Research Article

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The strong nil-cleaness of semigroup rings

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Abstract: In this paper, we study the strong nil-cleaness of certain classes of semigroup rings. For a completely 0-simple semigroup $M = \mathcal{M}^0(G; I, \Lambda; P)$, we show that the contracted semigroup ring $R_0[M]$ is strongly nil-clean if and only if either $|I| = 1$ or $|\Lambda| = 1$, and $R[G]$ is strongly nil-clean; as a corollary, we characterize the strong nil-cleaness of locally inverse semigroup rings. Moreover, let $S = [Y; S_\alpha, \varphi_{\alpha, \beta}]$ be a strong semilattice of semigroups, then we prove that $R[S]$ is strongly nil-clean if and only if $R[S_\alpha]$ is strongly nil-clean for each $\alpha \in Y$.

Keywords: strong nil-cleaness, semigroup rings, maximal subgroups, completely 0-simple semigroups, supplementary semilattice sum

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1 Introduction

Diesl [1] introduced the concept of nil-clean and strongly nil-clean rings and asked the question when a matrix ring is (strongly) nil-clean. As we have observed, it is difficult to characterize the arbitrary ring R such that $M_n(R)$ is nil-clean [2]. It is also known that the ring of all 2×2 -matrices over any commutative local ring is not strongly nil-clean [3]. On the other hand, the strong nil-cleaness of some generalizations of matrix rings has been considered. By [1], if R is a commutative ring with identity, then the formal block matrix ring $\begin{pmatrix} A & R \\ 0 & B \end{pmatrix}$ is strongly nil-clean if and only if A, B are strongly nil-clean. Moreover, the strong nil-cleaness of Morita contexts, formal matrix rings and generalized matrix rings has also been studied [2].

Let $M = \mathcal{M}^0(G; I, \Lambda; P)$ be a completely 0-simple semigroup. Then the contracted semigroup ring $R_0[M] \cong \mathcal{M}(R[G]; I, \Lambda; P)$ is a Munn ring, which is isomorphic to the matrix ring $M_{|\Lambda|}(R[G])$ if such M is a Brandt semigroup. Thus, general contracted completely 0-simple semigroup rings can be viewed as a generalization of matrix rings. Motivated by this, we want to characterize the strong nil-cleaness of the contracted semigroup rings $R_0[M]$ in terms of the structure of M and the strong nil-cleaness of the group ring $R[G]$. Noting that semigroup rings are also an extension of group rings, the strong nil-cleaness of certain kinds of group rings has been studied in [2, 4, 5].

However, for a finite completely 0-simple semigroup $M = \mathcal{M}^0(G; I, \Lambda; P)$, the contracted semigroup ring $R_0[M]$ contains an identity if and only if $|I| = |\Lambda|$ and P is an invertible matrix over G^0 , and if and only if $R_0[M]$ is isomorphic to the matrix ring $M_{|\Lambda|}(R[G])$ [6]. Note that the term “non-unital ring” or “general ring” refers to a ring that does not necessarily have an identity. It is known that Nicholson extended many of the results on the (strong) cleanness of rings with identity to non-unital rings [7], and that in [1, 2], the (strong) nil-cleaness of non-unital rings has been studied.

The paper is organized as follows. In Section 2, some preliminaries are given. In Section 3, we show that the strong nil-cleaness of the contracted semigroup ring $R_0[M]$ of a completely 0-simple semigroup

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$M = \mathcal{M}^0(G; I, \Lambda; P)$ can be characterized by $|I|, |\Lambda|$ and the strong nil-cleanliness of $R[G]$. As an application, we determine when the contracted semigroup ring of a locally inverse semigroup with idempotents set locally finite is strongly nil-clean. In Section 4, we consider the strong nil-cleanliness of certain classes of supplementary semilattice sums of rings, and the relationship between the strong nil-cleanliness of contracted semigroup rings and semigroup rings.

Throughout this paper, a ring always means an associate non-unital (or general) ring, and we always assume that R is a ring with identity (not necessarily commutative).

2 Preliminaries

In this section, the notations and definitions on semigroups and semigroup rings which will be used in the sequel are provided, see [6, 8, 9].

Unless otherwise stated, a semigroup S is always assumed to have a zero element (denoted by θ or θ_S). Denote by S^1 the semigroup obtained from S by adding an identity if S has no identity, otherwise, let $S^1 = S$. Green's equivalence relations play an important role in the theory of semigroups, which were introduced by Green (1951): for $a, b \in S$,

$$\begin{aligned} a \mathcal{L} b &\Leftrightarrow S^1 a = S^1 b, \\ a \mathcal{R} b &\Leftrightarrow a S^1 = b S^1, \\ a \mathcal{J} b &\Leftrightarrow S^1 a S^1 = S^1 b S^1 \end{aligned}$$

and $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, $\mathcal{D} = \mathcal{L} \vee \mathcal{R}$.

A semigroup is said to be *regular* if every $\mathcal{L}$ -class and every $\mathcal{R}$ -class of it contains an idempotent. An *inverse semigroup* is a regular semigroup with commutative idempotents. Let $E(S) = \{e \in S \mid e^2 = e\}$ be the set of idempotents of S . We call a regular semigroup S a *locally inverse semigroup* if eSe is an inverse subsemigroup of S for each $e \in E(S)$. It is clear that an inverse semigroup is locally inverse. We say the idempotent set $E(S)$ of a locally inverse semigroup S is *locally finite* if $E(eSe)$ is finite for each $e \in E(S)$.

