Research Article

Xinhong Zhang, Caijuan Xue, and Ruijuan Li*

The domination number of round digraphs

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Abstract: The concept of the domination number plays an important role in both theory and applications of digraphs. Let $D = (V, A)$ be a digraph. A vertex subset $T \subseteq V(D)$ is called a dominating set of D , if there is a vertex $t \in T$ such that $tv \in A(D)$ for every vertex $v \in V(D) \setminus T$. The dominating number of D is the cardinality of a smallest dominating set of D , denoted by $\gamma(D)$. In this paper, the domination number of round digraphs is characterized completely.

Keywords: domination number, round digraph, local tournament, purely local tournament

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1 Introduction

The domination theory of graphs was derived from a board game in ancient India. In 1962, Ore formally gave the definitions of the dominating set and the domination number in [1]. Due to the universality of its applications to both theoretical and practical problems, domination has become one of the important research topics in graph theory. A summary of most important results and applications can be found in [2]. Problems of resource allocations and scheduling in networks are frequently formulated as domination problems on underlying graphs (digraphs). By contrast, domination in digraphs has not yet gained the same amount of attention, although it has several useful applications as well. For example, it has been used in the study of answering skyline query in the database [3] and routing problems in networks [4]. The relationship among domination numbers of different orientations of a graph was studied in [5]. The relevant results about domination numbers of digraphs can be found in [6–10]. Recent studies on domination theory include [11–13].

We refer the reader to [14] for terminology and notation not defined in this paper. Let $D = (V, A)$ be a digraph, which means that V and A represent the vertex set and the arc set of D , respectively. The *order* of D is the number of vertices in D , denoted by $|V(D)|$. If uv is an arc, then we say that u *dominates* v (or v is dominated by u) and use the notation $u \rightarrow v$ to denote this. For a vertex v of a digraph D , we define the vertex set $N_D^+(v) = \{u \in V \setminus \{v\} | vu \in A\}$, $N_D^-(v) = \{w \in V \setminus \{v\} | vw \in A\}$. We also call the vertex set $N_D^+(v)$, $N_D^-(v)$ and the vertex set $N_D^+(v) \cup N_D^-(v)$ the *out-neighbourhood*, the *in-neighbourhood* and the *neighbourhood* of the vertex v , respectively. $d_D^+(v)$ is the number of all arcs with tail v , and we call $d_D^+(v)$ the *out-degree* of v . $d_D^-(v)$ is the number of all arcs with head v , and we call $d_D^-(v)$ the *in-degree* of v . If each arc of $A(D)$ with both end-vertices in $V(H)$ is in $A(H)$, we say that H is *induced* by $X = V(H)$ and denote it by $H = D\langle X \rangle$. We call H an *induced subdigraph* of D .

Let D be a digraph. Let $v_1, v_2, \dots, v_n$ be a vertex labelling of D . If there is always $i < j$ for every arc $v_i v_j$ in D , then we often refer to the vertex labelling as an *acyclic ordering* of D . A *walk* in D is an alternating sequence

* **Corresponding author: Ruijuan Li**, School of Mathematical Science, Shanxi, University, 030006 Taiyuan, People's Republic of China, e-mail: ruijuanli@sxu.edu.cn

Xinhong Zhang, Caijuan Xue: Department of Applied Mathematics, Taiyuan University of Science and Technology, 030024 Taiyuan, People's Republic of China

$W = v_1 a_1 v_2 a_2 v_3 \dots v_{k-1} a_{k-1} v_k$ of vertices v_i and arcs a_j from D such that the tail of a_i is v_i and the head of a_i is v_{i+1} for every $i = 1, 2, \dots, k-1$. A walk W is *closed* if $v_1 = v_k$, and *open* otherwise. The set of vertices $\{v_1, v_2, \dots, v_k\}$ is denoted by $V(W)$; the set of arcs $\{a_1, a_2, \dots, a_{k-1}\}$ is denoted by $A(W)$. If the vertices of W are distinct, W is a path. If the vertices $v_1, v_2, \dots, v_{k-1}$ are distinct, $k \geq 3$ and $v_1 = v_k$, W is a cycle. A walk (path, cycle) W is a *Hamiltonian walk (path, cycle)* if $V(W) = V(D)$. The digraph D is *strongly connected* (or *strong*) if, for each pair u and v of distinct vertices in D , there is a (u, v) -walk and a (v, u) -walk.

A *semicomplete digraph* is a digraph in which every pair of distinct vertices is adjacent. A *tournament* is a semicomplete digraph with no cycle of length two. A digraph D is *locally in-semicomplete* (*out-semicomplete*) if, for every vertex x of D , the in-neighbours (out-neighbours) of x induce a semicomplete digraph. A digraph D is *locally semicomplete* if it is both locally in-semicomplete and locally out-semicomplete. A locally semicomplete digraph with no 2-cycle is a local tournament. If a digraph is a locally semicomplete digraph (local tournament) but not a semicomplete digraph (tournament), then the digraph is a *purely local semicomplete digraph* (*purely local tournament*). Related surveys about the locally semicomplete digraphs can be found in [15,16].

