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Orthogonal polynomials for exponential weights $x^{2\alpha}(1-x^2)^{2\rho}e^{-2Q(x)}$ on $[0, 1]$

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Abstract: Let $W_{\alpha,\rho} = x^\alpha(1-x^2)^\rho e^{-Q(x)}$, where $\alpha > -\frac{1}{2}$ and Q is continuous and increasing on $[0, 1]$, with limit ∞ at 1. This paper deals with orthogonal polynomials for the weights $W_{\alpha,\rho}^2$ and gives bounds on orthogonal polynomials, zeros, Christoffel functions and Markov inequalities. In addition, estimates of fundamental polynomials of Lagrange interpolation at the zeros of the orthogonal polynomial and restricted range inequalities are obtained.

Keywords: orthonormal polynomials, exponential weights, Christoffel functions, Markov inequalities

MSC 2010: 42C05, 33C45

1 Introduction and results

In this paper, for $\alpha > -\frac{1}{2}$, we set

$$W_{\alpha,\rho}(x) = x^\alpha(1-x^2)^\rho W(x), \quad x \in [0, 1],$$

for which the moment problem possesses a unique solution, and discuss the orthogonal polynomials for the weight $W_{\alpha,\rho}^2$ on $[0, 1]$. The main results tell us that adding an even factor $(1-x^2)^{2\rho}$ to the weight $x^{2\alpha}e^{-2Q(x)}$, $\alpha > -\frac{1}{2}$, under sufficient conditions for ρ and $Q(x)$, its properties will be invariant. It is an important and meaningful extension to the case $\rho = 0$ (we can see [1, 2]).

Assume that

$$\mathbf{I} = [0, d], \quad 0 < d \leq \infty$$

and

$$W = e^{-Q},$$

where $Q : \mathbf{I} \rightarrow [0, \infty)$ is continuous. All power moments for W exist. Such W is called an exponential weight on $\mathbf{I}$. In the paper, for $0 < p \leq \infty$, $\|\cdot\|_{L_p(\mathbf{I})}$ is the usual L_p (quasi) norm on the interval $\mathbf{I}$.

Levin and Lubinsky [3, 4] discussed orthogonal polynomials for exponential weights W^2 on $[-1, 1]$ and (c, d) , $c < 0 < d$, respectively. Kasuga and Sakai [5] dealt with generalized Freud weights $|x|^{2\alpha}W(x)^2$ in $(-\infty, \infty)$. Liu and Shi [6] considered generalized Jacobi-Exponential weights UW , where $U(x)$ is generalized Jacobi weights on (c, d) , $c < 0 < d$, and gave the estimates of the zeros of orthogonal polynomials for UW . Meanwhile, Shi [7] gave the estimates of the L_p Christoffel functions for UW on (c, d) . In [8], Liu and Shi got further estimations of the L_p Christoffel functions for UW on $[-1, 1]$. In [9], Notarangelo stated analogues of the Mhaskar-Saff inequality for doubling-exponential weights on $(-1, 1)$. The above references dealt with exponential weights on a real interval (c, d) containing 0 in its interior. In [1, 2], Levin and Lubinsky dealt

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with exponential weights $x^{2\alpha}W(x)^2$, $\alpha > -1/2$, in $[0, d)$, since the results of [3, 4] cannot be applied through such one-sided weights. All the results on one-sided case and two-sided case are useful in polynomial approximation. Mastroianni and Notarangelo [10, 11] considered Lagrange interpolation processes based on the zeros for exponential weight on $(-1, 1)$ and the real semiaxis, respectively.

Levin and Lubinsky [1, 2] defined an even weight corresponding to the one-sided weight. The weight is denoted that

$$\mathbf{I}^* = (-\sqrt{d}, \sqrt{d})$$

and for $x \in \mathbf{I}^*$,

$$Q^*(x) := Q(x^2), \quad (1.1)$$

$$W^*(x) := \exp(-Q^*(x)). \quad (1.2)$$

Throughout, $c, C_0, C_1, \dots$ stand for positive constants independent of variables and indices, unless otherwise indicated and their values may be different at different occurrences, even in subsequent formulas. Moreover, $C_n \sim D_n$ means that there are two constants c_1 and c_2 such that $c_1 \leq C_n/D_n \leq c_2$ for the relevant range of n . We write $c = c(\lambda)$ or $c \neq c(\lambda)$ to indicate dependence on or independence of a parameter λ . $\mathbf{P}_n$ stands for the set of polynomials of degree at most n .