We call a semigroup S a *completely 0-simple semigroup* if S is 0-simple and all its non-zero idempotents are primitive. Let G be a group, I and Λ be two non-empty sets and let $P = (p_{\lambda i})$ be a $\Lambda \times I$ -matrix with entries in $G^0 (= G \cup \{0\})$, and suppose that P is regular, in the sense that no row or column of P consists entirely of zeros. Let $M = (I \times G \times \Lambda) \cup \{0\}$ and define a multiplication on M by

$$(i, x, \lambda)(j, y, \mu) = \begin{cases} (i, xp_{\lambda j}y, \mu), & \text{if } p_{\lambda j} \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$(i, x, \lambda)0 = 0(i, x, \lambda) = 0.$$

Then M is a completely 0-simple semigroup, denoted by $\mathcal{M}^0(G; I, \Lambda; P)$. Note that the zero element of a completely 0-simple semigroup is denoted by 0 instead of θ . Conversely, every completely 0-simple semigroup is isomorphic to one constructed in this way.

In particular, if P is with entries in G , then the set $(I \times G \times \Lambda)$ forms a subsemigroup of $\mathcal{M}^0(G; I, \Lambda; P)$, we denote this subsemigroup by $\mathcal{M}(G; I, \Lambda; P)$, and call it a *completely simple semigroup*.

Lemma 2.1. [8, Theorem 3.4.1] *Let $M_1 = \mathcal{M}^0(G; I, \Lambda; P)$ and $M_2 = \mathcal{M}^0(G; I, \Lambda; P)$ be two completely 0-simple semigroups. Then $M_1 \cong M_2$ if and only if there exist a $\Lambda \times \Lambda$ diagonal matrix X over G^0 and an $I \times I$ diagonal matrix Y over G^0 such that $P = XQY$.*

By Lemma 2.1, for a completely 0-simple semigroup $M = \mathcal{M}^0(G; I, \Lambda; P)$, we can always assume the sandwich matrix P satisfies $p_{11} = e$, where $1 \in I \cap \Lambda$, and where e is the identity of G .

We introduce the concept of semigroup rings. Let S be a semigroup. The *semigroup ring* $R[S]$ of S is defined to be the ring consisting of all finite formal sums $\sum r_i s_i$, where $r_i \in R$ and $s_i \in S$, with the obvious definition of addition and with multiplication induced by the given multiplication in R and S according to the rule

$$\sum r_i s_i \sum l_j t_j = \sum (r_i l_j) s_i t_j,$$

where $r_i, l_j \in R$ and $s_i, t_j \in S$. Define $\gamma(\sum r_i s_i)$ to be $\sum (\gamma r_i) s_i$, where $\gamma, r_i \in R$ and $s_i \in S$, and if 1_R is the identity of R , then $1_R s = s$ for all $s \in S$. If I is a subset of S , let $R[I]$ denote the set of all finite R -linear combinations of elements of I . We call the factor ring $R[S]/R\theta$ the *contracted semigroup ring* of S over R , denoted by $R_0[S]$. Note that if S^0 is the semigroup resulting from the adjunction of a zero element θ to S (whether or not S has a zero to begin with), then $R_0[S^0] \cong R[S]$. Thus, any semigroup ring can be regarded as a contracted semigroup ring.

Let S be a semigroup and $\sum r_i s_i \in R_0[S]$, where $r_i \in R$ and $s_i \in S$. Define the *support set* of $\sum r_i s_i$ to be the subset $\text{supp}(\sum r_i s_i) = \{s_i \in S \mid r_i \neq 0\}$ of S .

Recall the definition of a Munn ring. Let T be a ring, I, Λ be two non-empty sets. It will be necessary to describe a $I \times \Lambda$ -matrix over T in terms of its entries. In this case, we use such notations as

$$X = [a_{i\lambda}], \quad X = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad X = [a_1, a_2, \dots, a_n, \dots].$$

We use $[a]_{i\lambda}$ to represent a $I \times \Lambda$ -matrix with only one non-zero entry a in the position (i, λ) . Let $X = [a_{i\lambda}]$ be a $I \times \Lambda$ -matrix over T . We say that the matrix X is *bounded* if $a_{i\lambda} = 0$ for all but finitely many (i, λ) . Let P be a fixed $\Lambda \times I$ matrix over T (possibly not bounded) with the property that every row and every column of P contain an unit of T , and let $\mathcal{M}(T; I, \Lambda; P)$ be the set of all bounded $I \times \Lambda$ matrices over T . Then $\mathcal{M}(T; I, \Lambda; P)$ becomes a ring over R with the usual matrix addition and the multiplication $\circ$ defined by: $X \circ Y = XPY$, which is the usual matrix multiplication. We call $\mathcal{M}(T; I, \Lambda; P)$ a *Munn ring over T with the sandwich matrix P* , or briefly a *Munn ring* [9,10]. Let $S = \mathcal{M}^0(G; I, \Lambda; P)$ be a completely 0-simple semigroup over a group G . By [6, Lemma 5.17], $R_0[S] \cong \mathcal{M}(R[G]; I, \Lambda; P)$ is a Munn ring.

An element a in a ring U is said to be *left quasi-regular* if there exists $u \in U$ such that $u + a + ua = 0$. A left ideal I of U is said to be *left quasi-regular* if every element of I is left quasi-regular. The *Jacobson radical* $J(U)$ of U is defined to be the left quasi-regular left ideal which contains every left quasi-regular left ideal of U [11]. We provide a method for determining whether an element of U belongs to $J(U)$ or not.

Lemma 2.2. [11, Lemma 2.15(ii)] *Let U be a ring and $a \in U$. Then $a \in J(U)$ if and only if Ua is a left quasi-regular left ideal of U .*

Note that the aforementioned definitions and results on Jacobson radicals are valid with “left” replaced by “right.”

