Research Article

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SEMT valuation and strength of subdivided star of $K_{1,4}$

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Abstract: This study focuses on finding super edge-magic total (SEMT) labeling and deficiency of imbalanced fork and disjoint union of imbalanced fork with star, bistar and path; in addition, the SEMT strength for Imbalanced Fork is investigated.

Keywords: SEMT labeling, SEMT deficiency, SEMT strength, imbalanced fork

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1 Preliminaries

Labeling is a technique that allots labels to the components of a graph. Total labeling gives us both components (vertices and edges) labeled. A (v, ε) -graph G determines an edge-magic total (EMT) labeling when $\Gamma : V(G) \cup E(G) \rightarrow \{\overline{1, v + \varepsilon}\}$ is bijective so as the weights at every edge are the same constant (say) c , such a number c is interpreted as a magic constant. If all vertices gain the smallest of the labels, then an EMT labeling is called a super edge-magic total (SEMT) labeling. Kotzig and Rosa [1] and Enomoto et al. [2] found the concepts of EMT and SEMT graphs, respectively, and presented the conjectures: every tree is EMT [1] and every tree is SEMT [2].

If a graph G allows at least one SEMT labeling, then the smallest of the magic constants for all possible distinct SEMT labelings of G describes SEMT strength, $sm(G)$, of G . Avadayappan et al. first introduced the notion of SEMT strength [3] and found the exact values of SEMT strength for some graphs.

In [1], the notion of EMT deficiency was proposed, and Figueroa-Centeno et al. [4] continued it to SEMT graphs. For any graph G , the SEMT deficiency, signified as $\mu_s(G)$, is the least number n of isolated vertices that we have to take in union with G so that the resulting graph $G \cup nK_1$ is SEMT, and the case $+\infty$ will arise if no isolated vertex fulfills this criterion. More specifically,

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$$\mu_s(G) = \begin{cases} \min M(G) & \text{if } M(G) \neq \emptyset, \\ +\infty & \text{if } M(G) = \emptyset, \end{cases}$$

where $M(G) = \{n \geq 0 : G \cup nK_1 \text{ is an SEMT graph}\}$.

In [4,5], Figueroa-Centeno et al. proposed a conjecture about the confined deficiencies of the forests. In [6], an assumption was made as a special case of a previous one that says, the deficiency of each two-tree forest is not more than 1. The results in [7–22] are found to be useful in the aspect of examined labeling here. For more review, see the recent survey of graph labelings by Gallian [23].

In this paper, we formulated the results on SEMT labeling and deficiency of forests consisting of imbalanced fork and disjoint union of imbalanced fork with star, bistar and path, respectively. The values of parameters of the star, bistar and path are totally dependent on the parameters involved in the imbalanced fork.

2 Main results

Definition 2.1. An imbalanced fork, represented as $\text{Fr}(l, l+1, l+2)$, $l \in \mathbb{N}$, see Figure 1, is also a tree consisting of three paths of lengths l , $l+1$ and $l+2$, that is,

$$P_l : x_{1,j}; 1 \leq j \leq l,$$

$$P_{l+1} : x_{2,j}; 1 \leq j \leq l+1,$$

$$P_{l+2} : x_{3,j}; 1 \leq j \leq l+2.$$

A single new vertex $x_{2,0}$ is added to the path P_{l+1} through an edge, and these three paths are joined together by two edges that are $x_{i,1}x_{i+1,1}$, $1 \leq i \leq 2$. Precisely, the set of vertices and the set of edges of imbalanced fork are as follows:

$$V(\text{Fr}(l, l+1, l+2)) = \{x_{2,0}\} \cup \{x_{i,j} : i = 1, 1 \leq j \leq l\} \cup \{x_{i,j} : i = 2, 1 \leq j \leq l+1\} \cup \{x_{i,j} : i = 3, 1 \leq j \leq l+2\},$$

$$\begin{aligned} E(\text{Fr}(l, l+1, l+2)) = & \{x_{i,j}x_{i,j+1} : i = 1, 1 \leq j \leq l-1\} \cup \{x_{i,j}x_{i,j+1} : i = 2, 1 \leq j \leq l\} \\ & \cup \{x_{i,j}x_{i,j+1} : i = 3, 1 \leq j \leq l+1\} \cup \{x_{i,1}x_{i+1,1} : 1 \leq i \leq 2\} \cup \{x_{2,0}x_{2,1}\}, \end{aligned}$$

respectively.

Note 1. Another way of writing imbalanced fork $\text{Fr}(l, l+1, l+2)$ is $T(1, l, l, l+2)$, $l \in \mathbb{N}$ because of its extraction from a star by its subdivision. In our another paper [24], we have established SEMT labeling and strength, for all positive integers ≥ 2 , of fork which is also a subdivision of star $K_{1,4}$. Imbalanced Fork is basically a special case of the Fork.

The following lemma is an elementary tool for proving graphs to be SEMT. It will be used as a base in each result presented in this work.

Lemma 2.2. [25] A (v, ε) -graph G is SEMT if and only if $\exists$ a bijective map $\Gamma : V(G) \rightarrow \{\overline{1}, \overline{v}\}$ s.t. the set of edge-sums

$$S = \{\Gamma(l) + \Gamma(m) : lm \in E(G)\}$$

constructs ε consecutive Z^+ . In that case, G can extend to an SEMT labeling of G with magic constant $c = v + \varepsilon + \min(S)$ and

$$S = \{c - (v + \varepsilon), c - (v + \varepsilon) + 1, \dots, c - (v + 1)\}.$$

The following result of SEMT graphs also holds.

Figure 1: Imbalanced Fork $\text{Fr}(4, 5, 6)$.

