Research Article

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Boundary layer analysis for a 2-D Keller-Segel model

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Abstract: We study the boundary layer problem of a Keller-Segel model in a domain of two space dimensions with vanishing chemical diffusion coefficient. By using the method of matched asymptotic expansions of singular perturbation theory, we construct an accurate approximate solution which incorporates the effects of boundary layers and then use the classical energy estimates to prove the structural stability of the approximate solution as the chemical diffusion coefficient tends to zero.

Keywords: Keller-Segel model, boundary layer phenomenon, matched asymptotic expansions, energy estimates

MSC 2020: 35B25, 35B40, 35Q92, 92C17

1 Introduction

Chemotaxis is the directed movement of cells in response to certain chemical substances in their environment, and it plays an important role in many biological processes. In 1970, Keller and Segel [1] proposed the following system to describe the aggregation of cellular slime mold toward higher concentration of the chemical signal

$$\begin{cases} n_t = \nabla \cdot (D(n) \nabla n) - \nabla \cdot (\psi(n) \nabla c), & x \in \Omega, \quad t > 0, \\ c_t = \Delta c - n f(c), & x \in \Omega, \quad t > 0, \\ \partial_\nu n = \partial_\nu c = 0, & x \in \partial\Omega, \quad t > 0, \\ n(x, 0) = n_0(x), \quad c(x, 0) = c_0(x), & x \in \Omega, \end{cases} \quad (1)$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, ∂_ν denotes the directional derivative in the outer normal direction of $\partial\Omega$, $n(x, t)$ denotes the bacterial population density and $c(x, t)$ denotes the chemical concentration. The parameter $D(n)$ is the bacterial diffusion coefficient and $\psi(n)$ is the chemotactic coefficient. The quantity $f(c)$ denotes the consumption rate of the chemical substance by the bacteria.

System (1) has been widely studied in the literature and is well understood. For example, in the case $\psi(n) = n\chi(c)$ for some smooth function $\chi(c) > 0$, under different assumptions, Wang et al. have written a series of articles [2–4] on the existence and uniqueness of global classical solutions of (1). Liu et al. [5] considered system (1) with $f(c) = c$ and showed that the classical solutions are globally bounded.

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Wang et al. [6] considered system (1) where $\psi(n) = n$, $f(c) = c$ and $D(n)$ satisfies $D(n) \geq c_D n^{m-1}$ with $c_D > 0$ and $m > 1$ and proved that (1) admits a unique classical global solution when $m > 2 - \frac{6}{N+4}$. Fan and Jin [7] obtained similar results under the condition $m > \frac{3}{2} - \frac{1}{N}$. In the case $D(n) \equiv 1$, Tao and Winkler [8] proved that for arbitrarily large initial data and $N = 3$, this model admits at least one global weak solution $(n(x, t), c(x, t))$, which eventually becomes bounded and smooth, and also approaches the unique constant state $\left(\frac{1}{|\Omega|} \int_{\Omega} n_0(x) dx, 0\right)$ as $t \rightarrow \infty$ uniformly with respect to $x \in \Omega$. Tao [9] proved that for $N \geq 2$, system (1) admits a unique global classical solution provided that the chemotactic sensitivity constant χ satisfies $0 < \chi \leq \frac{1}{6(N+1) \|c_0\|_{L^\infty(\Omega)}}$. Zhang and Li [10] considered the same system and showed that if either $N \leq 2$ or $0 < \chi \leq \frac{1}{6(N+1) \|c_0\|_{L^\infty(\Omega)}}$, the global classical solution (n, c) of this model converges to the constant state $\left(\frac{1}{|\Omega|} \int_{\Omega} n_0(x) dx, 0\right)$ exponentially as $t \rightarrow \infty$.

Winkler [11] also analyzed a chemotaxis system with tensor-valued sensitivities $\psi = nS(x, n, c)$. Under a mild growth assumption on S and $D(n) \equiv 1$, the system has at least one global generalized solution for all sufficiently regular nonnegative initial data. Under suitable growth conditions on $D(n)$ and $S(x, n, c)$, Wang [12] showed that the system possesses at least one global bounded weak solution for any sufficiently smooth non-negative initial data. See also [13] for the case when the system contains an additional logistic source term.

There is also a large literature on the case when the chemical substance diffuses, degenerates and is produced by the bacteria so that the second equation in (1) is replaced by $c_t = \Delta c - c + n$, see, for example, [15–17].

In the current paper, we investigate model (1) with $D(n) \equiv 1$, $\psi(n) = n$, $f(c) = c$ under the effect of a small chemical diffusion coefficient $\varepsilon > 0$ so that (1) can be reformulated as

$$\begin{cases} n_t^\varepsilon = \Delta n^\varepsilon - \nabla \cdot (n^\varepsilon \nabla c^\varepsilon), & x \in \Omega, \quad t > 0, \\ c_t^\varepsilon = \varepsilon \Delta c^\varepsilon - n^\varepsilon c^\varepsilon, & x \in \Omega, \quad t > 0, \\ \partial_\nu n^\varepsilon = \partial_\nu c^\varepsilon = 0, & x \in \partial\Omega, \quad t > 0, \\ n^\varepsilon(x, 0) = n_0(x), \quad c^\varepsilon(x, 0) = c_0(x), & x \in \Omega, \end{cases} \quad (2)$$

where $n^\varepsilon = n$, $c^\varepsilon = c$.

The focus of this paper is on the vanishing viscosity limit of system (2) in two space dimensions. By taking the formal limit of system (2) as $\varepsilon \rightarrow 0$, we can obtain,

$$\begin{cases} n_t^0 = \Delta n^0 - \nabla \cdot (n^0 \nabla c^0), & x \in \Omega, \quad t > 0, \\ c_t^0 = -n^0 c^0, & x \in \Omega, \quad t > 0, \\ \partial_\nu n^0 - n^0 \partial_\nu c^0 = 0, & x \in \partial\Omega, \quad t > 0, \\ n^0(x, 0) = n_0^0(x), \quad c^0(x, 0) = c_0^0(x), & x \in \Omega. \end{cases} \quad (3)$$

The form of the boundary condition for system (3) is derived in the following section. For simplicity, we assume that the initial data $(n_0(x), c_0(x))$ in (2) do not depend on ε so that we may take $n_0^0(x) = n_0(x)$ and $c_0^0(x) = c_0(x)$ in (3). Corrias [18] studied this system and established the existence of global (in time) weak solution. Cong considered a similar system in her doctoral dissertation [19], see also [20–22] and further references therein.

Note that the second equation in (3) is an ordinary differential equation and requires no boundary condition for c^0 at $x \in \partial\Omega$. Since (3) is the formal limiting system of (2) as $\varepsilon \rightarrow 0$, we expect that for small $\varepsilon > 0$ the solution of (2) is well approximated by that of (3). However, due to the mismatch of the boundary conditions for systems (2) and (3), an important boundary layer correction term at the $O(\sqrt{\varepsilon})$ order has to be added. This is a pivotal observation and the starting point of our current work.

The boundary layer phenomenon in fluid mechanics is well-known and has been studied extensively [23–35]. In contrast, there are much fewer studies available for the boundary layer problems for the Keller-

Segel model. Hou et al. [36,37] have made some very important contributions to the study of boundary layers of chemotaxis systems related to the Keller-Segel model. Our research in [38] and in this work were partly motivated by and also benefited from the previous results in [36,37]. Although our paper and [36,37] are concerned with the boundary layer behaviors in chemotaxis models, there are a few key differences: first, we study the Keller-Segel model with linear sensitivity function, while the authors of [36,37] study the Keller-Segel model with the singular logarithmic sensitivity function. As a result, the equations considered in [36,37] and in our work are somewhat different. Second, the boundary conditions are also different. We are concerned with the Neumann boundary condition, while the authors of [36,37] treat the Dirichlet boundary condition. Finally, the domains of the equations are also different. Instead of the half plane case considered in [36,37], we consider the domain $\Omega = \mathbb{T} \times (0, 1)$, where $\mathbb{T}$ is the torus domain in $\mathbb{R}^1$.

The remainder of the paper is organized as follows. In Section 2, we use the method of matched asymptotic expansions of singular perturbation theory to construct an accurate approximate solution which takes into account the effects of the boundary layers. In Section 3, we state the main results of this paper and use the classical energy estimates to establish the structural stability of this approximate solution as the chemical diffusion coefficient ε tends to zero.

2 Asymptotic expansions

We consider system (2) in $\Omega = \mathbb{T} \times (0, 1)$, where $\mathbb{T}$ is the torus domain in $\mathbb{R}^1$ and look for solutions $(n^\varepsilon, c^\varepsilon)$ of system (2) in the form:

$$n^\varepsilon(x, y, t) = \sum_{i=0}^{\infty} \varepsilon^{i/2} \left[n^i(x, y, t) + f(y) n_{B+}^i \left(x, \frac{y}{\sqrt{\varepsilon}}, t \right) + g(y) n_{B-}^i \left(x, \frac{y-1}{\sqrt{\varepsilon}}, t \right) \right], \quad (4)$$

$$c^\varepsilon(x, y, t) = \sum_{i=0}^{\infty} \varepsilon^{i/2} \left[c^i(x, y, t) + f(y) c_{B+}^i \left(x, \frac{y}{\sqrt{\varepsilon}}, t \right) + g(y) c_{B-}^i \left(x, \frac{y-1}{\sqrt{\varepsilon}}, t \right) \right], \quad (5)$$

where the inner functions (n^i, c^i) are independent of ε , the boundary layer functions n_{B+}^i, c_{B+}^i corresponding to the left boundary at $y = 0$ depend on x, t , and the stretched variable $\xi = \frac{y}{\sqrt{\varepsilon}} \in \mathbb{R}_+$ while the boundary layer functions n_{B-}^i, c_{B-}^i corresponding to the right boundary at $y = 1$ depend on x, t , and the stretched variable $\eta = \frac{y-1}{\sqrt{\varepsilon}} \in \mathbb{R}_-$. They should model boundary layers of thickness $\sqrt{\varepsilon}$ near the boundaries $y = 0$ and $y = 1$, respectively.

The cut-off functions $f(y)$ and $g(y)$ are C^∞ smooth monotone functions on $\mathbb{R}$ satisfying $0 \leq f(y) \leq 1$, $0 \leq g(y) \leq 1$ and

$$f(y) = \begin{cases} 1, & \text{if } 0 \leq y \leq 1/4, \\ 0, & \text{if } 1/2 \leq y \leq 1, \end{cases} \quad g(y) = \begin{cases} 0, & \text{if } 0 \leq y \leq 1/2, \\ 1, & \text{if } 3/4 \leq y \leq 1. \end{cases}$$

It should be clear that $\frac{f^{(k)}(y)}{y^l}$ and $\frac{g^{(k)}(y)}{(y-1)^l}$ are also smooth and hence uniformly bounded on $[0, 1]$ for any integers $k \geq 1$ and $l \geq 0$, where $f^{(k)}(y)$ and $g^{(k)}(y)$ denote the k th derivative of f and g , respectively. For simplicity, we focus on the boundary layer functions corresponding to the left boundary at $y = 0$. The case for $y = 1$ can be treated similarly.

In the inner zone, i.e., away from the boundaries $y = 0$ and $y = 1$, the boundary layer corrections are expected to be negligible. This means

$$(n_{B+}^i, c_{B+}^i)(\cdot, \xi, \cdot) \rightarrow 0 \text{ as } \xi \rightarrow \infty \quad \text{and} \quad (n_{B-}^i, c_{B-}^i)(\cdot, \eta, \cdot) \rightarrow 0 \text{ as } \eta \rightarrow -\infty. \quad (6)$$

By plugging the aforementioned expansions (4) and (5) into (2) and matching terms that have the same powers $O(\varepsilon^{k/2})$ of ε for various integers of k , one obtains a collection of equations on the inner functions (n^i, c^i) .

