

Lemma 1.3 [3] Let $P_{2k+1} = \langle v_1, v_2, \dots, v_{2k+1} \rangle$. If P_{2k+1}^2 has 2^k pebbles and v_1 has an even number of pebbles, then a pebble can be moved to v_{2k+1} .

Define $p(H)$ to be the number of pebbles on a subgraph H of G and $p(v)$ to be the number of pebbles on a vertex v of G . Moreover, denote by $\tilde{p}(H)$ and $\tilde{p}(v)$ respectively, the number of pebbles on H and the number of pebbles on v after a specified sequence of pebbling moves, respectively.

2 The 2-pebbling property of P_n^2

Let $P_n = \langle v_1, v_2, \dots, v_n \rangle$. In this section, we first will show that P_{2k}^2 has the 2-pebbling property and then we shall prove that P_{2k+1}^2 also has the 2-pebbling property.

Theorem 2.1 P_{2k}^2 has the 2-pebbling property.

Proof. Suppose that $2^{k+1} - q + 1$ pebbles are placed arbitrarily at the vertices of P_{2k}^2 , where q is the number of vertices with at least one pebble. Suppose that our target vertex is v . We divide into two cases: $p(v) = 1$ or $p(v) = 0$.

Case 1. $p(v) = 1$. Then P_{2k}^2 has $2^{k+1} - q$ pebbles other than one pebble on v . Since $2^{k+1} - q \geq 2^k$, by Lemma 1.1, we can move one additional pebble to v so that $\tilde{p}(v) = 2$.

Case 2. $p(v) = 0$.

Subcase 2.1. If $q = 2k - 1$, then $p(P_{2k}^2) = 2^{k+1} - 2k + 2$. For $k = 2$, it is clearly that we can move 2 pebbles to v . For $k \geq 3$, we can move one pebble to v using at most $k + 1$ pebbles. This leaves $2^{k+1} - 3k + 1 \geq 2^k$ pebbles on P_{2k}^2 . By Lemma 1.1, we may move one additional pebble to v so that $\tilde{p}(v) = 2$.

Subcase 2.2. Now we assume $q \leq 2k - 2$. Without loss of generality, we may assume that our target vertex v is not v_{2k} (otherwise, relabeling). We claim that $p(v_{2k}) \geq 2^k + 2^{k-1} - q + 1 > 2^k$. Otherwise,

$$p(P_{2k-1}^2) \geq 2^{k+1} - q + 1 - (2^k + 2^{k-1} - q) = 2^{k-1} + 1.$$

By Lemma 1.1, we can move one pebble to v . This leaves

$$2^{k+1} - q + 1 - (2^{k-1} + 1) = 2^k + 2^{k-1} - q \geq 2^k$$

pebbles on P_{2k}^2 . By Lemma 1.1, we can move one additional pebble to v so that $\tilde{p}(v) = 2$. Using 2^k pebbles on v_{2k} , we can move at least one pebble to v by Lemma 1.1. This leaves at least $2^k - q + 1$ pebbles on P_{2k}^2 other than one pebble on v . Let $p(P_{2k-1}^2) = s$ and $s \leq 2^{k-1}$. Thus $\tilde{p}(v_{2k}) = 2^k - q - s + 1 \geq 2$. We can move $2^{k-1} - \lceil \frac{q+s-1}{2} \rceil$ pebbles to P_{2k-1}^2 pebbles from v_{2k} so that P_{2k-1}^2 has

$$s + 2^{k-1} - \left\lceil \frac{q+s-1}{2} \right\rceil = 2^{k-1} + \left\lfloor \frac{s-q+1}{2} \right\rfloor \geq 2^{k-1} \quad (s \geq q-1)$$

pebbles other than one pebble on v and v_{2k-1} has an even number of pebbles. By Lemma 1.3, we can move one additional pebble to v so that $\tilde{p}(v) = 2$. $\square$

Theorem 2.2 P_{2k+1}^2 has 2-pebbling property.

Proof. Suppose that $2^{k+1} - q + 3$ pebbles are placed arbitrarily at the vertices of P_{2k+1}^2 . Let our target vertex be v . If $p(v) = 1$, then P_{2k+1}^2 has $2^{k+1} - q + 2$ pebbles other than one pebble on v . Since $2^{k+1} - q + 2 \geq 2^k + 1$, by Lemma 1.1, we can move one additional pebble to v so that $\tilde{p}(v) = 2$. Hence we assume $p(v) = 0$.

