Research Article

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Controllability of fractional stochastic evolution equations with nonlocal conditions and noncompact semigroups

<https://doi.org/10.1515/math-2020-0034>

received September 11, 2019; accepted May 15, 2020

Abstract: This article deals with the exact controllability for a class of fractional stochastic evolution equations with nonlocal initial conditions in a Hilbert space under the assumption that the semigroup generated by the linear part is noncompact. Our main results are obtained by utilizing stochastic analysis technique, measure of noncompactness and the Mönch fixed point theorem. In the end, an example is presented to illustrate that our theorems guarantee the effectiveness of controllability results in the infinite dimensional spaces.

Keywords: noncompact semigroups, controllability, stochastic evolution equations, measure of noncompactness

MSC 2010: 93B05, 60H15, 47J35

1 Introduction

In this article, we shall be concerned with the controllability for the following fractional evolution equations with nonlocal initial conditions of the form:

$$\begin{cases} {}^c D_t^\alpha x(t) + Ax(t) = f(t, x(t)) + \sigma(t, x(t)) \frac{dW(t)}{dt} + Bu(t), & t \in J = [0, b], \\ x(0) + g(x) = x_0, \end{cases} \quad (1.1)$$

where ${}^c D_t^\alpha$ is the Caputo fractional derivatives of order $\frac{1}{2} < \alpha < 1$. Let $\mathbb{H}$ and $\mathbb{K}$ be two separable Hilbert spaces and the state $x(\cdot)$ takes its values in $\mathbb{H}$. $A: D(A) \subset \mathbb{H} \rightarrow \mathbb{H}$ is a closed linear operator, and $-A$ is the infinitesimal generator of a C_0 -semigroup $T(t) (t \geq 0)$ on $\mathbb{H}$. For convenience, we will use the same notation $\|\cdot\|$ to denote the norms in $\mathbb{H}$ and $\mathbb{K}$ and use $(\cdot, \cdot)$ to denote the inner product of $\mathbb{H}$ and $\mathbb{K}$ without any confusion. We are also employing the same notation $\|\cdot\|$ for the norm of $L(\mathbb{K}, \mathbb{H})$, which denotes the space of all bounded linear operators from $\mathbb{K}$ to $\mathbb{H}$. Suppose that $\{W(t): t \geq 0\}$ is a $\mathbb{K}$ -valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q \geq 0$ defined on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$. The control function $u(\cdot)$ belongs to the space $L^2_{\mathcal{F}}(J, \mathbb{U})$, a Banach space of admissible control functions, for a separable Hilbert space $\mathbb{U}$, $B: \mathbb{U} \rightarrow \mathbb{H}$ is a bounded linear operator. f , σ and g are appropriate functions to be given later. x_0 is an $\mathcal{F}_0$ measurable $\mathbb{H}$ -value random variable.

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Fractional evolution equations have gained considerable importance due to their application in various sciences [1–17]. On the other hand, controllability for various types of linear and nonlinear dynamical systems have been considered in many publications by using different approaches due to its applications in many fields of science and engineering [1–7, 18–29]. It should be emphasized that there are many different notions of controllability for dynamical systems, for example, approximate controllability, exact controllability, null controllability and so on. There have been many studies on the approximate controllability for semilinear evolution systems in abstract spaces (see [1–4, 20, 21, 30, 31] and references therein). Several authors have studied exact controllability for differential control systems (see [5–7, 22–24] and references therein). It should be pointed out that some studies on controllability of abstract control systems contain a similar technical error when the compactness of the semigroup generated by the linear part and the invertibility of the controllability operator are satisfied, and in this case the application of exact controllability results is just restricted to the finite dimensional space and it fails in infinite dimensional space [23].

In recent years, Stochastic differential equations have attracted great interest due to their successful applications to problems in mechanics, electricity, economics, physics and several fields in engineering. For details, see [8, 9, 32–38] and references therein. In particular, some researchers investigated controllability of stochastic dynamical control systems in infinite dimensional spaces [25–29]. It is generally known that nonlocal initial conditions can be applied in physics with better effect than the classical Cauchy problem traditional initial condition. However, to the best of our knowledge, the exact controllability of stochastic control systems with nonlocal conditions of the form (1.1) has not yet been studied. Therefore, a natural problem is as follows: how to investigate the exact controllability of stochastic evolution equations involving noncompact semigroups?

Motivated by this consideration, in this article we establish the exact controllability for fractional stochastic evolution equations with nonlocal conditions of the form (1.1) in a Hilbert space under the assumption that the semigroup is noncompact. By using some constructive control functions, we transfer the controllability problem into a fixed-point problem and then apply the measure of noncompactness and the Mönch fixed point theorem to discuss the controllability for problem (1.1). We delete the compactness of semigroup $T(t)(t > 0)$, so our theorems guarantee the effectiveness of controllability results in the infinite dimensional spaces. Furthermore, we give a useful way to discuss stochastic control systems with noncompact semigroups.

We organize the article in the following way. In Section 2, we introduce some useful definitions and preliminary results to be used in this article. In Section 3, we state and prove the exact controllability results for fractional stochastic evolution equations with nonlocal conditions. Finally, in Section 4, an example is provided to illustrate the applications of the obtained results.

2 Preliminaries

In this section, we introduce notations, definitions and preliminary facts, which are used throughout this article.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a filtered complete probability space satisfying the usual condition, which means that the filtration is a right continuous increasing family and $\mathcal{F}_0$ contains all P -null sets. Let $\{e_k, k \in \mathbb{N}\}$ be a complete orthonormal basis of $\mathbb{K}$. $\{W(t): t \geq 0\}$ is a cylindrical $\mathbb{K}$ -valued Brownian motion or Wiener process defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ with a finite trace nuclear covariance operator $Q \geq 0$, we denote $\text{Tr}(Q) = \sum_{k=1}^{\infty} \lambda_k = \lambda < \infty$, which satisfies that $Qe_k = \lambda_k e_k, k \in \mathbb{N}$. Let $\{W_k(t), k \in \mathbb{N}\}$ be a sequence of one-dimensional standard Wiener processes mutually independent on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ such that

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} W_k(t) e_k, \quad t \geq 0.$$

Furthermore, we assume that $\mathcal{F}_t = \sigma\{W(s), 0 \leq s \leq t\}$ is the σ -algebra generated by W and $\mathcal{F}_b = \mathcal{F}$.

Let $L_2^0 = L_2(Q^{1/2}\mathbb{K}, \mathbb{H})$ denote the space of all Hilbert-Schmidt operators from $Q^{1/2}\mathbb{K}$ into $\mathbb{H}$ with the inner product $(\phi, \varphi) = \text{Tr}(\phi Q \varphi^*)$. It also turns out to be a separable Hilbert space. The collection of all $\mathcal{F}_b$ -measurable, square-integrable $\mathbb{H}$ -valued random variables, denoted $L^2(\Omega, \mathbb{H})$, is a Banach space equipped with the norm $\|x\|_{L^2} = (\mathbb{E}\|x(\omega)\|^2)^{1/2}$, where $\mathbb{E}$ denotes the expectation with respect to the measure P . Let $C(J, L^2(\Omega, \mathbb{H}))$ be the Banach space of all continuous mappings from J to $L^2(\Omega, \mathbb{H})$ satisfying $\sup_{t \in J} (\mathbb{E}\|x(t)\|^2) < \infty$. We use $\mathcal{H}$ to denote the space of all $\mathcal{F}_t$ -adapted measurable processes $x \in C(J, L^2(\Omega, \mathbb{H}))$ endowed with the norm $\|x\|_{\mathcal{H}} = (\sup_{t \in J} \mathbb{E}\|x(t)\|^2)^{1/2}$. The theory of stochastic integrals in a Hilbert space can be found in [34, 36].