Thus we have, for $k = 0$,

$$\begin{cases} n_t^0 = \Delta n^0 - \operatorname{div}(n^0 \nabla c^0), \\ c_t^0 = -n^0 c^0 \end{cases}$$

and for $k = 1$

$$\begin{cases} n_t^1 = \Delta n^1 - \operatorname{div}(n^1 \nabla c^0 + n^0 \nabla c^1), \\ c_t^1 = -(n^0 c^1 + n^1 c^0) \end{cases} \quad (7)$$

and for all $k \geq 2$

$$\begin{cases} n_t^k = \Delta n^k - \operatorname{div}\left(\sum_{i=0}^k n^i \nabla c^{k-i}\right), \\ c_t^k = \Delta c^{k-2} - \left(\sum_{i=0}^k n^i c^{k-i}\right). \end{cases} \quad (8)$$

By plugging (4) and (5) into the boundary conditions $\partial_\nu n^\varepsilon|_{y=0} = \partial_\nu c^\varepsilon|_{y=0} = 0$ and matching terms of the same magnitude, we can also obtain the following boundary conditions at $y = 0$:

$$\partial_\xi n_{B+}^0(x, 0, t) = 0, \quad \partial_y n^i(x, 0, t) + \partial_\xi n_{B+}^{i+1}(x, 0, t) = 0, \quad i \geq 0, \quad (9)$$

$$\partial_\xi c_{B+}^0(x, 0, t) = 0, \quad \partial_y c^i(x, 0, t) + \partial_\xi c_{B+}^{i+1}(x, 0, t) = 0, \quad i \geq 0. \quad (10)$$

At $y = 1$, the boundary conditions are as follows:

$$\partial_\eta n_{B-}^0(x, 0, t) = 0, \quad \partial_y n^i(x, 1, t) + \partial_\eta n_{B-}^{i+1}(x, 0, t) = 0, \quad i \geq 0,$$

$$\partial_\eta c_{B-}^0(x, 0, t) = 0, \quad \partial_y c^i(x, 1, t) + \partial_\eta c_{B-}^{i+1}(x, 0, t) = 0, \quad i \geq 0.$$

Note that in the boundary region, the boundary layer correction terms are no longer negligible. By including the boundary layer terms in the expansions, inserting (4) and (5) into (2) and matching terms with the same powers of ε , we can now derive the equations for the corresponding boundary layer functions (n_{B+}^i, c_{B+}^i) and (n_{B-}^i, c_{B-}^i) . Also, in the boundary region, the inner functions can be expressed as power series of ε . For example, near $y = 0$, since $y = \varepsilon^{1/2}\xi$, we have the following Taylor expansion for the inner functions $n^i(x, y, t)$:

$$n^i(x, \varepsilon^{1/2}\xi, t) = n^i(x, 0, t) + \partial_y n^i(x, 0, t)\varepsilon^{1/2}\xi + \frac{1}{2!}\partial_y^2 n^i(x, 0, t)(\varepsilon^{1/2}\xi)^2 + \cdots + \frac{1}{k!}\partial_y^k n^i(x, 0, t)(\varepsilon^{1/2}\xi)^k + \cdots.$$

For clarity of exposition, for any function $f = f(x, t)$, we denote by $\Gamma_0 f$ the function $f(x, Z, t) \rightarrow f(x, 0, t)$ and denote by $\Gamma_1 f$ the function $f(x, Z, t) \rightarrow f(x, 1, t)$.

Thus, at the order $O(\varepsilon^{-1})$ in the first equation of (2) and the order $O(\varepsilon^0)$ in the second equation of (2), we obtain

$$\begin{cases} \partial_\xi^2 n_{B+}^0 - \Gamma_0 n^0 \partial_\xi^2 c_{B+}^0 - \partial_\xi(n_{B+}^0 \partial_\xi c_{B+}^0) = 0, \\ \partial_t c_{B+}^0 = \partial_\xi^2 c_{B+}^0 - (\Gamma_0 n^0 + n_{B+}^0)c_{B+}^0 + n_{B+}^0 \Gamma_0 c^0. \end{cases}$$

Thanks to the boundary conditions $\partial_\xi n_{B+}^0 = 0$, $\partial_\xi c_{B+}^0 = 0$ in (9) and (10), $c_{B+}^0 \rightarrow 0$ as $\xi \rightarrow \infty$ and the initial condition $c_{B+}^0|_{t=0} = 0$, we have $(n_{B+}^0, c_{B+}^0) = (0, 0)$.

At the order $O(\varepsilon^{-1/2})$ of the first equation of (2) and $(n_{B+}^0, c_{B+}^0) = (0, 0)$, we get

$$\partial_\xi^2 n_{B+}^1 - \Gamma_0 n^0 \partial_\xi^2 c_{B+}^1 = 0. \quad (11)$$

Then, together with the decay conditions (6) and the boundary condition $(\Gamma_0 \partial_y n^0, \Gamma_0 \partial_y c^0) = -(\Gamma_0 \partial_\xi n_{B+}^1, \Gamma_0 \partial_\xi c_{B+}^1)$, we obtain the boundary condition of (n^0, c^0) at $y = 0$: $\partial_y n^0 - \Gamma_0 n^0 \partial_y c^0 = 0$. The same as $y = 1$. We can get (n^0, c^0) by (3).

At the order $O(\varepsilon^{1/2})$ of the second equation of (2), we get

$$\begin{cases} \partial_t c_{B+}^1 = \partial_\xi^2 c_{B+}^1 - [\Gamma_0 n^0 + \Gamma_0 n^0 \Gamma_0 c^0] c_{B+}^1, & x \in \mathbb{T}, \quad \xi \in (0, \infty), \quad t > 0, \\ \partial_\xi c_{B+}^1|_{\xi=0} = -\partial_y c^0|_{y=0}, \quad c_{B+}^1 \rightarrow 0 \text{ as } \xi \rightarrow \infty, & x \in \mathbb{T}, \quad t > 0, \\ c_{B+}^1|_{t=0} = 0, & x \in \mathbb{T}, \quad \xi \in (0, \infty). \end{cases}$$

By solving (11), we can now get

$$n_{B+}^1(x, \xi, t) = \Gamma_0 n^0 c_{B+}^1(x, \xi, t).$$

Next, at the order $O(1)$ of the first equation of (2) and $(n_{B+}^0, c_{B+}^0) = (0, 0)$, we get

$$\partial_\xi^2 n_{B+}^2 - \Gamma_0 n^0 \partial_\xi^2 c_{B+}^2 - \Gamma_0 n^1 \partial_\xi^2 c_{B+}^1 = \Phi_2, \quad (12)$$

where

$$\Phi_2(x, \xi, t) = \xi(\Gamma_0 \partial_y n^0) \partial_\xi^2 c_{B+}^1 + (\Gamma_0 \partial_y n^0) \partial_\xi c_{B+}^1 + (\Gamma_0 \partial_y c^0) \partial_\xi n_{B+}^1 + \partial_\xi(n_{B+}^1 \partial_\xi c_{B+}^1).$$

Thanks to the decay conditions (6) and the boundary condition for $\partial_y n^1$ and $\partial_y c^1$ in (9), (10) we have,

$$(\partial_y n^1 - \Gamma_0 n^0 \partial_y c^1 - \Gamma_0 n^1 \partial_y c^0)|_{y=0} = \int_0^\infty \Phi_2(x, \xi, t) d\xi. \quad (13)$$

We can obtain the following boundary condition at $y = 0$ in the same way

$$(\partial_y n^1 - \Gamma_1 n^0 \partial_y c^1 - \Gamma_1 n^1 \partial_y c^0)|_{y=1} = - \int_{-\infty}^0 \Psi_2(x, \eta, t) d\eta,$$

where

$$\Psi_2(x, \eta, t) = \eta(\Gamma_1 \partial_y n^0) \partial_\eta^2 c_{B-}^1 + (\Gamma_1 \partial_y n^0) \partial_\eta c_{B-}^1 + (\Gamma_1 \partial_y c^0) \partial_\eta n_{B-}^1 + \partial_\eta(n_{B-}^1 \partial_\eta c_{B-}^1).$$

Then, (n^1, c^1) can be obtained by (7), (13) and the initial condition $(n^1, c^1)|_{t=0}$.

Similarly, we get the following equations for c_{B+}^2 :

$$\begin{cases} \partial_t c_{B+}^2 = \partial_\xi^2 c_{B+}^2 - [\Gamma_0 n^0 + \Gamma_0 n^0 \Gamma_0 c^0] c_{B+}^2 - A_2, \\ \partial_\xi c_{B+}^2|_{\xi=0} = -\partial_y c^1|_{y=0}, \quad c_{B+}^2 \rightarrow 0 \text{ as } \xi \rightarrow \infty, \\ c_{B+}^2|_{t=0} = 0, \end{cases}$$

where

$$\begin{aligned} A_2(x, \xi, t) = & \xi[(\Gamma_0 \partial_y n^0) c_{B+}^1 + (\Gamma_0 \partial_y c^0) n_{B+}^1] + (\Gamma_0 n^1 + n_{B+}^1) c_{B+}^1 + n_{B+}^1 \Gamma_0 c^1 \\ & + \Gamma_0 c^0 \Gamma_0 n^1 c_{B+}^1 + \Gamma_0 c^0 \int_\xi^\infty \left(\int_y^\infty \Phi_2(x, s, t) ds \right) dy. \end{aligned}$$

The expression of n_{B+}^2 can be obtained by integrating (12)

$$n_{B+}^2(x, \xi, t) = \Gamma_0 n^0 c_{B+}^2 + \Gamma_0 n^1 c_{B+}^1 + \int_\xi^\infty \left(\int_y^\infty \Phi_2(x, s, t) ds \right) dy.$$

In a similar manner, we can obtain the boundary conditions of the inner function (n^2, c^2)

$$(\partial_y n^2 - \Gamma_0 n^0 \partial_y c^2 - n^2 \partial_y \Gamma_0 c^0)|_{y=0} = \int_0^\infty \Phi_3(x, \xi, t) d\xi, \quad (14)$$

$$(\partial_y n^2 - \Gamma_1 n^0 \partial_y c^2 - n^2 \partial_y \Gamma_1 c^0)|_{y=1} = - \int_{-\infty}^0 \Psi_3(x, \eta, t) d\eta, \quad (15)$$

where

$$\begin{aligned} \Phi_3(x, \xi, t) &= \frac{1}{2} \xi^2 (\Gamma_0 \partial_y^2 n^0) \partial_\xi^2 c_{B+}^1 + \xi \Phi_{3,1} + \Phi_{3,2}, \\ \Phi_{3,1}(x, \xi, t) &= (\Gamma_0 \partial_y^2 n^0) \partial_\xi c_{B+}^1 + (\Gamma_0 \partial_y^2 c^0) \partial_\xi n_{B+}^1 + (\Gamma_0 \partial_y n^0) \partial_\xi^2 c_{B+}^2 + (\Gamma_0 \partial_y n^1) \partial_\xi^2 c_{B+}^1, \\ \Phi_{3,2}(x, \xi, t) &= \partial_x (\Gamma_0 n^0 \partial_x c_{B+}^1 + n_{B+}^1 \Gamma_0 \partial_x c^0) + (\Gamma_0 \partial_y^2 c^0) n_{B+}^1 \\ &\quad + (\Gamma_0 \partial_y n^0) \partial_\xi c_{B+}^2 + (\Gamma_0 \partial_y n^1) \partial_\xi c_{B+}^1 + (\Gamma_0 \partial_y c^1) \partial_\xi n_{B+}^1 + (\Gamma_0 \partial_y c^0) \partial_\xi n_{B+}^2 \\ &\quad + \Gamma_0 n^1 \partial_\xi^2 c_{B+}^2 + \partial_\xi (n_{B+}^1 \partial_\xi c_{B+}^2 + n_{B+}^2 \partial_\xi c_{B+}^1) + \partial_t n_{B+}^1 - \partial_x^2 n_{B+}^1 \end{aligned}$$

and

$$\begin{aligned} \Psi_3(x, \eta, t) &= \frac{1}{2} \eta^2 (\Gamma_1 \partial_y^2 n^0) \partial_\eta^2 c_{B-}^1 + \eta \Psi_{3,1} + \Psi_{3,2}, \\ \Psi_{3,1}(x, \eta, t) &= (\Gamma_1 \partial_y^2 n^0) \partial_\eta c_{B-}^1 + (\Gamma_1 \partial_y^2 c^0) \partial_\eta n_{B-}^1 + (\Gamma_1 \partial_y n^0) \partial_\eta^2 c_{B-}^2 + (\Gamma_1 \partial_y n^1) \partial_\eta^2 c_{B-}^1, \\ \Psi_{3,2}(x, \eta, t) &= \partial_x [(\Gamma_1 n^0) \partial_x c_{B-}^1 + n_{B-}^1 \Gamma_1 \partial_x c^0] + (\Gamma_1 \partial_y^2 c^0) n_{B-}^1 \\ &\quad + (\Gamma_1 \partial_y n^0) \partial_\eta c_{B-}^2 + (\Gamma_1 \partial_y n^1) \partial_\eta c_{B-}^1 + (\Gamma_1 \partial_y c^1) \partial_\eta n_{B-}^1 + (\Gamma_1 \partial_y c^0) \partial_\eta n_{B-}^2 \\ &\quad + (\Gamma_1 n^1) \partial_\eta^2 c_{B-}^2 + \partial_\eta (n_{B-}^1 \partial_\eta c_{B-}^2 + n_{B-}^2 \partial_\eta c_{B-}^1) + \partial_t n_{B-}^1 - \partial_x^2 n_{B-}^1. \end{aligned}$$

We can thus determine (n^2, c^2) by solving (8) (for $k = 2$), (14), (15) and the initial condition $(n^2, c^2)|_{t=0}$.