Note 2. [3] Let $c(\Gamma)$ be a magic constant of an SEMT labeling Γ of $G(V, E)$, then we end up on this statement:

$$\varepsilon \quad c(\Gamma) = \sum_{v \in V} \deg_G(v) \Gamma(v) + \sum_{p \in E} \Gamma(p), \quad \varepsilon = |E(G)|. \quad (1)$$

For a single graph, many SEMT labelings might exist and of course for a different labeling, there will be a different magic constant. For lower and upper bounds of the magic constants for subdivided stars, see [26,27]. Now we are concerned with evaluating the SEMT labeling and strength of imbalanced fork.

Theorem 2.3. For $l \geq 1$, the graph $G \cong \text{Fr}(l, l+1, l+2)$ is SEMT with magic constant:

$$a = \begin{cases} \frac{15l+22}{2}; & l \equiv 0(\text{mod } 2); \\ \frac{5(3l+5)}{2}; & l \equiv 1(\text{mod } 2). \end{cases}$$

Proof. Let $G \cong \text{Fr}(l, l+1, l+2)$, $l \geq 1$ and $p = |V(G)|$, $q = |E(G)|$, then $p = 3l+4$, $q = 3l+3$.

Consider the vertex labeling $f: V(G) \rightarrow \{1, 2, \dots, p\}$ as follows:

For $l \equiv 1(\text{mod } 2)$:

$$f(x_{2,0}) = l+2,$$

$$f(x_{i,j}) = \begin{cases} 1 + (l+2)\left(\frac{i-1}{2}\right) + \left(\frac{j-1}{2}\right); & i \equiv 1(\text{mod } 2), \quad i = 1, 3; \quad j \equiv 1(\text{mod } 2), \quad j \geq 1; \\ l - \frac{j-2}{2} + 1; & i \equiv 0(\text{mod } 2), \quad i = 2; \quad j \equiv 0(\text{mod } 2), \quad j \geq 2; \\ \frac{3l+1}{2} + l\left(\frac{i-1}{2}\right) + \frac{j-2}{2} + 4; & i \equiv 1(\text{mod } 2), \quad i = 1, 3; \quad j \equiv 0(\text{mod } 2), \quad j \geq 2; \\ \frac{3l+1}{2} + l - \frac{j-1}{2} + 3; & i \equiv 0(\text{mod } 2), \quad i = 2; \quad j \equiv 1(\text{mod } 2), \quad j \geq 1. \end{cases}$$

From the above labeling “ f ”, we obtain consecutive Z_+ from $h+1$ to $h+q$, where $h = 6 + 3\left(\frac{l-1}{2}\right)$. Thus, Lemma 2.2 gives the required result with magic constant $a = p + q + h + 1$, where $h+1 = \min(S)$.

For $l \equiv 0(\text{mod } 2)$:

$$f(x_{2,0}) = 2l+3,$$

$$f(x_{i,j}) = \begin{cases} 2(l+1) + (l+2)\left(\frac{i-1}{2}\right) - \left(\frac{j-1}{2}\right); & i \equiv 1(\text{mod } 2), \quad i = 1, 3; \quad j \equiv 1(\text{mod } 2), \quad j \geq 1; \\ 2(l+2) + \frac{j-2}{2}; & i \equiv 0(\text{mod } 2), \quad i = 2; \quad j \equiv 0(\text{mod } 2), \quad j \geq 2; \\ \frac{l}{2} + (l+2)\left(\frac{i-1}{2}\right) - \frac{j-2}{2}; & i \equiv 1(\text{mod } 2), \quad i = 1, 3; \quad j \equiv 0(\text{mod } 2), \quad j \geq 2; \\ \frac{l}{2} + \frac{j-1}{2} + 1; & i \equiv 0(\text{mod } 2), \quad i = 2; \quad j \equiv 1(\text{mod } 2), \quad j \geq 1. \end{cases}$$

From the above labeling “ f ”, we obtain consecutive Z_+ from $h' + 1$ to $h' + q$, where $h' = 6 + 3\left(\frac{l-2}{2}\right)$. Thus, Lemma 2.2 gives the required result with magic constant $a = p + q + h' + 1$, where $h' + 1 = \min(S)$. $\square$

This theorem gives us the magic constants $a(f) = \frac{15l+22}{2}$, for $l \equiv 0(\text{mod } 2)$, and $a(f) = \frac{5(3l+5)}{2}$, for $l \equiv 1(\text{mod } 2)$ of imbalanced fork, where $l \geq 1$ and using the lower bound of magic constants in lemma obtained in [27], we have $a(f) \geq \frac{5q^2+q+12}{2q}$, where $q = 3l + 3$, thus we can conclude:

Theorem 2.4. *The SEMT strength for Imbalanced Fork $\text{Fr}(l, l+1, l+2)$, $l \geq 1$ (subdivision of star $K_{1,4}$) is*

$$\frac{15l^2 + 31l + 20}{2(l+1)} \leq sm(\text{Fr}(l, l+1, l+2)) \leq \frac{15l+22}{2}, \quad l \equiv 0(\text{mod } 2),$$

$$\frac{15l^2 + 31l + 20}{2(l+1)} \leq sm(\text{Fr}(l, l+1, l+2)) \leq \frac{5(3l+5)}{2}, \quad l \equiv 1(\text{mod } 2).$$

2.1 SEMT labeling and deficiency of forests formed by imbalanced fork, star, bistar and path

In this section, it is shown that the forests consisting of imbalanced fork, star, bistar and path are SEMT with certain conditions on the parameters.

Theorem 2.6. *For $l \geq 1$,*

- (a) $\text{Fr}(l, l+1, l+2) \cup K_{1,\varpi}$ is SEMT,
- (b) $\mu_s(\text{Fr}(l, l+1, l+2) \cup K_{1,\varpi-1}) \leq 1$,

where

$$\varpi = \begin{cases} \frac{l+1}{2}; & l \equiv 1(\text{mod } 2); \\ l+1; & l \equiv 0(\text{mod } 2). \end{cases}$$

Proof.

