Research Article

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On the Gevrey ultradifferentiability of weak solutions of an abstract evolution equation with a scalar type spectral operator on the real axis

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Abstract: Given the abstract evolution equation $y'(t) = Ay(t)$, $t \in \mathbb{R}$, with a *scalar type spectral operator* A in a complex Banach space, we find conditions on A , formulated exclusively in terms of the location of its spectrum in the complex plane, *necessary and sufficient* for all *weak solutions* of the equation, which *a priori* need not be strongly differentiable, to be strongly Gevrey ultradifferentiable of order $\beta \geq 1$, in particular *analytic* or *entire*, on $\mathbb{R}$. We also reveal certain inherent smoothness improvement effects and show that if all weak solutions of the equation are Gevrey ultradifferentiable of orders less than one, then the operator A is necessarily *bounded*. The important particular case of the equation with a *normal operator* A in a complex Hilbert space follows immediately.

Keywords: weak solution, scalar type spectral operator, Gevrey classes

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1 Introduction

We find conditions on a scalar type spectral operator A in a complex Banach space, formulated exclusively in terms of the location of its *spectrum* in the complex plane, *necessary and sufficient* for all *weak solutions* of the evolution equation

$$y'(t) = Ay(t), \quad t \in \mathbb{R}, \quad (1.1)$$

which *a priori* need not be strongly differentiable, to be strongly Gevrey ultradifferentiable of order $\beta \geq 1$, in particular *analytic* or *entire*, on $\mathbb{R}$. We also reveal certain inherent smoothness improvement effects and show that if all weak solutions of the equation are Gevrey ultradifferentiable of orders less than one, then the operator A is necessarily *bounded*.

The important particular case of the equation with a *normal operator* A in a complex Hilbert space follows immediately.

We proceed along the path of developing the results of paper [1] on the strong differentiability of the weak solutions of equation (1.1) on $\mathbb{R}$ and of papers [2–4], where similar consideration is given to the Gevrey ultradifferentiability of the weak solutions of the evolution equation

$$y'(t) = Ay(t), \quad t \geq 0, \quad (1.2)$$

on $[0, \infty)$ and $(0, \infty)$.

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2 Preliminaries

Here, we briefly outline certain facts essential for the subsequent discourse (for more details, see, e.g., [1–3, 5]).

2.1 Weak solutions

Definition 2.1. (Weak solution)

Let A be a densely defined closed linear operator in a Banach space $(X, \|\cdot\|)$ and I be an interval of the real axis $\mathbb{R}$. A strongly continuous vector function $y : I \rightarrow X$ is called a *weak solution* of the evolution equation

$$y'(t) = Ay(t), \quad t \in I, \quad (2.3)$$

if, for any $g^* \in D(A^*)$,

$$\frac{d}{dt} \langle y(t), g^* \rangle = \langle y(t), A^* g^* \rangle, \quad t \in I,$$

where $D(\cdot)$ is the *domain* of an operator, A^* is the operator *adjoint* to A , and $\langle \cdot, \cdot \rangle$ is the *pairing* between the space X and its dual X^* (cf. [6]).

Remarks 2.1.

- Due to the *closedness* of A , a weak solution of equation (2.3) can be equivalently defined to be a strongly continuous vector function $y : I \rightarrow X$ such that, for all $t \in I$,

$$\int_{t_0}^t y(s) ds \in D(A) \quad \text{and} \quad y(t) = y(t_0) + A \int_{t_0}^t y(s) ds,$$

where t_0 is an arbitrary fixed point of the interval I , and is also called a *mild solution* (cf. [7, Ch. II, Definition 6.3], see also [5, Preliminaries]).

- Such a notion of *weak solution*, which need not be differentiable in the strong sense, generalizes that of *classical* one, strongly differentiable on I and satisfying the equation in the traditional plug-in sense, the classical solutions being precisely the weak ones strongly differentiable on I .
- As is easily seen $y : \mathbb{R} \rightarrow X$ is a weak solution of equation (1.1) *iff*