In the rest of the manuscript, we suppose that A generates an equicontinuous C_0 -semigroup $T(t) (t \geq 0)$ of uniformly bounded linear operator in $\mathbb{H}$. That is, there exists a positive constant $M \geq 1$ such that $\|T(t)\| \leq M$ for all $t \geq 0$. For any constant $r > 0$, let $B_r = \{x \in \mathcal{H} : \|x\|_{\mathcal{H}}^2 \leq r\}$. Evidently, B_r is a bounded closed convex set in $\mathcal{H}$.

By [29, Proposition 2.8], we have the following result which will be used throughout this article.

Lemma 2.1. *If $h: J \times \mathbb{H} \rightarrow L(\mathbb{K}, \mathbb{H})$ is continuous and $x \in C(J, L^2(\Omega, \mathbb{H}))$, then*

$$\mathbb{E} \left\| \int_J h(t, x(t)) dW(t) \right\|^2 \leq \text{Tr}(Q) \int_J \mathbb{E} \|h(t, x(t))\|^2 dt.$$

Definition 2.1. [10] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y: (0, +\infty) \rightarrow \mathbb{R}$ is given by

$$I_0^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds,$$

provided the right side is pointwise defined on $(0, +\infty)$.

Definition 2.2. [10] The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y: [0, +\infty) \rightarrow \mathbb{R}$ is given by

$$D_0^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$, provided that the right side is pointwise defined on $(0, +\infty)$.

Definition 2.3. [10] The Caputo fractional derivative of order $\alpha > 0$ of a function $y: [0, +\infty) \rightarrow \mathbb{R}$ is given by

$${}^c D_0^\alpha y(t) = D_0^\alpha \left[y(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} y^{(k)}(0) \right],$$

where $n = [\alpha] + 1$, provided that the right side is pointwise defined on $(0, +\infty)$.

Remark 2.1.

(i) In the case $y(t) \in C^n[0, +\infty)$, then

$${}^c D_0^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds = I_0^{n-\alpha} y^{(n)}(t).$$

- (ii) If $y(t)$ is an abstract function with values in E , then the integrals which appear in Definitions 2.1, 2.2 and 2.3 are taken in Bochner's sense.
- (iii) The Caputo derivative of a constant is equal to zero.

For $x \in \mathbb{H}$, we define two operators $\mathcal{T}_\alpha(t)$ ($t \geq 0$) and $S_\alpha(t)$ ($t \geq 0$) as follows:

$$\mathcal{T}_\alpha(t)x = \int_0^\infty \zeta_\alpha(\theta) T(t^\alpha \theta) x d\theta, \quad S_\alpha(t)x = \alpha \int_0^\infty \theta \zeta_\alpha(\theta) T(t^\alpha \theta) x d\theta, \quad (2.1)$$

where

$$\begin{aligned} \zeta_\alpha(\theta) &= \frac{1}{\alpha} \theta^{-1-1/\alpha} \rho_\alpha(\theta^{-1/\alpha}), \\ \rho_\alpha(\theta) &= \frac{1}{\pi} \sum_{k=0}^\infty (-1)^{n-1} \theta^{-\alpha n-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \quad \theta \in (0, \infty). \end{aligned}$$

$\zeta_\alpha(\theta)$ is a probability density function defined on $(0, +\infty)$, that is,

$$\zeta_\alpha(\theta) \geq 0, \quad \theta \in (0, \infty), \quad \int_0^\infty \zeta_\alpha(\theta) d\theta = 1, \quad \int_0^\infty \theta \zeta_\alpha(\theta) d\theta = \frac{1}{\Gamma(1+\alpha)}. \quad (2.2)$$

The following properties about the operators $\mathcal{T}_\alpha(t)$ ($t \geq 0$) and $S_\alpha(t)$ ($t \geq 0$), which can be found in [11,12], will be needed in our argument.

Lemma 2.2. *The operators $\mathcal{T}_\alpha(t)$ ($t \geq 0$) and $S_\alpha(t)$ ($t \geq 0$) satisfy the following properties:*

(i) *For any fixed $t \geq 0$, $\mathcal{T}_\alpha(t)$ and $S_\alpha(t)$ are linear and bounded operators in $\mathbb{H}$, i.e., for any $x \in \mathbb{H}$,*

$$\|\mathcal{T}_\alpha(t)x\| \leq M\|x\|, \quad \|S_\alpha(t)x\| \leq \frac{M}{\Gamma(\alpha)}\|x\|. \quad (2.3)$$

(ii) *For every $x \in \mathbb{H}$, $t \rightarrow \mathcal{T}_\alpha(t)x$ and $t \rightarrow S_\alpha(t)x$ are continuous functions from $[0, \infty)$ into $\mathbb{H}$.*

(iii) *The operators $\mathcal{T}_\alpha(t)$ and $S_\alpha(t)$ are strongly continuous.*

(iv) *If semigroup $T(t)$ is an equicontinuous semigroup, $\mathcal{T}_\alpha(t)$ and $S_\alpha(t)$ are also equicontinuous in $\mathbb{H}$ for $t > 0$.*

Now, we recall some properties of the measure of noncompactness, which will be used later.

Let $\mu(\cdot)$ denote the Kuratowski measure of noncompactness of the bounded set. For the details of the definition and properties of the measure of noncompactness [39]. Let X be a Banach space, for any $D \subset C(J, X)$ and $t \in J$, let $D(t) = \{x(t) : x \in D\} \subset X$. If D is bounded in $C(J, X)$, then $D(t)$ is bounded in X , and $\mu(D(t)) \leq \mu(D)$.

The following lemmas are needed in our arguments.

Lemma 2.3. [40] *Let X be a Banach space and $D \subset C(J, X)$ be bounded and equicontinuous, where J is a finite closed interval in $\mathbb{R}$. Then, $\mu(D(t))$ is continuous on J , and*

$$\mu(D) = \max_{t \in J} \mu(D(t)) = \mu(D(J)).$$

Lemma 2.4. [41] *Let $D = \{u_n\} \subset C(J, X)$ be a bounded and countable set. Then, $\mu(D(t))$ is the Lebesgue integral on J , and*

$$\mu\left(\left\{\int_J u_n(t) dt \mid n \in \mathbb{N}\right\}\right) \leq 2 \int_J \mu(D(t)) dt.$$

In this article, we adopt the following definition of the mild solution of (1.1) based on [27,36].