A function $f : [0, d) \rightarrow (0, \infty)$ is said to be *quasi-increasing* if there exists $C > 0$ such that

$$f(x) \leq Cf(y), \quad 0 < x \leq y < d.$$

Definition 1.1. (see [1, Definition 1.1]). Let $\mathbf{I} = [0, d)$. Assume that $W = e^{-Q}$ where $Q : \mathbf{I} \rightarrow [0, \infty)$ satisfies the following properties:

- (a) $\sqrt{x}Q'(x) \in C(\mathbf{I})$ with limit 0 at 0 and $Q(0) = 0$.
- (b) Q'' exists in $(0, d)$, while $Q^{*''}$ is positive in $(0, \sqrt{d})$.
- (c)

$$\lim_{x \rightarrow d-} Q(x) = \infty.$$

(d) The function

$$T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \in (0, d)$$

is quasi-increasing in $(0, d)$, with

$$T(x) \geq \Lambda > \frac{1}{2}, \quad x \in (0, d). \quad (1.3)$$

(e) There exists $C_1 > 0$ such that

$$\frac{|Q''(x)|}{Q'(x)} \leq C_1 \frac{Q'(x)}{Q(x)}, \quad \text{a.e. } x \in (0, d). \quad (1.4)$$

Then we write $W \in \mathcal{L}(C^2)$. If also there exists a compact subinterval $\mathbf{J}$ of $\mathbf{I}^*$, and $C_2 > 0$ such that

$$\frac{Q^{*''}(x)}{|Q^{*'}(x)|} \geq C_2 \frac{|Q^{*'}(x)|}{Q^*(x)}, \quad \text{a.e. } x \in \mathbf{I}^* \setminus \{\mathbf{J}\}, \quad (1.5)$$

then we write $W \in \mathcal{L}(C^2+)$.

For $W \in \mathcal{L}(C^2)$ and $t > 0$, the Mhaskar-Rahmanov-Saff number $0 < a_t := a_t(Q)$ is defined by the equation

$$t = \frac{1}{\pi} \int_0^1 \frac{a_t x Q'(a_t x)}{[x(1-x)]^{1/2}} dx.$$

Put for $t > 0$,

$$\Delta_t := \Delta_t(Q) := [0, a_t], \quad \eta_t := \eta_t(Q) := (tT(a_t))^{-2/3},$$

$$\varphi_t(x) := \varphi_t(Q; x) := \begin{cases} \frac{\sqrt{x+a_t t^{-2}(a_{2t}-x)}}{t\sqrt{a_t-x+a_t\eta_t}}, & x \in [0, a_t], \\ \varphi_t(a_t), & x > a_t, \\ \varphi_t(0), & x < 0. \end{cases}$$

We also need a modification of φ_t , namely

$$\varphi_t^\#(x) := \begin{cases} \sqrt{\frac{x}{x+a_t t^{-2}}} \varphi_t(x) = \frac{\sqrt{x(a_{2t}-x)}}{t\sqrt{a_t-x+a_t\eta_t}}, & x \in [0, a_t], \\ \varphi_t^\#(a_t), & x > a_t. \end{cases}$$

The orthogonal polynomial of degree n for $W_{\alpha,\rho}^2$ is denoted by $p_n(W_{\alpha,\rho}^2, x)$ or just $p_n(x)$. Thus

$$\int_{\mathbf{I}} p_n(x) p_m(x) W_{\alpha,\rho}^2 dx = \delta_{nm}$$

and

$$p_n(x) = \gamma_n x^n + \cdots,$$

where $\gamma_n = \gamma_n(W_{\alpha,\rho}^2) > 0$.

The zeros of $p_n(x)$ are denoted by

$$x_{nn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n},$$

and the corresponding fundamental polynomials of Lagrange interpolation are polynomials $\ell_{jn} \in \mathbf{P}_{n-1}$. The classical Christoffel function is

$$\lambda_n(W_{\alpha,\rho}^2; x) = \inf_{P \in \mathbf{P}_n} (\|PW_{\alpha,\rho}\|_{L_2(\mathbf{I})} / |P(x)|)^2.$$

Considering the factor $(1-x^2)^\rho$, we introduce the following weight

$$\hat{Q}(x) := Q(x) + \rho q(x), \quad q(x) := -\ln(1-x^2), \quad \hat{W}(x) := e^{-\hat{Q}(x)}.$$

Before stating our results, we need some corresponding notations,

$$\hat{\Delta}_t := \Delta_t(\hat{Q}) := [0, \hat{a}_t], \quad \hat{a}_t := a_t(\hat{Q}), \quad \hat{\eta}_t := \eta_t(\hat{Q}),$$

$$\hat{T}(x) := T(\hat{Q}; x), \quad \hat{\varphi}_t(x) := \varphi_t(\hat{Q}; x).$$

The following theorems are similar in spirit with their analogues for weights $(1-x^2)^\rho e^{-Q(x)}$ on two-sided intervals [12], while the results of [12] cannot be applied to one-sided case. Furthermore, the formulation of the results are different, just as there are between the Laguerre and Hermite weights.

Theorem 1.1. Let $\mathbf{I} = [0, 1)$ and $W \in \mathcal{L}(C^2)$ (or $W \in \mathcal{L}(C^2+)$). Suppose that there exists λ such that for $x \in \mathbf{I} \setminus \{0\}$,

$$Q''(x) \geq \lambda \frac{1+x^2}{(1-x^2)^2}, \quad (1.6)$$

where

$$\lambda > \begin{cases} \frac{2|\rho|\Lambda}{\Lambda-\frac{1}{2}}, & \rho < 0, \\ 0, & \rho \geq 0, \end{cases} \quad (1.7)$$

and Λ is defined by (1.3). Then we have $\hat{W} \in \mathcal{L}(C^2)$ (or $\mathcal{L}(C^2+)$).