(i)

$$y_+(t) := y(t), \quad t \geq 0,$$

is a weak solution of equation (1.2) and

$$y_-(t) := y(-t), \quad t \geq 0,$$

is a weak solution of the equation

$$y'(t) = -Ay(t), \quad t \geq 0, \quad (2.4)$$

or

(ii)

$$y_-(t) := y(-t), \quad t \in \mathbb{R},$$

is a weak solution of the equation

$$y'(t) = -Ay(t), \quad t \in \mathbb{R}.$$

- When a closed densely defined linear operator A in a complex Banach space X generates a strongly continuous group $\{T(t)\}_{t \in \mathbb{R}}$ of bounded linear operators (see, e.g., [7,8]), i.e., the associated *abstract Cauchy problem* (ACP)

$$\begin{cases} y'(t) = Ay(t), & t \in \mathbb{R}, \\ y(0) = f \end{cases} \quad (2.5)$$

is *well posed* (cf. [7, Ch. II, Definition 6.8]), the weak solutions of equation (1.1) are the orbits

$$y(t) = T(t)f, \quad t \in \mathbb{R}, \quad (2.6)$$

with $f \in X$ (cf. [7, Ch. II, Proposition 6.4], see also [6, Theorem]), whereas the classical ones are those with $f \in D(A)$ (see, e.g., [7, Ch. II, Proposition 6.3]).

2.2 Scalar type spectral operators

Henceforth, unless specified otherwise, A is a *scalar type spectral operator* in a complex Banach space $(X, \|\cdot\|)$ with strongly σ -additive *spectral measure* (the *resolution of the identity*) $E_A(\cdot)$ assigning to Borel sets of the complex plane $\mathbb{C}$ bounded projection operators on X and having the operator's *spectrum* $\sigma(A)$ as its *support* [9–11].

Observe that, in a complex Hilbert space, the scalar type spectral operators are precisely all those that are similar to the *normal* ones [12–14].

Associated with a scalar type spectral operator A is the *Borel operational calculus* assigning to each complex-valued Borel measurable function $F : \sigma(A) \rightarrow \mathbb{C}$ a scalar type spectral operator

$$F(A) := \int_{\sigma(A)} F(\lambda) dE_A(\lambda)$$

[11]. In particular,

$$A^n = \int_{\sigma(A)} \lambda^n dE_A(\lambda), \quad n \in \mathbb{Z}_+, \quad \text{and} \quad e^{tA} := \int_{\sigma(A)} e^{t\lambda} dE_A(\lambda), \quad t \in \mathbb{R}, \quad (2.7)$$

($\mathbb{Z}_+ := \{0, 1, 2, \dots\}$ is the set of nonnegative integers, $A^0 := I$, I is the *identity operator* on X).

Provided

$$\sigma(A) \subseteq \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leq \omega\}$$

with some $\omega \in \mathbb{R}$, the collection of exponentials $\{e^{tA}\}_{t \geq 0}$ is the C_0 -*semigroup* generated by A [15, Proposition 3.1] (cf. also [16,17]), and hence, if

$$\sigma(A) \subseteq \{\lambda \in \mathbb{C} \mid -\omega \leq \operatorname{Re} \lambda \leq \omega\}$$

with some $\omega \geq 0$, the collection of exponentials $\{e^{tA}\}_{t \in \mathbb{R}}$ is the *strongly continuous group* of bounded linear operators generated by A .

Being strongly σ -additive, the spectral measure is bounded, i.e., there exists an $M \geq 1$ such that, for any Borel set $\delta \subseteq \mathbb{C}$,

$$\|E_A(\delta)\| \leq M \quad (2.8)$$

[11,18].

Remark 2.2. The notation $\|\cdot\|$ is used here to designate the norm on the space $L(X)$ of all bounded linear operators on X . Henceforth, we adhere to this rather conventional economy of symbols adopting the same notation also for the norm on the dual space X^* .

For arbitrary Borel measurable function $F : \mathbb{C} \rightarrow \mathbb{C}$, $f \in D(F(A))$, $g^* \in X^*$, and Borel set $\delta \subseteq \mathbb{C}$,

$$\int_{\delta} |F(\lambda)| d\nu(f, g^*, \lambda) \leq 4M \|E_A(\delta) F(A) f\| \|g^*\|, \quad (2.9)$$

where $\nu(f, g^*, \cdot)$ is the *total variation measure* of the complex-valued Borel measure $\langle E_A(\cdot) f, g^* \rangle$, for which

$$\nu(f, g^*, \mathbb{C}) = \nu(f, g^*, \sigma(A)) \leq 4M \|f\| \|g^*\|, \quad (2.10)$$

where $M \geq 1$ in (2.9) and (2.10) is from (2.8) (see, e.g., [19,20]).

