Research Article

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Quasi-ideal Ehresmann transversals: The spined product structure

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Abstract: In any U -abundant semigroup with an Ehresmann transversal, two significant components R and L are introduced in this paper and described by Green's $\sim$ -relations. Some interesting properties associated with R and L are explored and some equivalent conditions for the Ehresmann transversal to be a quasi-ideal are acquired. Finally, a spined product structure theorem is established for a U -abundant semigroup with a quasi-ideal Ehresmann transversal by means of R and L .

Keywords: U -abundant semigroup, Ehresmann transversal, Green's $\sim$ -relations, quasi-ideal, spined product

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1 Introduction

Suppose that S is a regular semigroup and S^o a subsemigroup of S . We denote the intersection of $V(a)$ and S^o by $V_{S^o}(a)$ and that $I = \{aa^o : a \in S, a^o \in V_{S^o}(a)\}$ and $\Lambda = \{a^oa : a \in S, a^o \in V_{S^o}(a)\}$. An *inverse transversal* of the semigroup S is a subsemigroup S^o that contains exactly one inverse of every element of S , that is, S is regular and S^o inverse with $|V_{S^o}(a)| = 1$. This important concept was introduced by Blyth and McFadden [1]. Thereafter, this class of regular semigroups excited many semigroup researchers' attention and a good deal of important results were obtained (see [1–4] and references therein). Tang [4] has shown that for S a regular semigroup with an inverse transversal S^o , I and Λ are both bands with I left regular and Λ right regular. These two bands play an important role in the study of regular semigroups with inverse transversals. Other important subsets of S are $R = \{x \in S : x^ox = x^ox^oo\}$ and $L = \{x \in S : xx^o = x^oox^o\}$. They are subsemigroups with R and L left and right inverse, respectively.

The concept of an *adequate transversal* was introduced in the class of abundant semigroups by El-Qallali [5] as the generalisation of the concept of an inverse transversal. Chen, Guo and Shum [6,7] obtained some important results about a quasi-ideal adequate transversal. Kong [8] explored some properties about adequate transversals. Kong and Wang [9] considered the product of quasi-ideal adequate transversals. The concept of adequate transversals was generalised, and the product of quasi-ideal adequate transversals was generalised to the refined quasi-adequate transversals and quasi-Ehresmann transversals [10–14]. In 2008, Kong [15] introduced two important subsets R and L in the adequate transversal case and described them by Green's $*$ -relations. Furthermore, Kong established a spined product structure theorem for an abundant semigroup with a quasi-ideal adequate transversal by means of R and L . It was interesting that this spined product structure theorem was independently

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reobtained by Al-Bar and Renshaw [16] and later, they [17] noticed the fact that this structure theorem had been already obtained by Kong.

To extend the class of regular semigroups and the class of abundant semigroups, a new relation $\widetilde{\mathcal{L}}^U$ on a semigroup S is introduced in the following way. Let $E(S)$ be the set of all idempotents of a semigroup S and suppose that U is a non-empty subset of $E(S)$. Then the relation $\widetilde{\mathcal{L}}^U$ on S is defined as follows:

$$\widetilde{\mathcal{L}}^U = \{(a, b) \in S \times S \mid (\forall e \in U) ae = a \Leftrightarrow be = b\}.$$

It is easy to see that $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}^U$. In particular, if S is an abundant semigroup with $U = E(S)$, then $\widetilde{\mathcal{L}}^U = \mathcal{L}^*$. Furthermore, if S is a regular semigroup with $U = E(S)$, then $\mathcal{L} = \mathcal{L}^* = \widetilde{\mathcal{L}}^U$. Dually, a relation $\widetilde{\mathcal{R}}^U$ is defined, and for any results concerning $\widetilde{\mathcal{L}}^U$, we also have the dual results for $\widetilde{\mathcal{R}}^U$. Recall that a semigroup S is a weakly U -abundant semigroup if each $\widetilde{\mathcal{L}}^U$ -class and each $\widetilde{\mathcal{R}}^U$ -class of S contain an idempotent from U , and we denote such a weakly U -abundant semigroup by (S, U) . In this case, we call the set U a set of distinguished idempotents of (S, U) , and an element u of U is called a distinguished idempotent of (S, U) . It is well known that the relations $\mathcal{L}$ and $\mathcal{L}^*$ are always right congruences on a semigroup S , but the same need not be true for $\widetilde{\mathcal{L}}^U$. A weakly U -abundant semigroup (S, U) is said to satisfy the congruence condition (C) if $\widetilde{\mathcal{L}}^U$ and $\widetilde{\mathcal{R}}^U$ are a right congruence and a left congruence on (S, U) , respectively. We call a weakly U -abundant semigroup having the congruence condition (C) a U -abundant semigroup. Clearly, regular semigroups and abundant semigroups are all U -abundant semigroups with $U = E(S)$. A U -abundant semigroup (S, U) is called an Ehresmann semigroup (quasi-Ehresmann semigroup) if U is a semilattice (band). The class of Ehresmann semigroups, named by Lawson [18], is an important subclass of the class of U -abundant semigroups, which are generalizations of inverse semigroups in the class of regular semigroups and adequate semigroups in the class of abundant semigroups.

The concept of an *Ehresmann transversal* was introduced almost simultaneously in three papers by Ma, Ren, Gong [19], Wang [20] and Yang [21] in the class of U -abundant semigroups as the generalisation of the concept of an *inverse transversal* and an *adequate transversal*. The aforementioned three kinds of Ehresmann transversals were basically the same, with only slight difference. To achieve interesting properties of Ehresmann transversals which parallel to those of adequate transversals, we give the following definition of Ehresmann transversals combining the aforementioned three kinds of Ehresmann transversals. In [19–21], some basic properties about Ehresmann transversals are given.

In the present paper, we continue along the lines of [3,15,16] by exploring the properties associated with R and L of Ehresmann transversals of U -abundant semigroups. The main result of this paper is to establish a spined product structure theorem for a U -abundant semigroup with a quasi-ideal Ehresmann transversal by means of R and L . The related results concerning adequate transversals are generalised and enriched.