If $q = 2k$, then $p(P_{2k+1}^2) = 2^{k+1} - 2k + 3$. For $k = 2$, it is clearly that we can move 2 pebbles to v . For $k \geq 3$, we can move one pebble to v using at most $k + 1$ pebbles. This leaves $2^{k+1} - 3k + 2 \geq 2^k + 1$ pebbles on P_{2k+1}^2 . By Lemma 1.1, we may move one additional pebble. Now we assume $q \leq 2k - 1$. Without loss of generality, we may assume that our target vertex v is not v_{2k} or v_{2k+1} (otherwise, relabeling). We claim that $p(v_{2k}) + p(v_{2k+1}) \geq 2^k + 2^{k-1} - q + 3 > 2^k + 1$. Otherwise,

$$p(P_{2k-1}^2) \geq 2^{k+1} - q + 3 - (2^k + 2^{k-1} - q + 2) = 2^{k-1} + 1.$$

By Lemma 1.1, we can move one pebble to v . This leaves $2^{k+1} - q + 3 - (2^{k-1} + 1) \geq 2^k + 1$ pebbles on P_{2k+1}^2 . By Lemma 1.1, we can move one additional pebble to v so that $\tilde{p}(v) = 2$. Using $2^k + 1$ pebbles on the set $\{v_{2k}, v_{2k+1}\}$, we can move at least one pebble to v by Lemma 1.1. This leaves at least $2^k - q + 2$ pebbles on P_{2k+1}^2 other than one pebble on v . Let $p(P_{2k-1}^2) = s$ and $s \leq 2^{k-1}$. Thus $\tilde{p}(v_{2k}) + \tilde{p}(v_{2k+1}) = 2^k - q - s + 3 \geq 3$. We can move $2^{k-1} - \lceil \frac{q+s-2}{2} \rceil + 1$ pebbles to P_{2k-1}^2 from v_{2k} and v_{2k+1} so that

$$p(P_{2k-1}^2) = s + 2^{k-1} - \left\lceil \frac{q+s-2}{2} \right\rceil + 1 = 2^{k-1} + \left\lfloor \frac{s-q+2}{2} \right\rfloor + 1 \geq 2^{k-1} + 1 \quad (s \geq q - 2)$$

pebbles other than one pebble on v . By Lemma 1.1, we can move one additional pebble to v so that $\tilde{p}(v) = 2$. $\square$

3 Graham's conjecture on P_n^2

Given two graphs G and H , the Cartesian product of them is denoted by $G \times H$. Its vertex set

$$V(G \times H) = \{(u, v) | u \in V(G), v \in V(H)\},$$

and (u, v) is adjacent to (u', v') if and only if $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$.

We can depict $G \times H$ pictorially by drawing a copy of H at every vertex of G and joining each vertex in one copy of H to the corresponding vertex in an adjacent copy of H . We write $u(H)$ (respectively, $v(G)$) for the subgraph of vertices whose projection onto $V(G)$ is the vertex u , (respectively, whose projection onto $V(H)$ is the vertex v). Obviously, $u(H) \cong H$, $v(G) \cong G$.

Graham's conjecture or any two graphs G and H , we have

$$f(G \times H) \leq f(G)f(H).$$

Lemma 3.1 [5] Let P_n be a path with n vertices and G be a graph with the 2-pebbling property. Then $f(P_n \times G) \leq 2^{n-1}f(G)$.

Lemma 3.2 Let $P_{2k} = \langle v_1, v_2, \dots, v_{2k} \rangle$ ($k \geq 2$) and $p(v_i) = j_i$ ($1 \leq i \leq 2k$). If one of the following cases holds

- (1) $j_1 + j_2 + 1 = 2^k$ and $j_2 > 0$;
- (2) $j_1 + j_2 + 2 = 2^k$, $j_2 > 0$ and $\tilde{p}(v_1) = j_1 + 1$;
- (3) $j_1 + j_2 + 2 = 2^k$, $j_2 > 0$ and $\tilde{p}(v_2) = j_2 + 1$;
- (4) $j_1 + j_2 + 3 = 2^k$, $\tilde{p}(v_1) = j_1 + 1$ and $\tilde{p}(v_2) = j_2 + 1$,

then we can move one pebble to the vertex v of P_{2k}^2 , where $v \in \{v_3, v_4, \dots, v_{2k}\}$.