In particular, for $\delta = \sigma(A)$, $E_A(\delta) = I$ (see, e.g., [11]), (2.9) turns into

$$\int_{\sigma(A)} |F(\lambda)| d\nu(f, g^*, \lambda) \leq 4M \|F(A) f\| \|g^*\|. \quad (2.11)$$

Furthermore (see, e.g., [1,2]), for arbitrary Borel measurable function $F : \mathbb{C} \rightarrow [0, \infty)$, Borel set $\delta \subseteq \mathbb{C}$, sequence $\{\Delta_n\}_{n=1}^{\infty}$ of pairwise disjoint Borel sets in $\mathbb{C}$, $f \in X$, and $g^* \in X^*$,

$$\int_{\delta} F(\lambda) d\nu(E_A(\cup_{n=1}^{\infty} \Delta_n) f, g^*, \lambda) = \sum_{n=1}^{\infty} \int_{\delta \cap \Delta_n} F(\lambda) d\nu(E_A(\Delta_n) f, g^*, \lambda). \quad (2.12)$$

Remark 2.3. Subsequently, the frequently used term “*spectral measure*” is abbreviated to s.m.

The following statement characterizing the domains of Borel measurable functions of a scalar type spectral operator in terms of Borel measures is fundamental for our discourse.

Proposition 2.1. [21, Proposition 3.1]

Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$ with spectral measure $E_A(\cdot)$ and $F : \sigma(A) \rightarrow \mathbb{C}$ be a Borel measurable function. Then $f \in D(F(A))$ iff

- (i) for each $g^* \in X^*$, $\int_{\sigma(A)} |F(\lambda)| d\nu(f, g^*, \lambda) < \infty$ and
- (ii) $\sup_{\{g^* \in X^* : \|g^*\| = 1\}} \int_{\{\lambda \in \sigma(A) : |F(\lambda)| > n\}} |F(\lambda)| d\nu(f, g^*, \lambda) \rightarrow 0, \quad n \rightarrow \infty,$

where $\nu(f, g^*, \cdot)$ is the *total variation measure* of $\langle E_A(\cdot) f, g^* \rangle$.

The succeeding key theorem provides a description of the weak solutions of equation (1.1) with a scalar type spectral operator A in a complex Banach space.

Theorem 2.1. [1, Theorem 7]

Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$. A vector function $y : \mathbb{R} \rightarrow X$ is a weak solution of equation (1.1) iff there exists an $f \in \bigcap_{t \in \mathbb{R}} D(e^{tA})$ such that

$$y(t) = e^{tA} f, \quad t \in \mathbb{R}, \quad (2.13)$$

the operator exponentials understood in the sense of the Borel operational calculus (see (2.7)).

We also need the following characterization of a particular weak solution of equation (1.1) with a scalar type spectral operator A in a complex Banach space being strongly infinite differentiable on a subinterval I of $\mathbb{R}$.

Proposition 2.2. [1, Corollary 11]

Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$ and I be interval of the real axis $\mathbb{R}$. A weak solution $y(\cdot)$ of equation (1.1) is strongly infinite differentiable on I ($y(\cdot) \in C^{\infty}(I, X)$) iff, for each $t \in I$,

$$y(t) \in C^{\infty}(A) := \bigcap_{n=0}^{\infty} D(A^n),$$

in which case

$$y^{(n)}(t) = A^n y(t), \quad n \in \mathbb{N}, t \in I.$$

2.3 Gevrey classes of functions

Definition 2.2. (Gevrey classes of functions)

Let $(X, \|\cdot\|)$ be a (real or complex) Banach space, $C^\infty(I, X)$ be the space of all X -valued functions strongly infinite differentiable on an interval I of the real axis $\mathbb{R}$, and $0 \leq \beta < \infty$.

The following subspaces

$$\mathcal{E}^{(\beta)}(I, X) := \{g(\cdot) \in C^\infty(I, X) \mid \forall [a, b] \subseteq I \exists \alpha > 0 \exists c > 0: \max_{a \leq t \leq b} \|g^{(n)}(t)\| \leq c\alpha^n(n!)^\beta, n \in \mathbb{Z}_+\}$$

and

$$\mathcal{E}^{(\beta)}(I, X) := \{g(\cdot) \in C^\infty(I, X) \mid \forall [a, b] \subseteq I \forall \alpha > 0 \exists c > 0: \max_{a \leq t \leq b} \|g^{(n)}(t)\| \leq c\alpha^n(n!)^\beta, n \in \mathbb{Z}_+\}$$

of $C^\infty(I, X)$ are called the β th-order Gevrey classes of strongly ultradifferentiable vector functions on I of Roumieu and Beurling type, respectively (see, e.g., [22–25]).