Definition 2.4. For any given $u \in L^2_{\mathcal{F}}(J, \mathbb{U})$, a stochastic process $x \in \mathcal{H}$ is said to be a mild solution of (1.1) on J if x satisfies

- (i) $x(t)$ is measurable and adapted to $\mathcal{F}_t$;
- (ii) $x(t)$ satisfies the following integral equation:

$$\begin{aligned} x(t) = & \mathcal{T}_\alpha(t)(x_0 - g(x)) + \int_0^t (t-s)^{\alpha-1} \mathcal{S}_\alpha(t-s)[f(s, x(s)) + Bu(s)]ds \\ & + \int_0^t (t-s)^{\alpha-1} \mathcal{S}_\alpha(t-s)\sigma(s, x(s))dW(s). \end{aligned}$$

Let $x(b; u)$ be the state value of system (1.1) at terminal time b corresponding to control u . The set $\mathcal{R}(b) = \{x(b; u): u \in L^2_{\mathcal{F}}(J, \mathbb{U})\}$ is called the reachable set of (1.1) at the terminal time b .

Definition 2.5. [28] (Exact controllability) The fractional stochastic control system (1.1) is called exact controllable on the interval J if $\mathcal{R}(b) = L^2(\Omega, \mathbb{H})$.

To prove the controllability result, the following hypotheses are necessary throughout the article:

(H1) The linear operator $\Phi: L^2_{\mathcal{F}}(J, \mathbb{U}) \rightarrow \mathbb{H}$ defined by

$$\Phi u = \int_0^b (b-s)^{\alpha-1} \mathcal{S}_\alpha(b-s)Bu(s)ds$$

satisfies:

- (i) Φ has an inverse operator Φ^{-1} which takes values in $L^2_{\mathcal{F}}(J, \mathbb{U}) \setminus \ker \Phi$, and there exist two constants L_B and L_Φ such that $\|B\| \leq L_B$, $\|\Phi^{-1}\| \leq L_\Phi$;
- (ii) There exist a constant $q_0 \in (0, q)$ and a function $k_\Phi \in L^{1/q_0}(J, \mathbb{R}^+)$ such that

$$\mu(\Phi^{-1}(D)(t)) \leq k_\Phi(t)\mu(D)$$

for any countable subset $D \subset \mathbb{H}$.

(H2) The function $f: J \times \mathbb{H} \rightarrow \mathbb{H}$ satisfies:

- (i) For any $t \in J$, $f(t, \cdot)$ is continuous, and $f(\cdot, x)$ is strongly measurable for all $x \in \mathbb{H}$;
- (ii) For some $r > 0$, there exists a function $\xi_f \in L^1(J, \mathbb{R}^+)$ such that

$$\sup_{\|x\|^2 \leq r} \mathbb{E}\|f(t, x)\|^2 \leq \xi_f(t), \quad t \in J,$$

where $\xi_f(t)$ satisfies

$$\liminf_{r \rightarrow +\infty} \frac{\|\xi_f\|_{L^1}}{r} \triangleq \Lambda_1 < \infty;$$

- (iii) For any bounded and countable set $D \subset \mathbb{H}$, there exist a constant $q_1 \in (0, \alpha)$ and a function $k_f \in L^{1/q_1}(J, \mathbb{R}^+)$ such that

$$\mu(f(t, D)) \leq k_f(t)\mu(D), \quad t \in J.$$

(H3) The function $\sigma: J \times \mathbb{H} \rightarrow L^0_2$ satisfies the following conditions:

- (i) For $t \in J$, $\sigma(t, \cdot)$ is continuous, and $\sigma(\cdot, x)$ is strongly measurable for all $x \in \mathbb{H}$;
- (ii) For some $r > 0$, there exist a constant $q_2 \in (0, 2\alpha - 1)$ and a function $\xi_\sigma \in L^{1/q_2}(J, \mathbb{R}^+)$ such that

$$\sup_{\|x\|^2 \leq r} \mathbb{E}\|\sigma(t, x)\|^2_{L^0_2} \leq \xi_\sigma(t), \quad t \in J,$$

where $\xi_\sigma(t)$ satisfies

$$\liminf_{r \rightarrow +\infty} \frac{\|\xi_\sigma\|_{L^{1/q_2}}}{r} \triangleq \Lambda_2 < \infty;$$

(iii) For any bounded and countable set $D \subset \mathbb{H}$, there exist a constant $q_3 \in \left(0, \frac{2\alpha-1}{2}\right)$ and a function

$k_\sigma \in L^{1/q_3}(J, \mathbb{R}^+)$ such that

$$\mu(\sigma(t, D)) \leq k_\sigma(t)\mu(D), \quad t \in J.$$

(H4) $g: \mathcal{H} \rightarrow \mathbb{H}$ is continuous and there exists a constant $K > 0$ such that for any $r > 0$

$$\mathbb{E}\|g(x) - g(y)\|^2 \leq K\|x - y\|_{\mathcal{H}}^2, \quad \forall x, y \in B_r.$$

For any $x_1 \in L^2(\Omega, \mathbb{H})$, we introduce a control $u(t) := u(t; x)$ by

$$u(t; x) = \Phi^{-1} \left[x_1 - \mathcal{T}_\alpha(b)(x_0 - g(x)) - \int_0^b (b-s)^{\alpha-1} \mathcal{S}_\alpha(b-s) f(s, x(s)) ds - \int_0^b (b-s)^{\alpha-1} \mathcal{S}_\alpha(b-s) \sigma(s, x(s)) dW(s) \right] (t).$$

Next, we estimate some properties of control $u(t)$ defined above.

Lemma 2.5. Suppose that assumptions (H1)–(H4) are satisfied, then for any $x \in B_r$, the following conclusions hold:

- (i) $\mathbb{E}\|u(t; x)\|^2 \leq L_u$;
- (ii) $u(t; x)$ is continuous in B_r ,

where

$$L_u = L_\Phi^2 (4\mathbb{E}\|x_1\|^2 + 12M^2\mathbb{E}\|x_0\|^2 + 12M^2r + 12M^2\mathbb{E}\|g(\theta)\|^2 + 4c_0\|\xi_f\|_{L^1} + 4\text{Tr}(Q)c_2\|\xi_\sigma\|_{L^{1/q_2}}),$$

$$c_0 = \frac{M^2}{\Gamma^2(\alpha)} \frac{b^{2\alpha-1}}{2\alpha-1}, \quad c_2 = \frac{M^2}{\Gamma^2(\alpha)} \left(\frac{1-q_2}{2\alpha-1-q_2} \right)^{1-q_2} b^{2\alpha-1-q_2}.$$

Proof. For $t \in J$ and $x \in B_r$, by the Hölder inequality, conditions (H1)(i), (H2)(i)(ii), (H3)(i)(ii), (H4), (2.3) and Lemma 2.1, we get that

$$\begin{aligned} \mathbb{E}\|u(t; x)\|^2 &\leq L_\Phi^2 \left(4\mathbb{E}\|x_1\|^2 + 4M^2\mathbb{E}\|x_0 - g(x)\|^2 + 4\mathbb{E} \left\| \int_0^b (b-s)^{\alpha-1} \mathcal{S}_\alpha(b-s) f(s, x(s)) ds \right\|^2 \right. \\ &\quad \left. + 4 \left\| \mathbb{E} \int_0^b (b-s)^{\alpha-1} \mathcal{S}_\alpha(b-s) \sigma(s, x(s)) dW(s) \right\|^2 \right) \\ &\leq L_\Phi^2 (4\mathbb{E}\|x_1\|^2 + 12M^2\mathbb{E}\|x_0\|^2 + 12M^2\|x\|_{\mathcal{H}}^2 + 12M^2\mathbb{E}\|g(\theta)\|^2 \\ &\quad + \frac{4M^2}{\Gamma^2(\alpha)} \frac{b^{2\alpha-1}}{2\alpha-1} \int_0^b \xi_f(s) ds + \frac{4\text{Tr}(Q)M^2}{\Gamma^2(\alpha)} \int_0^b (b-s)^{2\alpha-2} \mathbb{E}\|\sigma(s, x(s))\|_{L_2^0}^2 ds) \\ &\leq L_\Phi^2 (4\mathbb{E}\|x_1\|^2 + 12M^2\mathbb{E}\|x_0\|^2 + 12M^2r + 12M^2\mathbb{E}\|g(\theta)\|^2 \\ &\quad + \frac{4M^2}{\Gamma^2(\alpha)} \frac{b^{2\alpha-1}}{2\alpha-1} \|\xi_f\|_{L^1} + \frac{4\text{Tr}(Q)M^2}{\Gamma^2(\alpha)} \left(\frac{1-q_2}{2\alpha-1-q_2} \right)^{1-q_2} b^{2\alpha-1-q_2} \|\xi_\sigma\|_{L^{1/q_2}}) = L_u. \end{aligned}$$

