Review Article

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# Free nonunitary Rota-Baxter family algebras and typed leaf-spaced decorated planar rooted forests

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**Abstract:** In this paper, we decorate leaves and edges of planar rooted forests simultaneously and use a part of them to construct free nonunitary Rota-Baxter family algebras. As a corollary, we obtain the construction of free nonunitary Rota-Baxter algebras.

**Keywords:** Rota-Baxter algebras, Rota-Baxter family algebras, typed decorated rooted forests, leaf-spaced, parallelly typed leaf-spaced decorated

**MSC 2020:** 16W99, 16S10, 08B20, 16T30, 81R15

## 1 Introduction

### 1.1 Rota-Baxter (family) algebras

An associative algebra $R$ together with a $\mathbf{k}$-linear operator $P : R \to R$ is called a Rota-Baxter algebra of weight $\lambda$, if

$$P(x)P(y) = P(P(x)y) + P(xP(y)) + \lambda P(xy), \quad \text{for } x, y \in R, \tag{1}$$

where $\lambda$ is a fixed element in the basis ring $\mathbf{k}$. When $\lambda = 0$, the operator $P$ is indeed an algebraic abstraction of the usual integration operation in analysis [1]. The mathematician Glen E. Baxter first studied the Rota-Baxter algebras [2] in 1960 in his probability study. Some combinatoric properties of Rota-Baxter algebras were studied by Rota [3] and Cartier [4]. Free Rota-Baxter associative algebras were constructed on both commutative and noncommutative cases by using different methods, which appeared in [1,3–8]. A Rota-Baxter algebra naturally carries a dendriform or tridendriform algebra structure [9]. Nowadays, Rota-Baxter algebra has become a new branch with broad connections to other objects in mathematics, such as pre-Lie algebras, pre-Poisson algebras [10,11], quantum field theory [12–14], Hopf algebras [15,16], commutative algebras [17,18], Loday's dendriform algebras [9,19], and Aguiar's associative analogue of the classical Yang-Baxter equation [20–22].

In 2007, K. Ebrahimi-Fard, J. Gracia-Bondia and F. Patras [23, Proposition 9.1] (see also [25, Theorem 3.7.2]) introduced the first example of Rota-Baxter family about algebraic aspects of renormalization in quantum field theory, where a "Rota-Baxter family" appears: this terminology was suggested to the authors by

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Li Guo (see Footnote following Proposition 9.2 therein), who further discussed the underlying structure under the name *Rota-Baxter family algebra* in [24]. Namely, let $\Omega$ be a semigroup and $\lambda \in \mathbf{k}$ be given. A Rota-Baxter family of weight $\lambda$ on an algebra $R$ is a collection of $\mathbf{k}$-linear operators $(P_\omega)_{\omega\in\Omega}$ on $R$ such that

$$P_\alpha(a)P_\beta(b) = P_{\alpha\beta}(P_\alpha(a)b + aP_\beta(b) + \lambda ab), \quad \text{for } a, b \in R \text{ and } \alpha, \beta \in \Omega. \tag{2}$$

Then the pair $(R, (P_\omega)_{\omega\in\Omega})$ is called a Rota-Baxter family algebra of weight $\lambda$. Rota-Baxter family algebra arises naturally in renormalization of quantum field theory. It is worthwhile to study the algebraic structure of Rota-Baxter family algebras. As the construction of free objects in a category is always interesting and important, the author in [26] constructed, respectively, free commutative unitary Rota-Baxter family algebras, and free noncommutative unitary Rota-Baxter family algebras by the method of Gröbner-Shirshov bases.

## 1.2 Algebraic structures on (typed decorated) rooted forests

Rooted trees/forests are a useful tool for studying many interesting algebraic structures. It appeared in the work of Arthur Cayley [27] in the 1850s considered rooted trees as a representation of combinatorial structures related to the free pre-Lie algebra. More than a century later, these structures formed the foundation of John Butcher's theory of B-series [28,29], which has become an indispensable tool in the analysis of numerical integration. Many Hopf algebraic structures have been built up on top of rooted forests, such as Connes-Kreimer Hopf algebra [15], Loday-Ronco [19], Grossman-Larson [30] and Foissy-Holtkamp [31,32]. In particular, the famous Connes-Kreimer Hopf algebra was employed to deal with renormalization in quantum field theory [13,14]. Pre-Lie structures on non-planar rooted trees lead to Hopf algebras of combinatorial nature, which appeared in the works in [15,30,33]. Free pre-Lie algebras can also be described as the space of non-planar rooted trees with product given by grafting of trees [34].

The multi pre-Lie structures were first introduced in [35] (see Section 4 and Appendix A) in a more general setting before [36] which considered only the non-deformed structures. A recent preprint [37] used typed decorated rooted forests in numerical analysis for developing a general scheme for dispersive partial differential equations (PDEs) which strengths the universal aspect of these structures. Typed decorated rooted forests also appeared in a context of low-dimension topology [38] and in a context of the description of combinatorial species [39].

## 1.3 Motivation and layout of the paper

Our motivations come from two points. The first point is that it is almost natural to construct free nonunitary Rota-Baxter family algebras, parallel to the unitary case done in [26]. The second point is along the line of typed decorated rooted forests. Recall that Guo [24] constructed free nonunitary Rota-Baxter algebras in terms of leaf-spaced decorated planar rooted forests. In the present paper, combining typed decorated and leaf-spaced decorated planar rooted forests, we construct free nonunitary Rota-Baxter family algebras, as a generalization of the work in [24].

The following is the outline of the paper. In Section 2.1, we recall some basic concepts of planar rooted forests used in this paper. Combining leaf decorated and typed decorated planar rooted forests, we propose the concept of (parallelly) typed leaf-spaced decorated planar rooted forests in Section 2.2. Based on this concept, we construct the free nonunitary Rota-Baxter family algebra on a set in Section 2.3. As a corollary, the construction of free nonunitary Rota-Baxter algebra on a set is obtained.

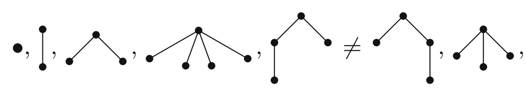
# 2 Free nonunitary Rota-Baxter family algebras

In this section, we first propose the concept of parallelly typed leaf-spaced decorated planar rooted forests and then use them to construct free nonunitary Rota-Baxter family algebras.

## 2.1 Planar rooted forests

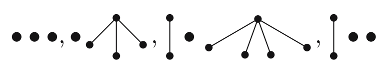
In this subsection, let us recall some basic concepts of planar rooted forests. A rooted tree is a connected and simply connected set of vertices and oriented edges such that there is precisely one distinguished vertex, called the root, with no incoming edge. The only vertex of the tree $\bullet$ is taken to be a leaf. If two vertices of a rooted tree are connected by an edge, then the vertex on the side of the root is called the **parent** and the vertex on the opposite side of the root is called a **child**.