2 Preliminaries

Throughout this paper, a U -abundant semigroup is always denoted as (S, U) and the $\widetilde{\mathcal{L}}^U$ -class and the $\widetilde{\mathcal{R}}^U$ -class of (S, U) containing the element a as $\widetilde{\mathcal{L}}_a^U$ and $\widetilde{\mathcal{R}}_a^U$, respectively. Moreover, for any $a \in (S, U)$ the distinguished idempotents in $\widetilde{\mathcal{L}}_a^U \cap U$ and $\widetilde{\mathcal{R}}_a^U \cap U$ are denoted by a^* and a^+ , respectively. In particular, when (S, U) is an Ehresmann semigroup, then there exists a unique distinguished idempotent in $\widetilde{\mathcal{L}}_a^U \cap U$ and a unique distinguished idempotent in $\widetilde{\mathcal{R}}_a^U \cap U$. Therefore, as for Ehresmann semigroups, for all $a, b \in (S, U)$, $a\widetilde{\mathcal{R}}^U b$ if and only if $a^+ = b^+$; $a\widetilde{\mathcal{L}}^U b$ if and only if $a^* = b^*$, and consequently $(ab)^* = (a^*b)^*$ and $(ab)^+ = (ab^+)^+$. We remark here that an excellent survey of investigations of restriction semigroups and Ehresmann semigroups was given by Gould [22] and the class of Ehresmann monoids were deeply investigated by Branco, Gomes and Gould [23,24].

Lemma 2.1. Suppose that S is a semigroup and U a non-empty subset of $E(S)$. Let $\text{Reg}_U S = \{a \in S \mid (\exists a' \in V(a)) aa', a'a \in U\}$. Then

- (i) $\widetilde{\mathcal{L}}^U \cap (\text{Reg}_U S \times \text{Reg}_U S) = \mathcal{L} \cap (\text{Reg}_U S \times \text{Reg}_U S)$.
- (ii) $\widetilde{\mathcal{R}}^U \cap (\text{Reg}_U S \times \text{Reg}_U S) = \mathcal{R} \cap (\text{Reg}_U S \times \text{Reg}_U S)$.

If $V^U(a) = \{a' \in V(a) \mid aa', a'a \in U\}$, then $a \in \text{Reg}_U S$ if and only if $V^U(a) \neq \emptyset$. The following result is immediate from the definition of $\widetilde{\mathcal{L}}^U$, and there is a dual result for $\widetilde{\mathcal{R}}^U$.

Lemma 2.2. Let a be an element of a weakly U -abundant semigroup (S, U) and e a distinguished idempotent of (S, U) . Then the following statements are equivalent:

- (i) $e\widetilde{\mathcal{L}}^U a$.
- (ii) $ae = a$, and for all $f \in U$, $af = a$ implies $ef = e$.

Let (S, U) be a U -abundant semigroup and (T, V) a U -abundant subsemigroup of (S, U) , that is, $T \subseteq S$ and $V = U \cap T$. Then the U -abundant subsemigroup (T, V) is called a $\sim$ -subsemigroup of (S, U) if for any $a \in T$, there exist $e \in \widetilde{\mathcal{L}}_a^U \cap V$ and $f \in \widetilde{\mathcal{R}}_a^U \cap V$, or equivalently, if $\widetilde{\mathcal{L}}^V(T) = \widetilde{\mathcal{L}}^U(S) \cap (T \times T)$ and $\widetilde{\mathcal{R}}^V(T) = \widetilde{\mathcal{R}}^U(S) \cap (T \times T)$.

Definition 2.3. An Ehresmann $\sim$ -subsemigroup (T, V) of a U -abundant semigroup (S, U) is called an *Ehresmann transversal* of (S, U) if U forms an order ideal of $E(S)$, and for any $x \in S$, there exist $e, f \in U$ and a unique $\bar{x} \in T$ such that $x = e\bar{x}f$, where $e\mathcal{L}\bar{x}^+$ and $f\mathcal{R}\bar{x}^*$. It can easily be shown that e and f are uniquely determined by x and T (see [15]). Hence, we denote e by e_x and f by f_x and we have $e_x\widetilde{\mathcal{R}}^U x, x\widetilde{\mathcal{L}}^U f_x$.

A subsemigroup (T, V) of a semigroup (S, U) is called a *quasi-ideal* if $TST \subseteq T$, and the corresponding Ehresmann transversal is called a quasi-ideal Ehresmann transversal. Let (T, V) be an Ehresmann transversal of the U -abundant semigroup (S, U) . Let

$$I = \{e_x \in U \mid x \in (S, U)\}, \quad \Lambda = \{f_x \in U \mid x \in (S, U)\}.$$

Lemma 2.4. Let (S, U) be a U -abundant semigroup with an Ehresmann transversal (T, V) . Then

- (1) $I \cap \Lambda = V$;
- (2) $I = \{e \in U \mid (\exists v \in V) v\mathcal{L}e\}$, $\Lambda = \{f \in U \mid (\exists h \in V) h\mathcal{R}f\}$;
- (3) $IV \subseteq I, V\Lambda \subseteq \Lambda$;
- (4) $x\widetilde{\mathcal{R}}^U y$ if and only if $e_x = e_y$; $x\widetilde{\mathcal{L}}^U y$ if and only if $f_x = f_y$;
- (5) If $e, f \in I$ and $e\mathcal{R}f$, then $e = f$; if $e, f \in \Lambda$ and $e\mathcal{L}f$, then $e = f$;
- (6) $(\forall x \in \text{Reg}_U S) |V^U(x) \cap T| \leq 1$;
- (7) If T is a quasi-ideal of S , then $(\forall x \in \text{Reg}_U S) |V^U(x) \cap T| = 1$.

Similarly to the adequate transversal case, one can easily show the following decomposition formula. It is worth remarking that if the condition “quasi-ideal” is strengthened as “multiplicative,” the proof was obtained in [15].