Proof. Let $A = \langle v_2, v_3, v_4, \dots, v_{2k} \rangle$. Then $A^2 \cong P_{2k-1}^2$.

(1) If j_1 is odd and j_2 is even, then we can move $\frac{j_1-1}{2}$ pebbles to v_3 from v_1 . Thus we have

$$\tilde{p}(A^2) = \frac{j_1-1}{2} + j_2 = \frac{(j_1+j_2+1) + (j_2-2)}{2} \geq 2^{k-1}.$$

By Lemma 1.3, we are done. If j_1 is even and j_2 is odd, then we can move $\frac{j_1-2}{2}$ pebbles to v_3 and one pebble to v_2 from v_1 so that $\tilde{p}(v_2) = j_2 + 1$ and

$$\tilde{p}(A^2) = \frac{j_1-2}{2} + j_2 + 1 = \frac{(j_1+j_2+1) + (j_2-1)}{2} \geq 2^{k-1}.$$

By Lemma 1.3, we are done.

(2) Let j_1 and j_2 be odd. If $\tilde{p}(v_1) = j_1 + 1$, then we can move one pebble to v_2 and $\frac{j_1-1}{2}$ pebbles to v_3 from v_1 so that $\tilde{p}(v_2) = j_2 + 1$ and

$$\tilde{p}(A^2) = j_2 + 1 + \frac{j_1-1}{2} = \frac{(j_1+j_2+2) + (j_2-1)}{2} \geq 2^{k-1}.$$

By Lemma 1.3, we are done. For even numbers j_1 and j_2 , we can move $\frac{j_1}{2}$ pebbles to v_3 from v_1 so that

$$\tilde{p}(A^2) = j_2 + \frac{j_1}{2} = \frac{(j_1+j_2+2) + (j_2-2)}{2} \geq 2^{k-1}.$$

By Lemma 1.3, we are done.

(3) The proof is similar to (2).

(4) If j_1 is even and j_2 is odd, then we can move $\frac{j_1}{2}$ pebbles to v_3 from v_1 so that

$$\tilde{p}(A^2) = j_2 + 1 + \frac{j_1}{2} = \frac{(j_1+j_2+3) + (j_2-1)}{2} \geq 2^{k-1}.$$

And $\tilde{p}(v_2) = j_2 + 1$, we are done by Lemma 1.3. If j_1 is odd and j_2 is even, then we can move one pebble to v_2 and $\frac{j_1-1}{2}$ pebbles to v_3 from v_1 so that $\tilde{p}(v_2) = j_2 + 2$ and

$$\tilde{p}(A^2) = j_2 + 2 + \frac{j_1-1}{2} = \frac{(j_1+j_2+3) + j_2}{2} \geq 2^{k-1}.$$

By Lemma 1.3, we are done. □

Theorem 3.3 Let G be a graph with the 2-pebbling property. Then $f(P_{2k}^2 \times G) \leq 2^k f(G)$.

Proof. For $k = 1$, $P_{2k}^2 = P_2$. By Lemma 3.1, the conclusion is right. We assume that the inequality $f(P_{2k}^2 \times G) \leq 2^k f(G)$ is right for $k > 1$. Next, we shall determine the above bound of the pebbling number of $P_{2k+2}^2 \times G$.

Suppose that $2^{k+1}f(G)$ pebbles have been distributed arbitrarily on the vertices of $P_{2k+2}^2 \times G$. Let $p_i = p(v_i(G))$, where $1 \leq i \leq 2k+2$. Let q_i be the number of occupied vertices in $v_i(G)$ ($1 \leq i \leq 2k+2$). Suppose that v is our target vertex in $P_{2k+2}^2 \times G$. Suppose that $v = (v_i, x) \in v_i(G)$ ($3 \leq i \leq 2k+2$) for $x \in V(G)$ (relabeling if necessary). For simplicity, let $A = \langle v_3, v_4, \dots, v_{2k+1}, v_{2k+2} \rangle$. By the induction hypothesis, $f(A^2 \times G) \leq 2^k f(G)$. We shall divide into the following two cases.