Next, we prove (ii). Let $x_n \rightarrow x$ in B_r , then we have

$$f(t, x_n(t)) \rightarrow f(t, x(t)), \quad \sigma(t, x_n(t)) \rightarrow \sigma(t, x(t)), \quad (n \rightarrow \infty).$$

Moreover, for all $t \in J$, using the Hölder inequality and Lebesgue dominated convergence theorem, we can get

$$\begin{aligned} & \mathbb{E} \left\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) [f(s, x_n(s)) - f(s, x(s))] ds \right\|^2 \\ & \leq \frac{M^2}{\Gamma^2(\alpha)} \int_0^t (t-s)^{2\alpha-2} ds \int_0^t \mathbb{E} \|f(s, x_n(s)) - f(s, x(s))\|^2 ds \\ & \leq \frac{M^2}{\Gamma^2(\alpha)} \frac{b^{2\alpha-1}}{2\alpha-1} \int_0^t \mathbb{E} \|f(s, x_n(s)) - f(s, x(s))\|^2 ds \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned} \quad (2.4)$$

On the other hand, from Lemma 2.1, the Hölder inequality and Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned} & \mathbb{E} \left\| \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) [\sigma(s, x_n(s)) - \sigma(s, x(s))] dW(s) \right\|^2 \\ & \leq \frac{4\text{Tr}(Q)M^2}{\Gamma^2(\alpha)} \int_0^t (t-s)^{2\alpha-2} \mathbb{E} \|\sigma(s, x_n(s)) - \sigma(s, x(s))\|^2 ds \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned} \quad (2.5)$$

According to the inequality obtained above, we obtain the following relation:

$$\begin{aligned} & \mathbb{E} \|u(t; x_n) - u(t; x)\|^2 \\ & \leq 3L_\phi^2 M^2 \mathbb{E} \|g(x_n) - g(x)\|^2 + 3L_\phi^2 \mathbb{E} \left\| \int_0^b (b-s)^{\alpha-1} S_\alpha(b-s) [f(s, x_n(s)) - f(s, x(s))] ds \right\|^2 \\ & \quad + 3L_\phi^2 \mathbb{E} \left\| \int_0^b (b-s)^{\alpha-1} S_\alpha(b-s) [\sigma(s, x_n(s)) - \sigma(s, x(s))] dW(s) \right\|^2 \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

Therefore, $u(t; x)$ is continuous in B_r . This completes the proof of Lemma 2.5. $\square$

At the end of this section, we present the Mönch fixed point theorem, which plays a key role in our proof of controllability for system (1.1).

Lemma 2.6. [42] *Let Ω be a closed convex subset of a Banach space X and $\theta \in \Omega$. Assume that $P: \Omega \rightarrow \Omega$ is a continuous map, which satisfies Mönch's condition, i.e., for $D \subset \Omega$ is countable and $D \subset \overline{\text{co}}(\{\theta\} \cup P(D)) \Rightarrow D$ is compact. Then, P has at least one fixed point in X .*

3 Main results

In this section, we shall discuss the exact controllability of the fractional stochastic dynamical control system (1.1) by using the measure of noncompactness and Mönch fixed point theorem.

Theorem 3.1. *Assume that $-A$ generates an equicontinuous C_0 -semigroup $T(t)(t \geq 0)$ of uniformly bounded operators in Hilbert space $\mathbb{H}$. If assumptions (H1)–(H4) are satisfied, then fractional stochastic control systems with nonlocal conditions (1.1) are exact controllability on J provided that*

$$9M^2 + 6c_0\Lambda_1 + 72c_0bM^2L_\Phi^2L_B^2 + 24c_0^2bL_\Phi^2L_B^2\Lambda_1 + 24c_0c_2b\text{Tr}(Q)L_\Phi^2L_B^2\Lambda_2 + 3c_2\text{Tr}(Q)\Lambda_2 < 1 \quad (3.1)$$

and

$$\rho \triangleq \left(\bar{M} + L_B \bar{M} b^{\alpha-q_0} \|\xi_\Phi\|_{L^{1/q_0}} \left(\frac{1-q_0}{\alpha-q_0} \right)^{1-q_0} \right) < 1, \quad (3.2)$$

where

$$\bar{M} = MK^{1/2} + \frac{2Mb^{1-q_1} \|k_f\|_{L^{1/q_1}} \left(\frac{1-q_1}{\alpha-q_1} \right)^{1-q_1}}{\Gamma(\alpha)} + \frac{M \|k_\sigma\|_{L^{1/q_3}} (2\text{Tr}(Q) b^{2\alpha-1-2q_3})^{1/2} \left(\frac{1-2q_3}{2\alpha-1-2q_3} \right)^{1-2q_3/2}}{\Gamma(\alpha)}.$$

Proof. To begin with, we define an operator $F: \mathcal{H} \rightarrow \mathcal{H}$ as follows:

$$\begin{aligned} (Fx)(t) &= \mathcal{T}_\alpha(t)(x_0 - g(x)) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) [f(s, x(s)) + Bu(s; x)] ds \\ &\quad + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \sigma(s, x(s)) dW(s). \end{aligned} \quad (3.3)$$

It is easy to see that the mild solution of control system (1.1) is equivalent to the fixed point of the operator F defined by (3.3). Next, we complete the proof by four steps.