Lemma 2.5. Let (S, U) be a U -abundant semigroup with a quasi-ideal Ehresmann transversal (T, V) . For any $x, y \in S$, then $\overline{xy} = \bar{x}f_x e_y \bar{y}$; $e_{xy} = e_x(\bar{x}f_x e_y)^+$; $f_{xy} = (f_x e_y \bar{y})^* f_y$.

The so-called Miller-Clifford theorem is crucial and frequently used in this paper.

Lemma 2.6. [25]

- (1) Let e and f be $\mathcal{D}$ -equivalent idempotents of a semigroup S . Then each element a of $R_e \cap L_f$ has a unique inverse a' in $R_f \cap L_e$, such that $aa' = e$ and $a'a = f$.
- (2) Let $a, b \in S$. Then $ab \in R_a \cap L_b$ if and only if $L_a \cap R_b$ contains an idempotent.

3 Some properties of R and L

The objective in this section is to give some properties about R and L . It is known that R and L play an important role in constructing regular semigroups with quasi-ideal inverse transversals and abundant semigroups with quasi-ideal adequate transversals. For any result concerning R , there is a dual result for L which we list but omit its proof.

Proposition 3.1. *Let (S^0, V) be an Ehresmann transversal of a U -abundant semigroup (S, U) . If $a, b \in S^0$ and $c \in S$ are such that $a\bar{R}^U c \bar{L}^U b$, then $c \in S^0$.*

Proof. Since $a, b \in S^0$ and S^0 is an Ehresmann semigroup, there exist $a^+, b^* \in V$, such that

$$a^+ \bar{R}^U a \bar{R}^U c \bar{L}^U b \bar{L}^U b^*.$$

From the definition of an Ehresmann transversal, we have $c = e_c \bar{c} f_c$, where $e_c \bar{L}^U \bar{c}^+$ and $f_c \bar{R}^U \bar{c}^*$. Moreover, $e_c \bar{R}^U c \bar{L}^U f_c$. It follows that $a^+ \bar{R}^U c \bar{R}^U e_c \bar{L}^U \bar{c}^+$, consequently $e_c \in \Lambda \cap I = V$. Similarly, $\bar{c}^* \bar{R}^U f_c \bar{L}^U c \bar{L}^U b^*$ and $f_c \in V$. Thus, $c = e_c \cdot \bar{c} \cdot f_c \in V \cdot S^0 \cdot V \subseteq S^0$. $\square$

Proposition 3.2. *Let (S^0, V) be an Ehresmann transversal of a U -abundant semigroup (S, U) . Then for any $x \in \text{Reg}_U S$, $|V^U(x) \cap S^0| = 1$ and we call $x^0 = V^U(x) \cap S^0$ the unique U -inverse in S^0 of x .*

Proof. For every $x \in \text{Reg}_U S$, since x, e_x and f_x are all U -regular, from $e_x \bar{R}^U x \bar{L}^U f_x$ we deduce that $e_x \bar{R} x \bar{L} f_x$, so by Lemma 2.6 x has an inverse $x' \in R_{f_x} \cap L_{e_x}$ and $xx' = e_x, x'x = f_x$. Thus, $\bar{x}^* \bar{R} f_x \bar{R} x' \bar{L} e_x \bar{L} \bar{x}^+$, so by Proposition 3.1 $x' \in S^0$. Now let $(x')^{-1}$ denote the unique U -inverse of x' in the Ehresmann semigroup S^0 . Then from $x = xx' \cdot (x')^{-1} \cdot x'x$ and Definition 2.3, the uniqueness of $x' \in S^0$ is obvious. That is $|V^U(x) \cap S^0| = 1$. $\square$

Proposition 3.3. *Let (S, U) be a U -abundant semigroup with an Ehresmann transversal (S^0, V) . If $U = E(S)$, then S is regular if and only if S^0 is an inverse semigroup; in this case, S^0 is an inverse transversal of S .*

Proof. Suppose that S is a regular semigroup. Then by $U = E(S)$ and $|V^U(x) \cap S^0| = 1$, every regular element $x \in S$ has a unique inverse in S^0 and, in particular, every element in S^0 has a unique U -inverse in itself. Hence, S^0 is an inverse semigroup and S^0 is an inverse transversal of S .

Conversely, assume that S^0 is an inverse semigroup. Then $V = U \cap E(S^0) = E(S) \cap E(S^0) = E(S^0)$ and for every x in S , there exists $y \in S^0$ such that $\bar{x} = \bar{x}y\bar{x}$. Consequently,

$$x(\bar{x}^*y\bar{x}^+)x = e_x \bar{x} (f_x \bar{x}^*) y (\bar{x}^+ e_x) \bar{x} f_x = e_x \bar{x} y \bar{x} f_x = e_x \bar{x} f_x = x,$$

and so x is regular in S . Thus, S is a regular semigroup. $\square$

Proposition 3.4. *Let (S, U) be a U -abundant semigroup with an Ehresmann transversal (S^0, V) . If S^0 is a right ideal of S , then $f_x \in V$ for every $x \in S$ and $U = I$. Consequently, $f_x = \bar{x}^*$ and thus $x = e_x \bar{x}$.*

Dually, if S^0 is a left ideal of S , then $e_a \in V$ for every $a \in S$ and $U = \Lambda$. Consequently, $e_a = \bar{a}^+$ and thus $a = \bar{a} f_a$.

Proof. By Definition 2.3, for every $x \in S$, $x = e_x \bar{x} f_x$, where $e_x \bar{L} \bar{x}^+$ and $f_x \bar{R} \bar{x}^*$. Since S^0 is a right ideal of S , $f_x = \bar{x}^* f_x \in S^0$ and thus $f_x \in V$. Let $e \in U$ and e^0 denote the unique U -inverse in S^0 of e . Since $e^0 e \in V(e) \cap S^0$, we have $e^0 e = e^0$. Thus, $ee^0 = e$ for every $e \in U$. Consequently, $U = I = \{e_x : x \in S\}$. $\square$

Proposition 3.5. *Suppose that (S, U) is a U -abundant semigroup with an Ehresmann transversal (S^0, V) . Let*

$$R = \{r \in S : f_r \in V\} \quad \text{and} \quad L = \{a \in S : e_a \in V\}.$$

Then

$$R = \{r \in S : (\exists v \in V) r \widetilde{\mathcal{L}}^U v\} \quad \text{and} \quad L = \{a \in S : (\exists h \in V) a \widetilde{\mathcal{R}}^U h\}.$$

Consequently, $R \cap L = S^0$, $U \cap R = I$ and $U \cap L = \Lambda$.