Remarks 2.4.

- In view of Stirling's formula, the sequence $\{(n!)^\beta\}_{n=0}^\infty$ can be replaced with $\{n^{\beta n}\}_{n=0}^\infty$.
- For $0 \leq \beta < \beta' < \infty$, the inclusions

$$\mathcal{E}^{(\beta)}(I, X) \subseteq \mathcal{E}^{(\beta)}(I, X) \subseteq \mathcal{E}^{(\beta')}(I, X) \subseteq \mathcal{E}^{(\beta')}(I, X) \subseteq C^\infty(I, X)$$

hold.

- For $1 < \beta < \infty$, the Gevrey classes $\mathcal{E}^{(\beta)}(I, X)$ and $\mathcal{E}^{(\beta)}(I, X)$ are *non-quasianalytic* (see, e.g., [24]).
- The first-order Roumieu-type Gevrey class $\mathcal{E}^{(1)}(I, X)$ consists of all *analytic* on I , i.e., *analytically continuable* into complex neighborhoods of I , vector functions and the first-order Beurling-type Gevrey class $\mathcal{E}^{(1)}(I, X)$ consists of all *entire*, i.e., allowing *entire* continuations, vector functions [26].
- For $0 \leq \beta < 1$, the Roumieu-type Gevrey class $\mathcal{E}^{(\beta)}(I, X)$ (the Beurling-type Gevrey class $\mathcal{E}^{(\beta)}(I, X)$) consists of all functions $g(\cdot) \in \mathcal{E}^{(1)}(I, X)$ such that, for some (any) $\gamma > 0$, there exists an $M > 0$, for which

$$\|g(z)\| \leq Me^{\gamma|z|^{1/(1-\beta)}}, \quad z \in \mathbb{C}, \quad (2.14)$$

[27]. In particular, the Gevrey classes $\mathcal{E}^{(0)}(I, X)$ and $\mathcal{E}^{(0)}(I, X)$ are the classes of entire vector functions of *exponential* and *minimal exponential type*, respectively (see, e.g., [28]).

2.4 Gevrey classes of vectors

Definition 2.3. (Gevrey classes of vectors)

Let A be a densely defined closed linear operator in a (real or complex) Banach space $(X, \|\cdot\|)$ and $0 \leq \beta < \infty$.

The following subspaces

$$\mathcal{E}^{(\beta)}(A) := \{f \in C^\infty(A) \mid \exists \alpha > 0 \exists c > 0 : \|A^n f\| \leq c\alpha^n(n!)^\beta, n \in \mathbb{Z}_+\}$$

and

$$\mathcal{E}^{(\beta)}(A) := \{f \in C^\infty(A) \mid \forall \alpha > 0 \exists c > 0 : \|A^n f\| \leq c\alpha^n(n!)^\beta, n \in \mathbb{Z}_+\}$$

of $C^\infty(A)$ are called the β th-order Gevrey classes of ultradifferentiable vectors of A of Roumieu and Beurling type, respectively (see, e.g., [29–31]).

Remarks 2.5.

- In view of *Stirling's formula*, the sequence $\{(n!)^\beta\}_{n=0}^\infty$ can be replaced with $\{n^{\beta n}\}_{n=0}^\infty$.
- For $0 \leq \beta < \beta' < \infty$, the inclusions

$$\mathcal{E}^{(\beta)}(A) \subseteq \mathcal{E}^{(\beta')}(A) \subseteq \mathcal{C}^\infty(A)$$

hold.

- In particular, $\mathcal{E}^{\{1\}}(A)$ and $\mathcal{E}^{(1)}(A)$ are the classes of *analytic* and *entire* vectors of A , respectively [32,33] and $\mathcal{E}^{\{0\}}(A)$ and $\mathcal{E}^{(0)}(A)$ are the classes of *entire* vectors of A of *exponential* and *minimal exponential type*, respectively (see, e.g., [31,34]).
- As is readily seen, in view of the *closedness* of A , the first-order Beurling-type Gevrey class $\mathcal{E}^{(1)}(A)$ forms the subspace of the initial values $f \in X$ generating the (classical) solutions of (1.1), which are entire vector functions represented by the power series

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} A^n f, \quad t \in \mathbb{R},$$

the classes $\mathcal{E}^{(\beta)}(A)$ and $\mathcal{E}^{(\beta')}(A)$ with $0 \leq \beta < 1$ being the subspaces of such initial values for which the solutions satisfy growth estimate (2.14) with some (any) $\gamma > 0$ and some $M = M(\gamma) > 0$, respectively (cf. [28]).