3 Strong nil-cleanness of completely 0-simple semigroup rings

The first part of this section presents some elementary definitions and results on strongly nil-clean rings. The remainder of this section is concerned with the strong nil-cleanness of the contracted semigroup rings of both completely 0-simple semigroups and locally inverse semigroups.

Let U be a ring. Following Diesl [1], an element $u \in U$ is said to be *nil-clean* if there is an idempotent $e \in U$ and a nilpotent $b \in U$ such that $u = e + b$. The element u is further said to be *strongly nil-clean* if such an idempotent and a nilpotent can be chosen to satisfy the equality $be = eb$. The ring U is called a *nil-clean* (respectively, *strongly nil-clean*) ring if each element in it is nil-clean (respectively, strongly nil-clean).

We establish a basic result which will be used later.

Lemma 3.1. [1, Proposition 2.9] *A left (resp., right) ideal of a strongly nil-clean ring is also strongly nil-clean.*

The following lemma provides a very useful characterization of strongly nil-clean rings, in terms of Jacobson radicals. A ring is said to be *boolean* if every element of the ring is an idempotent.

Lemma 3.2. [2] *A ring U is strongly nil-clean iff $U/J(U)$ is boolean and $J(U)$ is nil.*

By Lemma 3.2, it will be helpful to know the Jacobson radicals of contracted completely 0-simple semigroup rings.

Lemma 3.3. [12] *Let $M = \mathcal{M}^0(G; I, \Lambda; P)$ be a completely 0-simple semigroup. Then*

$$J(R_0[M]) = \{X \in R_0[M] \mid PXP \in (J(R[G]))_{\Lambda \times \Lambda}\}.$$

Noting that for a completely 0-simple semigroup $M = \mathcal{M}^0(G; I, \Lambda; P)$, if we denote the identity of G by e , then $eMe \cong G$, and thus $eR_0[M]e \cong R[G]$, whence $R[G]$ is a corner ring of $R_0[M]$. Diesl proved the inheritance of the strong nil-cleanness by corner subrings of a strong nil-cleanly ring with identity from the ring with identity. This remains true for non-unital rings by applying similar argument in the proof of [1, Proposition 3.25].

Lemma 3.4. *Let U be a ring and f be any idempotent in U . If U is strongly nil-clean, then the ring fUf is also strongly nil-clean.*

Proof. Since Uf is a left ideal of U and fUf is a right ideal of Uf , we apply Lemma 3.1 to obtain first that Uf is a strongly nil-clean ring, and then that fUf is a strongly nil-clean ring. The lemma is proved. $\square$

Later, Lemma 3.4 is used to give a necessary condition for a contracted completely 0-simple semigroup ring to be strongly nil-clean.

Lemma 3.5. *Let M be a completely 0-simple semigroup with exactly two $\mathcal{R}$ -classes and exactly two $\mathcal{L}$ -classes. Then its contracted semigroup ring $R_0[M]$ is not strongly nil-clean.*

Proof. By hypothesis, the completely 0-simple semigroup M is of the form $\mathcal{M}^0(G; \mathbf{2}, \mathbf{2}; P)$, where G is a group, $\mathbf{2} = \{1, 2\}$ and P is a 2×2 -matrix over G^0 . Since M is a regular semigroup, by Lemma 2.1, the sandwich matrix P must be one of the matrices in the set $\{\Delta, N_1, N'_1, N_2, N'_2, N_3\}$, up to isomorphic, where

$$\Delta = \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}, \quad N_1 = \begin{bmatrix} e & e \\ e & 0 \end{bmatrix}, \quad N'_1 = \begin{bmatrix} e & 0 \\ e & e \end{bmatrix},$$

$$N_2 = \begin{bmatrix} e & e \\ 0 & e \end{bmatrix}, \quad N'_2 = \begin{bmatrix} 0 & e \\ e & e \end{bmatrix}, \quad N_3 = \begin{bmatrix} e & e \\ e & g \end{bmatrix},$$

and where e is the identity of the group G and $g \in G$. We claim that the contracted semigroup ring $R_0[M]$ is not strongly nil-clean, no matter which of the above normalized matrix P is chosen. There are five cases to be considered. For convenience, let $\mathcal{F}$ denote the set consisting of all the fields of characteristic 2.

Case 1. $P = \Delta$. Then $R_0[M] \cong \mathcal{M}(R[G]; \mathbf{2}, \mathbf{2}; P) \cong \mathbb{M}_2(R[G])$, and thus $R_0[M]$ is not strongly nil-clean, see [3].

Case 2. $P = N_1$, $P = N'_1$, $P = N_2$ or $P = N'_2$. It is clear that P is an invertible matrix, and thus the map $R_0[M] \rightarrow \mathbb{M}_2(R[G])$ defined by $A \mapsto AP$ is an isomorphism of rings. Then, by Case 1, $R_0[M]$ is not strongly nil-clean.