Proof. It is clear that if $r \in R$, there exists $v = f_r \in V$ such that $r \widetilde{\mathcal{L}}^U v$.

Conversely, for $r \in S$, if there exists $v \in V$ such that $r \widetilde{\mathcal{L}}^U v$, then $f_r \widetilde{\mathcal{L}}^U r \widetilde{\mathcal{L}}^U v$. Hence, $f_r \in I$. Therefore, $f_r \in I \cap \Lambda = V$. $\square$

Proposition 3.6. Suppose that (S, U) is a U -abundant semigroup with an Ehresmann transversal (S^0, V) and S^0 is a quasi-ideal of S . Let $R = \{r \in S : f_r \in V\}$ and $L = \{a \in S : e_a \in V\}$. Then (R, I) and (L, Λ) are both U -abundant semigroups with an Ehresmann transversal (S^0, V) , where S^0 is a right ideal of R and a left ideal of L .

Moreover, for $r, s \in R$ and $a, b \in L$, if $\bar{r} = \bar{a}$, $\bar{s} = \bar{b}$ and $e_r a = e_s b$, then $r = s$ and $a = b$.

Proof. Obviously, $S^0 \subseteq R$ and $S^0 \subseteq L$. For any $r, s \in R$, by Lemma 2.5, we have $f_{rs} = (f_r e_s \bar{s})^* f_s = (f_r e_s \bar{s})^* \bar{s}^* \in V$ and so R is a subsemigroup of (S, U) .

For any $r \in R$ and $s^0 \in S^0$, we have $f_r \in V$ and consequently $s^0 r = s^0 r f_r \in S^0$ since S^0 is a quasi-ideal of S . Thus, S^0 is a right ideal of R and by Proposition 3.4, $U(R) = I$. Thus, (S^0, V) is an Ehresmann transversal of (R, I) .

If $\bar{r} = \bar{a}$, $\bar{s} = \bar{b}$ and $e_r a = e_s b$ for $r, s \in R$ and $a, b \in L$, then we have

$$\begin{aligned} \overline{e_r a} &= \overline{e_r} \cdot f_{e_r} \cdot e_a \bar{a} && \text{(by Lemma 2.5)} \\ &= \bar{r}^+ \bar{r}^+ e_a \bar{a} && \text{(by } e_r = e_r \cdot \bar{r}^+ \bar{r}^+) \\ &= \bar{a}^+ e_a \bar{a} = \bar{a}^+ \bar{a} = \bar{a}. \end{aligned}$$

Similarly, we have $\overline{e_s b} = \bar{b}$. Thus, $\bar{a} = \bar{b}$, $\bar{r} = \bar{s}$ and consequently $f_r = \bar{r}^* = \bar{s}^* = f_s$. Therefore,

$$\begin{aligned} r &= e_r \bar{r} f_r = e_r \bar{a} f_r = e_r (\bar{a}^+ a \bar{a}^*) f_r = (e_r \bar{r}^+) a (\bar{r}^+ f_r) \\ &= e_r a f_r = e_s b f_r = e_s b f_s = e_s \bar{b} f_s = e_s \bar{s} f_s = s. \end{aligned}$$

Similarly, we have $a = b$. $\square$

Proposition 3.7. Let (S^0, V) be an Ehresmann transversal of a U -abundant semigroup (S, U) . Then the following statements are equivalent:

- (1) S^0 is a quasi-ideal of S ;
- (2) $\Lambda I \subseteq S^0$;
- (3) $VI \subseteq S^0$ and $\Lambda V \subseteq S^0$;
- (4) $LR \subseteq S^0$;
- (5) S^0 is a left ideal of L and a right ideal of R ;
- (6) $S^0 I \subseteq S^0$ and $\Lambda S^0 \subseteq S^0$;
- (7) $S^0 I S^0 \subseteq S^0$ and $S^0 \Lambda S^0 \subseteq S^0$;
- (8) $SS^0 \subseteq R$, $S^0 S \subseteq L$;
- (9) R is a left ideal and L is a right ideal of S .

Proof. (1) $\Rightarrow$ (2). For any $f \in \Lambda$ and $e \in I$, there exist $f^+, e^* \in V$ such that $f \mathcal{R} f^+$ and $e \mathcal{L} e^*$. So we have $fe = f^+ \cdot fe \cdot e^* \in VSV \subseteq S^0 SS^0 \subseteq S^0$.

(2) $\Rightarrow$ (3). This is trivial.

(3) $\Rightarrow$ (4). For any $a \in L$ and $r \in R$, there exist $h, v \in V$ such that $a \widetilde{\mathcal{R}}^U h$ and $r \widetilde{\mathcal{L}}^U v$. Thus,

$$ar = h(ar)v = he_{ar} \bar{a} r f_{ar} v \in VI \bar{a} r \Lambda V \subseteq S^0 \bar{a} r S^0 \subseteq S^0.$$

(4) $\Rightarrow$ (5). This is clear since $S^0 \subseteq L, R$.

(5) $\Rightarrow$ (6). This is clear since $I \subseteq R$ and $\Lambda \subseteq L$.

(6) $\Rightarrow$ (7). This is obvious.

(7) $\Rightarrow$ (8). If (7) holds, then for any $a \in S$, $r^o \in S^o$, we have

$$ar^o = e_a \cdot \bar{a}f_a \cdot r^o \tilde{\mathcal{L}}^U \bar{a}^+ \bar{a}f_a r^o = \bar{a}f_a r^o \tilde{\mathcal{L}}^U (\bar{a}f_a r^o)^* \in V,$$

since $\bar{a}f_a r^o \in S^o \Lambda S^o \subseteq S^o$. Hence, $ar^o \in R$ by Proposition 3.5 and $SS^o \subseteq R$. Dually, $S^o S \subseteq L$.