A rooted tree is called planar rooted tree if it is endowed with an embedding in the plane. Here are some examples.



where the root in a planar rooted tree is at the top. A **subforest** of a planar rooted tree $T$ is the forest consisting of a set of vertices of $T$ together with their descents and edges connecting all these vertices.

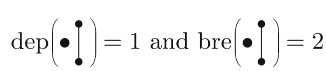
Let $\mathcal{T}$ be the set of planar rooted trees and $\mathcal{F} := S(\mathcal{T})$ the free semigroup generated by $\mathcal{T}$. Thus, an element in $S(\mathcal{T})$, called a planar rooted forest, is a noncommutative product of planar rooted trees in $\mathcal{T}$. Here are some examples of planar rooted forests.



The following concepts are standard.

**Definition 2.1.**
(a) For a planar rooted tree $T$, the **depth** $\mathrm{dep}(T)$ of $T$ is the maximal length of paths from the root to leaves of the tree. The **depth** $\mathrm{dep}(F)$ of a forest $F$ is the maximal depth of trees in $F$.
(b) For a planar rooted forest $F = T_1 \dots T_b$ with $T_1, \dots, T_b \in \mathcal{T}$, define $\mathrm{bre}(F) := b$ to be the **breadth** of $F$.

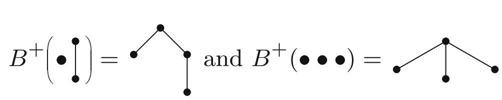
For example,



In the noncommutative version of the well-known Connes-Kreimer Hopf algebra [31,32], there is a linear grafting operation

$$B^+ : \mathbf{k}\mathcal{F} \to \mathbf{k}\mathcal{F}, \quad T_1 \dots T_n \mapsto B^+(T_1 \dots T_n), \tag{3}$$

where $B^+(T_1 \dots T_n)$ is obtained by adding a new root together with an edge from the new root to the root of each of the planar rooted trees $T_1, \dots, T_n$. For example,



**Notation:** Throughout this paper, let $\mathbf{k}$ be a nonunitary commutative ring which will be the base ring of all modules, algebras, as well as linear maps. Algebras are nonunitary associative algebras but not necessary commutative. For a set $Y$, we denote by $\mathbf{k}Y$ and $S(Y)$ the free $\mathbf{k}$-module with a basis $Y$ and free semigroup on $Y$, respectively.

## 2.2 Parallelly typed leaf-spaced decorated planar rooted forests

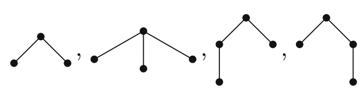
In this subsection, we first recall the concept of leaf-spaced decorated planar rooted forests [24] and then generalize it to parallelly typed version with an eye toward constructing free nonunitary Rota-Baxter family algebras.

Guo utilized leaf-spaced decorated rooted forests to construct free nonunitary Rota-Baxter algebras [24].

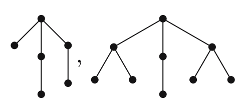
**Definition 2.2.** Let $X$ be a set and $T$ a planar rooted tree.
(a) The tree $T$ is called **leaf-decorated** if its each leaf is decorated by an element of $X$.
(b) A **subtree** starting from a vertex $v$ of $T$ is the planar rooted tree consisting of $v$, as the root, together with all descents of $v$ and edges connecting all these vertices. If we write the subtree starting from $v$ in the form $B^{+}(T_1 \ldots T_n)$ with $T_1, \ldots, T_n$ being planar rooted trees, we call $T_i$ and $T_{i+1}$ **adjacent branches** of $v$, where $1 \leq i \leq n-1$.
(c) A planar rooted tree is called **leaf-spaced** if it does not have a vertex with adjacent non-leaf branches.
(d) A **leaf-spaced decorated planar rooted tree** is a leaf-decorated planar rooted tree which is also leaf-spaced.

Let us expose some examples for better understanding.

**Example 2.3.** The following are leaf-spaced planar rooted trees

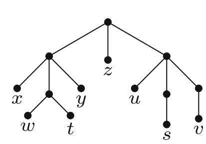


while

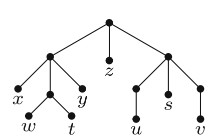


are not leaf-spaced planar rooted trees, since two right most branches are not separated by a leaf branch.

**Example 2.4.** The leaf-decorated planar rooted tree




is not leaf-spaced since two right most branches, with leaves decorated by $s$ and $v$, are not separated by a leaf branch. While the leaf-decorated planar rooted tree



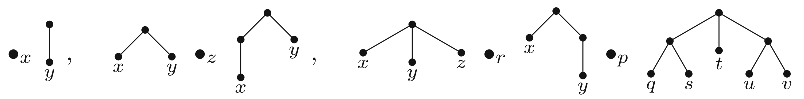

is leaf-spaced.

**Remark 2.5.** For a leaf-spaced planar rooted tree, it doesn't have a vertex with adjacent non-leaf branches. In the view point of rewriting system, it means that we rewrite the Rota-Baxter equation (1) from left to right. So it is natural to use leaf-spaced planar rooted trees to construct free nonunitary Rota-Baxter algebra in [25].

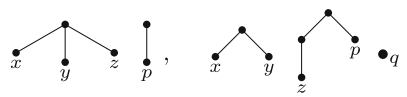
Now we generalize Definition 2.2 to planar rooted forests.

**Definition 2.6.** Let $X$ be a set and $F = T_1 \dots T_n$ a decorated planar rooted forest, where $T_1, \dots, T_n$ are decorated planar rooted trees. We call $F$ **leaf-spaced** if $T_1, \dots, T_n$ are leaf-spaced and for each $i$, at least one of $\operatorname{dep}(T_i)$ and $\operatorname{dep}(T_{i+1})$ is 0.

**Example 2.7.** The following are some examples of leaf-spaced decorated planar rooted forests:



while the following are some counterexamples:



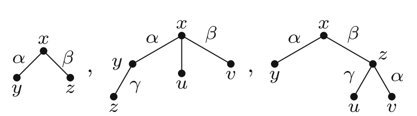
Here decorations are from $X$.

Typed decorated planar rooted trees are planar rooted trees with vertices decorated by elements of a set $X$ and edges decorated by elements of a set $\Omega$, which are applied to give a systematic description of a canonical renormalization procedure of stochastic PDEs [40]. Several algebraic structures have been built up on these planar rooted trees [36].

For a rooted tree $T$, denote by $V(T)$ (resp. $E(T)$) the set of its vertices (resp. edges).