At the next order, $\varepsilon^{3/2}$, we obtain:

$$\begin{cases} \partial_t c_{B+}^3 = \partial_\xi^2 c_{B+}^3 - (\Gamma_0 n^0 + \Gamma_0 n^0 \Gamma_0 c^0) c_{B+}^3 - A_3, \\ \partial_\xi c_{B+}^3|_{\xi=0} = -\partial_y c^2|_{y=0}, \quad c_{B+}^3 \rightarrow 0 \text{ as } \xi \rightarrow \infty, \\ c_{B+}^3|_{t=0} = 0, \end{cases}$$

where

$$\begin{aligned} A_3(x, \xi, t) &= \frac{1}{2} \xi^2 [(\Gamma_0 \partial_y^2 n^0) c_{B+}^1 + (\Gamma_0 \partial_y^2 c^0) n_{B+}^1] + \xi A_{3,1} + A_{3,2}, \\ A_{3,1}(x, \xi, t) &= (\Gamma_0 \partial_y n^0) c_{B+}^2 + (\Gamma_0 \partial_y n^1) c_{B+}^1 + (\Gamma_0 \partial_y c^0) n_{B+}^2 + (\Gamma_0 \partial_y c^1) n_{B+}^1, \\ A_{3,2}(x, \xi, t) &= (\Gamma_0 n^1 + n_{B+}^1) c_{B+}^2 + (\Gamma_0 n^2 + n_{B+}^2) c_{B+}^1 + n_{B+}^1 \Gamma_0 c^2 + n_{B+}^2 \Gamma_0 c^1 + \Gamma_0 c^0 \Gamma_0 n^2 c_{B+}^1 \\ &\quad - \partial_x^2 c_{B+}^1 + \Gamma_0 c^0 \int_\xi^\infty \left(\int_y^\infty \Phi_3(x, s, t) ds \right) dy. \end{aligned}$$

Then,

$$n_{B+}^3(x, \xi, t) = \Gamma_0 n^0 c_{B+}^3 + \Gamma_0 n^2 c_{B+}^1 + \int_\xi^\infty \left(\int_y^\infty \Phi_3(x, s, t) ds \right) dy.$$

Similarly, we can obtain the boundary conditions of the inner function (n^3, c^3) :

$$\begin{aligned} (\partial_y n^3 - \Gamma_0 n^0 \partial_y c^3 - n^3 \partial_y c^0)|_{y=0} &= \int_0^\infty \Phi_4(x, \xi, t) d\xi, \\ (\partial_y n^3 - \Gamma_1 n^0 \partial_y c^3 - n^3 \partial_y c^0)|_{y=1} &= - \int_{-\infty}^0 \Psi_4(x, \eta, t) d\eta, \end{aligned}$$

where

$$\begin{aligned}\Phi_4(x, \xi, t) &= \frac{1}{3!} \xi^3 (\Gamma_0 \partial_y^3 n^0) \partial_\xi^2 c_{B+}^1 + \frac{1}{2} \xi^2 \Phi_{4,1} + \xi \Phi_{4,2} + \Phi_{4,3}, \\ \Phi_{4,1}(x, \xi, t) &= (\Gamma_0 \partial_y^3 n^0) \partial_\xi c_{B+}^1 + (\Gamma_0 \partial_y^3 c^0) \partial_\xi n_{B+}^1 + (\Gamma_0 \partial_y^2 n^0) \partial_\xi^2 c_{B+}^2 + (\Gamma_0 \partial_y^2 n^1) \partial_\xi^2 c_{B+}^1, \\ \Phi_{4,2}(x, \xi, t) &= \partial_x [(\Gamma_0 \partial_y n^0) \partial_x c_{B+}^1 + n_{B+}^1 \partial_x (\Gamma_0 \partial_y c^0)] + (\Gamma_0 \partial_y^3 c^0) n_{B+}^1 + (\Gamma_0 \partial_y^2 n^0) \partial_\xi c_{B+x}^2 \\ &\quad + (\Gamma_0 \partial_y^2 n^1) \partial_\xi c_{B+}^1 + (\Gamma_0 \partial_y^2 c^0) \partial_\xi n_{B+}^1 + (\Gamma_0 \partial_y^2 c^0) \partial_\xi n_{B+}^2 + (\Gamma_0 \partial_y n^1) \partial_\xi^2 c_{B+}^2 + (\Gamma_0 \partial_y n^2) \partial_\xi^2 c_{B+}^1, \\ \Phi_{4,3}(x, \xi, t) &= \partial_x [\Gamma_0 n^0 \partial_x c_{B+}^2 + n_{B+}^1 (\Gamma_0 \partial_x c^1) + n_{B+}^2 (\Gamma_0 \partial_x c^0) + (\Gamma_0 n^1 + n_{B+}^1) \partial_x c_{B+}^1] \\ &\quad + (\Gamma_0 \partial_y^2 c^1) n_{B+}^1 + (\Gamma_0 \partial_y^2 c^0) n_{B+}^2 + (\Gamma_0 \partial_y n^0) \partial_\xi c_{B+}^3 + (\Gamma_0 \partial_y n^1) \partial_\xi c_{B+}^2 \\ &\quad + (\Gamma_0 \partial_y n^2) \partial_\xi c_{B+}^1 + (\Gamma_0 \partial_y c^2) \partial_\xi n_{B+}^1 + (\Gamma_0 \partial_y c^1) \partial_\xi n_{B+}^2 + (\Gamma_0 \partial_y c^2) \partial_\xi n_{B+}^3 \\ &\quad + \Gamma_0 n^1 \partial_\xi^2 c_{B+}^3 + \Gamma_0 n^2 \partial_\xi^2 c_{B+}^2 + \partial_\xi (n_{B+}^1 \partial_\xi c_{B+}^3 + n_{B+}^2 \partial_\xi c_{B+}^2 + n_{B+}^3 \partial_\xi c_{B+}^1) + \partial_t n_{B+}^2 - \partial_x n_{B+}^2.\end{aligned}$$

The expression of Ψ_4 is analogous to Φ_4 except that the inner functions are now all evaluated at $y = 1$ and the variable ξ is to be replaced with η in the aforementioned expressions. Similar to (n^2, c^2) , we can now get (n^3, c^3) .

We obtain the expression of c_{B+}^4 by $O(\varepsilon^3)$ magnitude of the second equation of (2)

$$\begin{cases} \partial_t c_{B+}^4 = \partial_\xi^2 c_{B+}^4 - [\Gamma_0 n^0 + \Gamma_0 n^0 \Gamma_0 c^0] c_{B+}^4 - A_4, \\ \partial_\xi c_{B+}^4|_{\xi=0} = -\partial_y c^3|_{y=0}, \quad c_{B+}^4 \rightarrow 0 \text{ as } \xi \rightarrow \infty, \\ c_{B+}^4|_{t=0} = 0, \end{cases}$$

where

$$\begin{aligned}A_4(x, \xi, t) &= \frac{1}{3!} \xi^3 [(\Gamma_0 \partial_y^3 n^0) c_{B+}^1 + (\Gamma_0 \partial_y^3 c^0) n_{B+}^1] + \frac{1}{2} \xi^2 A_{4,1} + \xi A_{4,2} + A_{4,3}, \\ A_{4,1}(x, \xi, t) &= (\Gamma_0 \partial_y^2 n^0) c_{B+}^2 + (\Gamma_0 \partial_y^2 n^1) c_{B+}^1 + (\Gamma_0 \partial_y^2 c^0) n_{B+}^2 + (\Gamma_0 \partial_y^2 c^1) n_{B+}^1, \\ A_{4,2}(x, \xi, t) &= (\Gamma_0 \partial_y n^0) c_{B+}^3 + (\Gamma_0 \partial_y n^1) c_{B+}^2 + (\Gamma_0 \partial_y n^2) c_{B+}^1 + (\Gamma_0 \partial_y c^0) n_{B+}^3 + (\Gamma_0 \partial_y c^1) n_{B+}^2 + (\Gamma_0 \partial_y c^2) n_{B+}^1, \\ A_{4,3}(x, \xi, t) &= (\Gamma_0 n^1 + n_{B+}^1) c_{B+}^3 + (\Gamma_0 n^2 + n_{B+}^2) c_{B+}^2 + (\Gamma_0 n^3 + n_{B+}^3) c_{B+}^1 + n_{B+}^1 \Gamma_0 c^3 + n_{B+}^2 \Gamma_0 c^2 \\ &\quad + n_{B+}^3 \Gamma_0 c^1 + \Gamma_0 c^0 \Gamma_0 n^3 c_{B+}^1 - \partial_x^2 c_{B+}^2 + \Gamma_0 c^0 \int_{\xi}^{\infty} \left(\int_y^{\infty} \Phi_4(x, s, t) ds \right) dy.\end{aligned}$$

Then,

$$n_{B+}^4(x, \xi, t) = \Gamma_0 n^0 c_{B+}^4 + \Gamma_0 n^3 c_{B+}^1 + \int_{\xi}^{\infty} \left(\int_y^{\infty} \Phi_4(x, s, t) ds \right) dy.$$

In the same way, one can derive the equations for higher-order boundary layer profiles (n^j, c^j) , (n_{B+}^j, c_{B+}^j) and (n_{B-}^j, c_{B-}^j) for $j \geq 4$.

Proposition 1. Assume the initial value $(n_0^0, c_0^0) \in \mathbb{H}^9 \times \mathbb{H}^9$ satisfies some compatibility conditions at each of the two boundaries $y = 0$ and $y = 1$. Then there exists $T^* \leq \infty$ such that for any $0 < T < T^*$, the following estimates hold for some positive constant $M = M(T) > 0$

$$\|\partial_t^{l_1} \partial_x^{l_2} \partial_y^{l_3} (n^j, c^j)\|_{L_{(x,y,t)}^{\infty}(\Omega \times [0, T])} \leq M,$$

where (n^j, c^j) are the inner functions and $l_1 = 0, 1, 2, l_2, l_3, j = 0, 1, 2, 3$.

The proof of Proposition 1 follows from the standard energy method, see Rascle [39], and also [18]. Hence, we omit the proof.