(a) Consider the graph $G \cong \text{Fr}(l, l+1, l+2) \cup K_{1,\varpi}$.

Let $p = |V(G)|$ and $q = |E(G)|$, then

$$p = 3l + \varpi + 5,$$

$$q = 3l + \varpi + 3.$$

For $l \equiv 1(\text{mod } 2)$:

We define a labeling $f: V(\text{Fr}(l, l+1, l+2)) \rightarrow \{1, 2, \dots, 3l+4\}$, as

$$f(x_{i,j}) = \begin{cases} l - \frac{l-1}{2}; & i \equiv 0(\text{mod } 2), \quad i = 2; \quad j \equiv 1(\text{mod } 2), \quad j \geq 1; \\ 1 + l\left(\frac{l-1}{2}\right) + \frac{l-2}{2}; & i \equiv 1(\text{mod } 2), \quad i = 1, 3; \quad j \equiv 0(\text{mod } 2), \quad j \geq 2. \end{cases}$$

Now consider the labeling $g: V(G) \rightarrow \{1, 2, \dots, p\}$.

For $1 \leq k \leq \varpi + 1$,

$$g(y_k) = \begin{cases} \frac{3(l+1)}{2}; & k = 1; \\ 3(l+1) + k; & k \neq 1. \end{cases}$$

Let $A = \frac{3(l+1)}{2}$, then

$$f(x_{i,j}) = \begin{cases} A + (l+1)\left(\frac{i-1}{2}\right) + \frac{j-1}{2} + 1; & i \equiv 1(\bmod 2), \quad i = 1, 3; \quad j \equiv 1(\bmod 2), \quad j \geq 1; \\ A + l - \frac{j-2}{2} + 1; & i \equiv 0(\bmod 2), \quad i = 2; \quad j \equiv 0(\bmod 2), \quad j \geq 2, \end{cases}$$

$$f(x_{2,0}) = g(x_{2,0}) = 3(l+1) + \varpi + 2,$$

$$g(x_{i,j}) = f(x_{i,j}); \quad 1 \leq i \leq 3, 1 \leq j \leq \varepsilon, l \leq \varepsilon \leq l+2.$$

From the above labeling “ f ”, we obtain consecutive Z_+ from $\hbar + 1$ to $\hbar + q$, where $\hbar = \frac{3(l+1)}{2} + 1$. Thus, Lemma 2.2 gives the required result with magic constant $a = p + q + \hbar + 1$, where $\hbar + 1 = \min(S)$.

For $l \equiv 0(\bmod 2)$:

We define a labeling $f : V(\text{Fr}(l, l+1, l+2)) \rightarrow \{1, 2, \dots, 3l+4\}$, as

$$f(x_{i,j}) = \begin{cases} \frac{l+2}{2} + \frac{j-1}{2}; & i \equiv 0(\bmod 2), \quad i = 2; \quad j \equiv 1(\bmod 2), \quad j \geq 1; \\ \frac{l}{2} + (l+2)\left(\frac{i-1}{2}\right) - \frac{j-2}{2}; & i \equiv 1(\bmod 2), \quad i = 1, 3; \quad j \equiv 0(\bmod 2), \quad j \geq 2. \end{cases}$$

Now consider the labeling $g : V(G) \rightarrow \{1, 2, \dots, p\}$.

For $1 \leq k \leq \varpi + 1$,

$$g(y_k) = \begin{cases} \frac{3(l+2)}{2}; & k = 1; \\ 3(l+1) + k; & k \neq 1. \end{cases}$$

Let $A = \frac{3(l+2)}{2}$, then

$$f(x_{i,j}) = \begin{cases} A + \frac{l}{2} + (l+1)\left(\frac{i-1}{2}\right) - \frac{j-1}{2}; & i \equiv 1(\bmod 2), \quad i = 1, 3; \quad j \equiv 1(\bmod 2), \quad j \geq 1; \\ A + \frac{l}{2} + \frac{j-2}{2} + 1; & i \equiv 0(\bmod 2), \quad i = 2; \quad j \equiv 0(\bmod 2), \quad j \geq 2, \end{cases}$$

$$f(x_{2,0}) = g(x_{2,0}) = 3(l+1) + \varpi + 2,$$

$$g(x_{i,j}) = f(x_{i,j}); \quad 1 \leq i \leq 3, 1 \leq j \leq \varepsilon, l \leq \varepsilon \leq l+2.$$

From the above labeling “ f ”, we obtain consecutive Z_+ from $\hbar' + 1$ to $\hbar' + q$, where $\hbar' = \frac{3(l+2)}{2} + 1$. Thus, Lemma 2.2 gives the required result with magic constant $a = p + q + \hbar' + 1$, where $\hbar' + 1 = \min(S)$.

(b) Let $\mathcal{U} \cong \text{Fr}(l, l+1, l+2) \cup K_{1,\varpi-1} \cup K_1$

$$V(\mathcal{U}) = V(\text{Fr}(l, l+1, l+2)) \cup V(K_{1,\varpi-1}) \cup \{z\}.$$

Let $p' = |V(\mathcal{U})|$ and $q' = |E(\mathcal{U})|$, then

$$p' = 3l + \varpi + 5$$

and

$$q' = 3l + \varpi + 2.$$

For $l \equiv 1(\bmod 2)$:

Keeping in mind the valuation f defined in (a), we describe the labeling $g' : V(\mathcal{U}) \rightarrow \{1, 2, \dots, p'\}$ as

$$f(x_{i,j}) = g(x_{i,j}) = g'(x_{i,j}); \quad 1 \leq i \leq 3, 1 \leq j \leq \varepsilon, l \leq \varepsilon \leq l+2$$

with $A = g(y_1) = g'(y_1) = \frac{3(l+1)}{2}$,

$$g'(y_k) = g(y_k); \quad 1 \leq k \leq \varpi,$$

$$g'(z) = 3(l+1) + \varpi + 1,$$

$$g'(x_{2,0}) = g(x_{2,0}) = f(x_{2,0}) = 3(l+1) + \varpi + 2.$$

From the above labeling “ g' ”, we obtain consecutive Z_+ from $\hbar + 1$ to $\hbar + q$, where $\hbar = \frac{3(l+1)}{2} + 1$. Thus, Lemma 2.2 gives the required result with magic constant $a = p' + q' + \hbar + 1$, where $\hbar + 1 = \min(S)$.