Step 1. We show that there exists a positive constant r such that $F(B_r) \subset B_r$. If this is not true, then for any $r > 0$, there would exist $x_r \in B_r$ and $t_r \in J$ such that $\mathbb{E}\|(Fx_r)(t_r)\|^2 > r$. From Lemma 2.5, we obtain

$$\begin{aligned} r &< \mathbb{E}\|(Fx_r)(t_r)\|^2 \leq 3\mathbb{E}\|\mathcal{T}_\alpha(t_r)(x_0 - g(x_r))\|^2 + 3\mathbb{E}\left\| \int_0^{t_r} (t_r-s)^{\alpha-1} S_\alpha(t_r-s) [f(s, x_r(s)) + Bu(s; x_r)] ds \right\|^2 \\ &\quad + 3\mathbb{E}\left\| \int_0^{t_r} (t_r-s)^{\alpha-1} S_\alpha(t_r-s) \sigma(s, x_r(s)) dW(s) \right\|^2 \\ &\leq 9M^2\mathbb{E}\|x_0\|^2 + 9M^2\|x_r\|_{\mathcal{H}}^2 + 9M^2\mathbb{E}\|g(\theta)\|^2 + \frac{3M^2}{\Gamma^2(\alpha)} \frac{b^{2\alpha-1}}{2\alpha-1} \\ &\quad \times \int_0^{t_r} \mathbb{E}\|f(s, x_r(s)) + Bu(s; x_r)\|^2 ds + \frac{3\text{Tr}(Q)M^2}{\Gamma^2(\alpha)} \int_0^{t_r} (t_r-s)^{2\alpha-2} \mathbb{E}\|\sigma(s, x_r(s))\|_{L_2^0}^2 ds \\ &\leq 9M^2\mathbb{E}\|x_0\|^2 + 9M^2r + 9M^2\mathbb{E}\|g(\theta)\|^2 + 3c_0(2\|\xi_f\|_{L^1} + 2bL_B^2L_u) + 3c_2\text{Tr}(Q)\|\xi_\sigma\|_{L^{1/q_1}}. \end{aligned}$$

Dividing both sides by r and taking the lower limit as $r \rightarrow +\infty$, we obtain

$$9M^2 + 6c_0\Lambda_1 + 72c_0bM^2L_\Phi^2L_B^2 + 24c_0^2bL_\Phi^2L_B^2\Lambda_1 + 24c_0c_2b\text{Tr}(Q)L_\Phi^2L_B^2\Lambda_2 + 3c_2\text{Tr}(Q)\Lambda_2 \geq 1,$$

which contradicts (3.1). Hence, for some positive integer $r > 0$, $F(B_r) \subset B_r$.

Step 2. we prove that $F: B_r \rightarrow B_r$ is a continuous operator. Let $x_n, x \in B_r$ and $x_n \rightarrow x$ ($n \rightarrow \infty$), then for any $t \in J$, we have

$$\begin{aligned} &\mathbb{E}\|(Fx_n)(t) - (Fx)(t)\|^2 \\ &\leq 4\mathbb{E}\|\mathcal{T}_\alpha(t)(g(x_n) - g(x))\|^2 + 4\mathbb{E}\left\| \int_0^b (b-s)^{\alpha-1} S_\alpha(b-s) [f(s, x_n(s)) - f(s, x(s))] ds \right\|^2 \\ &\quad + 4\mathbb{E}\left\| \int_0^b (b-s)^{\alpha-1} S_\alpha(b-s) [\sigma(s, x_n(s)) - \sigma(s, x(s))] dW(s) \right\|^2 \\ &\quad + 4\mathbb{E}\left\| \int_0^b (b-s)^{\alpha-1} S_\alpha(b-s) B[u(s, x_n) - u(s, x)] ds \right\|^2. \end{aligned}$$

For $t \in J$, (2.4), (2.5) and the Lebesgue dominated convergence theorem then give

$$\mathbb{E}\|(Fx_n)(t) - (Fx)(t)\|^2 \rightarrow 0 (n \rightarrow \infty), \quad t \in J.$$

Therefore, $\|(Fx_n)(t) - (Fx)(t)\|_{\mathcal{H}} \rightarrow 0 (n \rightarrow \infty)$, which means that F is continuous in B_r .

Step 3. We prove that $F(B_r)$ is an equicontinuous family of functions on $[0, b]$. For any $x \in B_r$ and $0 \leq t_1 < t_2 \leq b$, we get that

$$\begin{aligned} & \mathbb{E}\|(Fx)(t_2) - (Fx)(t_1)\|^2 \\ &= 7\mathbb{E}\|(\mathcal{T}_\alpha(t_2) - \mathcal{T}_\alpha(t_1))(x_0 - g(x))\|^2 + 7\mathbb{E}\left\|\int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \mathcal{S}_\alpha(t_2 - s)[f(s, x(s)) + Bu(s, x)] ds\right\|^2 \\ &+ 7\mathbb{E}\left\|\int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] \mathcal{S}_\alpha(t_2 - s)[f(s, x(s)) + Bu(s, x)] ds\right\|^2 \\ &+ 7\mathbb{E}\left\|\int_0^{t_1} (t_1 - s)^{\alpha-1} [\mathcal{S}_\alpha(t_2 - s) - \mathcal{S}_\alpha(t_1 - s)][f(s, x(s)) + Bu(s, x)] ds\right\|^2 \\ &+ 7\mathbb{E}\left\|\int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \mathcal{S}_\alpha(t_2 - s) \sigma(s, x(s)) dW(s)\right\|^2 \\ &+ 7\mathbb{E}\left\|\int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] \mathcal{S}_\alpha(t_2 - s) \sigma(s, x(s)) dW(s)\right\|^2 \\ &+ 7\mathbb{E}\left\|\int_0^{t_1} (t_1 - s)^{\alpha-1} [\mathcal{S}_\alpha(t_2 - s) - \mathcal{S}_\alpha(t_1 - s)] \sigma(s, x(s)) dW(s)\right\|^2 \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7. \end{aligned}$$

In order to prove that $\mathbb{E}\|(Fx)(t_2) - (Fx)(t_1)\|^2 \rightarrow 0 (t_2 - t_1 \rightarrow 0)$, we only need to check $I_i \rightarrow 0$ independently of $x \in B_r$ when $t_2 - t_1 \rightarrow 0$ for $i = 1, 2, \dots, 7$.

Applying Lemma 2.2, it is easy to see that $I_1 \rightarrow 0$ as $t_2 - t_1 \rightarrow 0$. For I_2 and I_5 , from (H2)(ii), (H3)(ii), Lemmas 2.1 and 2.5, we obtain the estimate

$$\begin{aligned} I_2 &= 7\mathbb{E}\left\|\int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \mathcal{S}_\alpha(t_2 - s)[f(s, x(s)) + Bu(s, x)] ds\right\|^2 \\ &\leq \frac{7M^2}{\Gamma^2(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{2\alpha-2} ds \int_{t_1}^{t_2} \mathbb{E}\|f(s, x(s)) + Bu(s, x)\|^2 ds \\ &\leq \frac{7M^2(2\|\xi_f\|_{L^1} + 2L_B^2 L_u b)}{\Gamma^2(\alpha)} \cdot \frac{(t_2 - t_1)^{2\alpha-1}}{2\alpha-1} \rightarrow 0 (t_2 - t_1 \rightarrow 0), \\ I_5 &= 7\mathbb{E}\left\|\int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \mathcal{S}_\alpha(t_2 - s) \sigma(s, x(s)) dW(s)\right\|^2 \\ &\leq \frac{7\text{Tr}(Q)M^2}{\Gamma^2(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{2\alpha-2} \mathbb{E}\|\sigma(s, x(s))\|^2 ds \\ &\leq \frac{7\text{Tr}(Q)M^2\|\xi_\sigma\|_{L^{1/q_1}}}{\Gamma^2(\alpha)} \left(\frac{1 - q_1}{2\alpha - 1 - q_1}\right)^{1-q_1} (t_2 - t_1)^{2\alpha-1-q_1} \rightarrow 0 (t_2 - t_1 \rightarrow 0). \end{aligned}$$