Proposition 2. Under the assumptions of Proposition 1, we have for $k_2 = 0, 1$, $k_4, k_6 = 0, 1, 2$, $k_1, k_3, k_5 = 0, 1, 2, 3$, $i = 1, 2, 3, 4$,

$$\begin{aligned} & \|\partial_t^{k_1} \partial_x^{k_2} (\xi^{k_3} \partial_\xi^{k_4} (n_{B+}^i, c_{B+}^i), \eta^{k_5} \partial_\eta^{k_6} (n_{B-}^i, c_{B-}^i))\|_{L_{(x,y,t)}^\infty(\Omega \times [0, T])} \leq M, \\ & \|\partial_t^{k_1} \partial_x^{k_2} (\xi^{k_3} \partial_\xi^{k_4} (n_{B+}^i, c_{B+}^i), \eta^{k_5} \partial_\eta^{k_6} (n_{B-}^i, c_{B-}^i))\|_{L_t^\infty([0, T]; L_x^\infty(\mathbb{T}^d \times L_y^2([0, 1])))} \leq M\varepsilon^{1/4}, \end{aligned}$$

where $n_{B+}^i, n_{B-}^i, c_{B+}^i, c_{B-}^i$ are the boundary layer functions.

Proposition 2 can also be proved by using the standard energy method, see Hou et al. [36,37], Lemmas 3.2–3.6, and also Wang et al. [30,31]. Hence, we omit the proof.

3 Error equations and main result

We decompose the solution $(n^\varepsilon, c^\varepsilon)$ of (2) as

$$n^\varepsilon(x, y, t) = n_a + n_R, \quad c^\varepsilon(x, y, t) = c_a + c_R, \quad (16)$$

where

$$\begin{aligned} n_a(x, y, t) &= \sum_{i=0}^3 \varepsilon^{i/2} n^i(x, y, t) + \sum_{i=1}^4 \varepsilon^{i/2} \left[f(y) n_{B+}^i \left(x, \frac{y}{\sqrt{\varepsilon}}, t \right) + g(y) n_{B-}^i \left(x, \frac{y-1}{\sqrt{\varepsilon}}, t \right) \right], \\ c_a(x, y, t) &= \sum_{i=0}^3 \varepsilon^{i/2} c^i(x, y, t) + \sum_{i=1}^4 \varepsilon^{i/2} \left[f(y) c_{B+}^i \left(x, \frac{y}{\sqrt{\varepsilon}}, t \right) + g(y) c_{B-}^i \left(x, \frac{y-1}{\sqrt{\varepsilon}}, t \right) \right]. \end{aligned}$$

Based on the aforementioned results, it can be checked that the approximate solution (n_a, c_a) satisfies the following properties:

$$\|\partial_t^{l_1} \partial_x^{l_2} \partial_y^{l_3} (n_a, c_a)\|_{L_{(x,y,t)}^\infty(\Omega \times [0, T])} \leq \frac{M}{(\sqrt{\varepsilon})^{l_3-1}}, \quad (17)$$

for some constant $M > 0$ and for $l_j = 0, 1, 2, 3$, $j = 1, 2, 3$.

Inserting (16) into (2) and using the boundary conditions (9), (10), we get the following system for $(n_R, c_R)(x, y, t)$:

$$\begin{cases} \partial_t n_R = \Delta n_R - \operatorname{div}(n_R \nabla c_R) + J_I + J_B + J_R, & (x, y) \in \Omega, \quad t \in [0, T], \\ \partial_t c_R = \varepsilon \Delta c_R - n_R c_R + K_I + K_B + K_R, & (x, y) \in \Omega, \quad t \in [0, T], \\ \partial_\nu n_R = \partial_\nu c_R = 0, & (x, y) \in \partial\Omega, \quad t > 0, \\ n_R(x, 0) = 0, \quad c_R(x, 0) = 0, & (x, y) \in \Omega, \end{cases} \quad (18)$$

where

$$\begin{aligned} J_I(x, y, t) &= -\varepsilon^2 \operatorname{div}(n^1 \nabla c^3 + n^2 \nabla c^2 + n^3 \nabla c^1) - \varepsilon^{5/2} \operatorname{div}(n^2 \nabla c^3 + n^3 \nabla c^2) - \varepsilon^3 \operatorname{div}(n^3 \nabla c^3), \\ J_B(x, \xi, \eta, t) &= \varepsilon^{-1/2} J_1 + J_2 + \varepsilon^{1/2} J_3 + \varepsilon J_4, \\ J_R(x, y, t) &= -\operatorname{div}(n_R \nabla c_a) - \operatorname{div}(n_a \nabla c_R), \\ K_I(x, y, t) &= \varepsilon^2 (\Delta c^2 - n^1 c^3 - n^2 c^2 - n^3 c^1) + \varepsilon^{5/2} (\Delta c^3 - n^2 c^3 - n^3 c^2) - \varepsilon^3 n^3 c^3, \\ K_B(x, \xi, \eta, t) &= \varepsilon^{1/2} K_1 + \varepsilon K_2 + \varepsilon^{3/2} K_3, \\ K_R(x, y, t) &= -n_a c_R - n_R c_a, \end{aligned}$$

and the functions J_j ($j = 1, 2, 3, 4$), K_j ($j = 1, 2, 3$) depend upon the boundary layer functions.

In the aforementioned expressions, we have

$$\sup_{0 \leq t \leq T} \|\partial_t^i (J_I, K_I)\|_{L^2(\Omega)} \leq M(T) \varepsilon^2, \quad i = 0, 1, 2 \quad (19)$$

and J_4, K_3 satisfy the following estimates:

$$\sup_{0 \leq t \leq T} \|\partial_t^i(J_4, K_3)\|_{L^2(\Omega)} \leq M(T)\varepsilon^{1/4}, \quad i = 0, 1, 2, 3.$$

We can also obtain the estimates

$$\sup_{0 \leq t \leq T} \|\partial_t^i J_B\|_{L^2(\Omega)}^2 \leq M(T)\varepsilon^{5/2}, \quad \sup_{0 \leq t \leq T} \|\partial_t^i K_B\|_{L^2(\Omega)}^2 \leq M(T)\varepsilon^{7/2}, \quad i = 0, 1, 2, 3 \quad (20)$$

by using the following estimates

$$\|2f'\partial_\xi n_{B+}^1\|_{L^2}^2 = 2 \left\| \frac{f'(y)}{y^2} \varepsilon \xi^2 \partial_\xi n_{B+}^1 \right\|_{L^2}^2 \leq M \left\| \frac{f'}{y^2} \right\|_{L^\infty}^2 \|\varepsilon \xi^2 \partial_\xi n_{B+}^1\|_{L^2}^2 \leq M \|\varepsilon \xi^2 \partial_\xi n_{B+}^1\|_{L^2}^2 \leq M\varepsilon^{5/2}$$

and

$$\|\varepsilon^{-1/2} J_1\|_{L^2}^2 \leq M \left\| \frac{\partial^3 n^0}{\partial y^3} \right\|_{L^\infty}^2 \|f\varepsilon \xi^3 \partial_{\xi\xi} c_{B+}^1\|_{L^2}^2 + M \left\| \frac{\partial^3 n^0}{\partial y^3} \right\|_{L^\infty}^2 \|g\varepsilon \eta^3 \partial_{\eta\eta} c_{B-}^1\|_{L^2}^2 \leq M\varepsilon^{5/2}.$$

Theorem 1. Denote by $(n^\varepsilon, c^\varepsilon)$ the unique smooth solution of (2) and (n^0, c^0) the unique smooth solution of (3). Let the initial value $(n_0^0, c_0^0) \in \mathbb{H}^9 \times \mathbb{H}^9$ satisfy the assumptions in Proposition 1. Then, for any fixed $0 < T < T^*$, there exists a positive constant $M = M(T)$ independent of ε such that

$$\|n_R\|_{L_{(x,y,t)}^\infty(\Omega \times [0, T])}^2 \leq M\varepsilon^{3/2-\delta}, \quad \|c_R\|_{L_{(x,y,t)}^\infty(\Omega \times [0, T])}^2 \leq M\varepsilon^{1/2-\delta} \quad (21)$$

for all $\delta \in (0, 1/2)$.

We divide the proof of Theorem 1 into several steps.

Lemma 1. If the assumptions in Theorem 1 are satisfied, and (n_R, c_R) is the solution of (18), then for any $0 < T < T^*$ and $t \in (0, T)$, it holds that

$$\begin{aligned} & \| (n_R, c_R)(\cdot, t) \|_{L^2}^2 + \int_0^t \|\nabla n_R(\cdot, s)\|_{L^2}^2 ds + \varepsilon \int_0^t \|\nabla c_R(\cdot, s)\|_{L^2}^2 ds \\ & \leq M \|(n_R, c_R)(\cdot, 0)\|_{L^2}^2 + M \int_0^t (\|n_R(\cdot, s)\|_{H^2}^2 \|c_R(\cdot, s)\|_{L^2}^2 + \|n_R(\cdot, s)\|_{H^2}^2 \|\nabla c_R(\cdot, s)\|_{L^2}^2) ds \\ & \quad + M \int_0^t \|(n_R, c_R)(\cdot, s)\|_{L^2}^2 ds + M\varepsilon^{5/2}, \end{aligned} \quad (22)$$

where $M = M(T)$ is a positive constant, independent of ε .

Proof. First, multiplying the first equation of (18) by n_R , and integrating over $\Omega = \mathbb{T} \times (0, 1)$, we obtain:

$$\frac{1}{2} \frac{d}{dt} \|n_R\|_{L^2}^2 + \|\nabla n_R\|_{L^2}^2 = - \int_{\Omega} \operatorname{div}(n_R \nabla c_R) \cdot n_R dx + \int_{\Omega} J_I \cdot n_R dx + \int_{\Omega} J_B \cdot n_R dx + \int_{\Omega} J_R \cdot n_R dx. \quad (23)$$

By using the divergence theorem, the boundary conditions of n_R, c_R , Sobolev's lemma and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} - \int_{\Omega} \operatorname{div}(n_R \nabla c_R) \cdot n_R dx &= \int_{\Omega} n_R \nabla c_R \cdot \nabla n_R dx \leq M \|n_R \nabla c_R\|_{L^2}^2 + 1/4 \|\nabla n_R\|_{L^2}^2 \\ &\leq M \|n_R\|_{L^\infty}^2 \|\nabla c_R\|_{L^2}^2 + 1/4 \|\nabla n_R\|_{L^2}^2 \leq M \|n_R\|_{H^2}^2 \|\nabla c_R\|_{L^2}^2 + 1/4 \|\nabla n_R\|_{L^2}^2. \end{aligned} \quad (24)$$

Thanks to the estimates (19), (20) and the Cauchy-Schwarz inequality, we obtain

$$\int_{\Omega} J_I \cdot n_R dx + \int_{\Omega} J_B \cdot n_R dx \leq M\varepsilon^{5/2} + M\|n_R\|_{L^2}^2. \quad (25)$$

By using (17), the Cauchy-Schwarz inequality and Sobolev's lemma, we get

$$\begin{aligned} \int_{\Omega} J_R \cdot n_R dx &= \int_{\Omega} (n_R \nabla c_a + n_a \nabla c_R) \cdot \nabla n_R dx \\ &\leq 1/4 \|\nabla n_R\|_{L^2}^2 + M\|n_R \nabla c_a\|_{L^2}^2 + M\|n_a \nabla c_R\|_{L^2}^2 \\ &\leq 1/4 \|\nabla n_R\|_{L^2}^2 + M\|\nabla c_a\|_{L^\infty}^2 \|n_R\|_{L^2}^2 + M\frac{1}{\varepsilon} \|n_a\|_{L^\infty}^2 \varepsilon \|\nabla c_R\|_{L^2}^2 \\ &\leq 1/4 \|\nabla n_R\|_{L^2}^2 + M\|n_R\|_{L^2}^2 + \varepsilon/4 \|\nabla c_R\|_{L^2}^2. \end{aligned} \quad (26)$$

Thus, by combining (23) with (24)–(26), we get

$$\frac{d}{dt} \|n_R\|_{L^2}^2 + \|\nabla n_R\|_{L^2}^2 \leq M\|n_R\|_{H^2}^2 \|\nabla c_R\|_{L^2}^2 + \varepsilon/4 \|\nabla c_R\|_{L^2}^2 + M\|n_R\|_{L^2}^2 + M\varepsilon^{5/2}. \quad (27)$$