For $l \equiv 0(\text{mod } 2)$:

Keeping in mind the valuation f defined in (a), we describe the labeling $g' : V(\mathcal{U}) \rightarrow \{1, 2, \dots, p'\}$ as

$$f(x_{i,j}) = g(x_{i,j}) = g'(x_{i,j}); \quad 1 \leq i \leq 3, 1 \leq j \leq \varepsilon, l \leq \varepsilon \leq l+2$$

with $A = g(y_1) = g'(y_1) = \frac{3(l+2)}{2}$,

$$g'(y_k) = g(y_k); \quad 1 \leq k \leq \varpi,$$

$$g'(z) = 3(l+1) + \varpi + 1,$$

$$g'(x_{2,0}) = g(x_{2,0}) = f(x_{2,0}) = 3(l+1) + \varpi + 2.$$

From the above labeling “ g' ”, we obtain consecutive Z_+ from $\hbar' + 1$ to $\hbar' + q'$, where $\hbar' = \frac{3(l+2)}{2} + 1$. Thus, Lemma 2.2 gives the required result with magic constant $a = p' + q' + \hbar' + 1$, where $\hbar' + 1 = \min(S)$.

Theorem 2.7. For $l \geq 1$,

(a) $\text{Fr}(l, l+1, l+2) \cup \text{BS}(\zeta, \xi)$ is SEMT,

(b) $\mu_s(\text{Fr}(l, l+1, l+2) \cup \text{BS}(\zeta, \xi-1)) \leq 1, \quad l \neq 1,$

where $\zeta \geq 0$ and

$$\xi = \begin{cases} \frac{l-1}{2}; & l \equiv 1(\text{mod } 2); \\ l; & l \equiv 0(\text{mod } 2). \end{cases}$$

Proof.

(a) Consider the graph $G \cong \text{Fr}(l, l+1, l+2) \cup \text{BS}(\zeta, \xi)$.

Let $p = |V(G)|$ and $q = |E(G)|$, then

$$p = 3l + \zeta + \xi + 6,$$

$$q = 3l + \zeta + \xi + 4.$$

For $l \equiv 1(\text{mod } 2)$:

Keeping in mind the valuation f defined in Theorem 2.6 with $A = \frac{3l+1}{2} + \zeta + 1$, we describe the labeling $g : V(G) \rightarrow \{1, 2, \dots, p\}$ as

$$g(z_{bt}) = \begin{cases} \frac{3l+1}{2} + t; & b = 1, \quad 1 \leq t \leq \zeta; \\ \frac{3l+1}{2} + \zeta + 1; & b = 2, \quad t = 0; \\ 3(l+1) + \zeta + 2; & b = 1, \quad t = 0; \\ 3(l+1) + \zeta + t + 2; & b = 2, \quad 1 \leq t \leq \xi, \end{cases}$$

$$f(x_{i,j}) = g(x_{i,j}); \quad 1 \leq i \leq 3, 1 \leq j \leq \varepsilon, l \leq \varepsilon \leq l+2,$$

$$f(x_{2,0}) = g(x_{2,0}) = 3(l+1) + \zeta + \xi + 3.$$

From the above labeling “ g ”, we obtain consecutive Z_+ from $\hbar + 1$ to $\hbar + q$, where $\hbar = \frac{3l+1}{2} + \zeta + 2$. Thus, Lemma 2.2 gives the required result with magic constant $a = p + q + \hbar + 1$, where $\hbar + 1 = \min(S)$.

For $l \equiv 0(\text{mod } 2)$:

Keeping in mind the valuation f defined in Theorem 2.6 with $A = \frac{3l+4}{2} + \zeta + 1$, we describe the labeling $g : V(G) \rightarrow \{1, 2, \dots, p\}$ as

$$g(z_{bt}) = \begin{cases} \frac{3l+4}{2} + t; & b = 1, \quad 1 \leq t \leq \zeta; \\ \frac{3l+4}{2} + \zeta + 1; & b = 2, \quad t = 0; \\ 3(l+1) + \zeta + 2; & b = 1, \quad t = 0; \\ 3(l+1) + \zeta + t + 2; & b = 2, \quad 1 \leq t \leq \xi, \end{cases}$$

$$f(x_{i,j}) = g(x_{i,j}); \quad 1 \leq i \leq 3, 1 \leq j \leq \varepsilon, l \leq \varepsilon \leq l+2,$$

$$f(x_{2,0}) = g(x_{2,0}) = 3(l+1) + \zeta + \xi + 3.$$

From the above labeling “ g ”, we obtain consecutive Z_+ from $\hbar' + 1$ to $\hbar' + q$, where $\hbar' = \frac{3l+4}{2} + \zeta + 2$. Thus, Lemma 2.2 gives the required result with magic constant $a = p + q + \hbar' + 1$, where $\hbar' + 1 = \min(S)$.

(b) Let $\mathcal{U} \cong \text{Fr}(l, l+1, l+2) \cup \text{BS}(\zeta, \xi-1) \cup K_1$.