According to the above Theorem and applying Theorem 1.5 in [1] and Theorem 1.4 and Theorem 1.5 in [2], we gain the following Theorem 1.2. We also get the following Theorem 1.3, by using Theorem 1.2 and Theorem 1.3 in [1].

Theorem 1.2. Let $\alpha > -\frac{1}{2}$, $W \in \mathcal{L}(C^2)$ and the other assumptions of Theorem 1.1 be valid. Assume that $0 < p \leq \infty$.
 (a) Let $L, \zeta \geq 0$. Let $\beta > -\frac{1}{p}$ if $p < \infty$ and $\beta \geq 0$ if $p = \infty$. There exist $C_1, n_0 > 0$ such that for $n \geq n_0$ and $P \in \mathbf{P}_n$,

$$\|(P\hat{W})(x)x^\beta\|_{L_p(\mathbf{I})} \leq C_1 \|(P\hat{W})(x)x^\beta\|_{L_p[L\hat{a}_n n^{-2}, \hat{a}_n(1-\zeta\hat{\eta}_n)]}.$$

Moreover, given $r > 1$, there exist $C_2, n_0, \nu > 0$ such that for $n \geq n_0$ and $P \in \mathbf{P}_n$,

$$\|(P\hat{W})(x)x^\beta\|_{L_p(\hat{a}_n, 1)} \leq \exp(-C_2 n^\nu) \|(P\hat{W})(x)x^\beta\|_{L_p(\hat{\Delta}_n)}.$$

(b) Let $\beta > -\frac{1}{p}$ if $p < \infty$ and $\beta \geq 0$ if $p = \infty$. Given $0 < r < 1$. Then for $n \geq 1, P \in \mathbf{P}_n$ and for some C ,

$$\|(P\hat{W})'(x)\hat{\phi}_n^\#(x)x^\beta\|_{L_p(\mathbf{I})} \leq C \|(P\hat{W})(x)x^\beta\|_{L_p(\mathbf{I})},$$

$$\|(P'\hat{W})(x)x^\beta\|_{L_p(\mathbf{I})} \leq C \frac{n^2}{\hat{a}_n} \|(P\hat{W})(x)x^\beta\|_{L_p(\mathbf{I})}$$

and

$$\|(P'\hat{W})(x)x^\beta\|_{L_p[\hat{a}_n, 1)} \leq C \frac{n}{\hat{a}_n} \sqrt{\hat{T}(\hat{a}_n)} \|(P\hat{W})(x)x^\beta\|_{L_p(\mathbf{I})}.$$

(c) Let $L > 0$. Then uniformly for $n \geq 1$ and $x \in [0, \hat{a}_n(1 + L\hat{\eta}_n)]$,

$$\lambda_n(W_{\alpha, \rho}^2; x) \sim \hat{\phi}_n(x)\hat{W}^2(x) \left(x + \frac{\hat{a}_n}{n^2}\right)^{2\alpha}.$$

Moreover, there exists $C > 0$ such that uniformly for $n \geq 1$ and $x \in \mathbf{I}$,

$$\lambda_n(W_{\alpha, \rho}^2; x) \geq C \hat{\phi}_n(x)\hat{W}^2(x) \left(x + \frac{\hat{a}_n}{n^2}\right)^{2\alpha}.$$

(d) Then uniformly for $n \geq 1$,

$$\sup_{x \in \mathbf{I}} \left| p_n(W_{\alpha, \rho}^2; x) \right| \hat{W}(x) \left(x + \frac{\hat{a}_n}{n^2}\right)^\alpha \left| \left(x + \frac{\hat{a}_n}{n^2}\right) (\hat{a}_n - x) \right|^{\frac{1}{4}} \sim 1.$$

(e) There exist $C_3, C_4 > 0$ such that for $n \geq 1$ and $1 \leq j \leq n-1$,

$$x_{jn} - x_{j+1, n} \leq C_3 \hat{\phi}_n(x_{jn}),$$

$$x_{nn} \sim \hat{a}_n n^{-2}$$

and

$$\hat{a}_n(1 - C_4 \hat{\eta}_n) \leq x_{1n} < \hat{a}_{n+\alpha+\frac{1}{4}}.$$

Furthermore, for each fixed j and n , x_{jn} is a non-decreasing function of α .

Theorem 1.3. Let $\alpha > -\frac{1}{2}$, $W \in \mathcal{L}(C^2+)$ and the other assumptions of Theorem 1.1 be valid.

(a) Let $0 < \beta < 1$, then uniformly for $n \geq 1$,

$$\sup_{x \in \mathbf{I}} |p_n(x)| \hat{W}(x) \left(x + \frac{\hat{a}_n}{n^2}\right)^\alpha \sim \left(\frac{n}{\hat{a}_n}\right)^{\frac{1}{2}},$$

$$\sup_{x \in [\hat{a}_{\beta n}, 1)} |p_n(x)| \hat{W}(x) \left(x + \frac{\hat{a}_n}{n^2}\right)^\alpha \sim \hat{a}_n^{-\frac{1}{2}} (n\hat{T}(\hat{a}_n))^{\frac{1}{6}}$$

and

$$x_{jn} - x_{j+1, n} \sim \hat{\phi}_n(x_{jn}), \quad 1 \leq j < n.$$

If $W \in \mathcal{L}(C^2)$, these estimates hold with $\sim$ replaced by $\leq C$.