Similarly, for I_3 and I_6 , we get

$$\begin{aligned}
 I_3 &= 7\mathbb{E} \left\| \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] \mathcal{S}_\alpha(t_2 - s) [f(s, x(s)) + Bu(s, x)] ds \right\|^2 \\
 &\leq \frac{7M^2}{\Gamma^2(\alpha)} \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}]^2 ds \int_0^{t_1} \mathbb{E} \|f(s, x(s)) + Bu(s, x)\|^2 ds \\
 &\leq \frac{7M^2(2\|\xi_f\|_{L^1} + 2L_B^2 L_u b)}{\Gamma^2(\alpha)} \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}]^2 ds \rightarrow 0 \quad (t_2 - t_1 \rightarrow 0), \\
 I_6 &= 7\mathbb{E} \left\| \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] \mathcal{S}_\alpha(t_2 - s) \sigma(s, x(s)) ds \right\|^2 \\
 &\leq \frac{7\text{Tr}(Q)M^2}{\Gamma^2(\alpha)} \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}]^2 \mathbb{E} \|\sigma(s, x(s))\|^2 ds \\
 &\leq \frac{7\text{Tr}(Q)M^2\|\xi_\sigma\|_{L^{1/q_2}}}{\Gamma^2(\alpha)} \left(\int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}]^{2/1-q_2} ds \right)^{1-q_2} \rightarrow 0 \quad (t_2 - t_1 \rightarrow 0).
 \end{aligned}$$

Furthermore, for I_4 and I_7 , if $t_1 = 0$, $0 < t_2 < b$, it is easy to see $I_4 = I_7 = 0$, so for $t_1 > 0$ and $0 < \varepsilon < t_1$ small enough, we obtain the following inequalities:

$$\begin{aligned}
 I_4 &= 7\mathbb{E} \left\| \int_0^{t_1} (t_1 - s)^{\alpha-1} [\mathcal{S}_\alpha(t_2 - s) - \mathcal{S}_\alpha(t_1 - s)] [f(s, x(s)) + Bu(s, x)] ds \right\|^2 \\
 &\leq 14\mathbb{E} \left\| \int_0^{t_1-\varepsilon} (t_1 - s)^{\alpha-1} [\mathcal{S}_\alpha(t_2 - s) - \mathcal{S}_\alpha(t_1 - s)] [f(s, x(s)) + Bu(s, x)] ds \right\|^2 \\
 &\quad + 14\mathbb{E} \left\| \int_{t_1-\varepsilon}^{t_1} (t_1 - s)^{\alpha-1} [\mathcal{S}_\alpha(t_2 - s) - \mathcal{S}_\alpha(t_1 - s)] [f(s, x(s)) + Bu(s, x)] ds \right\|^2 \\
 &\leq 14 \sup_{s \in [0, t_1-\varepsilon]} \|\mathcal{S}_\alpha(t_2 - s) - \mathcal{S}_\alpha(t_1 - s)\|^2 (2\|\xi_f\|_{L^1} + 2L_B^2 L_u b) \frac{t_1^{2\alpha-1} - \varepsilon^{2\alpha-1}}{2\alpha-1} \\
 &\quad + 14 \left(\frac{2M}{\Gamma(\alpha)} \right)^2 (2\|\xi_f\|_{L^1} + 2L_B^2 L_u b) \frac{\varepsilon^{2\alpha-1}}{2\alpha-1} \rightarrow 0 \quad (t_2 - t_1 \rightarrow 0 \text{ and } \varepsilon \rightarrow 0), \\
 I_7 &= 7\mathbb{E} \left\| \int_0^{t_1} (t_1 - s)^{\alpha-1} [\mathcal{S}_\alpha(t_2 - s) - \mathcal{S}_\alpha(t_1 - s)] \sigma(s, x(s)) dW(s) \right\|^2 \\
 &\leq 14\mathbb{E} \left\| \int_0^{t_1-\varepsilon} (t_1 - s)^{\alpha-1} [\mathcal{S}_\alpha(t_2 - s) - \mathcal{S}_\alpha(t_1 - s)] \sigma(s, x(s)) ds \right\|^2 \\
 &\quad + 14\mathbb{E} \left\| \int_{t_1-\varepsilon}^{t_1} (t_1 - s)^{\alpha-1} [\mathcal{S}_\alpha(t_2 - s) - \mathcal{S}_\alpha(t_1 - s)] \sigma(s, x(s)) ds \right\|^2 \\
 &\leq \frac{14\text{Tr}(Q)(1-q_2)}{2\alpha-1-q_2} \sup_{s \in [0, t_1-\varepsilon]} \|\mathcal{S}_\alpha(t_2 - s) - \mathcal{S}_\alpha(t_1 - s)\|^2 \|\xi_\sigma\|_{1/L^{q_2}} \left(t_1^{\frac{2\alpha-1-q_2}{1-q_2}} - \varepsilon^{\frac{2\alpha-1-q_2}{1-q_2}} \right)^{1-q_2} \\
 &\quad + 14\text{Tr}(Q) \left(\frac{2M}{\Gamma(\alpha)} \right)^2 \|\xi_\sigma\|_{1/L^{q_2}} \left(\frac{1-q_1}{2\alpha-1-q_2} \right)^{2\alpha-1-q_2} \varepsilon^{2\alpha-1-q_2} \rightarrow 0 \quad (t_2 - t_1 \rightarrow 0 \text{ and } \varepsilon \rightarrow 0).
 \end{aligned}$$

Above all, we have $I_i \rightarrow 0$ as $t_2 - t_1 \rightarrow 0$ and $\varepsilon \rightarrow 0$, which means $F(B_r)$ is equicontinuous.

Step 4. We will prove that F satisfies Mönch's condition. Let $F = F_1 + F_2$, where

$$F_1(x)(t) = \mathcal{T}_\alpha(t)(x_0 - g(x)),$$

$$F_2(x)(t) = \int_0^t (t-s)^{\alpha-1} \mathcal{S}_\alpha(t-s) [f(s, x(s)) + Bu(s; x)] ds + \int_0^t (t-s)^{\alpha-1} \mathcal{S}_\alpha(t-s) \sigma(s, x(s)) dW(s).$$

Suppose that $D \subset B_r$ be countable and $D \subset \overline{\mathcal{CO}}(\{0\} \cup F(D))$, for any $t \in J$, $x, y \in D$, by (H1), we get

$$\mathbb{E} \|F_1(x)(t) - F_1(y)(t)\|^2 = \mathbb{E} \|\mathcal{T}_\alpha(t)(g(x) - g(y))\|^2 \leq M^2 K \|x - y\|_{\mathcal{H}}^2.$$

Moreover, we obtain

$$\|F_1(x)(t) - F_1(y)(t)\|_{\mathcal{H}} \leq MK^{1/2} \|x - y\|_{\mathcal{H}}.$$

From the definition of the measure of noncompactness, it follows that

$$\mu(F_1(D)) \leq MK^{1/2} \mu(D). \quad (3.4)$$

Applying Lemma (2.1), we obtain the estimate

$$\begin{aligned} & \left\| \int_0^t (t-s)^{\alpha-1} \mathcal{S}_\alpha(t-s) [\sigma(s, x(s)) - \sigma(s, y(s))] dW(s) \right\| \\ & \leq \left(\left\| \int_0^t (t-s)^{\alpha-1} \mathcal{S}_\alpha(t-s) [\sigma(s, x(s)) - \sigma(s, y(s))] dW(s) \right\|^2 \right)^{1/2} \\ & \leq \frac{M}{\Gamma(\alpha)} \left(\text{Tr}(Q) \int_0^t (t-s)^{2\alpha-2} \|\sigma(s, x(s)) - \sigma(s, y(s))\|^2 ds \right)^{1/2}. \end{aligned}$$