(8) $\Rightarrow$ (9). For any $a \in S$, $r \in R$, $r = e_r \bar{r}f_r$ with $f_r \in V$, we have $ar = ae_r \cdot \bar{r}f_r \in SS^o \subseteq R$ and R is a left ideal of S . Dually, $S^o S \subseteq L$ implies that L is a right ideal of S .

(9) $\Rightarrow$ (1). For any $s, t \in S^o$ and $a \in S$, we have

$$sat = (sa)t \in SS^o \subseteq SR \subseteq R \quad \text{and} \quad sat = s(at) \in S^o S \subseteq LS \subseteq L.$$

Consequently, $sat \in R \cap L = S^o$ and S^o is a quasi-ideal of S . $\square$

4 The main theorem

The main objective of this paper is to establish a structure theorem for U -abundant semigroups with quasi-ideal Ehresmann transversals. In what follows, (R, I) denotes a U -abundant semigroup with a right ideal Ehresmann transversal (S^o, V) . Then, by Proposition 3.4, for every $r \in R$, $f_r = \bar{r}^* \in V$ and $r = e_r \bar{r}$. And (L, Λ) denotes a U -abundant semigroup with a left ideal Ehresmann transversal (S^o, V) , which has a dual result of (R, I) , especially, $a = \bar{a}f_a$ for every $a \in L$. Then R and L will be the components in such construction, which satisfy some compatible conditions.

Theorem 4.1. *Let (R, I) and (L, Λ) be a pair of U -abundant semigroups sharing a common Ehresmann transversal (S^o, V) . Suppose that S^o is a right ideal of R and a left ideal of L . Let $L \times R \rightarrow S^o$ described by $(a, r) \mapsto a * r$ be a mapping such that for any $r, s \in R$ and for any $a, b \in L$:*

- (1) $(a * r)s = a * rs$ and $b(a * r) = ba * r$;
- (2) If $\{r, a\} \cap S^o \neq \emptyset$, then $a * r = ar$;
- (3) For any $a, b \in L$, $r, s \in R$, if $a \tilde{\mathcal{L}}^\Lambda b$, then $(a * s) \tilde{\mathcal{L}}^V (b * s)$; if $r \tilde{\mathcal{R}}^I s$, then $(a * r) \tilde{\mathcal{R}}^V (a * s)$.

Define a multiplication on the set

$$\Gamma = R \mid \times \mid L = \{(r, a) \in R \times L : \bar{r} = \bar{a}\}$$

by

$$(r, a)(s, b) = (e_r(a * s), (a * s)f_b).$$

Then $(\Gamma, \overline{E(\Gamma)})$ is a U -abundant semigroup with a quasi-ideal Ehresmann transversal which is isomorphic to (S^o, V) .

Conversely, every U -abundant semigroup with a quasi-ideal Ehresmann transversal can be constructed in this manner.

To prove this theorem, in the following we will give a sequence of lemmas.

Lemma 4.1. *The multiplication on Γ is well-defined.*

Proof. We only need to prove $(e_r(a * s), (a * s)f_b) \in \Gamma$. It follows from $a * s \in S^o$ that $e_r(a * s) \in R$ and $(a * s)f_b \in L$. It is obvious that both a right ideal and a left ideal are quasi-ideals, and so we have

$$\begin{aligned}
\overline{e_r(a * s)} &= \overline{e_r f_{e_r} \cdot e_{(a * s)} a * s} && \text{(by Lemma 2.5)} \\
&= \bar{r}^+ \bar{r}^+(a * s)^+(a * s) && (a * s \in S^o) \\
&= \bar{r}^+(a * s) \\
&= \bar{a}^+(a * s) && (\bar{r} = \bar{a}) \\
&= e_a(a * s) && (\text{since } e_a \in V) \\
&= a * s.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\overline{(a * s)f_b} &= \overline{a * s f_{a * s} \cdot e_{f_b} \cdot \bar{f}_b} \\
&= (a * s)(a * s)^* \bar{b}^* \bar{b}^* \\
&= (a * s) \bar{b}^* \\
&= (a * s) \bar{s}^* && (\bar{s} = \bar{b}) \\
&= (a * s) f_s && (\text{since } f_s \in V) \\
&= a * s.
\end{aligned}$$

Therefore, $\overline{e_r(a * s)} = \overline{(a * s)f_b}$ and $(e_r(a * s), (a * s)f_b) \in \Gamma$. $\square$

Lemma 4.2. Γ is a semigroup.

Proof. Let $(r, a), (s, b), (t, c) \in \Gamma$. Then we have

$$\begin{aligned}
[(r, a)(s, b)](t, c) &= [e_r(a * s), (a * s)f_b](t, c) \\
&= (e_{e_r(a * s)}\{[(a * s)f_b] * t\}, \{[(a * s)f_b] * t\}f_c) \\
&= (e_r(a * s)^+[(a * s)(f_b * t)], [(a * s)(f_b * t)]f_c) \\
&= (e_r(a * s)(f_b * t), (a * s)(f_b * t)f_c).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(r, a)[(s, b)(t, c)] &= (r, a)[e_s(b * t), (b * t)f_c] \\
&= (e_r\{a * [e_s(b * t)]\}, \{a * [e_s(b * t)]\}f_{(b * t)f_c}) \\
&= (e_r[(a * e_s)(b * t)], [(a * e_s)(b * t)](b * t)^* f_c) \\
&= (e_r(a * e_s)(b * t), (a * e_s)(b * t)f_c) \\
&= (e_r(a * s)(f_b * t), (a * s)(f_b * t)f_c),
\end{aligned}$$

since $e_s b = e_s e_b \bar{b} f_b = e_s \bar{b}^+ \bar{b} f_b = e_s \bar{b} f_b = e_s \bar{s} f_b = s f_b$. Thus, $[(r, a)(s, b)](t, c) = (r, a)[(s, b)(t, c)]$ and so Γ is a semigroup. $\square$

Lemma 4.3. Let $(r, a) \in \Gamma$. Then $(r, a) \in E(\Gamma)$ if and only if $a * r = \bar{r} = \bar{a}$.