A digraph on n vertices is *round* if we can label its vertices $v_1, v_2, \dots, v_n$ so that for each i , we have $N_D^+(v_i) = \{v_{i+1}, \dots, v_{i+d^+(v_i)}\}$ and $N_D^-(v_i) = \{v_{i-d^-(v_i)}, \dots, v_{i-1}\}$ (all subscripts are taken modulo n). Let $D = (V, A)$ be a digraph, and let T be a subset of the vertices of D . If for every vertex $v \in V(D) \setminus T$, there is a vertex $t \in T$ such that $tv \in A(D)$, then we say that T is a *dominating set* of D and denote it by $T \rightarrow D$. The *dominating number* of D is the cardinality of a smallest dominating set of D , denoted by $\gamma(D)$.

We need the following lemma and theorem in order to prove the main theorems.

Lemma 1.1. [17] *Every round digraph is locally semicomplete.*

Theorem 1.2. [18] *Let D be a strong tournament on $n \geq 3$ vertices. For every $x \in V(T)$ and every integer $k \in \{3, 4, \dots, n\}$, there exists a k -cycle through x in D . In particular, a tournament is Hamiltonian if and only if it is strong.*

In this paper, the domination number of a round digraph is characterized by studying the round local tournament and the round non-local tournament, respectively.

2 The domination number of a round local tournament

2.1 The domination number of a round purely local tournament which is non-strong

Let D be a round purely local tournament which is non-strong. Let $P_n = v_0 v_1 \dots v_{n-1}$ be a directed path of D . If there is an arc $v_i v_j \in A(D)$ satisfying $|j - i| \geq 2$ for $i, j \in \{0, 1, \dots, n-1\}$, then the arc $v_i v_j$ is called a *cross arc* on P_n . If there is no cross arc $v_{i_\alpha} v_{j_\alpha}$ on P_n such that $i_\alpha < i < j < j_\alpha$, then the cross arc $v_i v_j$ is called a *maximal cross arc* on P_n . We call the vertex set $\{v_i, v_{i+1}, \dots, v_j\}$ *covered* by the maximal cross arc $v_i v_j$. We call the set G a *maximal cross-arc chain* on P_n , if there is a maximal cross-arc set $G = \{v_{i_t} v_{j_t} | v_{i_t} v_{j_t} \text{ is a maximal cross arc on } P_n, t \in \{0, 1, \dots, k-1\}\}$ on P_n satisfying one of the following conditions:

- (1) For $k = 1$,
 - (a) $v_{i_0} v_{j_0}$ is a maximal cross arc on P_n and there is no set $\{\alpha, \alpha', \beta, \beta'\} \subseteq \{0, 1, \dots, n-1\}$ such that $\alpha < i_0 < \alpha'$ and $\beta < j_0 < \beta'$, where $v_\alpha v_{\alpha'}$, $v_\beta v_{\beta'}$ are two cross arcs on P_n ; or
 - (b) $v_{i_0} v_{j_0}$ is a maximal cross arc on P_n . There is a maximal cross arc $v_\gamma v_{\gamma'}$ on P_n such that $\gamma < i_0 < \gamma'$ (or $\gamma < j_0 < \gamma'$) and $i_0 - \gamma = 1$ (or $\gamma' - j_0 = 1$), and there is no cross arc $v_\tau v_{\tau'}$ such that $\tau < i_0 < \tau'$ and $i_0 - \tau \geq 2$ (or $\tau < j_0 < \tau'$ and $\tau' - j_0 \geq 2$) (Figure 1(a)).

Figure 1: (a) $k = 1$: $G_1 = \{v_{i_0}v_{j_0}\}$, $G_2 = \{v_{i_2}v_{j_2}\}$ are two maximal cross-arc chains on $P = v_0v_1 \dots v_{16}$. $P_1 = v_0v_1$, $P_2 = v_6v_7v_8$, $P_3 = v_{15}v_{16}$ are all maximal pure subpaths on P ; (b) $k \geq 2$: v_2v_7 , v_4v_8 , v_6v_{10} , v_8v_{11} , $v_{11}v_{13}$ are all maximal cross arcs on path $P = v_0v_1 \dots v_{14}$. $G = \{v_{i_t}v_{j_t} \mid t = 0, 1, 2, 3\}$ is an only maximal cross-arc chain on path $P = v_0v_1 \dots v_{16}$, in which $v_{i_2}v_{j_2}$ is an only invalid cross arc in G . $P_{1'} = v_0v_1$, $P_{2'} = v_{14}v_{15}v_{16}$ are all maximal pure subpaths on P .

Figure 2: The vertex set partition of a round purely local tournament D which is non-strong. $V(D) = \bigcup_{i=0}^6 B_i$, where $B_0 = \{v_0\}$, $B_1 = \{v_1, v_2, v_3\}$, $B_2 = \{v_4\}$, $B_3 = \{v_5, v_6, v_7\}$, $B_4 = \{v_8\}$, $B_5 = \{v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$, $B_6 = \{v_{15}, v_{16}\}$.

(2) For $k \geq 2$,

- (a) $|i_1 - i_0| > 1$ and $|j_{k-1} - j_{k-2}| > 1$, and when $i_0 \neq 0$ ($j_{k-1} \neq n - 1$), there is at most one vertex $v_{i_0-1}(v_{j_{k-1}+1})$ such that $v_{i_0-1}v_{j_{k-1}+1}$ is a cross arc on P_n , where $i_0 < \beta$ ($\beta < j_{k-1} + 1$);
- (b) $i_t < i_{t+1} \leq j_t < j_{t+1}$, $t \in \{0, 1, \dots, k-2\}$; and
- (c) There is no maximal cross arc $v_{i_y}v_{j_y}$ on P_n such that $i_{t+1} < i_y \leq j_t < j_{t+1} < j_y$ for $t \in \{0, 1, \dots, k-2\}$ (Figure 1(b)).