**Definition 2.8.** [40] Let $X$ and $\Omega$ be two sets. An $X$-**decorated** $\Omega$-**typed** (**abbreviated typed decorated**) **rooted tree** is a triple $T = (T, \operatorname{dec}, \operatorname{type})$, where
(a) $T$ is a rooted tree,
(b) $\operatorname{dec} : V(T) \to X$ is a map,
(c) $\operatorname{type} : E(T) \to \Omega$ is a map.

Here are some examples of typed decorated rooted trees.

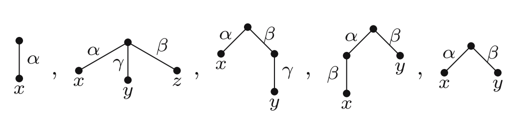


where $x, y, z, u, v \in X$ and $\alpha, \beta, \gamma \in \Omega$.

Combining Definitions 2.2 and 2.8, we propose the following concept. For a planar rooted tree $T$, denote by $L(T)$ the set of its leaves.

**Definition 2.9.** Let $X$ and $\Omega$ be two sets. A **typed leaf-spaced decorated planar rooted tree** (resp. **forest**) is a triple $T = (T, \operatorname{dec}, \operatorname{type})$, where
(a) $T$ is a leaf-spaced planar rooted tree (resp. forest),
(b) $\operatorname{dec} : L(T) \to X$ is a map,
(c) $\operatorname{type} : E(T) \to \Omega$ is a map.

**Example 2.10.** Let $X$ and $\Omega$ be two sets. For $x, y \in X$ and $\alpha, \beta, \gamma \in \Omega$,

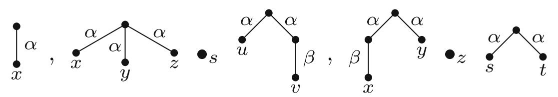


are typed leaf-spaced decorated planar rooted trees.

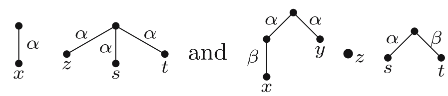
Now we come to the key concept used in this paper.

**Definition 2.11.** Let $X$ and $\Omega$ be two sets. A typed leaf-spaced decorated planar rooted tree (resp. forest) is called **parallelly typed** if $\mathrm{type}(e_1) = \mathrm{type}(e_2)$ whenever edges $e_1$ and $e_2$ share the same parent vertex. Denote by $\mathcal{T}_\ell(X, \Omega)$ (resp. $\mathcal{F}_\ell(X, \Omega)$) the set of parallelly typed leaf-spaced decorated planar rooted trees (resp. forests).

**Example 2.12.** The following are some elements in $\mathcal{F}_\ell(X, \Omega)$:



while



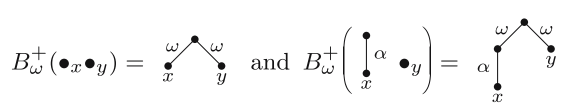
with $\alpha \neq \beta$ are two counterexamples not in $\mathcal{F}_\ell(X, \Omega)$. The first one is because it is not a leaf-spaced decorated planar rooted forest. The second one is because the right most two edges don't have the same edge decoration. Here $x, y, z, s, t, u, v \in X$ and $\alpha, \beta \in \Omega$.

**Remark 2.13.** Typed decorated planar rooted forests are allowed different decorations for edges sharing the same parent and are used to construct free Rota-Baxter family algebras [41] and free (tri)dendriform family algebras [42].

The classical grafting operation in Eq. (3) can be adapted to a linear operator $B^+_\omega$ for each $\omega \in \Omega$:

$$B^+_\omega : \mathbf{k}\mathcal{F}_\ell(X, \Omega) \to \mathbf{k}\mathcal{F}_\ell(X, \Omega), \quad T_1 \ldots T_n \mapsto B^+_\omega(T_1 \ldots T_n), \tag{4}$$

where $B^+_\omega(T_1 \ldots T_n)$ is the parallelly typed leaf-spaced decorated planar rooted tree obtained from $B^+(T_1 \ldots T_n)$ by decorating all the edges connecting the new root by $\omega$. For example,



Then $\mathbf{k}\mathcal{F}_\ell(X, \Omega)$ is closed under the operators $B^+_\omega$ with $\omega \in \Omega$.

## 2.3 Free nonunitary Rota-Baxter family algebras

This subsection is devoted to construct free nonunitary Rota-Baxter family algebras in terms of parallelly typed leaf-spaced decorated planar rooted forests.

Now we are going to equip $\mathbf{k}\mathcal{F}_\ell(X, \Omega)$ with a free nonunitary Rota-Baxter family algebra structure. Let us first define a multiplication

$$\diamond_\ell : \mathbf{k}\mathcal{F}_\ell(X, \Omega) \otimes \mathbf{k}\mathcal{F}_\ell(X, \Omega) \to \mathbf{k}\mathcal{F}_\ell(X, \Omega).$$

**Definition 2.14.** Let $X$ be a set and $\Omega$ a semigroup. Let $F, F' \in \mathcal{F}_\ell(X, \Omega)$.

(a) If $\mathrm{bre}(F) = 1 = \mathrm{bre}(F')$, then define

$$F \diamond_\ell F' := \begin{cases} FF' (\text{concatenation of planar rooted trees}), & \text{if } F = \bullet_x \text{ or } F' = \bullet_y; \\ B^+_{\alpha\beta}(\bar{F} \diamond_\ell F') + B^+_{\alpha\beta}(F \diamond_\ell \bar{F}') + \lambda B^+_{\alpha\beta}(\bar{F} \diamond_\ell \bar{F}'), & \text{if } F = B^+_\alpha(\bar{F}),\ F' = B^+_\beta(\bar{F}'). \end{cases} \tag{5}$$

(b) In general, if $\mathrm{bre}(F) = b$ and $\mathrm{bre}(F') = b'$, write

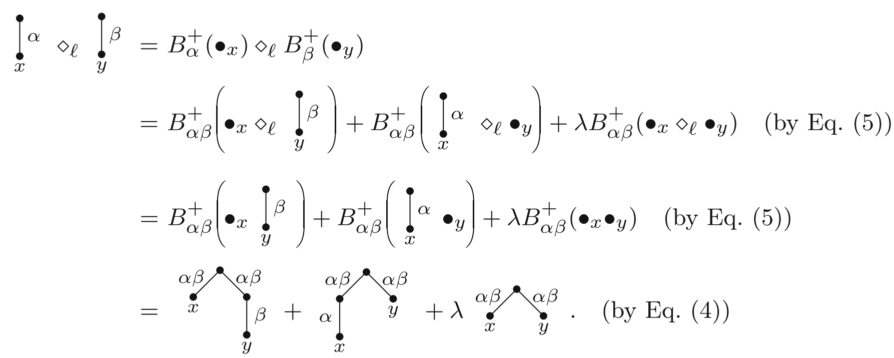
$$F = T_1 \dots T_b \ \text{ and } \ F' = T'_1 \dots T'_{b'}$$

and define

$$F \diamond_\ell F' := T_1 \dots T_{b-1}(T_b \diamond_\ell T'_1) T'_2 \dots T'_{b'}. \tag{6}$$

Let us give an example.