Here,

$$V(\mathcal{U}) = V(\text{Fr}(l, l+1, l+2)) \cup V(\text{BS}(\zeta, \xi-1)) \cup \{z\}.$$

Let $p' = |V(\mathcal{U})|$ and $q' = |E(\mathcal{U})|$, then

$$p' = 3l + \zeta + \xi + 6$$

and

$$q' = 3l + \zeta + \xi + 3.$$

For $l \equiv 1(\text{mod } 2)$:

$$f(x_{i,j}) = g(x_{i,j}) = g'(x_{i,j}); \quad 1 \leq i \leq 3, 1 \leq j \leq \varepsilon, l \leq \varepsilon \leq l+2$$

with $A = g(z_{20}) = g'(z_{20}) = \frac{3l+1}{2} + \zeta + 1$,

$$g'(z_{1t}) = g(z_{1t}); \quad 0 \leq t \leq \zeta,$$

$$g'(z_{2t}) = g(z_{2t}); \quad 0 \leq t \leq \xi-1,$$

$$g'(z) = 3(l+1) + \zeta + \xi + 2,$$

$$g'(x_{2,0}) = g(x_{2,0}) = f(x_{2,0}) = 3(l+1) + \zeta + \xi + 3.$$

From the above labeling “ g ”, we obtain consecutive Z_+ from $\hbar + 1$ to $\hbar + q'$, where $\hbar = \frac{3l+1}{2} + \zeta + 2$. Thus, Lemma 2.2 gives the required result with magic constant $a = p' + q' + \hbar + 1$, where $\hbar + 1 = \min(S)$.

For $l \equiv 0(\text{mod } 2)$:

$$f(x_{i,j}) = g(x_{i,j}) = g'(x_{i,j}); \quad 1 \leq i \leq 3, 1 \leq j \leq \varepsilon, l \leq \varepsilon \leq l+2$$

with $A = g(z_{20}) = g'(z_{20}) = \frac{3l+4}{2} + \zeta + 1$,

$$g'(z_{1t}) = g(z_{1t}); \quad 0 \leq t \leq \zeta,$$

$$g'(z_{2t}) = g(z_{2t}); \quad 0 \leq t \leq \xi-1,$$

$$g'(z) = 3(l+1) + \zeta + \xi + 2,$$

$$g'(x_{2,0}) = g(x_{2,0}) = f(x_{2,0}) = 3(l+1) + \zeta + \xi + 3.$$

From the above labeling “g”, we obtain consecutive Z_+ from $h' + 1$ to $h' + q'$, where $h' = \frac{3l+4}{2} + \zeta + 2$. Thus, Lemma 2.2 gives the required result with magic constant $a = p' + q' + h' + 1$, where $h' + 1 = \min(S)$. $\square$

In the following theorems, we present the two distinct SEMT labelings – which are non-dual of each other – for the same forest be composed of disjoint union of path P_m and imbalanced fork.

Theorem 2.8. For $l \geq 1$,

- (a)(i): $\text{Fr}(l, l+1, l+2) \cup P_r$ is SEMT,
- (a)(ii): $\text{Fr}(l, l+1, l+2) \cup P_{r-1}$ is SEMT,
- (b)(i): $\mu_s(\text{Fr}(l, l+1, l+2) \cup P_{r-2}) \leq 1$,
- (b)(ii): $\mu_s(\text{Fr}(l, l+1, l+2) \cup P_{r-3}) \leq 1$,

where

$$r = \begin{cases} l+2; & l \equiv 1(\text{mod } 2); \\ 2l+3; & l \equiv 0(\text{mod } 2). \end{cases}$$

Proof.

(a) Consider the graph $G \cong \text{Fr}(l, l+1, l+2) \cup P_0$, where

$$\varrho = \begin{cases} r; & \text{for } a(i); \\ r-1; & \text{for } a(ii). \end{cases}$$

Let $p = |V(G)|$ and $q = |E(G)|$, so we get

$$p = 3l + \varrho + 4,$$

$$q = 3l + \varrho + 2.$$

For $l \equiv 1(\text{mod } 2)$:

Keeping in mind the valuation f defined in Theorem 2.6 with $A = \left\lceil \frac{3l}{2} \right\rceil + \left\lfloor \frac{\varrho+1}{2} \right\rfloor$, we describe the labeling $g : V(G) \rightarrow \{1, 2, \dots, p\}$ as

$$g(x_t) = \begin{cases} \left\lceil \frac{3l}{2} \right\rceil + k; & t = 2k-1, \quad 1 \leq k \leq \left\lfloor \frac{\varrho+1}{2} \right\rfloor; \\ \frac{9(l+1)}{2} + k - l; & t = 2k, \quad 1 \leq k \leq \left\lfloor \frac{\varrho}{2} \right\rfloor, \text{ for } a(i); \\ \frac{9(l+1)}{2} + k - l - 1; & t = 2k, \quad 1 \leq k \leq \frac{\varrho}{2}, \text{ for } a(ii), \end{cases}$$

$$g(x_{2,0}) = f(x_{2,0}) = \begin{cases} \frac{9(l+1)}{2} + \left\lfloor \frac{\varrho}{2} \right\rfloor - l + 1; & \text{for } a(i); \\ \frac{9(l+1)}{2} + \frac{\varrho}{2} - l; & \text{for } a(ii), \end{cases}$$

$$g(x_{i,j}) = f(x_{i,j}); \quad 1 \leq i \leq 3, 1 \leq j \leq \varepsilon, l \leq \varepsilon \leq l+2.$$

From the above labeling “g”, we obtain consecutive Z_+ from $h + 1$ to $h + q$, where $h = \left\lceil \frac{3l}{2} \right\rceil + \left\lfloor \frac{\varrho+1}{2} \right\rfloor + 1$. Thus, Lemma 2.2 gives the required result with magic constant $a = p + q + \min(S)$, where $\min(S) = h + 1$.