The vertex set $\bigcup_{t=0}^{k-1} \{v_{i_t}, v_{i_t+1}, \dots, v_{j_t}\}$ is covered by the maximal cross-arc chain G . If there exists an arc $v_{i_\tau}v_{j_\tau} \in G$ satisfying $|i_{\tau+1} - j_{\tau-1}| = 1$, then we call the maximal cross arc $v_{i_\tau}v_{j_\tau}$ an *invalid cross arc* of G , where $1 \leq \tau \leq k-2$. In addition to the invalid cross arcs in G , the remaining maximal cross arcs are called the *valid cross arcs* of G . Let P_m be a subpath of P_n . If all the vertices on P_m are not covered by any maximal cross-arc chain, then P_m is called a *pure subpath* of P_n . If there is no $v_a \in V(P_n)$ such that $D\langle V(P_m) \cup \{v_a\} \rangle$ is a pure subpath of P_n , then call P_m a *maximal pure subpath* of P_n and all vertices in $V(P_m)$ covered by P_m . Figure 1 illustrates these definitions.

Subsequently, we show the partition problem of the vertices of a round purely local tournament which is non-strong. Let D be a round purely local tournament which is non-strong. Let $v_0, v_1, \dots, v_{n-1}$ be a round labelling of D . Since D is a round purely local tournament which is non-strong, $P = v_0v_1 \dots v_{n-1}$ is the only Hamilton path in D . If there exists a cross arc on P , then it can only be forward arc (that is, from the vertex with a small subscript to the vertex with a large subscript). Thus, $V(D)$ can form a partition. It means $V(D) = \bigcup_{i=0}^m B_i$, $B_i \cap B_j = \emptyset$ for any $i \neq j \in \{0, 1, \dots, m\}$, where B_i is covered by either some maximal pure subpath or some maximal cross-arc chain (Figure 2).

According to the partition about the vertex set above, the following conclusions can be obtained.

Lemma 2.1. Let D be a round purely local tournament which is non-strong on n vertices. Let $P = v_0v_1 \dots v_{n-1}$ be a Hamilton path in D . If $P_m = v_{i_0}v_{i_0+1} \dots v_{i_0+(m-1)}$ is a maximal pure subpath on P , then $\gamma(P_m) = \left\lceil \frac{m}{2} \right\rceil$.

Proof. Since P_m is a directed path, $v_j \rightarrow v_{j+1}$ for $j \in \{i_0, i_0 + 1, \dots, i_0 + (m - 2)\}$.

When m is even, let $T = \{v_{i_0}, v_{i_0+2}, v_{i_0+4}, \dots, v_{i_0+(m-2)}\}$. Since $v_j \rightarrow v_{j+1}$ for $j \in \{i_0, i_0 + 1, \dots, i_0 + (m - 2)\}$, we have $T \rightarrow P_m$. Thus, $\gamma(P_m) \leq |T| = \frac{m}{2}$. Choosing any vertex set $M = \{v_{k_1}, v_{k_2}, \dots, v_{k_{\frac{m}{2}-1}}\} \subseteq V(D)$, there must exist a vertex $v_\tau \in V(P_m)$ such that $\{v_\tau, v_{\tau+1}\} \not\subseteq M$. By maximal pure subpath P_m , we see that $v \rightarrow v_{\tau+1}$ for any vertex $v \in M$ since $N_D^-(v_{\tau+1}) = \{v_\tau\}$. Therefore, M is not a dominating set of P_m . For the arbitrariness of M , we have $\gamma(P_m) \geq \frac{m}{2}$. Thus, $\gamma(P_m) = \frac{m}{2} = \left\lceil \frac{m}{2} \right\rceil$.

When m is odd, let $T = \{v_{i_0}, v_{i_0+1}, v_{i_0+3}, \dots, v_{i_0+(m-2)}\}$. It is easy to see that $T \rightarrow P_m$ and $|T| = \left\lceil \frac{m}{2} \right\rceil$. So $\gamma(P_m) \leq \left\lceil \frac{m}{2} \right\rceil$. Choosing any vertex set $M' = \{v_{j_1}, v_{j_2}, \dots, v_{j_{\left\lceil \frac{m}{2} \right\rceil-1}}\} \subseteq V(D)$, there must exist a vertex $v_\tau \in V(P_m)$ such that $\{v_\tau, v_{\tau+1}\} \not\subseteq M'$. By maximal pure subpath P_m , we have $v \rightarrow v_{\tau+1}$ for any vertex $v \in M'$. Then M' is not a dominating set of P_m . It implies $\gamma(P_m) \geq \left\lceil \frac{m}{2} \right\rceil$. Thus, $\gamma(P_m) = \left\lceil \frac{m}{2} \right\rceil$. $\square$

Lemma 2.2. Let D be a round purely local tournament which is non-strong on n vertices. Let $v_0, v_1, \dots, v_{n-1}$ be a round labelling of D . $P = v_0 v_1 \dots v_{n-1}$ is the Hamiltonian path in D . If $B = \bigcup_{t=0}^{k-1} \{v_{i_t}, v_{i_t+1}, \dots, v_{j_t}\} \subseteq V(D)$ is covered by the maximal cross-arc chain $G = \{v_{i_t} v_{j_t} | t = 0, 1, 2, \dots, k-1\}$, then $\gamma(D\langle B \rangle) = \tau$, where τ is the number of all valid arcs in G , and all subscripts are ordered in the round ordering.