**Example 2.15.**




The concept of the free nonunitary Rota-Baxter family algebra is given as usual.

**Definition 2.16.** Let $X$ be a set and let $\Omega$ be a semigroup. Let $\lambda \in \mathbf{k}$ be given. A **free nonunitary Rota-Baxter family algebra** of weight $\lambda$ on $X$ is a nonunitary Rota-Baxter family algebra $F_{\mathrm{RBF}}(X)$ of weight $\lambda$ together with a set map $i_X : X \to F_{\mathrm{RBF}}(X)$ that satisfies the following universal property: for any nonunitary Rota-Baxter family algebra $(R, \diamond_R, (P_\omega)_{\omega\in\Omega})$ of weight $\lambda$ and any set map $f : X \to R$, there is a unique Rota-Baxter family algebra morphism $\bar{f} : F_{\mathrm{RBF}}(X) \to R$ such that $f = \bar{f} \circ i_X$.

We are ready for our main result. Let us define the set map $i_X$ by:

$$i_X : X \to \mathcal{F}_\ell(X, \Omega), \quad x \mapsto \bullet_x.$$

**Theorem 2.17.** *Let $X$ be a set and $\Omega$ a semigroup. The triple $(\mathbf{k}\mathcal{F}_\ell(X, \Omega), \diamond_\ell, (B^+_\omega)_{\omega\in\Omega})$, together with $i_X$, is the free nonunitary Rota-Baxter family algebra on $X$.*

**Proof.** We divide the proof into two steps.

**Step 1**: We prove that $(\mathbf{k}\mathcal{F}_\ell(X, \Omega), \diamond_\ell, (B^+_\omega)_{\omega\in\Omega})$ is a Rota-Baxter family algebra. From Eq. (5), we obtain immediately $(B^+_\omega)_{\omega\in\Omega}$ is a Rota-Baxter family of weight $\lambda$. It remains to prove that $\diamond_\ell$ satisfies the associativity:

$$(F_1 \diamond_\ell F_2) \diamond_\ell F_3 = F_1 \diamond_\ell (F_2 \diamond_\ell F_3), \quad \text{for } F_1, F_2, F_3 \in \mathcal{F}_\ell(X, \Omega).$$

We use induction on $\mathrm{dep}(F_1) + \mathrm{dep}(F_2) + \mathrm{dep}(F_3) \geq 0$. For the initial step of $\mathrm{dep}(F_1) + \mathrm{dep}(F_2) + \mathrm{dep}(F_3) = 0$, we have $\mathrm{dep}(F_1) = \mathrm{dep}(F_2) = \mathrm{dep}(F_3) = 0$. Let

$$\begin{aligned} F_1 &= \bullet_{x_{1,1}} \bullet_{x_{1,2}} \cdots \bullet_{x_{1,n_1}}, \\ F_2 &= \bullet_{x_{2,1}} \bullet_{x_{2,2}} \cdots \bullet_{x_{2,n_2}} \text{ and} \\ F_3 &= \bullet_{x_{3,1}} \bullet_{x_{3,2}} \cdots \bullet_{x_{3,n_3}} \text{ with } x_{i,n_i} \in X \text{ for } 1 \leq i \leq 3. \end{aligned}$$

Then by Eq. (6), we have

$$\begin{aligned} (F_1 \diamond_\ell F_2) \diamond_\ell F_3 &= (\bullet_{x_{1,1}} \bullet_{x_{1,2}} \cdots \bullet_{x_{1,n_1}} \diamond_\ell \bullet_{x_{2,1}} \bullet_{x_{2,2}} \cdots \bullet_{x_{2,n_2}}) \diamond_\ell \bullet_{x_{3,1}} \bullet_{x_{3,2}} \cdots \bullet_{x_{3,n_3}} \\ &= \bullet_{x_{1,1}} \bullet_{x_{1,2}} \cdots \bullet_{x_{1,n_1}} \bullet_{x_{2,1}} \bullet_{x_{2,2}} \cdots \bullet_{x_{2,n_2}} \bullet_{x_{3,1}} \bullet_{x_{3,2}} \cdots \bullet_{x_{3,n_3}} \\ &= \bullet_{x_{1,1}} \bullet_{x_{1,2}} \cdots \bullet_{x_{1,n_1}} \diamond_\ell (\bullet_{x_{2,1}} \bullet_{x_{2,2}} \cdots \bullet_{x_{2,n_2}} \diamond_\ell \bullet_{x_{3,1}} \bullet_{x_{3,2}} \cdots \bullet_{x_{3,n_3}}) \\ &= F_1 \diamond_\ell (F_2 \diamond_\ell F_3). \end{aligned}$$

For the inductive step of $\mathrm{dep}(F_1) + \mathrm{dep}(F_2) + \mathrm{dep}(F_3) \geq 1$, we have $\mathrm{dep}(F_i) \geq 1$ for some $i = 1, 2, 3$ and we use induction on $\mathrm{bre}(F_1) + \mathrm{bre}(F_2) + \mathrm{bre}(F_3) \geq 3$. For the initial step of $\mathrm{bre}(F_1) + \mathrm{bre}(F_2) + \mathrm{bre}(F_3) = 3$, we have $\mathrm{bre}(F_1) = \mathrm{bre}(F_2) = \mathrm{bre}(F_3)=1$. There are three cases to consider.