(b) There exists n_0 such that uniformly for $n > n_0$, $1 \leq j \leq n$,

$$|p'_n W_{\alpha, \rho}|(x_{jn}) \sim \hat{\varphi}_n(x_{jn})^{-1} [x_{jn}(\hat{a}_n - x_{jn})]^{-\frac{1}{4}}, \quad (1.8)$$

$$|p_{n-1} W_{\alpha, \rho}|(x_{jn}) \sim \hat{a}_n^{-1} [x_{jn}(\hat{a}_n - x_{jn})]^{\frac{1}{4}}, \quad (1.9)$$

$$\max_{x \in I} \left| \ell_{jn}(x) \hat{W}(x) \left(x + \frac{\hat{a}_n}{n^2} \right)^\alpha \right| (W_{\alpha, \rho})^{-1}(x_{jn}) \sim 1$$

and

$$1 - \frac{x_{1n}}{\hat{a}_n} \sim \hat{\eta}_n.$$

If we assume instead that $W \in \mathcal{L}(C^2)$, then (1.8) holds with $\sim$ replaced by $\leq C$ and (1.9) holds with $\sim$ replaced by $\geq C$.

(c) For $j \leq n-1$ and $x \in [x_{j+1, n}, x_{jn}]$,

$$|p_n W_{\alpha, \rho}|(x) \sim \min \{ |x - x_{jn}|, |x - x_{j+1, n}| \} \hat{\varphi}_n(x_{jn})^{-1} [x_{jn}(\hat{a}_n - x_{jn})]^{-\frac{1}{4}}.$$

Here we give the following two theorems as examples of Theorem 1.1.

Theorem 1.4. Let $I = [0, 1)$, $\tau > 0$, and

$$Q(x) = (1-x)^{-\tau} - 1, \quad x \in [0, 1).$$

(a) Then

$$T(x) \geq 1, \quad x \in (0, 1). \quad (1.10)$$

(b) If

$$\tau > \begin{cases} 4|\rho|, & \rho < 0, \\ 0, & \rho \geq 0, \end{cases} \quad (1.11)$$

then we have $\hat{W} \in \mathcal{L}(C^2+)$.

Theorem 1.5. Let $I = [0, 1)$, $k \geq 1$, $\tau > 0$, and

$$Q(x) = \exp_k((1-x)^{-\tau}) - \exp_k(1), \quad x \in [0, 1).$$

(a) Then the relation of (1.10) holds.

(b) If

$$\tau > \begin{cases} \frac{4|\rho|}{\prod_{j=1}^k \exp_j(1)}, & \rho < 0, \\ 0, & \rho \geq 0, \end{cases} \quad (1.12)$$

then we have $\hat{W} \in \mathcal{L}(C^2+)$.

Remark 1.1. For Theorem 1.4 and Theorem 1.5, Levin and Lubinsky in [1, 2] discussed the case when $\rho = 0$.

We shall give some technical lemmas in Section 2 and the proofs of Theorem 1.1, Theorem 1.4 and Theorem 1.5 in Section 3.

2 Auxiliary lemmas

Lemma 2.1. (a) Let the assumptions of Theorems 1.1 be valid. Then there exist $\mu > 0$ and

$$\lambda = \frac{2|\rho|}{\mu}, \quad \rho \neq 0$$

such that

$$\mu Q''(x) \geq |\rho| q''(x), \quad x \in \mathbf{I} \setminus \{0\} \quad (2.1)$$

and

$$\mu Q^{*''}(x) \geq |\rho| q^{*''}(x), \quad x \in \mathbf{I}^* \setminus \{0\}. \quad (2.2)$$

(b) Let $\mathbf{I} = [0, 1)$ and $W \in \mathcal{L}(C^2)$. Assume that there exists $\mu > 0$ such that (2.2) is valid. Then (2.1) holds. Moreover, there exists λ such that for $x \in \mathbf{I} \setminus \{0\}$, (1.6) and (1.7) hold.

Proof. The case when $\rho = 0$ is trivial. Let $\rho \neq 0$.

(a) Given any $\varepsilon > 0$, choose

$$\mu = \begin{cases} \frac{\Lambda - (\frac{1}{2} + \varepsilon)}{\Lambda}, & \rho < 0, \\ \frac{2\rho}{\varepsilon}, & \rho > 0. \end{cases} \quad (2.3)$$

Then

$$\lambda = \frac{2|\rho|}{\mu} > \begin{cases} \frac{2|\rho|\Lambda}{\Lambda - \frac{1}{2}}, & \rho < 0, \\ 0, & \rho > 0. \end{cases}$$

Clearly,

$$q'(x) = \frac{2x}{1-x^2}, \quad q''(x) = \frac{2(1+x^2)}{(1-x^2)^2}. \quad (2.4)$$

Then inequality (1.6) with the above relations may be written as

$$Q''(x) \geq \frac{|\rho|}{\mu} q''(x), \quad x \in (0, 1). \quad (2.5)$$

so the relation (2.1) follows from (2.5) directly.