Case 3. $P = N_3$ with $g \neq e$, and $R \notin \mathcal{F}$. Noting that each element in the Munn ring $R_0[M] = \mathcal{M}(R[G]; \mathbf{2}, \mathbf{2}; P)$ is of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a, b, c, d \in R[G]$. By Lemma 3.3, it is easy to verify that

$$J(R_0[M]) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R_0[M] \mid a + b + c + d, a + c + (b + d)g \in J(R[G]) \right\}. \quad (3.1)$$

By contrary, suppose that $R_0[M]$ is strongly nil-clean. This, together with Lemma 3.2, yields that $R_0[M]/J(R_0[M])$ is boolean. Let c be an arbitrary element of $R[G]$, define $N = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \in R_0[M]$. Then $N \circ N + J(R_0[M]) = (N + J(R_0[M]))^2 = N + J(R_0[M])$, which is equivalent to say that $N \circ N - N \in J(R_0[M])$. Note that $N \circ N = NN_3N = \begin{bmatrix} 0 & 0 \\ 0 & cgc \end{bmatrix}$, hence $N \circ N - N = \begin{bmatrix} 0 & 0 \\ 0 & cgc - c \end{bmatrix}$. It follows that $(cgc - c)g \in J(R[G])$ by (3.1). Next, we show that $J(R[G])$ is nil. For this, note that the group ring $R[G] \cong \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} R_0[M] \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$ is a corner subring of $R_0[M]$. Since $R_0[M]$ is strongly nil-clean by assumption, $R[G]$ is strongly nil-clean by Lemma 3.4. Then it follows from Lemma 3.2 that $J(R[G])$ is nil. Since g is an invertible element in G , we deduce that $(cg - e)c$ is a nilpotent element. Since c is arbitrary, we can take $c = -g^{-1}$, it follows that $-2g^{-1} \in R[G]$ is nilpotent. This, together with the fact $R \notin \mathcal{F}$, implies that the element $g^{-1} \in G$ is nilpotent, that is, $(g^{-1})^m = 0$ for some positive integer m , which is a contradiction. Consequently, $R_0[M]$ is not strongly nil-clean, as required.

Case 4. $P = N_3$ and $g = e$, and $R \notin \mathcal{F}$. By Case 3, if $J(R[G])$ is not nil, then $R_0[M]$ is not strong nil-clean. Suppose now that $J(R[G])$ is nil. We claim that $R_0[M]$ is not strongly nil-clean. By Lemma 3.2, it is sufficient to prove that $R_0[M]/J(R_0[M])$ is not boolean. Noting that, by Lemma 3.3, we have

$$J(R_0[M]) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R_0[M] \mid a + b + c + d \in J(R[G]) \right\}. \quad (3.2)$$

Let c, d be two arbitrary elements of $R[G]$. Define $N = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} \in R_0[M]$. Then

$$N \circ N = NN_3N = \begin{bmatrix} 0 & 0 \\ (c+d)c & (c+d)d \end{bmatrix},$$

thus

$$N \circ N - N = \begin{bmatrix} 0 & 0 \\ (c+d)c - c & (c+d)d - d \end{bmatrix}.$$

By contrary, assume that $N \circ N - N \in J(R_0[M])$. It follows that $(c+d)(c+d-e) \in J(R[G])$. If we take $c+d = -e$, then $2e = (c+d)(c+d-e) \in J(R[G])$. Since $J(R[G])$ is nil, there exists a positive integer n such that $2^n e = 0$, which is a contradiction, because $R \notin \mathcal{F}$. Therefore, $R_0[M]$ is not strongly nil-clean.

Case 5. $P = N_3$ and $R \in \mathcal{F}$. By contrary, suppose that $R_0[M]$ is a strongly nil-clean ring. By similar arguments to the above, we infer that $J(R[G])$ is nil, and $R_0[M]/J(R_0[M])$ is boolean. It is easy to check that

$$N_2 \circ N_2 - N_2 = \begin{bmatrix} 0 & e+g \\ e & e+g \end{bmatrix}. \text{ Since } \text{char} R = 2, \text{ we have that } e+g+(e+g)+e=e, \text{ which does not belong to}$$

$J(R[G])$. This, together with (3.2), implies that $N_2 + J(R_0[M])$ is not an idempotent in $R_0[M]/J(R_0[M])$. This is a contradiction, because $R_0[M]/J(R_0[M])$ is boolean. Therefore, $R_0[M]$ is not strongly nil-clean.

Consequently, in either case, the contracted semigroup ring $R_0[M]$ is not strongly nil-clean, as required. $\square$

If $M = \mathcal{M}^0(G; \mathbf{2}, \mathbf{2}; \Delta)$ is a Brandt semigroup, where Δ is the identity matrix over G^0 , by Lemma 3.5, $R_0[M] \cong \mathbb{M}_2(R[G])$ is not strongly nil-clean. This may be thought of as an extension of the result in [3].

By applying Lemmas 3.1 and 3.5, a necessary condition for a contracted completely 0-simple semigroup ring to be strongly nil-clean is obtained.

Proposition 3.6. *Let $M = \mathcal{M}^0(G; I, \Lambda; P)$ be a completely 0-simple semigroup. If $R_0[M]$ is strongly nil-clean, then either $|I| = 1$ or $|\Lambda| = 1$.*

Proof. Suppose that $R_0[M]$ is strongly nil-clean. Noting that $|I| \geq 1$ and $|\Lambda| \geq 1$. Then in order to show either $|I| = 1$ or $|\Lambda| = 1$, assume on the contrary that $|I| \geq 2$ and $|\Lambda| \geq 2$. It would then follow that the

$\Lambda \times I$ -matrix P may be written in the block form $\begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$, where P_1 is a 2×2 -matrix over G^0 . Define a subset