Proof. Now we distinguish two cases to prove this lemma.

Case 1. There is no invalid cross arc in G .

According to the definitions of the maximal cross arc and the round digraph, there must be $v_{i_t} \rightarrow v_l$ where $l \in \{i_t + 1, i_t + 2, \dots, j_t\}$ for $t \in \{0, 1, 2, \dots, k-1\}$. Since $i_t < i_{t+1} \leq j_t < j_{t+1}$ for $t \in \{0, 1, \dots, k-2\}$, we have $\{v_{i_t} | t = 0, 1, \dots, k-1\} \rightarrow D\langle B \rangle$. Thus, $\gamma(D\langle B \rangle) \leq k = \tau$.

For arbitrary $t \in \{1, 2, \dots, k-2\}$, $v_{i_t} v_{j_t}$ is a valid cross arc of G . By the definition of a valid cross arc, $|i_{t+1} - j_{t-1}| \geq 2$. Since G is a maximal cross-arc chain, $|i_1 - i_0| \geq 2$ and $|j_{k-1} - j_{k-2}| \geq 2$. Thus, there is at least a vertex v_{i_0} in the vertex set $\{v_{i_t}, v_{i_t+1}, \dots, v_{j_t}\}$ such that v_{i_0} is only covered by the maximal cross arc $v_{i_t} v_{j_t}$ in G , where $t \in \{0, 1, 2, \dots, k-1\}$. Let $A_t = \{v_{i_t}, v_{i_t+1}, \dots, v_{j_t}\}$, $t \in \{0, 1, \dots, k-1\}$. It is easy to see $A_i \cap A_j = \emptyset$, $i, j \in \{0, 1, \dots, k-1\}$. For any vertex set $M \subseteq B$ for $|M| = k-1$, there exists $\alpha \in \{0, 1, \dots, k-1\}$ such that $A_\alpha \cap M = \emptyset$. By the round purely local tournament D which is non-strong, we have $v \rightarrow v_\alpha \in A_\alpha$ for any $v \in A_\beta$ with $\beta \in \{\alpha + 1, \alpha + 2, \dots, k-1\}$. Since $v_{\alpha_0} \in A_\alpha$ is only covered by the maximal cross arc $v_{i_\alpha} v_{j_\alpha}$, we have $v_\gamma \rightarrow v_{\alpha_0}$ for $\gamma \in \{i_0, i_0 + 1, \dots, i_\alpha - 1\}$. Thus, $M \rightarrow v_{\alpha_0}$. According to the arbitrariness of M , $\gamma(D\langle B \rangle) \geq k = \tau$. So $\gamma(D\langle B \rangle) = k = \tau$ (Figure 3(a)).

Case 2. There is at least one invalid cross arc in G .

Let $v_{i_\alpha} v_{j_\alpha}$ be any invalid cross arc of G . According to the definition of an invalid cross arc, $|i_{\alpha+1} - j_{\alpha-1}| = 1$. It implies that $\{v_{i_\alpha}, v_{i_\alpha+1}, \dots, v_{j_{\alpha-1}}\}$ is covered by the maximal cross arc $v_{i_{\alpha-1}} v_{j_{\alpha-1}}$ and $\{v_{j_{\alpha-1}+1}, v_{j_{\alpha-1}+2}, \dots, v_{j_\alpha}\}$ is covered by the maximal cross arc $v_{i_{\alpha+1}} v_{j_{\alpha+1}}$. By the definitions of the cover and the round digraph, we obtain $\{v_{i_{\alpha-1}}, v_{i_{\alpha+1}}\} \rightarrow \{v_{i_\alpha}, v_{i_\alpha+1}, \dots, v_{j_\alpha}\}$.

For the arbitrariness of $v_{i_\alpha} v_{j_\alpha}$, all vertices covered by the invalid cross arcs in G can be covered by the valid cross arcs in G . This means $\gamma(D\langle B \rangle) \leq \tau$, where τ is the number of valid arcs in G . According to Case 1, $\gamma(D\langle B \rangle) \geq \tau$ can be proved similarly. Then, $\gamma(D\langle B \rangle) = \tau$ (Figure 3(b)). $\square$

Lemma 2.3. Let D be a round purely local tournament which is non-strong on n vertices. Let $v_0, v_1, \dots, v_{n-1}$ be a round labelling of D . Let $P = v_0 v_1 \dots v_{n-1}$ be the Hamilton path in D , $P_i = v_{i_0} v_{i_0+1} \dots v_{i_{t-1}}$ ($P_j = v_{j_{t-1}+1} v_{j_{t-1}+2} \dots v_{j_t}$) be a maximal subpath on P , and $G_i = \{v_{i_k} v_{j_k} | k = 1, 2, \dots, t\}$ ($G_j = \{v_{i_k} v_{j_k} | k = 0, 1, \dots, t-1\}$) be a maximal cross-arc chain on P adjacent to $P_i(P_j)$. If B and C represent the set of vertices covered by $P_i(P_j)$ and $G_i(G_j)$, respectively, then $\gamma(D\langle B \cup C \rangle) = \gamma(D\langle B \rangle) + \gamma(D\langle C \rangle)$.