If we introduce the notation

$$\overline{Q}(x) := Q(x) - \frac{|\rho|}{\mu} q(x), \quad x \in \mathbf{I}, \quad (2.6)$$

then according to (2.1),

$$\overline{Q}''(x) \geq 0, \quad x \in \mathbf{I} \setminus \{0\}. \quad (2.7)$$

With the help of (2.7) for $x \in (0, 1)$,

$$\overline{Q}'(x) = \int_0^x \overline{Q}''(z) dz \geq 0, \quad (2.8)$$

and

$$\overline{Q}(x) = \int_0^x \overline{Q}'(z) dz \geq 0. \quad (2.9)$$

Inequalities of (2.7), (2.8) and (2.9) give the estimates

$$\mu Q^{(i)}(x) \geq |\rho| q^{(i)}(x), \quad x \in (0, 1), \quad i = 0, 1, 2. \quad (2.10)$$

Here according to (1.1) and (1.2) we introduce

$$\overline{Q}^*(s) := Q^*(s) - \frac{|\rho|}{\mu} q^*(s), \quad s \in \mathbf{I}^*, \quad (2.11)$$

and then for $s \in \mathbf{I}^* \setminus \{0\} = (-1, 1) \setminus \{0\}$,

$$\overline{Q}^{*'''}(s) = Q^{*'''}(s) - \frac{|\rho|}{\mu} q^{*'''}(s) = 4s^2 [Q''(s^2) - \frac{|\rho|}{\mu} q''(s^2)] + 2[Q'(s^2) - \frac{|\rho|}{\mu} q'(s^2)]. \quad (2.12)$$

Since $s \in (-1, 1) \setminus \{0\}$ is equivalent to $s^2 \in (0, 1)$, so using (2.12) and (2.10) we obtain

$$\overline{Q}^{*'''}(s) \geq 0, \quad s \in (-1, 1) \setminus \{0\}.$$

This proves the relation (2.2).

(b) Given (2.2), using (2.12) it is shown that one of the following cases will hold:

Case 1:

$$Q''(x^2) - \frac{|\rho|}{\mu} q''(x^2) \geq 0$$

and

$$Q'(x^2) - \frac{|\rho|}{\mu} q'(x^2) \geq 0, \quad x \in (-1, 1) \setminus \{0\};$$

Case 2:

$$Q''(x^2) - \frac{|\rho|}{\mu} q''(x^2) \leq 0, \quad x \in (-1, 1) \setminus \{0\},$$

$$Q'(x^2) - \frac{|\rho|}{\mu} q'(x^2) \geq 0, \quad x \in (-1, 1) \setminus \{0\}$$

and

$$Q'(x^2) - \frac{|\rho|}{\mu} q'(x^2) \geq 2x^2 \left| Q''(x^2) - \frac{|\rho|}{\mu} q''(x^2) \right|, \quad x \in (-1, 1) \setminus \{0\}.$$

According to (2.7) and (2.8), it is clear that Case 1 gives (2.1).

Here Case 2 means that

$$\overline{Q}''(x) \leq 0, \quad x \in (0, 1)$$

and

$$\overline{Q}''(x) \geq 0, \quad x \in (0, 1).$$

Using the same arguments as (2.7) and (2.8), we see that Case 2 is contradictory. So we get (2.1).

Further, set $\lambda = \frac{2|\rho|}{\mu}$, where μ is defined by (2.3), then coupling with (2.1) we obtain (1.6) and (1.7). $\square$

Remark 2.1. According to the above Lemma (b), the result implies $Q''(x) > 0$, $x \in (0, 1)$. Moreover, we see that the assumptions of Theorems 1.1 and Lemma 2.1(b) are equivalent.

Lemma 2.2. Let

$$h(x) = (1 - x)^{-\tau} - 1 \quad (\text{or } (1 - x)^{-\tau}), \quad \tau > 0 \quad x \in [0, 1).$$

Then

$$h'(x) = \tau(1 - x)^{-\tau-1} \quad (2.13)$$

and

$$h''(x) = \tau(1 + \tau)(1 - x)^{-\tau-2} \geq \frac{\tau(1 + x^2)}{(1 - x^2)^2}. \quad (2.14)$$

Proof. By a short calculation we gain

$$h''(x) = \frac{\tau(1 + \tau)(1 + x)^{\tau+2}}{(1 - x^2)^{\tau+2}} \geq \frac{\tau(1 + \tau)(1 + x^2)^{\tau+2}}{(1 - x^2)^{\tau+2}} \geq \frac{\tau(1 + x^2)}{(1 - x^2)^2}, \quad x \in [0, 1).$$

$\square$

3 Proof of theorems

3.1. Proof of Theorem 1.1

Here let $\rho \neq 0$. The theorem for the case when $\rho = 0$ is trivial. In deed, the authors in [1, 2] discussed the case when $\rho = 0$. We use the idea of Theorem 1.1 in [12] with modification, and according to Lemma 2.1 it is more easier to get μ since we only require $\varepsilon > 0$ in formula (2.3).