For $l \equiv 0(\text{mod } 2)$:

Keeping in mind the valuation f defined in Theorem 2.6 with $A = \left\lceil \frac{3(l+1)}{2} \right\rceil + \left\lfloor \frac{\varrho+1}{2} \right\rfloor$, we describe the labeling $g : V(G) \rightarrow \{1, 2, \dots, p\}$ as

$$g(x_t) = \begin{cases} \left\lceil \frac{3(l+1)}{2} \right\rceil + k; & t = 2k - 1, \quad 1 \leq k \leq \left\lfloor \frac{q+1}{2} \right\rfloor; \\ 3(l+1) + \left\lfloor \frac{3(l+1)}{2} \right\rfloor + k - \frac{l}{2} + 1; & t = 2k, \quad 1 \leq k \leq \left\lfloor \frac{q}{2} \right\rfloor, \text{ for } a(i); \\ 3(l+1) + \left\lfloor \frac{3(l+1)}{2} \right\rfloor + k - \frac{l}{2}; & t = 2k, \quad 1 \leq k \leq \frac{q}{2}, \text{ for } a(ii), \end{cases}$$

$$g(x_{2,0}) = f(x_{2,0}) = \begin{cases} 3(l+1) + \left\lfloor \frac{3(l+1)}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor - \frac{l}{2} + 2; & \text{for } a(i); \\ 3(l+1) + \left\lfloor \frac{3(l+1)}{2} \right\rfloor + \frac{q}{2} - \frac{l}{2} + 1; & \text{for } a(ii), \end{cases}$$

$$g(x_{i,j}) = f(x_{i,j}); \quad 1 \leq i \leq 3, 1 \leq j \leq \varepsilon, l \leq \varepsilon \leq l+2.$$

From the above labeling “ g ”, we obtain consecutive Z_+ from $h' + 1$ to $h' + q$, where $h' = \left\lceil \frac{3(l+1)}{2} \right\rceil + \left\lfloor \frac{q+1}{2} \right\rfloor + 1$. Thus, Lemma 2.2 gives the required result with magic constant $a = p + q + \min(S)$, where $\min(S) = h' + 1$.

(a) Let $\mathcal{U} \cong \text{Fr}(l, l+1, l+2) \cup P_q \cup K_l$, where

$$V(\mathcal{U}) = V(\text{Fr}(l, l+1, l+2)) \cup V(P_q) \cup \{z\}.$$

Let $p' = |V(\mathcal{U})|$ and $q' = |E(\mathcal{U})|$, so we get

$$p' = 3l + q + 5,$$

$$q' = 3l + q + 2,$$

where

$$q = \begin{cases} r - 2; & \text{for } b(i); \\ r - 3; & \text{for } b(ii). \end{cases}$$

For $l \equiv 1 \pmod{2}$:

Keeping in mind the valuation f defined in Theorem 2.6, we describe the labeling $g' : V(\mathcal{U}) \rightarrow \{1, 2, \dots, p'\}$ as

$$f(x_{i,j}) = g(x_{i,j}) = g'(x_{i,j}); \quad 1 \leq i \leq 3, 1 \leq j \leq \varepsilon, l \leq \varepsilon \leq l+2, \text{ for both } b(i) \text{ and } b(ii)$$

with $A = \left\lceil \frac{3l}{2} \right\rceil + \left\lfloor \frac{q+1}{2} \right\rfloor$

$$g'(x_t) = g(x_t), \quad t \equiv 1 \pmod{2},$$

$$g'(x_t) = \begin{cases} \frac{9(l+1)}{2} + k - l - 1; & t = 2k, \quad 1 \leq k \leq \frac{q-1}{2}, \text{ for } b(i); \\ \frac{9(l+1)}{2} + k - l - 2; & t = 2k, \quad 1 \leq k \leq \left\lceil \frac{q-1}{2} \right\rceil, \text{ for } b(ii). \end{cases}$$

Let $B = \frac{9(l+1)}{2} + \frac{q-1}{2} - l - 1$ and $C = \frac{9(l+1)}{2} + \left\lceil \frac{q-1}{2} \right\rceil - l - 2$, then

$$g'(z) = \begin{cases} B + 1; & \text{for } b(i); \\ C + 1; & \text{for } b(ii), \end{cases}$$

$$g'(x_{2,0}) = \begin{cases} B + 2; & \text{for } b(i); \\ C + 2; & \text{for } b(ii). \end{cases}$$

From the above labeling “ g' ”, we obtain consecutive Z_+ from $h + 1$ to $h + q'$, where $h = \left\lceil \frac{3l}{2} \right\rceil + \left\lfloor \frac{q+1}{2} \right\rfloor + 1$. Thus, Lemma 2.2 gives the required result with magic constant $a = p' + q' + \min(S)$, where $\min(S) = h + 1$.

For $l \equiv 0(\text{mod } 2)$:

Keeping in mind the valuation f defined in Theorem 2.6, we describe the labeling $g' : V(\mathcal{U}) \rightarrow \{1, 2, \dots, p'\}$ as

$$f(x_{i,j}) = g(x_{i,j}) = g'(x_{i,j}); \quad 1 \leq i \leq 3, 1 \leq j \leq \varepsilon, l \leq \varepsilon \leq l+2, \quad \text{for both } b(i) \text{ and } b(ii)$$

$$\text{with } A = \left\lceil \frac{3(l+1)}{2} \right\rceil + \left\lfloor \frac{\varrho+1}{2} \right\rfloor,$$

$$g'(x_t) = g(x_t), \quad t \equiv 1(\text{mod } 2),$$

$$g'(x_t) = \begin{cases} 3(l+1) + \left\lfloor \frac{3(l+1)}{2} \right\rfloor + k - \frac{l}{2}; & t = 2k, \quad 1 \leq k \leq \frac{\varrho-1}{2}, \text{ for } b(i); \\ 3(l+1) + \left\lfloor \frac{3(l+1)}{2} \right\rfloor + k - \frac{l}{2} - 1; & t = 2k, \quad 1 \leq k \leq \left\lceil \frac{\varrho-1}{2} \right\rceil, \text{ for } b(ii). \end{cases}$$