Similarly, multiplying the second equation of (18) by c_R and integrating over $\Omega = \mathbb{T} \times (0,1)$, we have:

$$\frac{1}{2} \frac{d}{dt} \|c_R\|_{L^2}^2 + \varepsilon \|\nabla c_R\|_{L^2}^2 = - \int_{\Omega} (n_R c_R) \cdot c_R dx + \int_{\Omega} K_I \cdot c_R dx + \int_{\Omega} K_B \cdot c_R dx + \int_{\Omega} K_R \cdot c_R dx. \quad (28)$$

Using a similar procedure to estimate n_R , we get

$$- \int_{\Omega} (n_R c_R) \cdot c_R dx \leq M\|n_R\|_{H^2}^2 \|c_R\|_{L^2}^2 + M\|c_R\|_{L^2}^2$$

and

$$\int_{\Omega} K_I \cdot c_R dx + \int_{\Omega} K_B \cdot c_R dx \leq M\|c_R\|_{L^2}^2 + M\varepsilon^{7/2}.$$

Thanks to the Cauchy-Schwarz inequality and (17), we have

$$\int_{\Omega} K_R \cdot c_R dx \leq M\|c_R\|_{L^2}^2 + M\|c_a\|_{L^\infty}^2 \|n_R\|_{L^2}^2 + M\|n_a\|_{L^\infty}^2 \|c_R\|_{L^2}^2 \leq M\|c_R\|_{L^2}^2 + M\|n_R\|_{L^2}^2.$$

Plugging the aforementioned estimates into (28), we obtain the following estimate:

$$\frac{d}{dt} \|c_R\|_{L^2}^2 + \varepsilon \|\nabla c_R\|_{L^2}^2 \leq M\|n_R\|_{H^2}^2 \|c_R\|_{L^2}^2 + M\|(n_R, c_R)\|_{L^2}^2 + M\varepsilon^{7/2}. \quad (29)$$

The proof is completed by integrating (27) and (29) with respect to t over $[0, t]$. $\square$

Lemma 2. *If the assumptions in Theorem 1 are satisfied, and (n_R, c_R) is the solution of (18), then for any $0 < T < T^*$ and $t \in (0, T)$, it holds that*

$$\begin{aligned} &\|(\partial_t n_R, \partial_t c_R)\|_{L^2}^2 + \int_0^t \|\nabla(\partial_t n_R)\|_{L^2}^2 dt + \varepsilon \int_0^t \|\nabla(\partial_t c_R)\|_{L^2}^2 dt \\ &\leq M\|(\partial_t n_R, \partial_t c_R)|_{t=0}\|_{L^2}^2 + M \int_0^t (\|\partial_t n_R\|_{H^2}^2 \|\nabla c_R\|_{L^2}^2 + \|n_R\|_{H^2}^2 \|\nabla(\partial_t c_R)\|_{L^2}^2) dt \\ &\quad + M \int_0^t (\|\partial_t n_R\|_{H^1}^2 \|c_R\|_{H^1}^2 + \|n_R\|_{H^1}^2 \|\partial_t c_R\|_{H^1}^2) dt + M \int_0^t (\|(n_R, c_R, \partial_t n_R, \partial_t c_R)\|_{L^2}^2) dt + M\varepsilon^{5/2}, \end{aligned} \quad (30)$$

where $M = M(T)$ is a positive constant, independent of ε .

Proof. Differentiating (18) with respect to t , multiplying the first equation by $\partial_t n_R$ and integrating over $\Omega = \mathbb{T} \times (0,1)$, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_t n_R\|_{L^2}^2 + \|\nabla(\partial_t n_R)\|_{L^2}^2 &= - \int_{\Omega} \partial_t [\operatorname{div}(n_R \nabla c_R)] \cdot \partial_t n_R dx + \int_{\Omega} \partial_t J_I \cdot \partial_t n_R dx \\ &\quad + \int_{\Omega} \partial_t J_B \cdot \partial_t n_R dx + \int_{\Omega} \partial_t J_R \cdot \partial_t n_R dx. \end{aligned} \quad (31)$$

We can obtain the following estimates by similar arguments to those in the proof of Lemma 1

$$\begin{aligned} - \int_{\Omega} \partial_t [\operatorname{div}(n_R \nabla c_R)] \cdot \partial_t n_R dx &= \int_{\Omega} (\partial_t n_R \nabla c_R + n_R \nabla(\partial_t c_R)) \cdot \nabla(\partial_t n_R) dx \\ &\leq M \|\partial_t n_R\|_{H^2}^2 \|\nabla c_R\|_{L^2}^2 + M \|n_R\|_{H^2}^2 \|\nabla(\partial_t c_R)\|_{L^2}^2 + \frac{1}{4} \|\nabla(\partial_t n_R)\|_{L^2}^2 \end{aligned}$$

and

$$\int_{\Omega} \partial_t J_I \cdot \partial_t n_R dx + \int_{\Omega} \partial_t J_B \cdot \partial_t n_R dx \leq M \varepsilon^{5/2} + M \|\partial_t n_R\|_{L^2}^2.$$

For the last term of (31), we have

$$\begin{aligned} \int_{\Omega} \partial_t J_R \cdot \partial_t n_R dx &= \int_{\Omega} (\partial_t n_R \nabla c_a + n_R \nabla(\partial_t c_a) + \partial_t n_a \nabla c_R + n_a \nabla(\partial_t c_R)) \cdot \nabla(\partial_t n_R) dx \\ &\leq \frac{1}{4} \|\nabla(\partial_t n_R)\|_{L^2}^2 + M \|\nabla c_a\|_{L^\infty}^2 \|\partial_t n_R\|_{L^2}^2 + M \|\nabla(\partial_t c_a)\|_{L^\infty}^2 \|n_R\|_{L^2}^2 \\ &\quad + M \frac{1}{\varepsilon} \|n_a\|_{L^\infty}^2 \varepsilon \|\nabla c_R\|_{L^2}^2 + M \frac{1}{\varepsilon} \|\partial_t n_a\|_{L^\infty}^2 \varepsilon \|\nabla(\partial_t c_R)\|_{L^2}^2 \\ &\leq \frac{1}{4} \|\nabla(\partial_t n_R)\|_{L^2}^2 + M \|n_R, \partial_t n_R\|_{L^2}^2 + M_1 \varepsilon \|\nabla c_R, \nabla(\partial_t c_R)\|_{L^2}^2. \end{aligned}$$

Combining the aforementioned estimates, we get

$$\begin{aligned} \frac{d}{dt} \|\partial_t n_R\|_{L^2}^2 + \|\nabla(\partial_t n_R)\|_{L^2}^2 &\leq M \|\partial_t n_R\|_{H^2}^2 \|\nabla c_R\|_{L^2}^2 + M \|n_R\|_{H^2}^2 \|\nabla(\partial_t c_R)\|_{L^2}^2 + M \|n_R, \partial_t n_R\|_{L^2}^2 \\ &\quad + M_1 \varepsilon \|\nabla c_R, \nabla(\partial_t c_R)\|_{L^2}^2 + M \varepsilon^{5/2}, \end{aligned} \quad (32)$$

where M_1 is the positive constant and is independent of ε .

Similarly, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_t c_R\|_{L^2}^2 + \varepsilon \|\nabla(\partial_t c_R)\|_{L^2}^2 \\ = - \int_{\Omega} \partial_t (n_R c_R) \cdot \partial_t c_R dx + \int_{\Omega} \partial_t K_I \cdot \partial_t c_R dx + \int_{\Omega} \partial_t K_B \cdot \partial_t c_R dx + \int_{\Omega} \partial_t K_R \cdot \partial_t c_R dx. \end{aligned} \quad (33)$$

Thanks to the Cauchy-Schwarz inequality and Sobolev embedding inequality, we get

$$\begin{aligned} - \int_{\Omega} \partial_t (n_R c_R) \cdot \partial_t c_R dx &\leq M \|\partial_t n_R\|_{L^4}^2 \|c_R\|_{L^4}^2 + M \|n_R\|_{L^4}^2 \|\partial_t c_R\|_{L^4}^2 + M \|\partial_t c_R\|_{L^2}^2 \\ &\leq M \|\partial_t n_R\|_{H^1}^2 \|c_R\|_{H^1}^2 + M \|n_R\|_{H^1}^2 \|\partial_t c_R\|_{H^1}^2 + M \|\partial_t c_R\|_{L^2}^2, \\ \int_{\Omega} \partial_t K_I \cdot \partial_t c_R dx + \int_{\Omega} \partial_t K_B \cdot \partial_t c_R dx &\leq M \varepsilon^{7/2} + M \|\partial_t c_R\|_{L^2}^2 \end{aligned}$$

and

$$\int_{\Omega} \partial_t K_R \cdot \partial_t C_R dx = - \int_{\Omega} (\partial_t n_a C_R + n_a \partial_t C_R + \partial_t n_R C_a + n_R \partial_t C_a) \cdot \partial_t C_R dx \leq M \|n_R, \partial_t n_R, C_R, \partial_t C_R\|_{L^2}^2.$$

Then,

$$\begin{aligned} & \frac{d}{dt} \|\partial_t C_R\|_{L^2}^2 + \varepsilon \|\nabla(\partial_t C_R)\|_{L^2}^2 \\ & \leq M \|\partial_t n_R\|_{H^1}^2 \|C_R\|_{H^1}^2 + M \|n_R\|_{H^1}^2 \|\partial_t C_R\|_{H^1}^2 + M \|n_R, \partial_t n_R, C_R, \partial_t C_R\|_{L^2}^2 + M \varepsilon^{7/2}. \end{aligned} \quad (34)$$

Choosing δ_1 suitably small (but independent of ε) and combining the estimate $\delta_1 \times$ (32) with (34) gives us

$$\begin{aligned} & \frac{d}{dt} \|\partial_t n_R, \partial_t C_R\|_{L^2}^2 + \|\nabla(\partial_t n_R)\|_{L^2}^2 + \varepsilon \|\nabla(\partial_t C_R)\|_{L^2}^2 \\ & \leq M \|\partial_t n_R\|_{H^2}^2 \|\nabla C_R\|_{L^2}^2 + M \|n_R\|_{H^2}^2 \|\nabla(\partial_t C_R)\|_{L^2}^2 + M \|\partial_t n_R\|_{H^1}^2 \|C_R\|_{H^1}^2 \\ & \quad + M \|n_R\|_{H^1}^2 \|\partial_t C_R\|_{H^1}^2 + M \|n_R, \partial_t n_R, C_R, \partial_t C_R\|_{L^2}^2 + M \varepsilon^{5/2}. \end{aligned} \quad (35)$$

The proof of Lemma 2 is completed by integrating (35) with respect to t over $[0, t]$. $\square$

Lemma 3. *If the assumptions in Theorem 1 are satisfied, and (n_R, C_R) is the solution of (18), then for any $0 < T < T^*$ and $t \in (0, T)$, it holds that*

$$\begin{aligned} & \|\nabla n_R, \nabla C_R\|_{L^2}^2 + \int_0^t \|\Delta n_R\|_{L^2}^2 dt + \varepsilon \int_0^t \|\Delta C_R\|_{L^2}^2 dt \\ & \leq M \|(\nabla n_R, \nabla C_R)|_{t=0}\|_{L^2}^2 + M \int_0^t (\|\nabla n_R\|_{H^1}^2 \|C_R\|_{H^1}^2 + \|n_R\|_{H^2}^2 \|\nabla C_R\|_{L^2}^2) dt \\ & \quad + M \int_0^t \left(\left\| \frac{n_R}{\sqrt{\varepsilon}} \right\|_{H^2}^2 \|\sqrt{\varepsilon} \operatorname{div}(\nabla C_R)\|_{L^2}^2 + \left\| \frac{\nabla n_R}{\sqrt{\varepsilon}} \right\|_{H^1}^2 \|\sqrt{\varepsilon} \nabla C_R\|_{H^1}^2 \right) dt \\ & \quad + M \int_0^t (\|n_R, C_R, \nabla n_R, \nabla C_R\|_{L^2}^2) dt + M \int_0^t \varepsilon \|\operatorname{div}(\nabla C_R)\|_{L^2}^2 dt + M \varepsilon^{5/2}, \end{aligned} \quad (36)$$

where $M = M(T)$ is a positive constant, independent of ε .