We set

$$\hat{Q}(x) = Q(x) + \rho q(x), \quad q(x) := -\ln(1 - x^2), \quad x \in \mathbf{I}.$$

With Definition 1.1 (a) for Q we have

$$\hat{Q}(0) = 0$$

and by (2.4)

$$\sqrt{x}\hat{Q}'(x) = \sqrt{x}Q'(x) + \frac{2\rho x\sqrt{x}}{1-x^2}, \quad x \in \mathbf{I},$$

which shows that $\sqrt{x}\hat{Q}'(x)$ is continuous in $\mathbf{I}$, with limit 0 at 0.

This proves the properties listed in Definition 1.1 (a) with Q replaced by $\hat{Q}$.

By (2.4) and Definition 1.1 (b) for Q ,

$$\hat{Q}''(x) = Q''(x) + \rho \frac{2(1+x^2)}{(1-x^2)^2},$$

which means that $\hat{Q}''(x)$ exists in $(0, 1)$.

In the following parts, the notations μ , $\bar{Q}$ and $\bar{Q}^*$ are defined by (2.3), (2.6) and (2.11), respectively.

By the notation $\bar{Q}(x)$ in (2.6)

$$\hat{Q}(x) = \bar{Q}(x) + \left(\frac{|\rho|}{\mu} + \rho\right)q(x), \quad x \in \mathbf{I}, \quad (3.1)$$

then coupling with (2.9) and noticing that $0 < \mu < 1$ for $\rho < 0$, we see

$$\hat{Q}(x) > 0, \quad x \in (0, 1)$$

and

$$\lim_{x \rightarrow 1^-} \hat{Q}(x) = \infty.$$

Meanwhile, by (3.1), (2.10), (2.4) and the fact $(\frac{|\rho|}{\mu} + \rho) > 0$

$$\hat{Q}'(x) > 0, \quad x \in (0, 1). \quad (3.2)$$

Here according to the notation $\bar{Q}^*$ in (2.11)

$$\hat{Q}^{*''}(s) = \bar{Q}^{*''}(s) + \left(\frac{|\rho|}{\mu} + \rho\right)q^{*''}(s), \quad s \in (-1, 1) \setminus \{0\}. \quad (3.3)$$

By (2.4) and the fact $(\frac{|\rho|}{\mu} + \rho) > 0$,

$$\left(\frac{|\rho|}{\mu} + \rho\right)q^{*''}(s) = \left(\frac{|\rho|}{\mu} + \rho\right)(4s^2 q''(s^2) + 2q'(s^2)) > 0, \quad (3.4)$$

provided $s \in (-1, 1) \setminus \{0\}$, which is equivalent to $s^2 \in (0, 1)$.

Hence, combining (3.3), (2.2) and (3.4), we get

$$\hat{Q}^{*''}(s) > 0, \quad s \in (-1, 1) \setminus \{0\}.$$

This proves the properties listed in Definition 1.1 (b) and (c) with Q replaced by $\hat{Q}$.

In what follows we separate two cases. First, we give an inequality analogous to (2.10).

Using (2.2) and the same arguments as (2.8) and (2.9) with replaced $\bar{Q}(x)$ by $\bar{Q}^*(s)$ for $s \in (0, 1)$, we have $\bar{Q}^{*'}(s) \geq 0$ and $\bar{Q}^*(s) \geq 0$.

Similarly, by (2.2) for $s \in (-1, 0)$,

$$\bar{Q}^{*'}(s) = - \int_s^0 \bar{Q}^{*''}(z) dz \leq 0$$

and

$$\bar{Q}^*(s) = - \int_s^0 \bar{Q}^{*'}(z) dz \geq 0.$$

Hence, we have the estimates

$$\mu |Q^{*(i)}(s)| \geq |\rho q^{*(i)}(s)|, \quad s \in (-1, 1) \setminus \{0\}, \quad i = 0, 1, 2. \quad (3.5)$$

Case 1: $\rho < 0$.

(d) In this case by (3.2) and (2.10) for $x \in (0, 1)$,

$$\frac{\hat{Q}'(x)}{\hat{Q}(x)} \leq \frac{Q'(x) + |\rho|q'(x)}{Q(x) + \rho q(x)} \leq \left(\frac{1+\mu}{1-\mu} \right) \frac{Q'(x)}{Q(x)} \quad (3.6)$$

and

$$\frac{\hat{Q}'(x)}{\hat{Q}(x)} \geq \frac{Q'(x) - |\rho|q'(x)}{Q(x)} \geq (1-\mu) \frac{Q'(x)}{Q(x)}. \quad (3.7)$$

Thus, for the function $\hat{T}(x) = \frac{x\hat{Q}'(x)}{\hat{Q}(x)}$

$$(1-\mu)T(x) \leq \hat{T}(x) \leq \left(\frac{1+\mu}{1-\mu} \right) T(x), \quad x \in (0, 1). \quad (3.8)$$

According to Definition 1.1(d) for $T(x)$ and (3.8) for $0 < x \leq y < 1$,

$$\hat{T}(x) \leq \left(\frac{1+\mu}{1-\mu} \right) T(x) \leq C \left(\frac{1+\mu}{1-\mu} \right) T(y) \leq C \left(\frac{1+\mu}{(1-\mu)^2} \right) \hat{T}(y).$$

We see that $\hat{T}(x)$ is quasi-increasing in $(0, 1)$ and by (3.8), (1.3) and (2.3) for $x \in (0, 1)$,

$$\hat{T}(x) \geq (1-\mu)T(x) \geq \Lambda(1-\mu) = \hat{\Lambda} > \frac{1}{2}.$$

This proves Definition 1.1(d) with Q replaced by $\hat{Q}$.