Proof. Since $(r, a)(r, a) = (e_r(a * r), (a * r)f_a)$, and note that $r \in R$, $a \in L$ and $a * r \in S^o$, it is clear that if $a * r = \bar{r} = \bar{a}$, then

$$(e_r(a * r), (a * r)f_a) = (e_r \bar{r}, \bar{a} f_a) = (r, a).$$

Thus, $(r, a) \in E(\Gamma)$. Conversely, if $(r, a) \in E(\Gamma)$, then $e_r(a * r) = r = e_r \bar{r}$. It follows from $e_r \mathcal{L} \bar{r}^+$ that $\bar{r} = \bar{r}^+ \bar{r} = \bar{r}^+(a * r) = \bar{a}^+(a * r) = (e_a a * r) = (a * r)$. $\square$

Lemma 4.4. Let $\overline{E(\Gamma)} = \{(r, a) \in \Gamma \mid r \in I, a \in \Lambda \text{ and } a * r = \bar{a} = \bar{r}\} \subseteq E(\Gamma)$ be set of distinguished idempotents of Γ . Then Γ is a weakly abundant semigroup.

Proof. Let $(r, a) \in \Gamma$. Then $(e_r, \bar{r}^+) \in \Gamma$ and $\bar{r}^+ * e_r = \bar{r}^+ e_r = \bar{r}^+ = \bar{e}_r = \bar{\bar{r}}^+$ and so $(e_r, \bar{r}^+) \in \overline{E(\Gamma)}$. Thus,

$$(e_r, \bar{r}^+)(r, a) = (e_r \bar{r}^+ r, \bar{r}^+ r f_a) = (e_r r, \bar{r} f_a) = (r, \bar{a} f_a) = (r, a).$$

Suppose that $(s, b) \in \overline{E(\Gamma)}$ such that $(s, b)(r, a) = (r, a)$. Then, from $(s, b)(r, a) = (e_s(b * r), (b * r)f_a)$, we have $e_s(b * r) = r$ and $(b * r)f_a = a$. From $e_s(b * r) = r$, we deduce that $\bar{r} = \overline{e_s(b * r)} = \overline{e_s}f_{e_s}e(b * r)\bar{b} * r = \bar{s}^+\bar{s}^+(b * r)^+(b * r) = \bar{s}^+(b * r) = \bar{b}^+(b * r) = (e_b b) * r = b * r$. Thus, $\bar{r} = b * r = (b * e_r)\bar{r}$ and so $\bar{r}^+ = (b * e_r)\bar{r}^+ = b * (e_r\bar{r}^+) = b * e_r$. Therefore, $(s, b)(e_r, \bar{r}^+) = (e_s(b * e_r), (b * e_r)\bar{r}^+) = (e_s\bar{r}^+, \bar{r}^+\bar{r}^+) = (e_r, \bar{r}^+)$ and $(r, a)\widetilde{\mathcal{R}}^{\overline{E(\Gamma)}}(e_r, \bar{r}^+)$.

For $(r, a) \in \Gamma$, we have $(\bar{a}^*, f_a) \in \Gamma$ with $f_a * \bar{a}^* = f_a\bar{a}^* = \bar{a}^* = \overline{\bar{a}^*} = \overline{f_a}$ and so $(\bar{a}^*, f_a) \in \overline{E(\Gamma)}$. Thus,

$$(r, a)(\bar{a}^*, f_a) = (e_r(a * \bar{a}^*), (a * \bar{a}^*)f_a) = (e_r a\bar{a}^*, a\bar{a}^*f_a) = (e_r\bar{a}, af_a) = (e_r\bar{r}, a) = (r, a).$$

Suppose that $(s, b) \in \overline{E(\Gamma)}$ such that $(r, a)(s, b) = (r, a)$. Then $(e_r(a * s), (a * s)f_b) = (r, a)$ and $e_r(a * s) = r$, $(a * s)f_b = a$. That $e_r(a * s) = r$ implies that $\bar{r} = \overline{e_r(a * s)} = \overline{e_r}f_{e_r} \cdot e_{a * s}\bar{a} * s = \bar{r}^+\bar{r}^+(a * s)^+(a * s) = \bar{r}^+(a * s) = (\bar{a}^+a) * s = (\bar{a}^+e_a\bar{a}f_a) * s = (\bar{a}f_a) * s = a * s$. Thus, $\bar{a} = \bar{r} = a * s = (\bar{a}f_a) * s = \bar{a}(f_a * s)$ and so $\bar{a}^* = \bar{a}^*(f_a * s) = (\bar{a}^*f_a) * s = f_a * s$. Therefore, $(\bar{a}^*, f_a)(s, b) = (\bar{a}^*(f_a * s), (f_a * s)f_b) = (\bar{a}^*, f_a)$ and $(r, a)\widetilde{\mathcal{L}}^{\overline{E(\Gamma)}}(\bar{a}^*, f_a)$. $\square$

Lemma 4.5. Let $(r_1, a_1), (r_2, a_2) \in \Gamma$. Then

- (1) $(r_1, a_1)\widetilde{\mathcal{R}}^U(r_2, a_2)$ if and only if $e_{r_1} = e_{r_2}$ if and only if $r_1\widetilde{\mathcal{R}}^I r_2$.
- (2) $(r_1, a_1)\widetilde{\mathcal{L}}^U(r_2, a_2)$ if and only if $f_{a_1} = f_{a_2}$ if and only if $a_1\widetilde{\mathcal{L}}^A a_2$.

Proof.