So,

$$\mu \left(\int_0^t (t-s)^{\alpha-1} \sigma(s, D(s)) dW(s) \right) \leq \frac{M}{\Gamma(\alpha)} \left(2\text{Tr}(Q) \int_0^t (t-s)^{2\alpha-2} [\mu(\sigma(s, D(s)))]^2 ds \right)^{1/2}. \quad (3.5)$$

From the above inequalities (3.4), (3.5) and using assumptions (H1)(ii), (H2)(iii), (H3)(iii) and Lemma 2.4, we have

$$\begin{aligned} & \mu(u(t, D)) \\ & \leq k_\Phi(t) \mu \left(x_1 - \mathcal{T}_\alpha(b)(x_0 - g(x)) - \int_0^b (b-s)^{\alpha-1} \mathcal{S}_\alpha(b-s) f(s, D(s)) ds - \int_0^b (b-s)^{\alpha-1} \mathcal{S}_\alpha(b-s) \sigma(s, D(s)) dW(s) \right) \\ & \leq k_\Phi(t) \left(\mu(\mathcal{T}_\alpha(b)(x_0 - g(x))) + \mu \left(\int_0^b (b-s)^{\alpha-1} \mathcal{S}_\alpha(b-s) f(s, D(s)) ds \right) + \mu \left(\int_0^b (b-s)^{\alpha-1} \mathcal{S}_\alpha(b-s) \sigma(s, D(s)) dW(s) \right) \right) \\ & \leq k_\Phi(t) \left(MK^{1/2} \mu(D) + \frac{2M}{\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} \mu(f(s, D(s))) ds + \frac{M}{\Gamma(\alpha)} \left(2\text{Tr}(Q) \int_0^b (t-s)^{2\alpha-2} [\mu(\sigma(s, D(s)))]^2 ds \right)^{1/2} \right) \\ & \leq k_\Phi(t) \left(MK^{1/2} + \frac{2Mb^{\alpha-q_1} \|k_f\|_{L^{q_1}} \left(\frac{1-q_1}{\alpha-q_1} \right)^{1-q_1}}{\Gamma(\alpha)} + \frac{M \|k_\sigma\|_{L^{1/q_3}} (2\text{Tr}(Q) b^{2\alpha-1-2q_3})^{1/2} \left(\frac{1-2q_3}{2\alpha-1-2q_3} \right)^{1-2q_3/2}}{\Gamma(\alpha)} \right) \mu(D) \\ & = k_\Phi(t) \tilde{M} \mu(D). \end{aligned}$$

Now we estimate the measure of noncompactness of $F(D)$. For $t \in J$, we have

$$\begin{aligned}
 \mu((FD)(t)) &= \mu((F_1D)(t)) + \mu((F_2D)(t)) \\
 &\leq \mu(F_1D) + \mu\left(\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) [f(s, D(s)) + Bu(s, D)] ds + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \sigma(s, D(s)) dW(s)\right) \\
 &\leq MK_2^1 \mu(D) + \frac{2M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mu(f(s, D(s)) + Bu(s, D)) ds \\
 &\quad + \frac{M}{\Gamma(\alpha)} \left(2\text{Tr}(Q) \int_0^t (t-s)^{2\alpha-2} [\mu(\sigma(s, D(s)))]^2 ds \right)^{1/2} \\
 &\leq MK_2^{1/2} \mu(D) + \frac{2M}{\Gamma(\alpha)} \left(b^{\alpha-q_1} \|k_f\|_{L^{1/q_1}} \left(\frac{1-q_1}{\alpha-q_1} \right)^{1-q_1} + L_B \bar{M} b^{\alpha-q_0} \|\xi_\Phi\|_{L^{1/q_0}} \left(\frac{1-q_0}{\alpha-q_0} \right)^{1-q_0} \right) \mu(D) \\
 &\quad + \left(\frac{M \|k_\sigma\|_{L^{1/q_3}} (2\text{Tr}(Q) b^{2\alpha-1-2q_3})^{1/2}}{\Gamma(\alpha)} \left(\frac{1-2q_3}{2\alpha-1-2q_3} \right)^{\frac{1-2q_3}{2}} \right) \mu(D) \\
 &= \left(\bar{M} + L_B \bar{M} b^{\alpha-q_0} \|\xi_\Phi\|_{L^{1/q_0}} \left(\frac{1-q_0}{\alpha-q_0} \right)^{1-q_0} \right) \mu(D) = \rho \mu(D).
 \end{aligned}$$

It follows from Lemma 2.3 that

$$\mu(F(D)) = \max_{t \in J} \mu((FD)(t)) \leq \rho \mu(D).$$

Hence, we get

$$\mu(D) \leq \mu(\overline{\text{co}}(\{0\} \cup F(D))) \leq \mu(FD) \leq \rho \mu(D).$$

From assumption (3.2), we know that $\rho < 1$, which means $\mu(D) = 0$, that is, D is relatively compact. By Lemma 2.6, F has at least one fixed point $x \in B_r$, which is a mild solution of the fractional stochastic control system (1.1) and it satisfies $x(b) = x_1$ for any $x_1 \in L^2(\Omega, \mathbb{H})$. Therefore, the fractional stochastic control system (1.1) is exact controllable on J .

This completes the proof of Theorem 3.1. $\square$

Remark 3.2. Impulsive effects exist in many evolution processes, so it is necessary to study impulsive stochastic evolution equations. Upon making some appropriate assumptions, by applying the ideas and techniques as in this article, one can obtain the exact controllability results for a class of fractional impulsive stochastic evolution equations with nonlocal conditions.

4 Application

Example 4.1. Consider the following fractional stochastic control system of the form:

$$\begin{cases} \frac{\partial^{\frac{2}{3}}}{\partial t^{\frac{2}{3}}} x(z, t) - \frac{\partial x(z, t)}{\partial z} = \frac{x(z, t)}{e^t(1+x^2(z, t))} + \frac{e^{-2t} x(z, t)}{1+e^t} \frac{dW(t)}{dt} + u(z, t), & t \in J, z \in (0, 1), \\ x(0, t) = x(1, t), & t \in J, \\ x(0) = - \sum_{k=1}^m (-1)^k c_k x(z, k), & z \in (0, 1), \end{cases} \quad (4.1)$$

where $c_k \in \mathbb{R}$, and $\sum_{k=1}^m |c_k| \leq 1$. $W(t)$ is a standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$. To write the above system (4.1) into the abstract form of (1.1), let $\mathbb{H} = \mathbb{E} = \mathbb{U} = L^2[0, 1]$ with the norm $\|\cdot\|$, define the operator $A: D(A) \subset X \rightarrow \mathbb{H}$ by

$$Av = -v', \quad v \in D(A),$$

$$D(A) = \{v \in \mathbb{H}, v' \in X, v(0) = v(1) = 0\}.$$

We know that $-A$ generates an equicontinuous semigroup $T(t)(t \geq 0)$ in $\mathbb{H}$. For any $v \in \mathbb{H}$, $T(t)v(s) = v(t+s)$. Then, $T(t)$ is not a compact semigroup and $\|T(t)\| \leq 1$.