(e) By (2.10), (1.4) and (3.7) for a.e. $x \in (0, 1)$,

$$\frac{|\hat{Q}''(x)|}{\hat{Q}'(x)} \leq \frac{|Q''(x)| + |\rho q''(x)|}{|Q'(x) - |\rho|q'(x)|} \leq \left(\frac{1+\mu}{1-\mu} \right) \frac{|Q''(x)|}{|Q'(x)|} \leq C_1 \left(\frac{1+\mu}{1-\mu} \right) \frac{Q'(x)}{Q(x)} \leq C_1 \left(\frac{1+\mu}{(1-\mu)^2} \right) \frac{\hat{Q}'(x)}{\hat{Q}(x)}.$$

This proves $\hat{W} \in \mathcal{L}(C^2)$.

Replacing $x \in (0, 1)$ with s^2 , $s \in (-1, 1) \setminus \{0\}$ and multiplying by $2s$ in the equality of (3.6) we get

$$\frac{|\hat{Q}^{*'}(s)|}{\hat{Q}^*(s)} \leq \left(\frac{1+\mu}{1-\mu} \right) \frac{|Q^{*'}(s)|}{Q^*(s)}, \quad s \in (-1, 1) \setminus \{0\}. \quad (3.9)$$

By (3.5), (1.5) and (3.9) for a.e. $t \in \mathbf{I}^* \setminus \{\mathbf{J}\}$,

$$\frac{\hat{Q}^{*''}(s)}{|\hat{Q}^{*'}(s)|} \geq \frac{|Q^{*''}(s)| - |\rho q^{*''}(s)|}{|Q^{*'}(s)|} \geq (1-\mu) \frac{Q^{*''}(s)}{|Q^{*'}(s)|} \geq C_2(1-\mu) \frac{|Q^{*'}(s)|}{Q^*(s)} \geq C_2 \left(\frac{(1-\mu)^2}{1+\mu} \right) \frac{|\hat{Q}^{*'}(s)|}{\hat{Q}^*(s)}.$$

This proves $\hat{W} \in \mathcal{L}(C^2+)$.

Case 2: $\rho > 0$.

(d) By (3.2) and (2.10) for $x \in (0, 1)$,

$$\frac{\hat{Q}'(x)}{\hat{Q}(x)} \leq \frac{Q'(x) + \rho q'(x)}{Q(x)} \leq (1 + \mu) \frac{Q'(x)}{Q(x)} \quad (3.10)$$

and

$$\frac{\hat{Q}'(x)}{\hat{Q}(x)} \geq \frac{Q'(x)}{Q(x) + \rho q(x)} \geq \frac{Q'(x)}{(1 + \mu)Q(x)}. \quad (3.11)$$

Thus for the function $\hat{T}(x) = \frac{x\hat{Q}'(x)}{\hat{Q}(x)}$,

$$\frac{1}{1 + \mu} T(x) \leq \hat{T}(x) \leq (1 + \mu) T(x), \quad x \in (0, 1). \quad (3.12)$$

According to Definition 1.1(d) for $T(x)$ and (3.12) for $0 < x \leq y < 1$,

$$\hat{T}(x) \leq (1 + \mu) T(x) \leq C(1 + \mu) T(y) \leq C(1 + \mu)^2 \hat{T}(y),$$

which shows that $\hat{T}(x)$ is quasi-increasing in $(0, 1)$.

Now, we set a function $K(x) = xq'(x) - q(x)$. By (2.4)

$$K'(x) \geq 0, \quad x \in [0, 1)$$

and hence $K(x) \geq K(0) = 0$, which means

$$x\rho q'(x) \geq \rho q(x), \quad x \in [0, 1). \quad (3.13)$$

By (3.13) and (1.3)

$$x\hat{Q}'(x) \geq xQ'(x) + \rho xq'(x) \geq \Lambda Q(x) + \rho q(x) \geq \min\{\Lambda, 1\}[Q(x) + \rho q(x)] = \min\{\Lambda, 1\}\hat{Q}(x),$$

which gives $\hat{T}(x) \geq \min\{\Lambda, 1\} > 1/2$, $x \in (0, 1)$.

This proves Definition 1.1(d) with Q replaced by $\hat{Q}$.