Let $B = 3(l+1) + \left\lfloor \frac{3(l+1)}{2} \right\rfloor + \frac{\varrho-1}{2} - \frac{l}{2}$ and $C = 3(l+1) + \left\lfloor \frac{3(l+1)}{2} \right\rfloor + \left\lceil \frac{\varrho-1}{2} \right\rceil - \frac{l}{2} - 1$, then

$$g'(z) = \begin{cases} B+1; & \text{for } b(i); \\ C+1; & \text{for } b(ii), \end{cases}$$

$$g'(x_{2,0}) = \begin{cases} B+2; & \text{for } b(i); \\ C+2; & \text{for } b(ii). \end{cases}$$

From the above labeling “ g ”, we obtain consecutive Z_+ from $h' + 1$ to $h' + q'$, where $h' = \left\lceil \frac{3(l+1)}{2} \right\rceil + \left\lfloor \frac{\varrho+1}{2} \right\rfloor + 1$. Thus, Lemma 2.2 gives the required result with magic constant $a = p' + q' + \min(S)$, where $\min(S) = h' + 1$. $\square$

Theorem 2.9. For $l \geq 1$,

(a)(i): $\text{Fr}(l, l+1, l+2) \cup P_r$ is SEMT,

(a)(ii): $\text{Fr}(l, l+1, l+2) \cup P_{r-1}$ is SEMT, $l \neq 1$,

(b)(i): $\mu_s(\text{Fr}(l, l+1, l+2) \cup P_{r-2}) \leq 1$,

(b)(ii): $\mu_s(\text{Fr}(l, l+1, l+2) \cup P_{r-3}) \leq 1, \quad l \neq 1$,

where

$$r = \begin{cases} l+1; & l \equiv 1(\text{mod } 2); \\ 2(l+1); & l \equiv 0(\text{mod } 2). \end{cases}$$

Proof.

(a) Consider the graph $G \cong \text{Fr}(l, l+1, l+2) \cup P_r$, where

$$\varrho = \begin{cases} r; & \text{for } a(i); \\ r-1; & \text{for } a(ii). \end{cases}$$

Let $p = |V(G)|$ and $q = |E(G)|$, so we get

$$p = 3l + \varrho + 4,$$

$$q = 3l + \varrho + 2.$$

For $l \equiv 1(\text{mod } 2)$:

Keeping in mind the valuation f defined in Theorem 2.6 with $A = \left\lceil \frac{3l}{2} \right\rceil + \left\lceil \frac{\varrho-1}{2} \right\rceil$, we describe the labeling $g : V(G) \rightarrow \{1, 2, \dots, p\}$ as

$$g(x_t) = \begin{cases} \left\lceil \frac{3l}{2} \right\rceil + k; & t = 2k, \quad 1 \leq k \leq \left\lceil \frac{q-1}{2} \right\rceil; \\ \frac{9(l+1)}{2} - l + k - 1; & t = 2k - 1, \quad 1 \leq k \leq \frac{q}{2}, \text{ for } a(i); \\ \frac{9(l+1)}{2} - l + k - 2; & t = 2k - 1, \quad 1 \leq k \leq \left\lceil \frac{q}{2} \right\rceil, \text{ for } a(ii), \end{cases}$$

$$g(x_{2,0}) = f(x_{2,0}) = \begin{cases} \frac{9(l+1)}{2} - l + \frac{q}{2}; & \text{for } a(i); \\ \frac{9(l+1)}{2} - l + \left\lceil \frac{q}{2} \right\rceil - 1; & \text{for } a(ii), \end{cases}$$

$$g(x_{i,j}) = f(x_{i,j}); \quad 1 \leq i \leq 3, 1 \leq j \leq \varepsilon, l \leq \varepsilon \leq l + 2, \text{ for both } a(i) \text{ and } a(ii).$$

From the above labeling “ g ”, we obtain consecutive Z_+ from $h + 1$ to $h + q$, where $h = \left\lceil \frac{3l}{2} \right\rceil + \left\lceil \frac{q-1}{2} \right\rceil + 1$. Thus, Lemma 2.2 gives the required result with magic constant $a = p + q + \min(S)$, where $\min(S) = h + 1$.

For $l \equiv 0 \pmod{2}$:

Keeping in mind the valuation f defined in Theorem 2.6 with $A = \left\lceil \frac{3(l+1)}{2} \right\rceil + \left\lceil \frac{q-1}{2} \right\rceil$, we describe the labeling $g : V(G) \rightarrow \{1, 2, \dots, p\}$ as

$$g(x_t) = \begin{cases} \left\lceil \frac{3(l+1)}{2} \right\rceil + k; & t = 2k, \quad 1 \leq k \leq \left\lceil \frac{q-1}{2} \right\rceil; \\ 3(l+1) + \left\lfloor \frac{3(l+1)}{2} \right\rfloor + k - \frac{l}{2}; & t = 2k - 1, \quad 1 \leq k \leq \frac{q}{2}, \text{ for } a(i); \\ 3(l+1) + \left\lfloor \frac{3(l+1)}{2} \right\rfloor + k - \frac{l}{2} - 1; & t = 2k - 1, \quad 1 \leq k \leq \left\lceil \frac{q}{2} \right\rceil, \text{ for } a(ii), \end{cases}$$

$$g(x_{2,0}) = f(x_{2,0}) = \begin{cases} 3(l+1) + \left\lfloor \frac{3(l+1)}{2} \right\rfloor + \frac{q}{2} - \frac{l}{2} + 1; & \text{for } a(i); \\ 3(l+1) + \left\lfloor \frac{3(l+1)}{2} \right\rfloor + \left\lceil \frac{q}{2} \right\rceil - \frac{l}{2}; & \text{for } a(ii), \end{cases}$$

$$g(x_{i,j}) = f(x_{i,j}); \quad 1 \leq i \leq 3, 1 \leq j \leq \varepsilon, l \leq \varepsilon \leq l + 2, \text{ for both } a(i) \text{ and } a(ii).$$

From the above labeling “ g ”, we obtain consecutive Z_+ from $h' + 1$ to $h' + q$, where $h' = \left\lceil \frac{3(l+1)}{2} \right\rceil + \left\lceil \frac{q-1}{2} \right\rceil + 1$. Thus, Lemma 2.2 gives the required result with magic constant $a = p + q + \min(S)$, where $\min(S) = h' + 1$.