Case 1. $p_1 + p_2 \leq (2^{k+1} - 2)f(G)$. Then $A^2 \times G$ can obtain at least $\frac{1}{2} \left[(2^{k+1} - 2)f(G) - 2f(G) \right] + 2f(G) = 2^k f(G)$ pebbles by a sequence of pebbling moves and we are done.

Case 2. $p_1 + p_2 \geq (2^{k+1} - 2)f(G) + 1$. Thus $p(A^2 \times G) < 2f(G)$. Let $p(A^2 \times G) = (j_0 + \alpha_0)f(G)$, where $0 \leq j_0 \leq 1$ and $0 \leq \alpha_0 < 1$, and let $p_i = (j_i + \alpha_i)f(G)$, where $j_i \geq 0$ and $0 \leq \alpha_i < 1$ ($i = 1, 2$). Now we claim that $\sum_{i=1}^2 q_i > (j_0 + \alpha_0)f(G)$.

Otherwise, we have $\sum_{i=1}^2 q_i \leq (j_0 + \alpha_0)f(G)$. So $\tilde{p}(A^2 \times G) \geq \frac{1}{2} \left[2^{k+1}f(G) - (j_0 + \alpha_0)f(G) - (j_0 + \alpha_0)f(G) \right] + (j_0 +$

$\alpha_0)f(G) = 2^k f(G)$ and we are done. Let $\sum_{i=0}^2 \alpha_i = s \leq 2$. Then $\sum_{i=0}^2 j_i = 2^{k+1} - s$. Next, we discuss two following cases. Let $B = \langle (v_2, x), (v_3, x), \dots, (v_{2k+2}, x) \rangle$. Then $B^2 \cong P_{2k+1}^2$. We first prove the following claim.

Claim: Suppose $j_2 = 0$. If one of the following cases holds: (1) $j_0 = 0, s = 1$; (2) $j_0 = 0, s = 2$; (3) $j_0 = 1, s = 1$, then we can move one pebble to any vertex of B^2 .

In fact, since $j_2 = 0$, $p_1 = (2^{k+1} - j_0 - s + \alpha_1)f(G)$. We move $(2^k - 2)f(G)$ pebbles to $v_3(G)$ so that $\tilde{p}_3 \geq (2^k - 2)f(G)$ using $(2^{k+1} - 4)f(G)$ pebbles on $v_1(G)$, and $(1 - \alpha_2)f(G)$ to $v_2(G)$ so that $\tilde{p}_2 = f(G)$ using $(2 - 2\alpha_2)f(G)$ pebbles on $v_1(G)$. So we can move one pebble to (v_2, x) and $(2^k - 2)$ pebbles to (v_3, x) so that $\tilde{p}(B^2) = 2^k - 1$. And the remaining $(2 - j_0 - s + \alpha_1 + 2\alpha_2)f(G)$ pebbles on $v_1(G)$, i.e., $\tilde{p}_1 = (2 - j_0 - s + \alpha_1 + 2\alpha_2)f(G)$.

(1) $\tilde{p}_1 = (1 + \alpha_1 + 2\alpha_2)f(G)$. If $\alpha_0 \leq \alpha_2$, then $\tilde{p}_1 \geq (1 + \alpha_0 + \alpha_1 + \alpha_2)f(G) = (1 + s)f(G) = 2f(G)$. we move two pebbles to (v_1, x) , and move one pebble to (v_2, x) so that $\tilde{p}((v_2, x)) = 2$. By Lemma 1.3, we are done. If $\alpha_0 > \alpha_2$, note that $q_1 + q_2 > \alpha_0 f(G)$ and $q_2 \leq \alpha_2 f(G)$, then $q_1 > (\alpha_0 - \alpha_2)f(G)$. Thus $\tilde{p}_1 + q_1 > (1 + \alpha_0 + \alpha_1 + \alpha_2)f(G) = 2f(G)$. By the 2-pebbling property, we move two pebbles to (v_1, x) , and move one pebble to (v_2, x) so that $\tilde{p}((v_2, x)) = 2$ and $\tilde{p}(B^2) = 2^k$. By Lemma 1.3, we are done.