$A = \{(a_{i\lambda}) \in R_0[M] \mid a_{i\lambda} = 0 \text{ for all } i \neq 1, 2\}$ of $R_0[M]$. It is clear that A is closed under multiplication. Moreover, let $(a_{i\lambda}) \in A$ and $(b_{j\mu}) \in R_0[M]$, then the entry in the (i, μ) -position of the product $(a_{i\lambda}) \circ (b_{j\mu})$ is $\sum_{\lambda \in \Lambda, j \in I} a_{i\lambda} p_{\lambda j} b_{j\mu}$. Note that $\sum_{\lambda \in \Lambda, j \in I} a_{i\lambda} p_{\lambda j} b_{j\mu} = 0$ for all $i \notin \{1, 2\}$ because $a_{i\lambda} = 0$ for all such i . It follows that $(a_{i\lambda}) \circ (b_{j\mu}) \in A$, whence A is a right ideal of $R_0[M]$. Since $R_0[M]$ is strongly nil-clean, A is strongly nil-clean by Lemma 3.1. Moreover, A is isomorphic to the Munn ring $\mathcal{M}\left(R[G]; 2, \Lambda; \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}\right)$. Then a similar argument shows that the Munn ring $U = \mathcal{M}(R[G]; 2, 2; P_1)$ is isomorphic to a left ideal of A , and hence U is strongly nil-clean by Lemma 3.1 again. This is a contradiction, because $U \cong R_0[\mathcal{M}^0(G; 2, 2; P_1)]$ is not strongly nil-clean by Lemma 3.5. Therefore, we must have either $|I| = 1$ or $|\Lambda| = 1$. $\square$

The interesting case occurs when a completely 0-simple semigroup $M = \mathcal{M}^0(G; I, \Lambda; P)$ contains a unique $\mathcal{L}$ -class (or, a unique $\mathcal{R}$ -class).

Lemma 3.7. *Let $M = \mathcal{M}^0(G; I, \Lambda; P)$ be a completely 0-simple semigroup. If $|I| = 1$ or $|\Lambda| = 1$, then $R_0[M]$ is strongly nil-clean if and only if $R[G]$ is strongly nil-clean.*

Proof. It is sufficient to consider the case of $|\Lambda| = 1$, since the case of $|I| = 1$ is dual. Assume that $|\Lambda| = 1$. Before moving on, we explore the structure of M and determine $J(R_0[M])$. For this, since M is a completely 0-simple semigroup, S is regular, whence every $\mathcal{R}$ -class of M contains at least one idempotent. Since M has only one $\mathcal{L}$ -class, it follows that each $\mathcal{R}$ -class of M is a maximal group in M , and hence the $1 \times I$ -matrix P must be of the form $[e, e, \dots, e, \dots]$ whose entries are all equal to e , where e is the identity of the group G . Denote $\mathbf{N}^*$ by the set $\{1, 2, \dots\}$ consisting of all positive integers. Let $x = [x_1, x_2, \dots, x_n, 0, \dots]^T \in R_0[M]$ with $x_1, x_2, \dots, x_n \in R[G]$ and whose other entries are all equal to 0, where $n \in \mathbf{N}^*$. Then it is easy to check that $PxP = (\sum_{i=1}^n x_i)[e, e, \dots, e, \dots]$. Thus, by Lemma 3.3,

$$J(R_0[M]) = \left\{ x = [x_1, x_2, \dots, x_n, 0, \dots]^T \in R_0[M] \mid \sum_{i=1}^n x_i \in R[G] \right\}. \quad (3.3)$$

Now we suppose that $R[G]$ is strongly nil-clean. By Lemma 3.2, $R[G]/J(R[G])$ is boolean and $J(R[G])$ is nil. We want to show that $R_0[M]$ is strongly nil-clean. It suffices to prove that $J(R_0[M])$ is nil and $R_0[M]/J(R_0[M])$ is boolean, by Lemma 3.2 again. For this, let $x = [x_1, x_2, \dots, x_n, 0, \dots]^T \in J(R_0[M])$, where $x_1, x_2, \dots, x_n \in R[G]$ and $n \in \mathbf{N}^*$. By (3.3), $\sum_{i=1}^n x_i \in J(R[G])$, and because $J(R[G])$ is nil, it follows that $\sum_{i=1}^n x_i$ is a nilpotent element.

Note that, for any element $y = [y_1, y_2, \dots, y_t, 0, \dots]^T \in R_0[M]$, where $y_1, y_2, \dots, y_t \in R[G]$ and $t \in \mathbf{N}^*$, we have $y \circ y = yPy = y(\sum_{i=1}^t y_i)$, and thus for any integer $k \geq 2$, $y^k = y^{k-2} \circ y \circ y = y^{k-2} \circ y(\sum_{i=1}^t y_i) = \dots = y \cdot (\sum_{i=1}^t y_i)^{k-1}$.

Therefore, x is a nilpotent element when $x \in J(R_0[M])$, which yields that $J(R_0[M])$ is nil.

On the other hand, for any $z = [z_1, z_2, \dots, z_m, 0, \dots]^T \in R_0[M] \setminus J(R_0[M])$, where $z_1, \dots, z_m \in R[G]$ and $m \in \mathbf{N}^*$, we have $\sum_{j=1}^m z_j \notin J(R[G])$, whence $\sum_{j=1}^m z_j \in R[G] \setminus J(R[G])$. Note that

$$\begin{aligned} z^2 - z &= \left[z_1 \sum_{j=1}^m z_j - z_1, \dots, z_m \sum_{j=1}^m z_j - z_m, 0, \dots \right]^T \\ &= \left[z_1 \left(\sum_{j=1}^m z_j - e \right), \dots, z_m \left(\sum_{j=1}^m z_j - e \right), 0, \dots \right]^T \\ &= [z_1, \dots, z_m, 0, \dots]^T \left(\sum_{j=1}^m z_j - e \right), \end{aligned}$$

where e is the identity of G . Note also that $\sum_{i=1}^m z_i \left(\sum_{j=1}^m z_j - e \right) = \left(\sum_{j=1}^m z_j \right)^2 - \sum_{j=1}^m z_j \in J(R[G])$, because $\sum_{j=1}^m z_j \in R[G] \setminus J(R[G])$ and $R[G]/J(R[G])$ is boolean. Then, by (3.3), $z^2 - z \in J(R_0[M])$. It follows that $R_0[M]/J(R_0[M])$ is boolean.