Figure 3: (a) An example for Case 1 in Lemma 2.2, where there is no invalid cross arc; (b) an example for Case 2 in Lemma 2.2, where there is an invalid cross arc v_3v_5 .

Figure 4: An example for Lemma 2.3.

Proof. Let T_B and T_C be the minimum dominating set of $D\langle B \rangle$ and $D\langle C \rangle$, respectively. It implies $\gamma(D\langle B \rangle) = |T_B|$, $\gamma(D\langle C \rangle) = |T_C|$. It is easy to see $T_B \cup T_C \rightarrow D\langle B \cup C \rangle$. Thus, we have $\gamma(D\langle B \cup C \rangle) \leq |T_B| + |T_C|$. Without loss of generality, suppose that all the cross arcs in $G_i(G_j)$ are valid. Choose any vertex set $M \subseteq V(D)$ satisfying $|M| = |T_B| + |T_C| - 1$. According to the proof of Lemma 2.2 and the structure of the round purely local tournament which is non-strong, the minimum dominating set of $D\langle C \rangle$ must be $E = \{v_{t_k} | k = 1, 2, \dots, t\}$ ($E = \{v_{t_k} | k = 0, 1, \dots, t-1\}$), where $v_{t_k} \in \{v_{i_k}, v_{i_k+1}, \dots, v_{i_k+m}\}$, v_{i_k+m} is a vertex in C with the smallest subscript which is only covered by the valid cross arc $v_{i_k}v_{j_k}$. It is not difficult to see that $|E| = |T_C|$. Thus, if $|M \cap E| \leq |T_C| - 1$, then M cannot dominate $D\langle C \rangle$, which means M is not a dominating set of D . Otherwise, M contains $|T_B| - 1$ vertices in B at most. As we know from Lemma 2.1, there exists $v_\alpha \in \{v_{i_0}, v_{i_0+1}, \dots, v_{i_{t-2}}\} (\{v_{j_{t-1}+1}, v_{j_{t-1}+2}, \dots, v_{j_{t-1}}\})$ such that $\{v_\alpha, v_{\alpha+1}\} \cap M = \emptyset$. By the structure of D and the pure subpath $D\langle B \rangle$, we have $M \not\rightarrow v_{\alpha+1}$, i.e. M is not a dominating set of $D\langle B \cup C \rangle$. Due to the arbitrariness of M , $\gamma(D\langle B \cup C \rangle) \geq |T_B| + |T_C|$. Thus, $\gamma(D\langle B \cup C \rangle) = |T_B| + |T_C| = \gamma(D\langle B \rangle) + \gamma(D\langle C \rangle)$ (Figure 4). $\square$

According to the aforementioned results, we can get the following result about the dominating number of a round purely local tournament which is non-strong.

Theorem 2.4. Let D be a round purely local tournament which is non-strong on n vertices. Let $v_0, v_1, \dots, v_{n-1}$ be a round labelling of D . $P = v_0v_1 \dots v_{n-1}$ is the Hamiltonian path of D . If there exist k maximal cross-arc chains G_i ($i = 1, 2, \dots, k$) and l maximal pure subpaths P_j ($j = 1, 2, \dots, l$) on P , then $\gamma(D) = \sum_{i=1}^k \tau_i + \sum_{j=1}^l \lceil \frac{n_j}{2} \rceil$, where $l \in \{k-1, k, k+1\}$, τ_i is the number of valid cross arcs in G_i , and n_j is the number of vertices contained in P_j .

Proof. According to the structure of D , we obtain the vertices of D are either covered by a maximal cross-arc chain or covered by a maximal pure subpath. According to the definition of the maximal cross-arc chain and the maximal pure subpath, one can get $\gamma(D) \leq \sum_{i=1}^k \tau_i + \sum_{j=1}^l \lceil \frac{n_j}{2} \rceil$ by Lemmas 2.1 and 2.2.

We give a proof for $\gamma(D) \geq \sum_{i=1}^k \tau_i + \sum_{j=1}^l \lceil \frac{n_j}{2} \rceil$ as follows. Choose any vertex set $M \subseteq V(D)$ for $|M| \sum_{i=1}^k \tau_i + \sum_{j=1}^l \lceil \frac{n_j}{2} \rceil$, then one of the following two cases holds at least:

Case 1. There is a maximal cross-arc chain $G_\alpha = \{v_{i_t} v_{j_t} \mid t = 1, 2, \dots, \tau_\alpha\}$ such that $|M \cap \{v_\beta \mid \beta = i_1, i_1 + 1, \dots, j_{\tau_\alpha}\}| \leq \tau_\alpha - 1$.

By the proof of Case 1 in Lemma 2.2, there must be $v_{\alpha_0} \in \{v_\beta \mid \beta = i_1, i_1 + 1, \dots, j_{\tau_\alpha}\}$ such that $M \not\rightarrow v_{\alpha_0}$. Then M cannot dominate D .

Case 2. There is a maximal pure subpath P_β such that $|M \cap V(P_\beta)| \leq \lceil \frac{n_\beta}{2} \rceil - 1$, where $n_\beta = |V(P_\beta)|$.

According to Lemma 2.1, there must be a vertex set $\{v_{i_\beta}, v_{i_\beta+1}\} \subseteq V(P_\beta)$ such that $M \cap \{v_{i_\beta}, v_{i_\beta+1}\} = \emptyset$. Due to the structure of D , it is obvious $N_D^-(v_{i_\beta+1}) = \{v_{i_\beta}\}$, which means $M \not\rightarrow v_{i_\beta+1}$. Then M cannot dominate D .