Proof. First, multiplying the first equation of (18) by $-\Delta n_R$ and integrating over $\Omega = \mathbb{T} \times (0, 1)$, we obtain:

$$\frac{1}{2} \frac{d}{dt} \|\nabla n_R\|_{L^2}^2 + \|\Delta n_R\|_{L^2}^2 = \int_{\Omega} \operatorname{div}(n_R \nabla C_R) \cdot \Delta n_R dx - \int_{\Omega} J_I \cdot \Delta n_R dx - \int_{\Omega} J_B \cdot \Delta n_R dx - \int_{\Omega} J_R \cdot \Delta n_R dx. \quad (37)$$

Thanks to the Cauchy-Schwarz inequality and Sobolev embedding, we have

$$\begin{aligned} \int_{\Omega} \operatorname{div}(n_R \nabla C_R) \cdot \Delta n_R dx &= \int_{\Omega} n_R \operatorname{div}(\nabla C_R) \cdot \nabla n_R dx + \int_{\Omega} \nabla n_R \nabla C_R \cdot \nabla n_R dx \\ &\leq M \|\sqrt{\varepsilon} \operatorname{div}(\nabla C_R)\|_{L^2}^2 \left\| \frac{n_R}{\sqrt{\varepsilon}} \right\|_{L^\infty}^2 + M \|\sqrt{\varepsilon} \nabla C_R\|_{L^4}^2 \left\| \frac{\nabla n_R}{\sqrt{\varepsilon}} \right\|_{L^4}^2 + \frac{1}{4} \|\Delta n_R\|_{L^2}^2 \\ &\leq M \left\| \frac{n_R}{\sqrt{\varepsilon}} \right\|_{H^2}^2 \|\sqrt{\varepsilon} \operatorname{div}(\nabla C_R)\|_{L^2}^2 + \left\| \frac{\nabla n_R}{\sqrt{\varepsilon}} \right\|_{H^1}^2 \|\sqrt{\varepsilon} \nabla C_R\|_{H^1}^2 + \frac{1}{4} \|\nabla n_R\|_{L^2}^2. \end{aligned}$$

By a similar procedure to that of the proof of Lemma 1, we have the following estimates:

$$-\int_{\Omega} J_I \cdot \Delta n_R dx - \int_{\Omega} J_B \cdot \Delta n_R dx \leq M\varepsilon^{5/2} + \frac{1}{8} \|\Delta n_R\|_{L^2}^2$$

and

$$\begin{aligned} -\int_{\Omega} J_R \cdot \Delta n_R dx &\leq \frac{1}{8} \|\Delta n_R\|_{L^2}^2 + M \|\operatorname{div}(\nabla c_a)\|_{L^\infty}^2 \|n_R\|_{L^2}^2 + M \|\nabla c_a\|_{L^\infty}^2 \|\nabla n_R\|_{L^2}^2 \\ &\quad + M \|\nabla n_a\|_{L^\infty}^2 \|\nabla c_R\|_{L^2}^2 + M \|\nabla n_a\|_{L^\infty}^2 \|\operatorname{div}(\nabla c_R)\|_{L^2}^2 \\ &\leq \frac{1}{8} \|\Delta n_R\|_{L^2}^2 + M \left\| \frac{n_R}{\sqrt{\varepsilon}} \right\|_{L^2}^2 + M \|\nabla n_R, \nabla c_R\|_{L^2}^2 + M\varepsilon \|\operatorname{div}(\nabla c_R)\|_{L^2}^2. \end{aligned}$$

Plugging the aforementioned estimates into (37), we get

$$\begin{aligned} \frac{d}{dt} \|\nabla n_R\|_{L^2}^2 + \|\Delta n_R\|_{L^2}^2 &\leq M \left\| \frac{n_R}{\sqrt{\varepsilon}} \right\|_{H^2}^2 \|\sqrt{\varepsilon} \operatorname{div}(\nabla c_R)\|_{L^2}^2 + M \left\| \frac{\nabla n_R}{\sqrt{\varepsilon}} \right\|_{H^1}^2 \|\sqrt{\varepsilon} \nabla c_R\|_{H^1}^2 \\ &\quad + M \left\| \frac{n_R}{\sqrt{\varepsilon}} \right\|_{L^2}^2 + M \|\nabla n_R, \nabla c_R\|_{L^2}^2 + M\varepsilon \|\operatorname{div}(\nabla c_R)\|_{L^2}^2 + M\varepsilon^{5/2}. \end{aligned} \quad (38)$$

Similarly, multiplying the second equation of (18) by $-\Delta c_R$ and reforming integration, we have:

$$\frac{1}{2} \frac{d}{dt} \|\nabla c_R\|_{L^2}^2 + \varepsilon \|\Delta c_R\|_{L^2}^2 = \int_{\Omega} (n_R c_R) \cdot \Delta c_R dx - \int_{\Omega} K_I \cdot \Delta c_R dx - \int_{\Omega} K_B \cdot \Delta c_R dx - \int_{\Omega} K_R \cdot \Delta c_R dx.$$

Thanks to the Cauchy-Schwarz inequality, we get

$$\int_{\Omega} (n_R c_R) \cdot \Delta c_R dx \leq M \|\nabla n_R\|_{H^1}^2 \|c_R\|_{H^1}^2 + M \|n_R\|_{H^2}^2 \|\nabla c_R\|_{L^2}^2 + M \|\nabla c_R\|_{L^2}^2$$

and

$$-\int_{\Omega} K_I \cdot \Delta c_R dx - \int_{\Omega} K_B \cdot \Delta c_R dx \leq \frac{1}{4} \|\sqrt{\varepsilon} \Delta c_R\|_{L^2}^2 + M \left\| \frac{K_I}{\sqrt{\varepsilon}} \right\|_{L^2}^2 + M \left\| \frac{K_B}{\sqrt{\varepsilon}} \right\|_{L^2}^2 \leq \frac{1}{4} \|\sqrt{\varepsilon} \Delta c_R\|_{L^2}^2 + M\varepsilon^{5/2}.$$

Similar procedure further gives

$$\begin{aligned} -\int_{\Omega} K_R \cdot \Delta c_R dx &= -\int_{\Omega} \frac{1}{\sqrt{\varepsilon}} (n_a c_R + n_R c_a) \cdot \sqrt{\varepsilon} \Delta c_R dx \\ &\leq \frac{1}{4} \|\sqrt{\varepsilon} \Delta c_R\|_{L^2}^2 + M \left\| \frac{c_a}{\sqrt{\varepsilon}} \right\|_{L^\infty}^2 \|n_R\|_{L^2}^2 + M \left\| \frac{n_a}{\sqrt{\varepsilon}} \right\|_{L^\infty}^2 \|c_R\|_{L^2}^2 \\ &\leq \frac{1}{4} \|\sqrt{\varepsilon} \Delta c_R\|_{L^2}^2 + \|n_R, c_R\|_{L^2}^2. \end{aligned}$$

Then, we have

$$\frac{d}{dt} \|\nabla c_R\|_{L^2}^2 + \varepsilon \|\Delta c_R\|_{L^2}^2 \leq M \|\nabla n_R\|_{H^1}^2 \|c_R\|_{H^1}^2 + M \|n_R\|_{H^2}^2 \|\nabla c_R\|_{L^2}^2 + M \|\nabla c_R\|_{L^2}^2 + M \|n_R, c_R\|_{L^2}^2 + M\varepsilon^{5/2}. \quad (39)$$

The proof is completed by integrating (38) and (39) with respect to t over $[0, t]$. $\square$

Lemma 4. *If the assumptions in Theorem 1 are satisfied, and (n_R, c_R) is the solution of (18), then for any $0 < T < T^*$ and $t \in (0, T)$, it holds that*

$$\begin{aligned}
& \|\nabla(\partial_t n_R), \nabla(\partial_t c_R)\|_{L^2}^2 + \int_0^t \|\Delta(\partial_t n_R)\|_{L^2}^2 dt + \varepsilon \int_0^t \|\Delta(\partial_t c_R)\|_{L^2}^2 dt \\
& \leq M \|(\nabla(\partial_t n_R), \nabla(\partial_t c_R))\|_{t=0}^2 + M \int_0^t \left(\left\| \frac{n_R}{\sqrt{\varepsilon}} \right\|_{H^2}^2 \|\sqrt{\varepsilon} \operatorname{div}(\nabla(\partial_t c_R))\|_{L^2}^2 \right) dt \\
& \quad + M \int_0^t \left(\left\| \frac{\partial_t n_R}{\sqrt{\varepsilon}} \right\|_{H^2}^2 \|\sqrt{\varepsilon} \operatorname{div}(\nabla(c_R))\|_{L^2}^2 + \left\| \frac{\nabla(\partial_t n_R)}{\sqrt{\varepsilon}} \right\|_{H^1}^2 \|\sqrt{\varepsilon} \nabla(c_R)\|_{H^1}^2 \right) dt \\
& \quad + M \int_0^t \left(\left\| \frac{\nabla n_R}{\sqrt{\varepsilon}} \right\|_{H^1}^2 \|\sqrt{\varepsilon} \nabla(\partial_t c_R)\|_{H^1}^2 + \|\nabla(\partial_t n_R)\|_{H^1}^2 \|c_R\|_{H^1}^2 \right) dt \\
& \quad + M \int_0^t (\|\nabla n_R\|_{H^1}^2 \|\partial_t c_R\|_{H^1}^2 + \|\partial_t n_R\|_{H^2}^2 \|\nabla c_R\|_{L^2}^2 + \|n_R\|_{H^2}^2 \|\nabla(\partial_t c_R)\|_{L^2}^2) dt \\
& \quad + M \int_0^t (\|n_R, \partial_t n_R, \nabla n_R, \nabla(\partial_t n_R), c_R, \partial_t c_R, \nabla c_R, \nabla(\partial_t c_R)\|_{L^2}^2) dt \\
& \quad + M \int_0^t (\|\sqrt{\varepsilon} \operatorname{div}(\nabla(\partial_t c_R))\|_{L^2}^2 + \|\sqrt{\varepsilon} \operatorname{div}(\nabla(c_R))\|_{L^2}^2) dt \\
& \quad + M \int_0^t \left(\left\| \frac{\partial_t n_R}{\sqrt{\varepsilon}} \right\|_{L^2}^2 + \left\| \frac{n_R}{\sqrt{\varepsilon}} \right\|_{L^2}^2 \right) dt + M\varepsilon^{5/2},
\end{aligned} \tag{40}$$

where $M = M(T)$ is a positive constant, independent of ε .