Conversely, suppose that $R_0[M]$ is strongly nil-clean. We claim that $R[G]$ is strongly nil-clean. Indeed, let f denote the element $[e, 0, \dots, 0, \dots]^T$ in $R_0[M]$, where e is the identity of G . Then f is an idempotent, and moreover $f \circ R_0[M] \circ f \cong R[G]$. This, together with Lemma 3.4, implies that $R[G]$ is strongly nil-clean. $\square$

We have obtained all the preliminaries needed to prove the following principal theorem in this section.

Theorem 3.8. *Let $M = \mathcal{M}^0(G; I, \Lambda; P)$ be a completely 0-simple semigroup. Then $R_0[M]$ is strongly nil-clean if and only if $R[G]$ is strongly nil-clean, and either $|I| = 1$ or $|\Lambda| = 1$.*

Proof. Note that if $R_0[M]$ is strongly nil-clean, then $|I| = 1$ or $|\Lambda| = 1$ by Lemma 3.6. Then the result follows immediately from Lemma 3.7. $\square$

Note that for a completely 0-simple semigroup $M = \mathcal{M}^0(G; I, \Lambda; P)$, if $R_0[M]$ is strongly nil-clean, it follows from the proof of Theorem 3.8 that all entries of P must be non-zero, and hence M must be of the form $\mathcal{M}(G; I, \Lambda; P) \cup \{0\}$.

In the following, we provide two corollaries of Theorem 3.8.

Corollary 3.9. *Let $M = \mathcal{M}(G; I, \Lambda; P)$ be a completely simple semigroup. Then $R[M]$ is strongly nil-clean if and only if $R[G]$ is strongly nil-clean, and either $|I| = 1$ or $|\Lambda| = 1$.*

Proof. Note that the set $M^0 := \mathcal{M}(G; I, \Lambda; P) \cup \{0\}$ forms a completely 0-simple semigroup with $R_0[M^0] \cong R[M]$. The result then follows from Theorem 3.8. $\square$

Corollary 3.10. *Let S be a locally inverse semigroup with idempotent set locally finite and finitely many $\mathcal{D}$ -classes. Then $R_0[S]$ is strongly nil-clean if and only if the following two conditions hold.*

- (i) *Each $\mathcal{D}$ -class D of S has either a unique $\mathcal{R}$ -class or a unique $\mathcal{L}$ -class;*
- (ii) *For each maximal subgroup G of S , $R[G]$ is strongly nil-clean.*

Proof. By [13, Theorem 5.1], $R_0[S]$ is a finite direct product of the contracted completely 0-simple semigroup rings $R_0[\overline{D_\alpha}^0]$, where α runs over the set $(S/\mathcal{D})^*$ consisting all non-zero elements in $S/\mathcal{D}$. Then by [1, Proposition 3.13], $R_0[S]$ is strongly nil-clean if and only if $R_0[\overline{D_\alpha}^0]$ is strongly nil-clean for each $\alpha \in (S/\mathcal{D})^*$. Furthermore, for each $\alpha \in (S/\mathcal{D})^*$, the maximal subgroup of $\overline{D_\alpha}^0$ is isomorphic to the maximal subgroup in D_α , and the number of $\mathcal{L}$ -classes (resp., $\mathcal{R}$ -classes) in $\overline{D_\alpha}^0$ is equal to the number of $\mathcal{L}$ -classes (resp., $\mathcal{R}$ -classes) in D_α , see [13]. The result then follows from Theorem 3.8. $\square$

4 Strong nil-cleanness and strong semilattice of semigroups

We recall the definition of strong semilattice of semigroups. Let Y be a semilattice with natural partial order $\leq$ and S_α be a family of semigroups indexed by Y . We say a semigroup S is a *strong semilattice of semigroups* S_α ($\alpha \in Y$) if S is the disjoint union of the semigroups S_α , and for all $\alpha > \beta$ in Y there exists a homomorphism $\varphi_{\alpha,\beta} : S_\alpha \rightarrow S_\beta$ satisfying the following conditions:

- (i) $\varphi_{\alpha,\alpha}$ is the identity map of S_α ;
- (ii) If $\alpha > \beta > \gamma$, then $\varphi_{\beta,\gamma} \varphi_{\alpha,\beta} = \varphi_{\alpha,\gamma}$;
- (iii) If $a \in S_\alpha$ and $b \in S_\beta$, then the multiplication in S is given by $ab = \varphi_{\alpha,\alpha\beta}(a) \varphi_{\beta,\alpha\beta}(b)$.

If this is the case, denote the semigroup S by $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$. Note that each $\varphi_{\alpha,\beta}$ can be R -linearly extended into a semigroup ring homomorphism from $R[S_\alpha]$ to $R[S_\beta]$. Note also that $R[S]$ is always a supplementary semilattice sum of subrings $R[S_\alpha]$ ($\alpha \in Y$).

For the rest of this section, we always assume that S is a strong semilattice Y of semigroups S_α ($\alpha \in Y$), where Y is finite. The main aim of this section is to construct a relationship between the strong nil-cleaness of $R_0[S]$ and the strong nil-cleaness of $R[S_\alpha]$ ($\alpha \in Y$).