(2) $\tilde{p}_1 = (\alpha_1 + 2\alpha_2)f(G)$. If $\alpha_0 \leq \alpha_2$, then $\tilde{p}_1 \geq (\alpha_0 + \alpha_1 + \alpha_2)f(G) = sf(G) = 2f(G)$. If $\alpha_0 \geq \alpha_2$, then $q_1 > (\alpha_0 - \alpha_2)f(G)$. Thus $\tilde{p}_1 + q_1 > (\alpha_0 + \alpha_1 + \alpha_2)f(G) = 2f(G)$. By (1), we are done.

(3) $\tilde{p}_1 = (\alpha_1 + 2\alpha_2)f(G)$. Note that $q_1 + q_2 > (1 + \alpha_0)f(G)$ and $q_2 \leq \alpha_2 f(G)$, so $q_1 > (1 + \alpha_0 - \alpha_2)f(G)$. Thus $\tilde{p}_1 + q_1 > (1 + \alpha_0 + \alpha_1 + \alpha_2)f(G) = (1 + s)f(G) = 2f(G)$. By (1), we are done.

Next, we continue to complete the proof of the theorem with the following two cases.

Case 1: $\alpha_0 = 0$. Obviously, $0 \leq s \leq 1$.

Suppose that $j_0 = 0$, $p_1 + p_2 = 2^{k+1}f(G)$. By Lemma 3.1, we can move 2^k pebbles to the vertex (v_2, x) of $v_2(G)$ so that $\tilde{p}(B^2) = 2^k$. By Lemma 1.3, we are done. Suppose $j_0 = 1$, $p_1 + p_2 = 2^{k+1}f(G) - f(G)$. If $s = 0$, then $p_1 + p_2 = (j_1 + j_2)f(G)$. This implies that $j_1 + j_2 + 1 = 2^{k+1}$. For $j_2 = 0$, we can move at least $(2^k - 1)f(G)$ pebbles to $A^2 \times G$ from $v_1(G)$ so that

$$\tilde{p}(A^2 \times G) = 2^k f(G).$$

For $j_2 > 0$, note that we can move at least j_i pebbles to the vertex (v_i, x) of $v_i(G)$ ($i = 1, 2$). By Lemma 3.2, we are done. If $s = 1$, then $p_1 + p_2 = (j_1 + j_2 + 1)f(G)$. This implies that $j_1 + j_2 + 2 = 2^{k+1}$. For $j_2 = 0$, $p_1 = (2^{k+1} - 2 + \alpha_1)f(G)$ pebbles. By Claim, we are done. For $j_2 > 0$, we claim that at least one of $(\alpha_1 f(G) + q_1)$ and $(\alpha_2 f(G) + q_2)$ is greater than $f(G)$. For suppose not, we have $\sum_{i=1}^2 (\alpha_i f(G) + q_i) \leq 2f(G)$. But

$$\sum_{i=1}^2 (\alpha_i f(G) + q_i) = \sum_{i=1}^2 \alpha_i f(G) + \sum_{i=1}^2 q_i > \sum_{i=1}^2 \alpha_i f(G) + f(G) = (s + 1)f(G) = 2f(G).$$

This is a contradiction. Suppose $\alpha_1 f(G) + q_1 > f(G)$. By the 2-pebbling property we can move at least $j_1 + 1$ pebbles to the vertex (v_1, x) of $v_1(G)$. Note that we can move at least j_2 pebbles to the vertex (v_2, x) of $v_2(G)$. By Lemma 3.2, we are done. Suppose $\alpha_2 f(G) + q_2 > f(G)$. Then we can move at least $j_2 + 1$ pebbles to the vertex (v_2, x) of $v_2(G)$. Note that we can move at least j_1 pebbles to the vertex (v_1, x) of $v_1(G)$. By Lemma 3.2, we are done.

Case 2: $0 < \alpha_0 < 1$. Obviously, $1 \leq s \leq 2$.