**Case 1**: $F_1$, $F_2$ and $F_3$ are of depth greater than zero. Write $F_1 = B_\alpha^+(\overline{F}_1)$, $F_2 = B_\beta^+(\overline{F}_2)$ and $F_3 = B_\gamma^+(\overline{F}_3)$. Then

$$\begin{aligned}
(F_1 \diamond_\ell F_2) \diamond_\ell F_3 &= (B_\alpha^+(\bar{F}_1) \diamond_\ell B_\beta^+(\bar{F}_2)) \diamond_\ell B_\gamma^+(\bar{F}_3) \\
&= (B_{\alpha\beta}^+(B_\alpha^+(\bar{F}_1) \diamond_\ell \bar{F}_2) + B_{\alpha\beta}^+(\bar{F}_1 \diamond_\ell B_\beta^+(\bar{F}_2)) + \lambda B_{\alpha\beta}^+(\bar{F}_1 \diamond_\ell \bar{F}_2)) \diamond_\ell B_\gamma^+(\bar{F}_3) \\
&= B_{\alpha\beta}^+(B_\alpha^+(\bar{F}_1) \diamond_\ell \bar{F}_2) \diamond_\ell B_\gamma^+(\bar{F}_3) + B_{\alpha\beta}^+(\bar{F}_1 \diamond_\ell B_\beta^+(\bar{F}_2)) \diamond_\ell B_\gamma^+(\bar{F}_3) + \lambda B_{\alpha\beta}^+(\bar{F}_1 \diamond_\ell \bar{F}_2) \diamond_\ell B_\gamma^+(\bar{F}_3) \\
&= B_{\alpha\beta\gamma}^+(B_{\alpha\beta}^+(B_\alpha^+(\bar{F}_1) \diamond_\ell \bar{F}_2) \diamond_\ell \bar{F}_3) + B_{\alpha\beta\gamma}^+((B_\alpha^+(\bar{F}_1) \diamond_\ell \bar{F}_2) \diamond_\ell B_\gamma^+(\bar{F}_3)) + \lambda B_{\alpha\beta\gamma}^+((B_\alpha^+(\bar{F}_1) \diamond_\ell \bar{F}_2) \diamond_\ell \bar{F}_3) \\
&\quad + B_{\alpha\beta\gamma}^+(B_{\alpha\beta}^+(\bar{F}_1 \diamond_\ell B_\beta^+(\bar{F}_2)) \diamond_\ell \bar{F}_3) + B_{\alpha\beta\gamma}^+((\bar{F}_1 \diamond_\ell B_\beta^+(\bar{F}_2)) \diamond_\ell B_\gamma^+(\bar{F}_3)) + \lambda B_{\alpha\beta\gamma}^+((\bar{F}_1 \diamond_\ell B_\beta^+(\bar{F}_2)) \diamond_\ell \bar{F}_3) \\
&\quad + \lambda B_{\alpha\beta\gamma}^+(B_{\alpha\beta}^+(\bar{F}_1 \diamond_\ell \bar{F}_2) \diamond_\ell \bar{F}_3) + \lambda B_{\alpha\beta\gamma}^+((\bar{F}_1 \diamond_\ell \bar{F}_2) \diamond_\ell B_\gamma^+(\bar{F}_3)) + \lambda^2 B_{\alpha\beta\gamma}^+((\bar{F}_1 \diamond_\ell \bar{F}_2) \diamond_\ell \bar{F}_3) \\
&= B_{\alpha\beta\gamma}^+(B_{\alpha\beta}^+(B_\alpha^+(\bar{F}_1) \diamond_\ell \bar{F}_2) \diamond_\ell \bar{F}_3) + B_{\alpha\beta\gamma}^+((B_\alpha^+(\bar{F}_1) \diamond_\ell \bar{F}_2) \diamond_\ell B_\gamma^+(\bar{F}_3)) + \lambda B_{\alpha\beta\gamma}^+((B_\alpha^+(\bar{F}_1) \diamond_\ell \bar{F}_2) \diamond_\ell \bar{F}_3) \\
&\quad + B_{\alpha\beta\gamma}^+(B_{\alpha\beta}^+(\bar{F}_1 \diamond_\ell B_\beta^+(\bar{F}_2)) \diamond_\ell \bar{F}_3) + B_{\alpha\beta\gamma}^+(\bar{F}_1 \diamond_\ell B_{\beta\gamma}^+(B_\beta^+(\bar{F}_2) \diamond_\ell (\bar{F}_3))) + B_{\alpha\beta\gamma}^+(\bar{F}_1 \diamond_\ell B_{\beta\gamma}^+(\bar{F}_2 \diamond_\ell B_\gamma^+(\bar{F}_3))) \\
&\quad + \lambda B_{\alpha\beta\gamma}^+(\bar{F}_1 \diamond_\ell B_{\beta\gamma}^+(\bar{F}_2 \diamond_\ell \bar{F}_3)) + \lambda B_{\alpha\beta\gamma}^+((\bar{F}_1 \diamond_\ell B_\beta^+(\bar{F}_2)) \diamond_\ell \bar{F}_3) + \lambda B_{\alpha\beta\gamma}^+(B_{\alpha\beta}^+(\bar{F}_1 \diamond_\ell \bar{F}_2) \diamond_\ell \bar{F}_3) \\
&\quad + \lambda B_{\alpha\beta\gamma}^+((\bar{F}_1 \diamond_\ell \bar{F}_2) \diamond_\ell B_\gamma^+(\bar{F}_3)) + \lambda^2 B_{\alpha\beta\gamma}^+((\bar{F}_1 \diamond_\ell \bar{F}_2) \diamond_\ell \bar{F}_3) \\
&\quad \text{(by the induction on } \mathrm{dep}(F_1) + \mathrm{dep}(F_2) + \mathrm{dep}(F_3) \text{ and Eq. (5)).}
\end{aligned}$$

By a similar calculation, we have

$$\begin{aligned}
F_1 \diamond_\ell (F_2 \diamond_\ell F_3) &= B_\alpha^+(\bar{F}_1) \diamond_\ell (B_\beta^+(\bar{F}_2) \diamond_\ell B_\gamma^+(\bar{F}_3)) \\
&= B_{\alpha\beta\gamma}^+(B_{\alpha\beta}^+(B_\alpha^+(\bar{F}_1) \diamond_\ell \bar{F}_2) \diamond_\ell \bar{F}_3) + B_{\alpha\beta\gamma}^+(B_{\alpha\beta}^+(\bar{F}_1 \diamond_\ell B_\beta^+(\bar{F}_2)) \diamond_\ell \bar{F}_3) + \lambda B_{\alpha\beta\gamma}^+(B_{\alpha\beta}^+(\bar{F}_1 \diamond_\ell \bar{F}_2) \diamond_\ell \bar{F}_3) \\
&\quad + B_{\alpha\beta\gamma}^+(\bar{F}_1 \diamond_\ell B_{\beta\gamma}^+(B_\beta^+(\bar{F}_2) \diamond_\ell \bar{F}_3)) + \lambda B_{\alpha\beta\gamma}^+(\bar{F}_1 \diamond_\ell (B_\beta^+(\bar{F}_2) \diamond_\ell \bar{F}_3)) + B_{\alpha\beta\gamma}^+(B_\alpha^+(\bar{F}_1) \diamond_\ell (\bar{F}_2 \diamond_\ell B_\gamma^+(\bar{F}_3))) \\
&\quad + B_{\alpha\beta\gamma}^+(\bar{F}_1 \diamond_\ell B_{\beta\gamma}^+(\bar{F}_2 \diamond_\ell B_\gamma^+(\bar{F}_3))) + \lambda B_{\alpha\beta\gamma}^+(\bar{F}_1 \diamond_\ell (\bar{F}_2 \diamond_\ell B_\gamma^+(\bar{F}_3))) + \lambda B_{\alpha\beta\gamma}^+(B_\alpha^+(\bar{F}_1) \diamond_\ell (\bar{F}_2 \diamond_\ell \bar{F}_3)) \\
&\quad + \lambda B_{\alpha\beta\gamma}^+(\bar{F}_1 \diamond_\ell B_{\beta\gamma}^+(\bar{F}_2 \diamond_\ell \bar{F}_3)) + \lambda^2 B_{\alpha\beta\gamma}^+(\bar{F}_1 \diamond_\ell (\bar{F}_2 \diamond_\ell \bar{F}_3)).
\end{aligned}$$