Let β be a fixed maximal element in Y , and then denote the set $S \setminus \{S_\beta\}$ by T . Thus, T is an ideal of S , whence $R_0[T]$ is an ideal of $R_0[S]$. We introduce some preliminary results and definitions which are used to prove the main result of this section later.

Lemma 4.1. *Every idempotent of $R_0[S]/R_0[T]$ can be lifted to an idempotent of $R_0[S]$.*

Proof. Suppose that $x_1 + x_2 + R_0[T] \in R_0[S]/R_0[T]$ is an idempotent, where $x_1 \in R[S_\beta]$ and $x_2 \in R_0[T]$. Then $(x_1^2 - x_1) + (x_1x_2 + x_2x_1 + x_2^2 - x_2) \in R_0[T]$. Since β is maximal in Y and $R[S_\beta]$ is a subring of $R_0[S]$, we infer that $\text{supp}(x_1^2 - x_1) \cap \text{supp}(x_1x_2 + x_2x_1 + x_2^2 - x_2) = \emptyset$, which yields that $x_1 = x_1^2 \in R[S_\beta]$ is an idempotent. Since $x_1 + x_2 + R_0[T] = x_1 + R_0[T]$, the result follows. $\square$

Lemma 4.2. *Let $x = x_1 + x_2 \in R_0[S]$, where $x_1 \in R[S_\beta]$ and $x_2 \in R_0[T]$. If $x \in J(R_0[S])$, then $x_1 \in J(R[S_\beta])$.*

Proof. Suppose that $0 \neq x = x_1 + x_2 \in J(R_0[S])$. We claim that $x_1 \in J(R[S_\beta])$. For this, let z be an arbitrary element in $R[S_\beta]$, then by hypothesis, there exists an element $y \in R_0[S]$ such that $zx + y + y(zx) = 0$. Write $y = y_1 + y_2$, where $y_1 \in R[S_\beta]$ and $y_2 \in R_0[T]$. It follows from the maximality of β that $zx_1 + y_1 + y_1(zx_1) = 0$, whence $R[S_\beta]x_1$ is a left quasi-regular left ideal of $R[S_\beta]$. By Lemma 2.2, $x_1 \in J(R[S_\beta])$, as required. $\square$

Let E be a semilattice and $\alpha, \gamma \in E$. Then γ is said to be *maximal under α* [14] if $\alpha > \gamma$ and there is no $\nu \in E$ such that $\alpha > \nu > \gamma$. Denote by $\hat{\alpha}$ the set $\{\gamma \in E : \gamma \text{ is maximal under } \alpha\}$. Let α be a non-zero element in the finite semilattice Y . For each $a \in R[S_\alpha]$, by [14], define an element

$$a^* = a + \sum_{\substack{\{\alpha_{i_1}, \dots, \alpha_{i_t}\} \subseteq \hat{\alpha}, \\ t \geq 1}} (-1)^t \varphi_{\alpha, \alpha_{i_1}, \dots, \alpha_{i_t}}(a) \in R_0[S], \quad (4.1)$$

where $\{\alpha_{i_1}, \dots, \alpha_{i_t}\}$ runs over all non-empty subsets of $\hat{\alpha}$. Noting that, in (4.1), each element $\alpha_{i_1}, \dots, \alpha_{i_t} \in Y$ is strictly smaller than α , and thus the element $\varphi_{\alpha, \alpha_{i_1}, \dots, \alpha_{i_t}}(a) \in R_0[S]$ actually belongs to $\sum_{\gamma < \alpha} R[S_\gamma]$.

Lemma 4.3. [14] *Let α be a non-zero element in Y . If A is an ideal of $R[S_\alpha]$, then $A^* = \{a^* \mid a \in A\}$ is an ideal of $R_0[S]$. Moreover, the map $A \rightarrow A^* : a \mapsto a^*$ is a ring isomorphism from A to A^* .*

Note that Lemma 4.3 is true for the more general case when Y is taken to be a pseudofinite semilattice [14].

Lemma 4.4. $J(R[S_\beta]) + R_0[T] \subseteq J(R_0[S]) + R_0[T]$.

Proof. If $J(R_0[S]) = 0$, then $J(R[S_\alpha]) = 0$ for each $\alpha \in Y$, because $J(R_0[S]) = 0$ if and only if $J(R[S_\alpha]) = 0$ for each $\alpha \in Y$ by [14, Theorem 2]. Hence, in this case, $J(R[S_\beta]) + R_0[T] \subseteq J(R_0[S]) + R_0[T]$ holds true. If $J(R_0[S]) \neq 0$, then we claim that $J(R[S_\beta]) + R_0[T] \subseteq J(R_0[S]) + R_0[T]$. Indeed, for any element $0 \neq x \in J(R[S_\beta])$, the element $x^* \in R_0[S]$ defined by (4.1) is non-zero, and furthermore belongs to $J(R[S_\beta])^*$. Since $J(R[S_\beta])^*$ is an ideal of $R_0[S]$ and is isomorphic to $J(R[S_\beta])$ by Lemma 4.3, we infer that $J(R[S_\beta])^*$ is a left quasi-regular ideal of $R_0[S]$. Because $J(R_0[S])$ contains any left quasi-regular left ideal of $R_0[S]$, $J(R[S_\beta])^* \subseteq J(R_0[S])$. It follows that $x^* \in J(R_0[S])$. Therefore, by the definition of x^* again, $x + R_0[T] = x^* + R_0[T] \in J(R_0[S]) + R_0[T]$. Consequently, we have $J(R[S_\beta]) + R_0[T] \subseteq J(R_0[S]) + R_0[T]$. $\square$

Lemma 4.5. $J(R_0[S]/R_0[T]) = (J(R_0[S]) + R_0[T])/R_0[T]$.