- (1) To prove the first part of this Lemma, by Lemma 4.4, it then suffices to show that $(e_{r_1}, \bar{r}_1^+)\widetilde{\mathcal{R}}^{\overline{E(\Gamma)}}(e_{r_2}, \bar{r}_2^+)$ if and only if $e_{r_1} = e_{r_2}$. If $e_{r_1} = e_{r_2}$, then $\bar{r}_1^+ = \bar{r}_2^+$ and $(e_{r_1}, \bar{r}_1^+) = (e_{r_2}, \bar{r}_2^+)$. Conversely, if $u_1 = (e_{r_1}, \bar{r}_1^+)\widetilde{\mathcal{R}}^{\overline{E(\Gamma)}}(e_{r_2}, \bar{r}_2^+) = u_2$, then $u_1 u_2 = u_2$ and $u_2 u_1 = u_1$, and this implies

$$\begin{aligned} e_{r_2} &= e_{r_1}e_{r_2}, & \bar{r}_2^+ &= \bar{r}_1^+e_{r_2}; \\ e_{r_1} &= e_{r_2}e_{r_1}, & \bar{r}_1^+ &= \bar{r}_2^+e_{r_1}. \end{aligned}$$

Thus, $e_{r_1}\mathcal{R}e_{r_2}$, and from Lemma 2.4 we deduce that $e_{r_1} = e_{r_2}$. That $e_{r_1} = e_{r_2}$ if and only if $r_1\widetilde{\mathcal{R}}^I r_2$ is obvious. (2) is dual to (1). $\square$

Lemma 4.6. $(\Gamma, \overline{E(\Gamma)})$ is an $\overline{E(\Gamma)}$ -abundant semigroup.

Proof. We need only to prove that $(\Gamma, \overline{E(\Gamma)})$ satisfies the congruence conditions. Suppose that $(r, a), (s, b) \in \Gamma$ with $(r, a)\widetilde{\mathcal{L}}^{\overline{E(\Gamma)}}(s, b)$. It follows from Lemma 4.5 that $a\widetilde{\mathcal{L}}^A b$ and $f_a = f_b$, $\bar{a}^* = \bar{b}^*$. For any $(t, c) \in \Gamma$, we have

$$\begin{aligned} (r, a)(t, c) &= (e_r(a * t), (a * t)f_c), \\ (s, b)(t, c) &= (e_s(b * t), (b * t)f_c). \end{aligned}$$

Since $a\widetilde{\mathcal{L}}^A b$ and so $(a * t)\widetilde{\mathcal{L}}^V(b * t)$ by (3), thus $(a * t)f_c\widetilde{\mathcal{L}}^V(b * t)f_c$. Consequently, by Lemma 4.5 again $(r, a)(t, c)\widetilde{\mathcal{L}}^{\overline{E(\Gamma)}}(s, b)(t, c)$. Therefore, $\widetilde{\mathcal{L}}^{\overline{E(\Gamma)}}$ is a right congruence on $(\Gamma, \overline{E(\Gamma)})$, and dually, $\widetilde{\mathcal{R}}^{\overline{E(\Gamma)}}$ is a left congruence. That is, $(\Gamma, \overline{E(\Gamma)})$ is an $\overline{E(\Gamma)}$ -abundant semigroup. $\square$

Lemma 4.7. Let $W = \{(r, r) : r \in (S^0, V)\}$ with the distinguished idempotent set $\overline{W} = \{(r, r) : r \in V\}$. Then $(W, \overline{W})$ is an Ehresmann $\sim$ -subsemigroup of $(\Gamma, \overline{E(\Gamma)})$, and $(W, \overline{W})$ is isomorphic to (S^0, V) .

Proof. Clearly, $(W, \overline{W}) \subseteq (\Gamma, \overline{E(\Gamma)})$. For any $(r, r), (s, s) \in (W, \overline{W})$, we have $(r, r)(s, s) = (e_r(r * s), (r * s)f_s) = (e_r rs, rsf_s) = (rs, rs) \in (W, \overline{W})$ and $(W, \overline{W})$ is a subsemigroup. For any $s \in (S^0, V)$, we naturally define a mapping $\varphi : (S^0, V) \rightarrow (W, \overline{W})$ by $s\varphi = (s, s)$. It is easy to see that φ is bijective. For any $s, t \in (S^0, V)$, $s\varphi t\varphi = (s, s)(t, t) = (st, st) = (st)\varphi$. Thus, φ is a homomorphism and $(S^0, V) \cong (W, \overline{W})$. By Lemmas 4.4 and 4.5, $(r^+, r^+), (r^*, r^*) \in \overline{W}$ with $(r^+, r^+)\widetilde{\mathcal{R}}^{\overline{E(\Gamma)}}(r, r)\widetilde{\mathcal{L}}^{\overline{E(\Gamma)}}(r^*, r^*)$. Therefore, $(W, \overline{W})$ is a $\sim$ -subsemigroup of $(\Gamma, \overline{E(\Gamma)})$. $\square$

Lemma 4.8. $(W, \overline{W})$ is an Ehresmann transversal of $(\Gamma, \overline{E(\Gamma)})$.

Proof. For any $\lambda = (r, a) \in \Gamma$, let $\bar{\lambda} = (\bar{r}, \bar{r}) \in W$, $e_\lambda = (e_r, \bar{r}^+)$, $f_\lambda = (\bar{a}^*, f_a)$. It is obvious that $\lambda = e_\lambda \bar{\lambda} f_\lambda$ with $e_\lambda \mathcal{L} \bar{\lambda}^+ = (\bar{r}^+, \bar{r}^+)$, $f_\lambda \mathcal{R} \bar{\lambda}^* = (\bar{a}^*, \bar{a}^*)$.