(b) Let $\mathcal{U} \cong \text{Fr}(l, l + 1, l + 2) \cup P_q \cup K_1$, where

$$V(\mathcal{U}) = V(\text{Fr}(l, l + 1, l + 2)) \cup V(P_q) \cup \{z\},$$

where

$$q = \begin{cases} r - 2; & \text{for } b(i); \\ r - 3; & \text{for } b(ii). \end{cases}$$

For $l \equiv 1 \pmod{2}$:

Keeping in mind the valuation f defined in Theorem 2.6, we describe the labeling $g' : V(\mathcal{U}) \rightarrow \{1, 2, \dots, p'\}$ as

$$f(x_{i,j}) = g(x_{i,j}) = g'(x_{i,j}); \quad 1 \leq i \leq 3, 1 \leq j \leq \varepsilon, l \leq \varepsilon \leq l + 2, \text{ for both } b(i) \text{ and } b(ii)$$

with $A = \left\lceil \frac{3l}{2} \right\rceil + \left\lceil \frac{q-1}{2} \right\rceil$,

$$g'(x_t) = g(x_t), \quad t \equiv 0 \pmod{2},$$

$$g'(x_t) = \begin{cases} \frac{9(l+1)}{2} + k - l - 2; & t = 2k - 1, \quad 1 \leq k \leq \left\lfloor \frac{\varrho+1}{2} \right\rfloor, \text{ for } b(i); \\ \frac{9(l+1)}{2} + k - l - 3; & t = 2k - 1, \quad 1 \leq k \leq \frac{\varrho+1}{2}, \text{ for } b(ii). \end{cases}$$

Let $B = \frac{9(l+1)}{2} + \left\lfloor \frac{\varrho+1}{2} \right\rfloor - l - 2$ and $C = \frac{9(l+1)}{2} + \frac{\varrho+1}{2} - l - 3$, then

$$g'(z) = \begin{cases} B + 1; & \text{for } b(i); \\ C + 1; & \text{for } b(ii), \end{cases}$$

$$g'(x_{2,0}) = \begin{cases} B + 2; & \text{for } b(i); \\ C + 2; & \text{for } b(ii). \end{cases}$$

From the above labeling “ g' ”, we obtain consecutive Z_+ from $\hbar + 1$ to $\hbar + q'$, where $\hbar = \left\lceil \frac{3l}{2} \right\rceil + \left\lceil \frac{\varrho-1}{2} \right\rceil + 1$. Thus, Lemma 2.2 gives the required result with magic constant $a = p' + q' + \min(S)$, where $\min(S) = \hbar + 1$.

For $l \equiv 0 \pmod{2}$:

Keeping in mind the valuation f defined in Theorem 2.6, we describe the labeling $g' : V(\mathcal{U}) \rightarrow \{1, 2, \dots, p'\}$ as

$$f(x_{i,j}) = g(x_{i,j}) = g'(x_{i,j}); \quad 1 \leq i \leq 3, 1 \leq j \leq \varepsilon, l \leq \varepsilon \leq l + 2, \text{ for both } b(i) \text{ and } b(ii)$$

with $A = \left\lceil \frac{3(l+1)}{2} \right\rceil + \left\lceil \frac{\varrho-1}{2} \right\rceil$

$$g'(x_t) = g(x_t), \quad t \equiv 0 \pmod{2},$$

$$g'(x_t) = \begin{cases} 3(l+1) + \left\lfloor \frac{3(l+1)}{2} \right\rfloor + k - \frac{l}{2} - 1; & t = 2k - 1, \quad 1 \leq k \leq \left\lfloor \frac{\varrho+1}{2} \right\rfloor, \text{ for } b(i); \\ 3(l+1) + \left\lfloor \frac{3(l+1)}{2} \right\rfloor + k - \frac{l}{2} - 2; & t = 2k - 1, \quad 1 \leq k \leq \frac{\varrho+1}{2}, \text{ for } b(ii). \end{cases}$$

Let $B = 3(l+1) + \left\lfloor \frac{3(l+1)}{2} \right\rfloor + \left\lfloor \frac{\varrho+1}{2} \right\rfloor - \frac{l}{2} - 1$ and $C = 3(l+1) + \left\lfloor \frac{3(l+1)}{2} \right\rfloor + \frac{\varrho+1}{2} - \frac{l}{2} - 2$, then

$$g'(z) = \begin{cases} B + 1; & \text{for } b(i); \\ C + 1; & \text{for } b(ii), \end{cases}$$

$$g'(x_{2,0}) = \begin{cases} B + 2; & \text{for } b(i); \\ C + 2; & \text{for } b(ii). \end{cases}$$

From the above labeling “ g' ,” we obtain consecutive Z_+ from $\hbar' + 1$ to $\hbar' + q'$, where $\hbar' = \left\lceil \frac{3(l+1)}{2} \right\rceil + \left\lceil \frac{\varrho-1}{2} \right\rceil + 1$. Thus, Lemma 2.2 gives the required result with magic constant $a = p' + q' + \min(S)$, where $\min(S) = \hbar' + 1$. $\square$

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