The $i$th term in the expansion of $(F_1 \diamond_\ell F_2) \diamond_\ell F_3$ matches with the $\sigma(i)$th term in the expansion of $F_1 \diamond_\ell (F_2 \diamond_\ell F_3)$. Here $\sigma$ is a permutation of order 11:

$$\begin{pmatrix} i \\ \sigma(i) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 6 & 9 & 2 & 4 & 7 & 10 & 5 & 3 & 8 & 11 \end{pmatrix}.$$

**Case 2:** Exactly two of $F_1$, $F_2$ and $F_3$ are of depth greater than zero. There are three subcases to consider.

**Subcase 2.1**: $F_1 = B_\alpha^+(\bar{F}_1)$, $F_2 = B_\beta^+(\bar{F}_2)$ and $F_3 = \bullet_x$. Then

$$\begin{aligned}
(F_1 \diamond_\ell F_2) \diamond_\ell F_3 &= (B_\alpha^+(\bar{F}_1) \diamond_\ell B_\beta^+(\bar{F}_2)) \diamond_\ell \bullet_x \\
&= (B_\alpha^+(\bar{F}_1) \diamond_\ell B_\beta^+(\bar{F}_2)) \bullet_x \quad \text{(by Eq. (5))} \\
&= B_\alpha^+(\bar{F}_1) \diamond_\ell (B_\beta^+(\bar{F}_2) \bullet_x) \quad \text{(by Eq. (6))} \\
&= B_\alpha^+(\bar{F}_1) \diamond_\ell (B_\beta^+(\bar{F}_2) \diamond_\ell \bullet_x) \quad \text{(by Eq. (5))} \\
&= F_1 \diamond_\ell (F_2 \diamond_\ell F_3).
\end{aligned}$$

**Subcase 2.2**: $F_1 = B_\alpha^+(\bar{F}_1)$, $F_2 = \bullet_x$ and $F_3 = B_\beta^+(\bar{F}_3)$. Then

$$\begin{aligned}
(F_1 \diamond_\ell F_2) \diamond_\ell F_3 &= (B_\alpha^+(\bar{F}_1) \diamond_\ell \bullet_x) \diamond_\ell B_\beta^+(\bar{F}_3) \\
&= (B_\alpha^+(\bar{F}_1) \bullet_x) \diamond_\ell B_\beta^+(\bar{F}_3) \quad \text{(by Eq. (5))} \\
&= B_\alpha^+(\bar{F}_1) \bullet_x B_\beta^+(\bar{F}_3) \quad \text{(by Eqs. (5) and (6))} \\
&= B_\alpha^+(\bar{F}_1) \diamond_\ell (\bullet_x \diamond_\ell B_\beta^+(\bar{F}_3)) \quad \text{(by Eqs. (5) and (6))} \\
&= F_1 \diamond_\ell (F_2 \diamond_\ell F_3).
\end{aligned}$$

**Subcase 2.3**: $F_1 = \bullet_x$, $F_2 = B_\alpha^+(\bar{F}_2)$ and $F_3 = B_\beta^+(\bar{F}_3)$. This case is similar to Subcase 2.1.

**Case 3**: Exactly one of $F_1$, $F_2$ and $F_3$ is of depth greater than zero. We have three subcases to consider.
**Subcase 3.1:** $F_1 = B^+_\omega(\bar{F}_1)$, $F_2 = \bullet_x$ and $F_3 = \bullet_y$. Then

$$\begin{aligned}(F_1 \diamond_\ell F_2)\diamond_\ell F_3 &= (B^+_\omega(\bar{F}_1)\diamond_\ell \bullet_x)\diamond_\ell \bullet_y\\ &= (B^+_\omega(\bar{F}_1)\bullet_x)\diamond_\ell \bullet_y \quad \text{(by Eq. (5))}\\ &= B^+_\omega(\bar{F}_1)\bullet_x\ \bullet_y \quad \text{(by Eqs. (5) and (6))}\\ &= B^+_\omega(\bar{F}_1)\diamond_\ell(\bullet_x \diamond_\ell \bullet_y) \quad \text{(by Eqs. (5) and (6))}\\ &= F_1\diamond_\ell(F_2\diamond_\ell F_3).\end{aligned}$$

**Subcase 3.2:** $F_1 = \bullet_x$, $F_2 = B^+_\omega(\bar{F}_2)$ and $F_3 = \bullet_y$. Then

$$\begin{aligned}(F_1 \diamond_\ell F_2)\diamond_\ell F_3 &= (\bullet_x \diamond_\ell B^+_\omega(\bar{F}_2))\diamond_\ell \bullet_y\\ &= \bullet_x B^+_\omega(\bar{F})\bullet_y \quad \text{(by Eqs. (5) and (6))}\\ &= \bullet_x\diamond_\ell(B^+_\omega(\bar{F}_2)\diamond_\ell \bullet_y) \quad \text{(by Eqs. (5) and (6))}\\ &= F_1\diamond_\ell(F_2\diamond_\ell F_3).\end{aligned}$$

**Subcase 3.3**: $F_1 = \bullet_x$, $F_2 = \bullet_y$ and $F_3 = B^+_\omega(\bar{F}_3)$. This case is similar to Subcase 3.1.

For the inductive step of $\mathrm{bre}(F_1) + \mathrm{bre}(F_2) + \mathrm{bre}(F_3) > 3$, we have $\mathrm{bre}(F_i) \geq 2$ for some $i = 1, 2, 3$. There are three cases to consider.