Proof. First, we show that $J(R_0[S]/R_0[T]) \subseteq (J(R_0[S]) + R_0[T])/R_0[T]$. For this, let $x + R_0[T] \in J(R_0[S]/R_0[T])$ with $x \in R[S_\beta]$. Then for each $y \in R[S_\beta]$, $yx + R_0[T]$ is a left quasi-regular element of $R_0[S]/R_0[T]$. This, together

with the maximality of β , implies that $R[S_\beta]x$ is a left quasi-regular left ideal of $R[S_\beta]$, whence $x \in J(R[S_\beta])$. By Lemma 4.4, we infer that $x \in J(R_0[S]) + R_0[T]$, and hence $x + R_0[T] \in (J(R_0[S]) + R_0[T])/R_0[T]$, as required. Next, we prove that $(J(R_0[S]) + R_0[T])/R_0[T] \subseteq J(R_0[S]/R_0[T])$. Assume that $x + R_0[T] \in (J(R_0[S]) + R_0[T])/R_0[T]$, where $x \in J(R_0[S])$. Then x is a left quasi-regular element in $R_0[S]$, which is equivalent to say that $R_0[S]x$ is a left quasi-regular left ideal of $R_0[S]$, whence $(R_0[S]/R_0[T])(x + R_0[T])$ is also a left quasi-regular left ideal of $R_0[S]/R_0[T]$. This implies that $x + R_0[T] \in J(R_0[S]/R_0[T])$, as required. $\square$

The following result provides us an equivalent characterization for a non-unital ring to be strongly nil-clean, which plays an important role in the below proof.

Lemma 4.6. [2] *Let A be a non-unital ring and U be an ideal of A . Then A is strongly nil-clean if and only if the following three conditions hold:*

- (i) U and A/U are strongly nil-clean;
- (ii) Every idempotent of A/U can be lifted to an idempotent of A ;
- (iii) $J(A/U) = (U + J(A))/U$.

We are now ready to prove the main result of this section concerning the strong nil-cleaness of the contracted semigroup ring $R_0[S]$.

Theorem 4.7. *Let S be a strong semilattice Y of semigroups S_α ($\alpha \in Y$), where Y is finite. Then $R_0[S]$ is strongly nil-clean if and only if $R_0[S_{\theta_Y}]$ is strong nil-clean and $R[S_\alpha]$ is strongly nil-clean for each $\theta_Y \neq \alpha \in Y$.*

Proof. We prove the result by induction on the cardinality of the set Y . If $Y = \{\theta_Y\}$, this is trivial. If $|Y| \geq 2$, then let β be a maximal element in Y and define Y' be the subset $Y \setminus \{\beta\}$ of Y . By induction, the result holds for $T = \cup_{\alpha \in Y' \cup \{\beta\}} S_\alpha$, which means that $R_0[T]$ is strongly nil-clean if and only if $R[S_\alpha]$ is strongly nil-clean for each $\alpha \notin \{\theta_Y, \beta\}$ and $R_0[S_{\theta_Y}]$ is strongly nil-clean. Thus, in order to show that the result holds for S , it is sufficient to prove that $R_0[S]$ is strongly nil-clean if and only if both $R[S_\beta] = R_0[S]/R_0[T]$ and $R_0[T]$ are strongly nil-clean. Note that Lemma 4.6(ii) and (iii) holds true if we replace A , U and A/U by $R_0[S]$, $R_0[T]$ and $R_0[S]/R_0[T]$ ($\cong R[S_\beta]$), respectively, by Lemmas 4.1 and 4.5. Then the result is obtained immediately by Lemma 4.6. $\square$

A similar argument in the proof of Theorem 4.7 provides us a characterization of the strong nil-cleaness of the semigroup ring $R[S]$.

Corollary 4.8. *Let S be a strong semilattice Y of semigroups S_α ($\alpha \in Y$), where Y is finite. Then $R[S]$ is strongly nil-clean if and only if $R[S_\alpha]$ is strongly nil-clean for each $\alpha \in Y$.*

We end this paper with a brief discussion of the relationship between the strong nil-cleaness of semigroup rings and the strong nil-cleaness of contracted semigroup rings. Let N be a semigroup. Since $R_0[N] = R[N]/R\theta$, we get that $R_0[N]$ is strongly nil-clean whenever $R[N]$ is strongly nil-clean by Lemma 4.6. However, if $R_0[N]$ is strongly nil-clean, it does not follow that $R[N]$ is strongly nil-clean. The following example illustrates this fact.

Example 4.1. Let K be a field and $N = \{a, \theta\}$ be a semigroup with multiplication given by $a^2 = a\theta = \theta a = \theta^2 = \theta$.

- (i) Note that each non-zero element of $R_0[N]$ is of the form ka , where $0 \neq k \in K$. Then in $R_0[N]$, $(ka)^2 = k^2a^2 = 0$. Hence, $R_0[N]$ is nil, whence $R_0[N]$ is strongly nil-clean.
- (ii) It is clear that the set $\{ka - k\theta \mid k \in K\}$ (resp., $\{\theta\}$) collects all nilpotent elements (resp., idempotents) in $R[N]$. We claim that $R[N]$ is not strongly nil-clean. For this, it suffices to show that the element $2a \in R[N]$ is not strongly nil-clean. By contrary, suppose that $2a = \theta + (ka - k\theta)$ for some $k \in K$, then k must be equal to 1_K , whence $2a = a$, which is a contradiction.

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