Therefore, M is not a dominating set of D anyway. By the arbitrariness of M , $\gamma(D) \geq \sum_{i=1}^k \tau_i + \sum_{j=1}^l \lceil \frac{n_j}{2} \rceil$. Thus, $\gamma(D) = \sum_{i=1}^k \tau_i + \sum_{j=1}^l \lceil \frac{n_j}{2} \rceil$. $\square$

2.2 The domination number of a round purely local tournament which is strong

Theorem 2.5. Let D be a directed cycle $v_0 v_1 \dots v_{n-1} v_0$, then $\gamma(D) = \lceil \frac{n}{2} \rceil$, where all subscripts are taken modulo n .

Proof. Since D is a directed cycle, we have $v_i \rightarrow v_{i+1}$ for $0 \leq i \leq n-1$.

When n is even, let $T = \{v_0, v_2, v_4, \dots, v_{n-2}\}$. By the condition $v_i \rightarrow v_{i+1}$ for $0 \leq i \leq n-1$, we obtain $T \rightarrow D$, and then $\gamma(D) \leq \frac{n}{2}$. Choose arbitrary vertex set $M = \{v_{i_1}, v_{i_2}, \dots, v_{i_{\frac{n}{2}-1}}\}$, there must exist $\alpha \in \{0, 1, 2, \dots, n-1\}$ such that $\{v_\alpha, v_{\alpha+1}\} \not\subseteq M$. Since D is a cycle, we have $M \not\rightarrow v_{\alpha+1}$. So M is not a dominating set of D . By the arbitrariness of M , we have $\gamma(D) \geq \frac{n}{2}$. Therefore, $\gamma(D) = \frac{n}{2}$.

When n is odd, it is easy to see the vertex set $T = \{v_0, v_1, v_3, \dots, v_{n-2}\}$ is a dominating set of D because D is a directed cycle. Thus, $\gamma(D) \leq \lceil \frac{n}{2} \rceil$. Choose any vertex set $M = \{v_{j_1}, v_{j_2}, \dots, v_{j_{\frac{n}{2}-1}}\}$, there must exist $\beta \in \{0, 1, 2, \dots, n-1\}$ such that $\{v_\beta, v_{\beta+1}\} \not\subseteq M$. Since D is a cycle, $M \not\rightarrow v_{\beta+1}$. Therefore, M is not a dominating set of D . For the arbitrariness of M , we get $\gamma(D) \geq \lceil \frac{n}{2} \rceil$. Thus, $\gamma(D) = \lceil \frac{n}{2} \rceil$. $\square$

For the convenience of proof, we define as follows similar to that of Section 2.1. Let D be a round purely local tournament which is strong. Let $v_0, v_1, \dots, v_{n-1}$ be a round labelling of D . Let $C_n = v_0 v_1 \dots v_{n-1} v_0$ be the Hamilton cycle in D . The arc connecting any two non-adjacent vertices on the cycle is called a *cross arc* on C_n . Let $v_i v_j$ be a cross arc on C_n . If there is no cross arc $v_{i_\alpha} v_{j_\alpha}$ on C_n such that $i_\alpha < i < j < j_\alpha$ ($j_\alpha < j < i < i_\alpha$), then the cross arc $v_i v_j$ is called a *maximal cross arc* on C_n . We call the vertex set $\{v_i, v_{i+1}, \dots, v_j\}$ covered by the maximal cross arc $v_i v_j$. We call the set G a *maximal cross-arc chain* on C_n and the vertex set $\bigcup_{t=0}^{k-1} \{v_{i_t}, v_{i_t+1}, \dots, v_{j_t}\}$ covered by the maximal cross-arc chain G , if there is a maximal cross-arc set $\{v_{i_t} v_{j_t} \mid v_{i_t} v_{j_t} \text{ is a maximal cross arc on the cycle } C_n, t \in \{0, 1, \dots, k-1\}\}$ on C_n satisfied:

- (1) For $k = 1$,
 - (a) $v_{i_0} v_{j_0}$ is a maximal cross arc on C_n and there is no set $\{\alpha, \alpha', \beta, \beta'\} \subseteq \{0, 1, \dots, n-1\}$ such that $\alpha < i_0 < \alpha'$ and $\beta < j_0 < \beta'$, where $v_\alpha v_{\alpha'}$, $v_\beta v_{\beta'}$ are two cross arcs on C_n ; or
 - (b) $v_{i_0} v_{j_0}$ is a maximal cross arc on C_n . There is a maximal cross arc $v_\gamma v_{\gamma'}$ on C_n such that $\gamma < i_0 < \gamma'$ (or $\gamma < j_0 < \gamma'$) and $i_0 - \gamma = 1$ (or $\gamma' - j_0 = 1$), and there is no cross arc $v_\tau v_{\tau'}$ such that $\tau < i_0 < \tau'$ and $i_0 - \tau \geq 2$ (or $\tau < j_0 < \tau'$ and $\tau' - j_0 \geq 2$) (Figure 5(a)).

Figure 5: (a) v_2v_6 is a maximal cross arc on $C = v_0v_1 \dots v_{11}v_0$, and $G = \{v_2v_6\}$ is a maximal cross-arc chain on C ; (b) $G = \{v_2v_6, v_5v_8, v_7v_{10}\}$ is a maximal cross-arc chain on $C = v_0v_1 \dots v_{11}v_0$, where v_2v_6, v_7v_{10} is a valid cross arc in G , v_5v_8 is an invalid arc in G .