Proof. Differentiating (18) with respect to t , multiplying the first equation by $-\Delta(\partial_t n_R)$ and integrating over $\Omega = \mathbb{T} \times (0,1)$, we get

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\nabla(\partial_t n_R)\|_{L^2}^2 + \|\Delta(\partial_t n_R)\|_{L^2}^2 &= \int_{\Omega} \partial_t [\operatorname{div}(\partial_t (n_R \nabla c_R))] \cdot \Delta(\partial_t n_R) dx \\
&\quad - \int_{\Omega} \partial_t J_I \cdot \Delta(\partial_t n_R) dx - \int_{\Omega} \partial_t J_B \cdot \Delta(\partial_t n_R) dx - \int_{\Omega} \partial_t J_R \cdot \Delta(\partial_t n_R) dx.
\end{aligned} \tag{41}$$

We can obtain the following estimates by a similar procedure to that in the proof of Lemma 2

$$\begin{aligned}
& \int_{\Omega} \partial_t [\operatorname{div}(\partial_t (n_R \nabla c_R))] \cdot \Delta(\partial_t n_R) dx \\
&= \int_{\Omega} [\operatorname{div}(\nabla(\partial_t c_R)) \cdot n_R + \operatorname{div}(\nabla(c_R)) \cdot \partial_t n_R + \nabla(c_R) \cdot \nabla(\partial_t n_R) + \nabla(n_R) \cdot \nabla(\partial_t c_R)] \cdot \Delta(\partial_t n_R) dx \\
&\leq M \left\| \frac{n_R}{\sqrt{\varepsilon}} \right\|_{H^2}^2 \|\sqrt{\varepsilon} \operatorname{div}(\nabla(\partial_t c_R))\|_{L^2}^2 + \left\| \frac{\partial_t n_R}{\sqrt{\varepsilon}} \right\|_{H^2}^2 \|\sqrt{\varepsilon} \operatorname{div}(\nabla(c_R))\|_{L^2}^2 \\
&\quad + M \left\| \frac{\nabla(\partial_t n_R)}{\sqrt{\varepsilon}} \right\|_{H^1}^2 \|\sqrt{\varepsilon} \nabla(c_R)\|_{H^1}^2 + M \left\| \frac{\nabla n_R}{\sqrt{\varepsilon}} \right\|_{H^1}^2 \|\sqrt{\varepsilon} \nabla(\partial_t c_R)\|_{H^1}^2 + \frac{1}{8} \|\Delta(\partial_t n_R)\|_{L^2}^2
\end{aligned}$$

and

$$- \int_{\Omega} \partial_t J_I \cdot \Delta(\partial_t n_R) dx - \int_{\Omega} \partial_t J_B \cdot \Delta(\partial_t n_R) dx \leq M\varepsilon^{5/2} + \frac{1}{8} \|\Delta(\partial_t n_R)\|_{L^2}^2.$$

For the last term of (41), we get

$$\begin{aligned} - \int_{\Omega} \partial_t J_R \cdot \Delta(\partial_t n_R) dx &\leq \frac{1}{4} \|\Delta(\partial_t n_R)\|_{L^2}^2 + M \left\| \frac{\partial_t n_R}{\sqrt{\varepsilon}}, \frac{n_R}{\sqrt{\varepsilon}} \right\|_{L^2}^2 + M \|\sqrt{\varepsilon} \operatorname{div}(\nabla(\partial_t c_R))\|_{L^2}^2 \\ &\quad + M \|\sqrt{\varepsilon} \operatorname{div}(\nabla(c_R))\|_{L^2}^2 + M \|\nabla n_R, \nabla(\partial_t n_R), \nabla c_R, \nabla(\partial_t c_R)\|_{L^2}^2. \end{aligned}$$

Then, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla(\partial_t n_R)\|_{L^2}^2 + \|\Delta(\partial_t n_R)\|_{L^2}^2 &\leq \left\| \frac{n_R}{\sqrt{\varepsilon}} \right\|_{H^2}^2 \|\sqrt{\varepsilon} \operatorname{div}(\nabla(\partial_t c_R))\|_{L^2}^2 + \left\| \frac{\partial_t n_R}{\sqrt{\varepsilon}} \right\|_{H^2}^2 \|\sqrt{\varepsilon} \operatorname{div}(\nabla(c_R))\|_{L^2}^2 \\ &\quad + M \left\| \frac{\nabla(\partial_t n_R)}{\sqrt{\varepsilon}} \right\|_{H^1}^2 \|\sqrt{\varepsilon} \nabla(c_R)\|_{H^1}^2 + M \left\| \frac{\nabla n_R}{\sqrt{\varepsilon}} \right\|_{H^1}^2 \|\sqrt{\varepsilon} \nabla(\partial_t c_R)\|_{H^1}^2 \\ &\quad + M \left\| \frac{\partial_t n_R}{\sqrt{\varepsilon}}, \frac{n_R}{\sqrt{\varepsilon}} \right\|_{L^2}^2 + M \|\sqrt{\varepsilon} \operatorname{div}(\nabla(\partial_t c_R))\|_{L^2}^2 + M \varepsilon^{5/2} \\ &\quad + M \|\sqrt{\varepsilon} \operatorname{div}(\nabla(c_R))\|_{L^2}^2 + M \|\nabla n_R, \nabla(\partial_t n_R), \nabla c_R, \nabla(\partial_t c_R)\|_{L^2}^2. \end{aligned} \quad (42)$$

Similarly, differentiating (18) with respect to t , multiplying the first equation by $-\Delta(\partial_t c_R)$ and integrating over $\Omega = \mathbb{T} \times (0, 1)$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla(\partial_t c_R)\|_{L^2}^2 + \varepsilon \|\Delta(\partial_t c_R)\|_{L^2}^2 \\ = \int_{\Omega} \partial_t(n_R c_R) \cdot \Delta(\partial_t c_R) dx - \int_{\Omega} \partial_t K_I \cdot \Delta(\partial_t c_R) dx - \int_{\Omega} \partial_t K_B \cdot \Delta(\partial_t c_R) dx - \int_{\Omega} \partial_t K_R \cdot \Delta(\partial_t c_R) dx. \end{aligned} \quad (43)$$

By the Cauchy-Schwarz inequality and the Sobolev embedding inequality, we have

$$\begin{aligned} \int_{\Omega} \partial_t(n_R c_R) \cdot \Delta(\partial_t c_R) dx \\ \leq M \|\nabla(\partial_t n_R)\|_{H^1}^2 \|c_R\|_{H^1}^2 + M \|\nabla n_R\|_{H^1}^2 \|\partial_t c_R\|_{H^1}^2 + M \|\partial_t n_R\|_{H^2}^2 \|\nabla c_R\|_{L^2}^2 + M \|n_R\|_{H^2}^2 \|\nabla(\partial_t c_R)\|_{L^2}^2 + M \|\nabla(\partial_t c_R)\|_{L^2}^2, \\ - \int_{\Omega} \partial_t K_I \cdot \Delta(\partial_t c_R) dx - \int_{\Omega} \partial_t K_B \cdot \Delta(\partial_t c_R) dx \leq M \varepsilon^{5/2} + \frac{1}{4} \|\sqrt{\varepsilon} \Delta(\partial_t c_R)\|_{L^2}^2 \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega} \partial_t K_R \cdot \Delta(\partial_t c_R) dx &= - \int_{\Omega} (c_R \partial_t n_a + n_a \partial_t c_R + c_a \partial_t n_R + n_R \partial_t c_a) \cdot \Delta(\partial_t c_R) dx \\ &\leq M \|n_R, \partial_t n_R, c_R, \partial_t c_R\|_{L^2}^2 + \frac{1}{4} \|\sqrt{\varepsilon} \Delta(\partial_t c_R)\|_{L^2}^2. \end{aligned}$$

Combining the aforementioned estimates now leads to

$$\begin{aligned} \frac{d}{dt} \|\nabla(\partial_t c_R)\|_{L^2}^2 + \varepsilon \|\Delta(\partial_t c_R)\|_{L^2}^2 &\leq M \|\nabla(\partial_t n_R)\|_{H^1}^2 \|c_R\|_{H^1}^2 + M \|\nabla n_R\|_{H^1}^2 \|\partial_t c_R\|_{H^1}^2 + M \|\partial_t n_R\|_{H^2}^2 \|\nabla c_R\|_{L^2}^2 \\ &\quad + M \|n_R\|_{H^2}^2 \|\nabla(\partial_t c_R)\|_{L^2}^2 + M \|n_R, \partial_t n_R, c_R, \partial_t c_R, \nabla(\partial_t c_R)\|_{L^2}^2 + M \varepsilon^{5/2}. \end{aligned} \quad (44)$$

We end the proof by combining (43) with (44) and integrating with respect to t over $[0, t]$. $\square$

Lemma 5. *If the assumptions in Theorem 1 are satisfied, and (n_R, c_R) is the solution of (18), then for any $0 < T < T^*$ and $t \in (0, T)$, it holds that*

$$\begin{aligned}
\|(\nabla n_R, \Delta n_R)\|_{L^2}^2 + \varepsilon \|\nabla c_R, \Delta c_R\|_{L^2}^2 &\leq M \|n_R\|_{H^2}^2 \|c_R\|_{L^2}^2 + M \|n_R\|_{H^2}^2 \|\nabla c_R\|_{L^2}^2 + M \left\| \frac{n_R}{\sqrt{\varepsilon}} \right\|_{H^2}^2 \|\sqrt{\varepsilon} \operatorname{div}(\nabla c_R)\|_{L^2}^2 \\
&\quad + M \left\| \frac{\nabla n_R}{\sqrt{\varepsilon}} \right\|_{H^1}^2 \|\sqrt{\varepsilon} \nabla c_R\|_{H^1}^2 + M \|\nabla n_R\|_{H^1}^2 \|c_R\|_{H^1}^2 + M \|n_R\|_{H^2}^2 \|\nabla c_R\|_{L^2}^2 \\
&\quad + M \varepsilon \|\operatorname{div}(\nabla c_R)\|_{L^2}^2 + M \|n_R, c_R, \nabla n_R, \nabla c_R, \partial_t n_R, \partial_t c_R, \nabla(\partial_t n_R), \nabla(\partial_t c_R)\|_{L^2}^2 \\
&\quad + M \left\| \frac{n_R}{\sqrt{\varepsilon}} \right\|_{L^2}^2 + M \varepsilon^{5/2},
\end{aligned} \tag{45}$$

where $M = M(T)$ is a positive constant, independent of ε .

Proof. By (27) and (29), we can easily obtain

$$\begin{aligned}
\|\nabla n_R\|_{L^2}^2 + \varepsilon \|\nabla c_R\|_{L^2}^2 &\leq -\frac{d}{dt} \|n_R, c_R\|_{L^2}^2 + M \|n_R\|_{H^2}^2 \|\nabla c_R\|_{L^2}^2 + M \|n_R\|_{H^2}^2 \|c_R\|_{L^2}^2 + M \|n_R, c_R\|_{L^2}^2 + M \varepsilon^{5/2} \\
&\leq M \|n_R\|_{H^2}^2 \|c_R\|_{L^2}^2 + M \|n_R, c_R, \partial_t n_R, \partial_t c_R\|_{L^2}^2 + M \|n_R\|_{H^2}^2 \|\nabla c_R\|_{L^2}^2 + M \varepsilon^{5/2}.
\end{aligned} \tag{46}$$

We can also obtain the following estimate by (38) and (39)

$$\begin{aligned}
\|\Delta n_R\|_{L^2}^2 + \varepsilon \|\Delta c_R\|_{L^2}^2 &\leq -\frac{d}{dt} \|\nabla n_R, \nabla c_R\|_{L^2}^2 + M \left\| \frac{n_R}{\sqrt{\varepsilon}} \right\|_{H^2}^2 \|\sqrt{\varepsilon} \operatorname{div}(\nabla c_R)\|_{L^2}^2 + M \left\| \frac{n_R}{\sqrt{\varepsilon}} \right\|_{L^2}^2 \\
&\quad + M \left\| \frac{\nabla n_R}{\sqrt{\varepsilon}} \right\|_{H^1}^2 \|\sqrt{\varepsilon} \nabla c_R\|_{H^1}^2 + M \|\nabla n_R\|_{H^1}^2 \|c_R\|_{H^1}^2 + M \|n_R\|_{H^2}^2 \|\nabla c_R\|_{L^2}^2 \\
&\quad + M \varepsilon \|\operatorname{div}(\nabla c_R)\|_{L^2}^2 + M \|n_R, c_R, \nabla n_R, \nabla c_R\|_{L^2}^2 + M \varepsilon^{5/2} \\
&\leq M \left\| \frac{n_R}{\sqrt{\varepsilon}} \right\|_{H^2}^2 \|\sqrt{\varepsilon} \operatorname{div}(\nabla c_R)\|_{L^2}^2 + M \left\| \frac{n_R}{\sqrt{\varepsilon}} \right\|_{L^2}^2 \\
&\quad + M \left\| \frac{\nabla n_R}{\sqrt{\varepsilon}} \right\|_{H^1}^2 \|\sqrt{\varepsilon} \nabla c_R\|_{H^1}^2 + M \|\nabla n_R\|_{H^1}^2 \|c_R\|_{H^1}^2 + M \|n_R\|_{H^2}^2 \|\nabla c_R\|_{L^2}^2 \\
&\quad + M \varepsilon \|\operatorname{div}(\nabla c_R)\|_{L^2}^2 + M \|n_R, c_R, \nabla n_R, \nabla c_R, \nabla(\partial_t n_R), \nabla(\partial_t c_R)\|_{L^2}^2 + M \varepsilon^{5/2}.
\end{aligned} \tag{47}$$