For any $t \in J$, let $x(t)(z) = x(z, t)$, $f(t, x(t))(z) = \frac{x(z, t)}{e^t(1+x^2(z, t))}$, $\sigma(t, x(t))(z) = \frac{e^{-2t}x(z, t)}{1+e^t}$, $Bu(t)(z) = u(z, t)$ and $g(x) = \sum_{k=1}^m (-1)^k c_k x(z, k)$. Then, problem (4.1) can be rewritten into the abstract form of (1.1). From the definitions of nonlinear terms f and σ , for any $r > 0$ and $x(t) \in \mathbb{H}$, when $\|x(t)\|^2 \leq r$, $t > 0$, we can easily verify that assumptions (H2)(ii) and (H3)(ii) hold with $\xi_f(t) = \frac{r}{e^{2t}}$, $\xi_\sigma(t) = \frac{r}{e^{4t}}$ and $\Lambda_1 = \Lambda_2 = 1$. Moreover, (H2)(iii) and (H3)(iii) hold with $k_f(t) = k_\sigma(t) = \frac{1}{e^{2t}}$. (H4) holds with $K = \sum_{k=1}^m |c_k|^2$.

For $z \in (0, 1)$, the linear operator Φ is given by

$$(\Phi u)(z) = \int_0^b (b-s)^{-\frac{1}{3}} S_{\frac{2}{3}}(b-s) \omega u(z, s) ds,$$

where

$$S_{\frac{2}{3}}(s)\beta(s) = \frac{2}{3} \int_0^\infty \theta \zeta_{\frac{2}{3}}(\theta) \beta\left(t^{\frac{2}{3}}\theta + s\right) d\theta.$$

Suppose that Φ satisfies assumption (H1), then applying Theorem 3.1, we get that stochastic control system (4.1) is exact controllable on J provided that (3.1) and (3.2) hold.

Example 4.2. Consider the following fractional stochastic control system of the form:

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha} x(z, t) - \frac{\partial x(z, t)}{\partial z} = \frac{tx(z, t)}{2(1+|x(z, t)|)} + x(z, t) + e^t x(z, t) \frac{dW(t)}{dt} + u(z, t), & t \in J, z \in (0, 1), \\ x(0, t) = x(1, t), & t \in J, \\ x(0) = \int_0^b tx(z, t) dt, & z \in (0, 1), \end{cases} \quad (4.2)$$

where $0 < \alpha < 1$, $c_k \in \mathbb{R}$ and $\sum_{k=1}^m |c_k| \leq 1$. $W(t)$ is a standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$.

Let $\mathbb{H} = \mathbb{E} = \mathbb{U} = L^2[0, 1]$ with the norm $\|\cdot\|$. Define the operator $A: D(A) \subset X \rightarrow \mathbb{H}$ by

$$Av = -v', \quad v \in D(A),$$

$$D(A) = \{v \in \mathbb{H}, v' \in X, v(0) = v(1) = 0\}.$$

Then, $-A$ generates an equicontinuous semigroup $T(t)(t \geq 0)$ in $\mathbb{H}$, and $T(t)$ is not a compact semigroup.

For any $t \in J$, let $x(t)(z) = x(z, t)$, and

$$f(t, x(t))(z) = \frac{tx(z, t)}{2(1+|x(z, t)|)} + x(z, t),$$

$$\sigma(t, x(t))(z) = e^t x(z, t),$$

$$Bu(t)(z) = u(z, t), \quad g(x) = \int_0^b tx(z, t) dt.$$

Then, problem (4.2) can be rewritten into the abstract form of (1.1). For any $r > 0$ and $\|x(t)\|^2 \leq r$, we have

$$\|f(t, x)\|^2 = \int_0^1 \left| \frac{tx(z, t)}{2(1 + |x(z, t)|)} + x(z, t) \right|^2 dz \leq \left(\frac{t^2}{2} + 2 \right) \int_0^1 |x(z, t)|^2 dz \leq \left(\frac{t^2}{2} + 2 \right) r.$$

Therefore,

$$\sup_{\|x\|^2 \leq r} \mathbb{E} \|f(t, x)\|^2 \leq \left(\frac{t^2}{2} + 2 \right) r, \quad t \in J.$$

In addition,

$$\|\sigma(t, x)\|^2 = \int_0^1 |e^t x(z, t)|^2 dz \leq e^{2t} \int_0^1 |x(z, t)|^2 dz \leq e^{2t} r.$$

So assumptions (H2)(ii) and (H3)(ii) hold with $\xi_f(t) = \left(\frac{t^2}{2} + 2 \right) r$, $\xi_\sigma(t) = e^{2t} r$ and $\Lambda_1 = \Lambda_2 = 1$. Moreover, for any $x_1(t), x_2(t) \in \mathbb{H}$

$$\begin{aligned} \|f(t, x_2(t)) - f(t, x_1(t))\|^2 &= \int_0^1 \left| \frac{tx_2(z, t)}{2(1 + |x_2(z, t)|)} + x_2(z, t) - \frac{tx_1(z, t)}{2(1 + |x_1(z, t)|)} + x_1(z, t) \right|^2 dz \\ &\leq \left(\frac{t^2}{2} + 2 \right) \int_0^1 |x_2(z, t) - x_1(z, t)|^2 dz \leq \left(\frac{t^2}{2} + 2 \right) \|x_2(t) - x_1(t)\|^2, \end{aligned}$$

$$\|\sigma(t, x_2(t)) - \sigma(t, x_1(t))\|^2 = \int_0^1 |e^t x_2(z, t) - e^t x_1(z, t)|^2 dz \leq e^{2t} \|x_2(t) - x_1(t)\|^2.$$

Hence, (H2)(iii) and (H3)(iii) hold with $k_f(t) = \left(\frac{t^2}{2} + 2 \right)^{1/2}$, $k_\sigma(t) = e^t$. For any $x, y \in B_r$,

$$\begin{aligned} \|g(x) - g(y)\|^2 &= \int_0^1 \left| \int_0^b tx(z, t) - ty(z, t) dt \right|^2 dz \\ &\leq \frac{b^3}{3} \int_0^1 \int_0^b |x_2(z, t) - x_1(z, t)|^2 dt dz \leq \frac{b^4}{3} \sup \|x_2(t) - x_1(t)\|^2, \end{aligned}$$

which means

$$\mathbb{E} \|g(x) - g(y)\|^2 \leq K \|x - y\|_{\mathcal{H}}^2,$$

where $K = \frac{b^4}{3}$.

Suppose that Φ satisfies assumption (H1), then applying Theorem 3.1, we get that stochastic control system (4.2) is exact controllable on J provided that (3.1) and (3.2) hold.

5 Conclusion

In this article, the exact controllability for a class of fractional stochastic evolution equations with nonlocal initial conditions in a Hilbert space is studied. We delete the compactness of semigroup $T(t)(t > 0)$, so our theorems guarantee the effectiveness of controllability results in the infinite dimensional spaces. Moreover, the regularity of mild solution for fractional stochastic evolution equations with nonlocal initial conditions will be the topic of our future work.

Acknowledgments: This research was supported by the National Natural Science Foundation of China (grant numbers 11661071, 11261053 and 11361055).

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