- (2) For $k \geq 2$,
- (a) $|i_1 - i_0| > 1$ and $|j_{k-1} - j_{k-2}| > 1$, and when $i_0 \neq 0$ (or $j_{k-1} \neq n - 1$), there is at most one vertex v_{i_0-1} (or $v_{j_{k-1}+1}$) such that $v_{i_0-1}v_\beta$ (or $v_\beta v_{j_{k-1}+1}$) is a cross arc on C_n , where $i_0 < \beta$ (or $\beta < j_{k-1}$); and
 - (b) $i_t < i_{t+1} \leq j_t < j_{t+1}$, $t \in \{0, 1, \dots, k-2\}$; and
 - (c) There is no maximal cross arc $v_{i_t}v_{j_t}$ on C_n such that $i_{t+1} < i_y \leq j_t < j_{t+1} < j_y$, $t \in \{0, 1, \dots, k-2\}$ (Figure 5(b)).

If there is a maximal cross arc $v_{i_\tau}v_{j_\tau} \in G$ satisfying $|i_{\tau+1} - j_{\tau-1}| = 1$, then the maximal cross arc $v_{i_\tau}v_{j_\tau}$ is an *invalid cross arc* of G , where $1 \leq \tau \leq k-2$. In addition to the invalid cross arcs in G , we call the remaining maximal cross arcs the *valid cross arcs* of G . Let P_m be a subpath on C_n . If all the vertices on P_m are not covered by any maximal cross-arc chain, then call P_m a *pure subpath* on C_n . If there is no vertex $v_\alpha \in V(C_n)$ such that $D(V(P_m) \cup \{v_\alpha\})$ is a pure subpath on C_n , then P_m is a *maximal pure subpath* of C_n and $V(P_m)$ is covered by P_m . Figure 5 illustrates these definitions.

Next, we show the partition problem of the vertices of a round purely local tournament which is strong. Let D be a round purely local tournament which is strong. Let $v_0, v_1, \dots, v_{n-1}$ be a round labelling of D . Then $C = v_0v_1 \dots v_{n-1}v_0$ is the only Hamilton cycle in D . Similar to the round purely local tournament which is non-strong, $V(D)$ can form a partition, which means $V(D) = \bigcup_{i=0}^m B_i$, $B_i \cap B_j = \emptyset$ for any $i \neq j \in \{0, 1, \dots, m\}$, where B_i is covered by either some maximal pure subpath or some maximal cross-arc chain (Figure 5).

Theorem 2.6. Let D be a round purely local tournament which is strong on n vertices. Let $v_0, v_1, \dots, v_{n-1}$ be a round labelling of D . $C = v_0v_1 \dots v_{n-1}v_0$ is a Hamilton cycle in D . If there are k maximal cross-arc chains G_i on C , then $\gamma(D) = \sum_{i=0}^{k-1} \left(\tau_i + \lceil \frac{n_i}{2} \rceil \right)$, where τ_i indicates the number of valid cross arcs in G_i , n_i indicates the number of vertices contained in the maximal pure subpath P_i on C for $i \in \{0, 1, \dots, k-1\}$.

Proof. Obviously, there are k maximal pure subpaths on C . Without loss of generality, let $P_0 = v_0v_1 \dots v_{i_0-1}$, $G_0 = \{v_{i_0}v_{j_0} | t = 1, 2, \dots, \tau_0\}$, $P_1 = v_{j_0\tau_0+1}v_{j_0\tau_0+2} \dots v_{i_1-1}$, $G_1 = \{v_{i_1}v_{j_1} | t = 1, 2, \dots, \tau_1\}$, $\dots$, $P_{k-1} = v_{j_{k-2}\tau_{k-2}+1}v_{j_{k-2}\tau_{k-2}+2} \dots v_{i_{k-1}-1}$, $G_{k-1} = \{v_{i_{k-1}}v_{j_{k-1}} | t = 1, 2, \dots, \tau_{k-1}\}$, where $\tau_{k-1} = n-1$ (Figure 6). Let $D' = D - v_{n-1}v_0$. By Theorem 2.4, we have $\gamma(D') = \sum_{i=0}^{k-1} \left(\tau_i + \lceil \frac{n_i}{2} \rceil \right)$. Since $N_D^-(v_0) = \emptyset$, v_0 must be contained in any dominating set of D' . Also, owing to $N_D^-(v_0) = \{v_{n-1}\}$ and the proof of Lemma 2.2, one can obtain that v_{n-1} has been dominated by $v_{i_{k-1}\tau_{k-1}}$. Thus, $\gamma(D) = \gamma(D') = \sum_{i=0}^{k-1} \left(\tau_i + \lceil \frac{n_i}{2} \rceil \right)$. $\square$

Figure 6: A round purely local tournament D which is strong with k maximal cross-arc chains and k maximal pure subpaths.

2.3 The domination number of a round tournament

Theorem 2.7. Let D be a strong round tournament with n vertices. Let $v_0, v_1, \dots, v_{n-1}$ be a round labelling of D . Then $\gamma(D) = 2$, where all subscripts are taken modulo n .