We end the proof by combining (46) and (47). $\square$

Lemma 6. If the assumptions in Theorem 1 are satisfied, and (n_R, c_R) is the solution of (18), then for any $0 < T < T^*$ and $t \in (0, T)$, there exists a positive constant $M = M(T)$ such that

$$\begin{aligned}
&\|\partial_t^2 n_R, \partial_t^2 c_R\|_{L^2}^2 + \int_0^t \|\nabla(\partial_t^2 n_R)\|_{L^2}^2 dt + \varepsilon \int_0^t \|\nabla(\partial_t^2 c_R)\|_{L^2}^2 dt \\
&\leq M \|(\partial_t^2 n_R, \partial_t^2 c_R)|_{t=0}\|_{L^2}^2 + M \int_0^t (\|\partial_t^2 n_R\|_{H^2}^2 \|\nabla c_R\|_{L^2}^2 + \|\partial_t n_R\|_{H^2}^2 \|\nabla(\partial_t c_R)\|_{L^2}^2) dt \\
&\quad + M \int_0^t (\|\partial_t^2 n_R\|_{H^1}^2 \|c_R\|_{H^1}^2 + \|\partial_t n_R\|_{H^1}^2 \|\partial_t c_R\|_{H^1}^2 + \|n_R\|_{H^2}^2 \|\partial_t^2 c_R\|_{L^2}^2) dt \\
&\quad + M \int_0^t (\varepsilon \|c_R, \partial_t c_R, \partial_t^2 c_R, \nabla c_R, \nabla(\partial_t c_R), \nabla(\partial_t^2 c_R)\|_{L^2}^2) dt \\
&\quad + M \int_0^t (\|n_R, \partial_t n_R, \partial_t^2 n_R\|_{L^2}^2) dt + M \varepsilon^{5/2}.
\end{aligned} \tag{48}$$

Proof. Differentiating (18) with respect to t twice, multiplying the first equation by $\partial_t^2 n_R$ and integrating over $\Omega = \mathbb{T} \times (0,1)$, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_t^2 n_R\|_{L^2}^2 + \|\nabla(\partial_t^2 n_R)\|_{L^2}^2 = & - \int_{\Omega} \partial_t^2 [\operatorname{div}(n_R \nabla c_R)] \cdot \partial_t^2 n_R dx + \int_{\Omega} \partial_t^2 J_I \cdot \partial_t^2 n_R dx \\ & + \int_{\Omega} \partial_t^2 J_B \cdot \partial_t^2 n_R dx + \int_{\Omega} \partial_t^2 J_R \cdot \partial_t^2 n_R dx, \end{aligned} \quad (49)$$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_t^2 c_R\|_{L^2}^2 + \varepsilon \|\nabla(\partial_t^2 c_R)\|_{L^2}^2 = & - \int_{\Omega} \partial_t^2 (n_R c_R) \cdot \partial_t^2 c_R dx + \int_{\Omega} \partial_t^2 K_I \cdot \partial_t^2 c_R dx \\ & + \int_{\Omega} \partial_t^2 K_B \cdot \partial_t^2 c_R dx + \int_{\Omega} \partial_t^2 K_R \cdot \partial_t^2 c_R dx. \end{aligned} \quad (50)$$

All terms on the right hand side of (49), (50) can be estimated by a similar procedure to that in the proof of Lemma 2. For example,

$$\begin{aligned} - \int_{\Omega} \partial_t^2 [\operatorname{div}(n_R \nabla c_R)] \cdot \partial_t^2 n_R dx & \leq M \|\partial_t^2 n_R\|_{H^2}^2 \|\nabla c_R\|_{L^2}^2 + \|\partial_t n_R\|_{H^2}^2 \|\nabla(\partial_t c_R)\|_{L^2}^2 + \|n_R\|_{H^2}^2 \|\partial_t^2 c_R\|_{L^2}^2 + \frac{1}{4} \|\nabla(\partial_t^2 n_R)\|_{L^2}^2, \\ \int_{\Omega} \partial_t^2 J_R \cdot \partial_t^2 n_R dx & \leq M \|n_R, \partial_t n_R, \partial_t^2 n_R\|_{L^2}^2 + M \varepsilon \|\nabla c_R, \nabla(\partial_t c_R), \nabla(\partial_t^2 c_R)\|_{L^2}^2 \end{aligned}$$

and

$$- \int_{\Omega} \partial_t^2 (n_R c_R) \cdot \partial_t^2 c_R dx \leq M \|\partial_t^2 n_R\|_{H^1}^2 \|c_R\|_{H^1}^2 + \|\partial_t n_R\|_{H^1}^2 \|\partial_t c_R\|_{H^1}^2 + \|n_R\|_{H^2}^2 \|\partial_t^2 c_R\|_{L^2}^2 + \|\partial_t^2 c_R\|_{L^2}^2.$$

This completes the proof of Lemma 6. $\square$

Finally, to finish the proof of Theorem 1, we also need the following estimates, which can be established easily by using the regularity theory of elliptic partial differential equations with Dirichlet or Neumann boundary conditions.

Proposition 3. Under the assumptions of Proposition 1, and for $i = 0, 1$, we get

$$\|\partial_t^i n_R\|_{H^2}^2 \leq M (\|\partial_t^i n_R\|_{L^2}^2 + \|\Delta(\partial_t^i n_R)\|_{L^2}^2), \quad \|\partial_t^i c_R\|_{H^2}^2 \leq M (\|\partial_t^i c_R\|_{L^2}^2 + \|\Delta(\partial_t^i c_R)\|_{L^2}^2).$$

Next, we introduce the following ε -weighted functional for the remainder terms:

$$\begin{aligned} E_0^\varepsilon(t) &= \sum_{i=0}^2 \|\partial_t^i(n_R, \sqrt{\varepsilon} c_R)\|_{H^{2-i}}^2 + \|c_R, \partial_t c_R, \partial_t^2 c_R, \nabla c_R, \nabla(\partial_t c_R)\|_{L^2}^2, \\ E_1^\varepsilon(t) &= \sum_{i=0}^2 \|\partial_t^i(n_R, c_R)\|_{L^2}^2, \quad E_2^\varepsilon(t) = \sum_{i=0}^1 \|\partial_t^i(\nabla n_R, \nabla c_R)\|_{L^2}^2, \\ E^\varepsilon(t) &= E_1^\varepsilon(t) + E_2^\varepsilon(t) + \|\Delta n_R, \sqrt{\varepsilon} \Delta c_R\|_{L^2}^2, \\ G_1^\varepsilon(t) &= \sum_{i=0}^2 \|\partial_t^i(\nabla n_R, \sqrt{\varepsilon} \nabla c_R)\|_{L^2}^2, \quad G_2^\varepsilon(t) = \sum_{i=0}^1 \|\partial_t^i(\Delta n_R, \sqrt{\varepsilon} \Delta c_R)\|_{L^2}^2, \\ G^\varepsilon(t) &= \sum_{i=0}^1 \|\partial_t^i(\Delta n_R, \sqrt{\varepsilon} \Delta c_R)\|_{H^1}^2 + \|\partial_t^2(\Delta n_R, \sqrt{\varepsilon} \Delta c_R)\|_{L^2}^2, \quad G_3^\varepsilon(t) = G_1^\varepsilon(t) + G_2^\varepsilon(t). \end{aligned}$$

We get by Lemmas 1–6 that

$$\begin{aligned}
 E^\varepsilon(t) + \int_0^t G_3^\varepsilon(s) ds &\leq M(E_1^\varepsilon|_{t=0} + E_2^\varepsilon|_{t=0}) + M(E_0^\varepsilon(s))^2 + M\varepsilon^{5/2} \\
 &\quad + M \int_0^t \left(E^\varepsilon(s) + M(E_0^\varepsilon(s))^2 + M \frac{E_0^\varepsilon(s)}{\varepsilon} G^\varepsilon(s) + E_0^\varepsilon(s) G^\varepsilon(s) \right) ds \\
 &\quad + M \left\| \frac{n_R}{\sqrt{\varepsilon}}, \frac{\partial_t n_R}{\sqrt{\varepsilon}} \right\|_{L^2}^2 + M \int_0^t \left\| \frac{n_R}{\sqrt{\varepsilon}}, \frac{\partial_t n_R}{\sqrt{\varepsilon}} \right\|_{L^2}^2(\cdot, s) ds.
 \end{aligned} \tag{51}$$

Thanks to Lemmas 1, 2 and 6, we have

$$\begin{aligned}
 \left\| \frac{n_R}{\sqrt{\varepsilon}}, \frac{\partial_t n_R}{\sqrt{\varepsilon}} \right\|_{L^2}^2 + \int_0^t \left\| \frac{n_R}{\sqrt{\varepsilon}}, \frac{\partial_t n_R}{\sqrt{\varepsilon}} \right\|_{L^2}^2 ds &\leq M \left\| \left(\frac{n_R}{\sqrt{\varepsilon}}, \frac{\partial_t n_R}{\sqrt{\varepsilon}} \right) \right\|_{t=0}^2 \\
 &\quad + M \int_0^t \left(\frac{(E_0^\varepsilon(s))^2}{\varepsilon} + \frac{E_0^\varepsilon(s)}{\varepsilon} G^\varepsilon(s) \right) ds + M\varepsilon^{3/2}.
 \end{aligned} \tag{52}$$

Using Proposition 3, we know that there exists the positive constants $M_2 > 0$ and $M_3 > 0$ independent of ε , such that

$$M_1 E_0^\varepsilon(t) \leq E^\varepsilon(t) \leq M_2 E_0^\varepsilon(t), \quad M_1 G_3^\varepsilon(t) \leq G^\varepsilon(t) \leq M_2 G_3^\varepsilon(t). \tag{53}$$

Then, we can obtain the following estimate by (51)–(53)

$$\begin{aligned}
 E^\varepsilon(t) + \int_0^t G^\varepsilon(s) ds &\leq M E_1^\varepsilon|_{t=0} + M \frac{\left\| \left(\frac{n_R}{\sqrt{\varepsilon}}, \frac{\partial_t n_R}{\sqrt{\varepsilon}} \right) \right\|_{t=0}^2}{\varepsilon} + M(E_0^\varepsilon(s))^2 + M \frac{(E_0^\varepsilon(s))^2}{\varepsilon} \\
 &\quad + M \int_0^t \left(E^\varepsilon(s) + (E_0^\varepsilon(s))^2 + \frac{(E_0^\varepsilon(s))^2}{\varepsilon} + \frac{E_0^\varepsilon(s)}{\varepsilon} G^\varepsilon(s) + E_0^\varepsilon(s) G^\varepsilon(s) \right) ds + M\varepsilon^{3/2},
 \end{aligned} \tag{54}$$

where M is a positive constant independent of ε and $0 \leq t \leq T$.

The above inequality is an ε -weighted generalized entropy-production inequality. By using a Gronwall-type estimate in [29] (see also [30]), we can obtain

$$E^\varepsilon(t) \leq M\varepsilon^{3/2-\delta} \tag{55}$$

for any $0 < \delta < 1/2$ provided $E_1^\varepsilon \leq \varepsilon^{3/2}$ and $\|(n_R, \partial_t n_R)|_{t=0}\|_{L^2}^2 \leq \varepsilon^{5/2}$.

With (55), Theorem 1 now follows easily.

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