**Case 4**: $\mathrm{bre}(F_1) \geq 2$. Let $F_1 = T_{1,1}T_{1,2}\dots T_{1,s_1}$ with $T_{1,1},\dots,T_{1,s_1} \in \mathcal{T}_\ell(X,\Omega)$ and $s_1 \geq 2$. Then

$$\begin{aligned}(F_1 \diamond_\ell F_2)\diamond_\ell F_3 &= ((T_{1,1}T_{1,2}\dots T_{1,s_1})\diamond_\ell F_2)\diamond_\ell F_3\\ &= (T_{1,1}T_{1,2}\dots T_{1,s_1-1}(T_{1,s_1}\diamond_\ell F_2))\diamond_\ell F_3 \quad \text{(by Eq. (6))}\\ &= T_{1,1}T_{1,2}\dots T_{1,s_1-1}((T_{1,s_1}\diamond_\ell F_2)\diamond_\ell F_3) \quad \text{(by Eq. (6))}\\ &= T_{1,1}T_{1,2}\dots T_{1,s_1-1}(T_{1,s_1}\diamond_\ell(F_2\diamond_\ell F_3))\\ &\quad \text{(by the induction on } \mathrm{bre}(F_1)+\mathrm{bre}(F_2)+\mathrm{bre}(F_3))\\ &= (T_{1,1}T_{1,2}\dots T_{1,s_1})\diamond_\ell(F_2\diamond_\ell F_3) \quad \text{(by Eq. (6))}\\ &= F_1\diamond_\ell(F_2\diamond_\ell F_3).\end{aligned}$$

**Case 5**: $\mathrm{bre}(F_2) \geq 2$. Let $F_2 = T_{2,1}T_{2,2}\dots T_{2,s_2}$ with $T_{2,1},\dots, T_{2,s_2} \in \mathcal{T}_\ell(X,\Omega)$ and $s_2 \geq 2$. Then

$$\begin{aligned}(F_1 \diamond_\ell F_2)\diamond_\ell F_3 &= (F_1\diamond_\ell T_{2,1})T_{2,2}\dots T_{2,s_2-1}(T_{2,s_2}\diamond_\ell F_3) \quad \text{(by Eq. (6))}\\ &= F_1\diamond_\ell(F_2\diamond_\ell F_3).\end{aligned}$$

**Case 6**: $\mathrm{bre}(F_3) \geq 2$. This case is similar to Case 4.
**Step 2**: We show that $(\mathbf{k}\mathcal{F}_\ell(X,\Omega), \diamond_\ell, (B^+_\omega)_{\omega\in\Omega})$ satisfies the universal property. For this, let $(R, \diamond_R, (P_\omega)_{\omega\in\Omega})$ be a nonunitary Rota-Baxter family algebra and let $f: X \to R$ be a set map.

(Existence) We define a linear map

$$\bar{f} : \mathbf{k}\mathcal{F}_\ell(X,\Omega) \to R,\ F \mapsto \bar{f}(F),$$

by induction on $\mathrm{dep}(F) \geq 0$. Consider the initial step of $\mathrm{dep}(F) = 0$. If $\mathrm{bre}(F) = 1$, then $F = \bullet_x$ for some $x \in X$ and define

$$\bar{f}(\bullet_x) := \bar{f} \circ i_X(x) := f(x). \tag{7}$$

If $\mathrm{bre}(F) \geq 2$, then $F = \bullet_{x_1}\bullet_{x_2}\dots\bullet_{x_b}$ for some $x_1,\dots,x_b \in X$, and we define

$$\bar{f}(F) := f(x_1)\diamond_R\cdots\diamond_R f(x_b). \tag{8}$$

Assume that $\bar{f}(F)$ has been defined for $F \in \mathcal{F}_\ell(X,\Omega)$ with $\mathrm{dep}(F) \leq k$ for a $k \geq 0$ and consider $F \in \mathcal{F}_\ell(X,\Omega)$ with $\mathrm{dep}(F) = k+1 \geq 1$. If $\mathrm{bre}(F) = 1$, we have $F = B^+_\omega(\overline{F})$ for some $\omega \in \Omega$ and $\overline{F} \in \mathcal{F}_\ell(X,\Omega)$ with $\mathrm{dep}(\overline{F}) = k$. We then define

$$\bar{f}(F) := \bar{f}(B^+_\omega(\bar{F})) := P_\omega(\bar{f}(\bar{F})). \tag{9}$$

If $\operatorname{bre}(F) > 1$, let $F = T_1 \dots T_b$ with $T_1, \dots, T_b \in \mathcal{T}_\ell(X, \Omega)$ and define

$$\bar{f}(F) := \bar{f}(T_1) \diamond_R \cdots \diamond_R \bar{f}(T_b), \tag{10}$$

where each $\bar{f}(T_i)$, $1 \le i \le b$ is defined by Eq. (7) or Eq. (9).

Now we prove that $\bar{f}$ is an algebra homomorphism:

$$\bar{f}(F_1 \diamond_\ell F_2) = \bar{f}(F_1) \diamond_R \bar{f}(F_2) \quad \text{for } F_1, F_2 \in \mathcal{F}_\ell(X, \Omega), \tag{11}$$

by induction on the sum of depth $\operatorname{dep}(F_1) + \operatorname{dep}(F_2) \ge 0$. Write

$$F_1 = T_{1,1} \dots T_{1,s} \text{ and } F_2 = T_{2,1} \dots T_{2,t}.$$

If $\operatorname{dep}(F_1) + \operatorname{dep}(F_2) = 0$, then

$$F_1 = \bullet_{x_{1,1}} \dots \bullet_{x_{1,s}} \text{ and } F_2 = \bullet_{x_{2,1}} \dots \bullet_{x_{2,t}},$$

and so

$$F_1 \diamond_\ell F_2 = \bullet_{x_{1,1}} \dots \bullet_{x_{1,s}} \bullet_{x_{2,1}} \dots \bullet_{x_{2,t}}.$$

It follows from Eq. (8) that

$$\begin{aligned}
\bar{f}(F_1 \diamond_\ell F_2) &= \bar{f}(\bullet_{x_{1,1}} \dots \bullet_{x_{1,s}} \bullet_{x_{2,1}} \dots \bullet_{x_{2,t}}) \\
&= f(x_{1,1}) \diamond_R \cdots \diamond_R f(x_{1,s}) \diamond_R f(x_{2,1}) \diamond_R \cdots \diamond_R f(x_{2,t}) \\
&= (f(x_{1,1}) \diamond_R \cdots \diamond_R f(x_{1,s})) \diamond_R (f(x_{2,1}) \diamond_R \cdots \diamond_R f(x_{2,t})) \\
&= \bar{f}(F_1) \diamond_R \bar{f}(F_2).
\end{aligned}$$

For the inductive step, assume that Eq. (11) holds when $\operatorname{dep}(F_1) + \operatorname{dep}(F_2) \le k$ for a given $k \ge 0$ and consider the case of $\operatorname{dep}(F_1) + \operatorname{dep}(F_2) = k + 1$. We reduce to the induction on $\operatorname{bre}(F_1) + \operatorname{bre}(F_2) \ge 2$. For the initial step of $\operatorname{bre}(F_1) + \operatorname{bre}(F_2) = 2$, we have $\operatorname{bre}(F_1) = 1 = \operatorname{bre}(F_2)$ and $F_1 = B_\alpha^+(\overline{F}_1)$, $F_2 = B_\beta^+(\overline{F}_2)$.