If λ can be written in another form $\lambda = e'_\lambda \bar{\lambda}' f'_\lambda$, where $e'_\lambda = (s_1, b_1) \in \overline{E(\Gamma)}$, $f'_\lambda = (s_2, b_2) \in \overline{E(\Gamma)}$, $\bar{\lambda}' = (\bar{s}, \bar{s}) \in W$ and $e'_\lambda \mathcal{L} \bar{\lambda}'^+ = (\bar{s}^+, \bar{s}^+)$, $f'_\lambda \mathcal{R} \bar{\lambda}'^* = (\bar{s}^*, \bar{s}^*)$. By Lemma 4.7, we have $f_{b_1} = \bar{s}^+$, $\bar{b}_1^* = \bar{s}^+$ and $\bar{a}^* = e_{s_2}$, $\bar{s}_2^+ = \bar{a}^*$. From $e_\lambda \bar{\mathcal{R}}^U \lambda \bar{\mathcal{R}}^U e_{\lambda'}$, we deduce that $e_r = e_{s_1}$, $\bar{r}^+ = \bar{s}_1^+$. Similarly, we have $f_a = f_{b_2}$, $\bar{a}^* = \bar{b}_2^*$. Therefore,

$$b_1 = e_{b_1} \bar{b}_1 f_{b_1} = \bar{b}_1 \cdot \bar{b}_1^* = \bar{b}_1 \in S^o,$$

$$s_2 = e_{s_2} \bar{s}_2 f_{s_2} = e_{s_2} \bar{s}_2 = \bar{s}_2^+ \bar{s}_2 = \bar{s}_2 \in S^o,$$

and so $e'_\lambda = (s_1, \bar{b}_1)$, $f'_\lambda = (\bar{s}_2, b_2)$. It follows from $e'_\lambda, f'_\lambda \in \overline{E(\Gamma)}$ and Lemma 4.3 that $\bar{b}_1 s_1 = \bar{s}_1$, $b_2 \bar{s}_2 = \bar{b}_2$. From $\bar{b}_1 s_1 = \bar{s}_1$, we deduce that $\bar{b}_1 e_{s_1} \bar{s}_1 = \bar{s}_1$ since $s_1 = e_{s_1} \bar{s}_1$. Thus, $\bar{b}_1 e_{s_1} \bar{s}_1^+ = \bar{s}_1^+$ since $\bar{s}_1 \bar{\mathcal{R}}^V \bar{s}_1^+$. Note that $\bar{s}_1 = \bar{b}_1$ and $s_1 \in I$, we have $\bar{b}_1 = \bar{s}_1 = \bar{e}_{s_1} = \bar{s}_1^+ = \bar{r}^+$ and similarly $\bar{s}_2 = \bar{a}^*$. Consequently, $e'_\lambda = (s_1, \bar{b}_1) = (e_{s_1}, \bar{r}^+) = (e_r, \bar{r}^+) = e_\lambda$ and similarly $f'_\lambda = f_\lambda$. Hence,

$$\lambda = e'_\lambda \bar{\lambda}' f'_\lambda = (e_r, \bar{r}^+) (\bar{s}, \bar{s}) (\bar{a}^*, f_a) = (e_r \bar{s}, \bar{r}^+ \bar{s} f_a).$$

This implies $r = e_r \bar{s}$, and combining with $r = e_r \bar{r}$ implies that $e_r \bar{s} = e_r \bar{r}$. Premultiplying by $\bar{r}^+$, we obtain $\bar{r}^+ \bar{s} = \bar{r}$. Note that $\bar{r}^+ = \bar{b}_1 = \bar{b}_1^* = \bar{s}^+$. Therefore, $\bar{r} = \bar{r}^+ \bar{s} = \bar{s}^+ \bar{s} = \bar{s}$ and by Definition 2.3, $(W, \overline{W})$ is an Ehresmann Transversal of $(\Gamma, \overline{E(\Gamma)})$. $\square$

Lemma 4.9. W is a quasi-ideal of Γ .

Proof. For any $(s, s), (t, t) \in W$, $(r, a) \in \Gamma$, we have

$$(s, s)(r, a)(t, t) = (e_s s r f_a t, s r f_a t f_t) = (s r f_a t, s r f_a t) \in W,$$

since both a right ideal and a left ideal are quasi-ideals. $\square$

Now we shall prove the converse half of Theorem 4.1. Assume that (S, U) is a U -abundant semigroup with a quasi-ideal Ehresmann transversal (S^o, V) . Let

$$R = \{r \in S : f_r \in V\}, \quad L = \{a \in S : e_a \in V\}.$$

Then by Proposition 3.6, R and L are both U -abundant semigroups with an Ehresmann transversal (S^o, V) , where S^o is a right ideal of R and a left ideal of L .

For every $(a, r) \in L \times R$, let $a * r = ar$. Then $e_a, f_r \in V$ and so $a * r = ar = e_a a \cdot r f_r \in S^o$ since S^o is a quasi-ideal of S . Obviously, the map satisfies (1), (2) and (3). Therefore, we may acquire a U -abundant semigroup Γ in the way of the direct part of Theorem 4.1. In the following, we prove that Γ is isomorphic to (S, U) .

For every $(r, a) \in \Gamma$, we define $\theta : \Gamma \rightarrow S$ by $(r, a)\theta = e_r a$. It is clear that θ is well-defined and by Proposition 3.6, θ is injective.

For any $(r, a), (s, b) \in \Gamma$, we have

$$\begin{aligned} [(r, a)(s, b)]\theta &= [e_r(as), (as)f_b]\theta = e_{e_r(as)} \cdot (as)f_b = e_r (\bar{e}_r f_{e_r}(as)^+)^+ \cdot (as)f_b \\ &= e_r \cdot \bar{r}^+(as)^+ \cdot (as)f_b = e_r \cdot (as)f_b = e_r a \cdot s f_b \\ &= e_r a \cdot e_s b \quad (s f_b = e_s \bar{s} f_b = e_s \cdot \bar{b} f_b = e_s b) \\ &= (r, a)\theta(s, b)\theta, \end{aligned}$$

and so θ is a homomorphism.

For any $a \in S$, it is obvious that $a\bar{a}^* \in R$ and $\bar{a}^+ a \in L$ with $\overline{a\bar{a}^*} = \bar{a} f_a \cdot \bar{a}^* \bar{a}^* = \bar{a} \bar{a}^* = \bar{a}$ and $\overline{\bar{a}^+ a} = \bar{a}^+ \bar{a}^+ e_a \bar{a} = \bar{a}^+ \bar{a} = \bar{a}$. Consequently, $(a\bar{a}^*, \bar{a}^+ a) \in \Gamma$ and $(a\bar{a}^*, \bar{a}^+ a)\theta = e_{a\bar{a}^*} \cdot \bar{a}^+ a = e_a \cdot \bar{a}^+ \cdot a = e_a a = a$, that is, θ is surjective. Therefore, θ is an isomorphism.

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