Proof. Since D is a strong tournament, $d^+(v) \leq n - 2$ for any vertex $v \in V(D)$. It implies any vertex in D cannot form a dominating set of D , i.e. $\gamma(D) \geq 2$. Furthermore, we have $|V(D)| \geq 3$ and v_0 is adjacent to v_{n-2} since D is a strong tournament. Then the following two cases will be considered.

Case 1. $v_0 \rightarrow v_{n-2}$.

According to the definition of the round digraph, we have $v_0 \rightarrow v_t$ as well as $v_{n-2} \rightarrow v_{n-1}$, where $t \in \{1, 2, \dots, n-2\}$. Therefore, $\{v_0, v_{n-2}\} \rightarrow D$, which means $\gamma(D) \leq 2$. Thus, we have $\gamma(D) = 2$.

Case 2. $v_{n-2} \rightarrow v_0$.

Since D is a strong tournament, at this time we get $|V(D)| \geq 4$ for $v_{n-2} \rightarrow v_0$. If $v_0 \rightarrow v_{n-3}$, then $\{v_0, v_{n-2}\}$ is a dominating set of D according to $v_0 \rightarrow \{v_1, \dots, v_{n-3}\}$ and $v_{n-2} \rightarrow v_{n-1}$, i.e. $\gamma(D) \leq 2$. It implies $\gamma(D) = 2$. Conversely, if $v_{n-3} \rightarrow v_0$, then we consider v_{n-4} . Proceeding in this manner, if there exists a vertex $v_{t_0} \in \{v_2, \dots, v_{n-2}\}$ such that $v_0 \rightarrow v_{t_0}$, then $\{v_0, v_{t_0+1}\}$ is a dominating set of D . Otherwise, there will be $v_{t_0} \rightarrow v_0$, $v_{t_0} \in \{v_2, \dots, v_{n-2}\}$. If $v_2 \rightarrow v_0$, then $v_2 \rightarrow v_{u_0}$ for $u_0 \in \{v_3, v_4, \dots, v_{n-1}\}$ according to $v_2 \rightarrow v_3$ and the definition of a round digraph. From $v_0 \rightarrow v_1$, we know $\{v_0, v_2\}$ is a dominating set of D , which means $\gamma(D) \leq 2$. Then $\gamma(D) = 2$ (Figure 7). $\square$

Theorem 2.8. Let D be a non-strong round tournament with n vertices. $v_0, v_1, \dots, v_{n-1}$ is a round labelling of D . Then $\gamma(D) = 1$, where all subscripts are taken modulo n .

Proof. Since D is a round tournament which is non-strong, $v_0 \rightarrow v_i$ for $i \in \{1, 2, \dots, n-1\}$. Therefore, $\gamma(D) = 1$. $\square$

Figure 7: An example for Theorem 2.7.

Figure 8: The two cases corresponding to Theorem 3.1. (a) There exists $v_i \in V(D)$ such that $v_{i+1} \rightarrow v_i$. (b) There is a vertex set $\{v_\alpha, v_\beta\} \subseteq V(D)$ such that $\{v_\alpha v_\beta, v_\beta v_\alpha\} \subseteq A(D)$.

3 The domination number of a round non-local tournament

If a round digraph D is a non-local tournament, then there is a 2-cycle in D by Lemma 1.1. It implies D is a semicomplete digraph or a purely local semicomplete digraph. According to the definition of the round digraph, one can know that D is strongly connected.

Theorem 3.1. Let D be a round non-local tournament with n vertices. Let $v_0, v_1, \dots, v_{n-1}$ be a round labelling of D . Then

$$\gamma(D) = \begin{cases} 1, & \text{if there exists } v_i \in V(D) \text{ such that } \{v_i v_{i+1}, v_{i+1} v_i\} \subseteq A(D), \\ 2, & \text{otherwise.} \end{cases}$$

Proof. Since D is a strong round digraph, we have $v_0 v_1 \dots v_{n-1} v_0$ is a Hamilton cycle of D . If there exists $v_i \in V(D)$ such that $v_{i+1} \rightarrow v_i$, then $v_{i+1} \rightarrow v_j$ where $j \in \{i+2, i+3, \dots, i-1, i\}$ according to the definition of the round digraph. Therefore, $\{v_{i+1}\}$ is a dominating set of D . Thus, $\gamma(D) = 1$ (Figure 8(a)).

Otherwise, there is a vertex set $\{v_\alpha, v_\beta\} \subseteq V(D)$ such that $\{v_\alpha v_\beta, v_\beta v_\alpha\} \subseteq A(D)$ satisfying $|\alpha - \beta| \neq 1$ according to the round non-locally tournament D which is strong. By $v_\alpha \rightarrow v_\beta$, we have $v_\alpha \rightarrow v_\tau$ for $\tau \in \{\alpha+1, \alpha+2, \dots, \beta-1, \beta\}$. Similarly, we have $v_\beta \rightarrow v_\nu$ because $v_\beta \rightarrow v_\alpha$ for $\nu \in \{\beta+1, \beta+2, \dots, \alpha-1, \alpha\}$. Thus, $\{v_\alpha, v_\beta\}$ is a dominating set for D . Then $\gamma(D) \leq 2$. Choosing any vertex $v_\gamma \in V(D)$, we can see that $v_\gamma \rightarrow v_{\gamma-1}$ by the known conditions, which means that $\{v_\gamma\}$ cannot form a dominating set of D . For the arbitrariness of v_γ , $\gamma(D) \geq 2$. Therefore, $\gamma(D) = 2$ (Figure 8(b)). $\square$

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