Suppose that $j_0 = 0$, $p_1 + p_2 = 2^{k+1}f(G) - \alpha_0 f(G)$. If $s = 1$, then $\alpha_1 + \alpha_2 = 1 - \alpha_0$. Thus $p_1 + p_2 = (j_1 + j_2 + 1)f(G) - \alpha_0 f(G)$. This implies that $j_1 + j_2 + 1 = 2^{k+1}$. For $j_2 = 0$, $p_1 = (2^{k+1} - 1 + \alpha_1)f(G)$. By Claim, we are done. For $j_2 > 0$, note that we can move at least j_i pebbles to the vertex (v_i, x) of $v_i(G)$ ($i = 1, 2$). By Lemma 3.2, we are done. If $s = 2$, then $\alpha_1 + \alpha_2 = 2 - \alpha_0$. Thus $p_1 + p_2 = (j_1 + j_2 + 2)f(G) - \alpha_0 f(G)$. This implies that $j_1 + j_2 + 2 = 2^{k+1}$. For $j_2 = 0$, then $p_1 = (2^{k+1} - 2 + \alpha_1)f(G)$. By Claim, we are done. For $j_2 > 0$, now we claim that at least one of $(\alpha_1 f(G) + q_1)$ and $(\alpha_2 f(G) + q_2)$ is greater than $f(G)$. For suppose not, we have

$\sum_{i=1}^2 (\alpha_i f(G) + q_i) \leq 2f(G)$. But

$$\sum_{i=1}^2 (\alpha_i f(G) + q_i) = \sum_{i=1}^2 \alpha_i f(G) + \sum_{i=1}^2 q_i > \sum_{i=1}^2 \alpha_i f(G) + \alpha_0 f(G) = (\alpha_1 + \alpha_2 + \alpha_0) f(G) = 2f(G).$$

This is a contradiction. By Case 1, we are done.

Suppose that $j_0 = 1, p_1 + p_2 = 2^{k+1}f(G) - (1 + \alpha_0)f(G) = (2^{k+1} - 1)f(G) - \alpha_0 f(G)$. If $s = 1$, then $\alpha_1 + \alpha_2 = 1 - \alpha_0$. Thus

$$p_1 + p_2 = (j_1 + j_2 + 1)f(G) - \alpha_0 f(G).$$

This implies that $j_1 + j_2 + 2 = 2^{k+1}$. For $j_2 = 0, p_1 = (2^{k+1} - 2 + \alpha_1)f(G)$. By Claim, we are done. For $j_2 > 0$, we claim that at least one of $(\alpha_1 f(G) + q_1)$ and $(\alpha_2 f(G) + q_2)$ is greater than $f(G)$. For suppose not, we have $\sum_{i=1}^2 (\alpha_i f(G) + q_i) \leq 2f(G)$. But

$$\sum_{i=1}^2 (\alpha_i f(G) + q_i) = \sum_{i=1}^2 \alpha_i f(G) + \sum_{i=1}^2 q_i > \sum_{i=1}^2 \alpha_i f(G) + (1 + \alpha_0)f(G) = (1 + \alpha_1 + \alpha_2 + \alpha_0)f(G) = 2f(G).$$

This is a contradiction. By Case 1, we are done.

If $s = 2$, then $\alpha_1 + \alpha_2 = 2 - \alpha_0$. Thus $p_1 + p_2 = (j_1 + j_2 + 2)f(G) - \alpha_0 f(G)$. This implies that $j_1 + j_2 + 3 = 2^{k+1}$. Note that $\alpha_i f(G) + q_i < 2f(G)$ for $1 \leq i \leq 2$. We claim that $\alpha_2 f(G) + q_2 > f(G)$ and $\alpha_1 f(G) + q_1 > f(G)$. For suppose not, we have $\sum_{i=1}^2 (\alpha_i f(G) + q_i) \leq 3f(G)$. But

$$\sum_{i=1}^2 (\alpha_i f(G) + q_i) = \sum_{i=1}^2 \alpha_i f(G) + \sum_{i=1}^2 q_i > \sum_{i=1}^2 \alpha_i f(G) + (1 + \alpha_0)f(G) = (1 + \alpha_1 + \alpha_2 + \alpha_0)f(G) = 3f(G).$$

This is a contradiction. By the 2-pebbling property we can move at least $j_1 + 1$ pebbles to the vertex (v_1, x) of $v_1(G)$ and at least $j_2 + 1$ pebbles to the vertex (v_2, x) of $v_2(G)$ By Lemma 3.2, we are done. $\square$

By Theorems 2.1, 2.2 and 3.3, we have the following

Corollary 3.4 $f(P_{2k}^2 \times P_{2k}^2) \leq 2^{2k}, f(P_{2k}^2 \times P_{2k+1}^2) \leq 2^{2k} + 2^k$.

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