(e) By (2.10), (1.4) and (3.11) for a.e. $x \in (0, 1)$,

$$\frac{|\hat{Q}''(x)|}{|\hat{Q}'(x)|} \leq \frac{Q''(x) + \rho q''(x)}{Q'(x)} \leq (1 + \mu) \frac{Q''(x)}{Q'(x)} \leq C_1(1 + \mu) \frac{Q'(x)}{Q(x)} \leq C_1(1 + \mu)^2 \frac{\hat{Q}'(x)}{\hat{Q}(x)}.$$

Now we obtain $\hat{W} \in \mathcal{L}(C^2)$.

By (3.10) and with the same argument as (3.9) we have

$$\frac{|\hat{Q}^{*'}(s)|}{|\hat{Q}^*(s)|} \leq (1 + \mu) \frac{|Q^{*'}(s)|}{|Q^*(s)|}, \quad s \in (-1, 1) \setminus \{0\}. \quad (3.14)$$

By (3.5), (1.5) and (3.14) for a.e. $t \in \mathbf{I}^* \setminus \{\mathbf{J}\}$,

$$\frac{\hat{Q}^{*''}(s)}{|\hat{Q}^{*'}(s)|} \geq \frac{Q^{*''}(s)}{|Q^{*'}(s) + \rho q^{*'}(s)|} \geq \frac{1}{1 + \mu} \cdot \frac{Q^{*''}(s)}{|Q^{*'}(s)|} \geq \frac{C_2}{1 + \mu} \cdot \frac{|Q^{*'}(s)|}{|Q^*(s)|} \geq \frac{C_2}{(1 + \mu)^2} \cdot \frac{|\hat{Q}^{*'}(s)|}{|\hat{Q}^*(s)|}.$$

So we conclude that $\hat{W} \in \mathcal{L}(C^2+)$. □

3.2. Proof of Theorem 1.4

(a) Set $f(x) = xQ'(x) - Q(x)$, and then applying Lemma 2.2 we have

$$f'(x) = xQ''(x) = \tau(1 + \tau)x(1 - x)^{-\tau-2} \geq 0, \quad x \in [0, 1),$$

which means that $f(x) \geq f(0) = 0$. It is equivalent to

$$T(x) \geq 1, \quad x \in (0, 1). \quad (3.15)$$

(b) Using (2.14) we have

$$Q''(x) \geq \frac{\tau(1+x^2)}{(1-x^2)^2}, \quad x \in [0, 1).$$

Then by (1.11)

$$\lambda = \tau > \begin{cases} 4|\rho|, & \rho < 0, \\ 0, & \rho > 0. \end{cases}$$

By (3.15) we see $\Lambda = 1$ and hence coupling with the above relation we obtain that (1.6) and (1.7) are valid. Thus applying Theorem 1.1 we get $W \in \mathcal{L}(C^2+)$. $\square$

3.3. Proof of Theorem 1.5

(a) Put

$$h(x) = (1-x)^{-\tau},$$

and

$$g_j(x) = \exp_j(h(x)), \quad x \in [0, 1).$$

By calculation

$$\begin{aligned} Q'(x) &= g'_k(x) = h'(x) \prod_{j=1}^k g_j(x) = h'(x) \prod_{j=1}^k \exp_j(h(x)), \\ Q''(x) &= h''(x) \prod_{j=1}^k g_j(x) + h'(x) \sum_{j=1}^k g'_j(x) \prod_{\substack{i=1 \\ i \neq j}}^k g_i(x) \\ &= h''(x) \prod_{j=1}^k g_j(x) + h'(x)^2 \sum_{j=1}^k \prod_{i=1}^{j-1} g_i(x) \prod_{j=1}^k g_j(x) \\ &= \prod_{j=1}^k \exp_j(h(x)) \left[h''(x) + h'(x)^2 \sum_{j=1}^k \prod_{i=1}^{j-1} \exp_i(h(x)) \right]. \end{aligned} \quad (3.16)$$

By (3.16), (2.13) and (2.14) for $x \in [0, 1)$,

$$\begin{aligned} f'(x) &= (xQ'(x) - Q(x))' = xQ''(x) \\ &= x \prod_{j=1}^k \exp_j(h(x)) \left[\tau(1+\tau)(1-x)^{-\tau-2} + h'(x)^2 \sum_{j=1}^k \prod_{i=1}^{j-1} \exp_i(h(x)) \right] \geq 0, \end{aligned}$$

and hence by the same argument as Theorem 1.4 (a) inequality (1.10) holds.

(b) By (3.16) and (2.14)

$$Q''(x) \geq \prod_{j=1}^k \exp_j(h(x)) h''(x) \geq \prod_{j=1}^k \exp_j(1) \tau(1+\tau)(1-x)^{-\tau-2} \geq \prod_{j=1}^k \exp_j(1) \tau \frac{1+x^2}{(1-x^2)^2}.$$

Then by (1.12)

$$\lambda = \tau \prod_{j=1}^k \exp_j(1) > \begin{cases} 4|\rho|, & \rho < 0, \\ 0, & \rho > 0. \end{cases}$$

By the statements of (a), we see $\Lambda = 1$ and hence coupling with the above relation (1.6) and (1.7) are valid. Thus applying Theorem 1.1 we get $W \in \mathcal{L}(C^2+)$. $\square$

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