$$\begin{aligned}
\bar{f}(F_1 \diamond_\ell F_2) &= \bar{f}(B_\alpha^+(\bar{F}_1) \diamond_\ell B_\beta^+(\bar{F}_2)) \\
&= \bar{f}(B_{\alpha\beta}^+(B_\alpha^+(\bar{F}_1) \diamond_\ell \bar{F}_2) + B_{\alpha\beta}^+(\bar{F}_1 \diamond_\ell B_\beta^+(\bar{F}_2)) + \lambda B_{\alpha\beta}^+(\bar{F}_1 \diamond_\ell \bar{F}_2)) \quad \text{(by Eq. (5))} \\
&= \bar{f}(B_{\alpha\beta}^+(B_\alpha^+(\bar{F}_1) \diamond_\ell \bar{F}_2)) + \bar{f}(B_{\alpha\beta}^+(\bar{F}_1 \diamond_\ell B_\beta^+(\bar{F}_2))) + \bar{f}(\lambda B_{\alpha\beta}^+(\bar{F}_1 \diamond_\ell \bar{F}_2)) \\
&= P_{\alpha\beta}(\bar{f}(B_\alpha^+(\bar{F}_1) \diamond_\ell \bar{F}_2)) + P_{\alpha\beta}(\bar{f}(\bar{F}_1 \diamond_\ell B_\beta^+(\bar{F}_2))) + \lambda P_{\alpha\beta}(\bar{f}(\bar{F}_1 \diamond_\ell \bar{F}_2)) \quad \text{(by Eq. (9))} \\
&= P_{\alpha\beta}(\bar{f}(B_\alpha^+(\bar{F}_1)) \diamond_R \bar{f}(\bar{F}_2)) + P_{\alpha\beta}(\bar{f}(\bar{F}_1) \diamond_R \bar{f}(B_\beta^+(\bar{F}_2))) + \lambda P_{\alpha\beta}(\bar{f}(\bar{F}_1) \diamond_R \bar{f}(\bar{F}_2)) \\
&\quad \text{(by the induction on } \operatorname{dep}(F_1) + \operatorname{dep}(F_2)) \\
&= P_{\alpha\beta}(P_\alpha(\bar{f}(\bar{F}_1)) \diamond_R \bar{f}(\bar{F}_2)) + P_{\alpha\beta}(\bar{f}(\bar{F}_1) \diamond_R P_\beta(\bar{f}(\bar{F}_2))) + \lambda P_{\alpha\beta}(\bar{f}(\bar{F}_1) \diamond_R \bar{f}(\bar{F}_2)) \quad \text{(by Eq. (9))} \\
&= P_\alpha(\bar{f}(\bar{F}_1)) \diamond_R P_\beta(\bar{f}(\bar{F}_2)) \quad \text{(by Eq. (2))} \\
&= \bar{f}(B_\alpha^+(\bar{F}_1)) \diamond_R \bar{f}(B_\beta^+(\bar{F}_2)) \quad \text{(by Eq. (9))} \\
&= \bar{f}(F_1) \diamond_R \bar{f}(F_2).
\end{aligned}$$

For the induction step of $\operatorname{bre}(F_1) + \operatorname{bre}(F_2) \ge 3$, we write

$$F_1 = T_{1,1} \dots T_{1,s} \quad \text{and} \quad F_2 = T_{2,1} \dots T_{2,t}.$$

Then

$$\begin{aligned}
\bar{f}(F_1 \diamond_\ell F_2) &= \bar{f}((T_{1,1} \dots T_{1,s}) \diamond_\ell (T_{2,1} \dots T_{2,t})) \\
&= \bar{f}(T_{1,1} \dots T_{1,s-1}(T_{1,s} \diamond_\ell T_{2,1}) T_{2,2} \dots T_{2,t}) \quad \text{(by Eq. (6))} \\
&= \bar{f}(T_{1,1}) \diamond_R \cdots \diamond_R \bar{f}(T_{1,s-1}) \diamond_R \bar{f}(T_{1,s} \diamond_\ell T_{2,1}) \diamond_R \bar{f}(T_{2,2}) \diamond_R \cdots \diamond_R \bar{f}(T_{2,t}) \quad \text{(by Eq. (10))} \\
&= \bar{f}(T_{1,1}) \diamond_R \cdots \diamond_R \bar{f}(T_{1,s-1}) \diamond_R \bar{f}(T_{1,s}) \diamond_R \bar{f}(T_{2,1}) \diamond_R \bar{f}(T_{2,2}) \diamond_R \cdots \diamond_R \bar{f}(T_{2,t}) \\
&\quad \text{(by the induction hypothesis on } \operatorname{bre}(F_1) + \operatorname{bre}(F_2)) \\
&= \bar{f}(T_{1,1} \dots T_{1,s}) \diamond_R \bar{f}(T_{2,1} \dots \bar{f}(T_{2,t})) \quad \text{(by Eq. (10))} \\
&= \bar{f}(F_1) \diamond_R \bar{f}(F_2).
\end{aligned}$$

Thus, $\bar{f}$ is an algebra homomorphism. Further by Eq. (9),

$$\bar{f} \circ B^+_\omega = P_\omega \circ \bar{f} \quad \text{for } \omega \in \Omega,$$

whence $\bar{f}$ is a Rota-Baxter family algebra morphism. Finally, it follows from Eq. (7) that $\bar{f} \circ i_X = f$. This completes the proof of existence.

(Uniqueness) Suppose that such $\bar{f}$ exists. Then, since $\bar{f}$ is a Rota-Baxter family algebra morphism such that $\bar{f} \circ i_X = f$, $\bar{f}(F)$ must be of the forms in Eqs. (7)–(10) for $F \in \mathcal{F}_\ell(X, \Omega)$. □

If $\Omega$ is a trivial semigroup, that is, $\Omega$ has only one element, then all edges of an element $F$ in $\mathcal{F}_\ell(X, \Omega)$ are of the same decoration. So we can view $F$ has no edge decoration. In this case, denote $\mathcal{F}_\ell(X) := \mathcal{F}_\ell(X, \Omega)$ and notice that Rota-Baxter family Eq. (2) reduces to Rota-Baxter Eq. (1).

**Corollary 2.18.** *Let $X$ be a set. Then the triple $(\mathbf{k}\mathcal{F}_\ell(X), \diamond_\ell, B^+)$, together with the $i_X$, is the free nonunitary Rota-Baxter algebra of weight $\lambda$ on $X$.*

**Proof.** It follows from Theorem 2.17 by taking $\Omega$ to be a trivial semigroup. □

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