As is shown in [29] (see also [30,31]), for a *normal operator* A in a complex Hilbert space and any $0 < \beta < \infty$,

$$\mathcal{E}^{(\beta)}(A) = \bigcup_{t>0} D(e^{t|A|^{1/\beta}}) \quad \text{and} \quad \mathcal{E}^{(\beta')}(A) = \bigcap_{t>0} D(e^{t|A|^{1/\beta'}}), \quad (2.15)$$

the operator exponentials $e^{t|A|^{1/\beta}}$, $t > 0$, understood in the sense of the Borel operational calculus (see, e.g., [12,13]).

In [20,35], descriptions (2.15) are extended to *scalar type spectral operators* in a complex Banach space. In [35], similar nature descriptions of the classes $\mathcal{E}^{\{0\}}(A)$ and $\mathcal{E}^{(0)}(A)$ ($\beta = 0$), known for a normal operator A in a complex Hilbert space (see, e.g., [31]), are also generalized to scalar type spectral operators in a complex Banach space. In particular [35, Theorem 5.1],

$$\mathcal{E}^{\{0\}}(A) = \bigcup_{\alpha>0} E_A(\Delta_\alpha)X,$$

where

$$\Delta_\alpha := \{\lambda \in \mathbb{C} \mid |\lambda| \leq \alpha\}, \quad \alpha > 0.$$

2.5 Gevrey ultradifferentiability of a particular weak solution of (1.2)

We also need the following characterization of a particular weak solution of equation (1.1) with a scalar type spectral operator A in a complex Banach space being strongly Gevrey ultradifferentiable on a subinterval I of $[0, \infty)$.

Proposition 2.3. [2, Proposition 3.1]

Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$, $0 \leq \beta < \infty$, and I be a subinterval of $[0, \infty)$. The restriction of a weak solution $y(\cdot)$ of equation (1.2) to I belongs to the Gevrey class $\mathcal{E}^{(\beta)}(I, X)$ ($\mathcal{E}^{(\beta)}(I, X)$) iff, for each $t \in I$,

$$y(t) \in \mathcal{E}^{(\beta)}(A)(\mathcal{E}^{(\beta)}(A), \text{ respectively}),$$

in which case

$$y^{(n)}(t) = A^n y(t), \quad n \in \mathbb{N}, \quad t \in I.$$

3 Gevrey ultradifferentiability of a particular weak solution

Proposition 3.1. (Gevrey ultradifferentiability of a particular weak solution)

Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$, $0 \leq \beta < \infty$, and I be an interval of the real axis $\mathbb{R}$. The restriction of a weak solution $y(\cdot)$ of equation (1.1) to I belongs to the Gevrey class $\mathcal{E}^{(\beta)}(I, X)$ ($\mathcal{E}^{(\beta)}(I, X)$) iff, for each $t \in I$,

$$y(t) \in \mathcal{E}^{(\beta)}(A)(\mathcal{E}^{(\beta)}(A), \text{ respectively}),$$

in which case

$$y^{(n)}(t) = A^n y(t), \quad n \in \mathbb{N}, \quad t \in I.$$

Proof. As is noted in Remarks 2.1, $y : \mathbb{R} \rightarrow X$ is a weak solution of (1.1) iff

$$y_+(t) := y(t), \quad t \geq 0,$$

is a weak solution of equation (1.2) and

$$y_-(t) := y(-t), \quad t \geq 0,$$

is a weak solution of equation (2.4).

The statement immediately follows from Proposition 2.3 applied to

$$y_+(t) := y(t), \quad t \geq 0, \quad \text{and} \quad y_-(t) := y(-t), \quad t \geq 0,$$

for an arbitrary weak solution $y(\cdot)$ of equation (1.1) in view of

$$\mathcal{E}^{(\beta)}(-A) = \mathcal{E}^{(\beta)}(A) \quad \text{and} \quad \mathcal{E}^{(\beta)}(-A) = \mathcal{E}^{(\beta)}(A).$$

□

4 Gevrey ultradifferentiability of order $\beta \geq 1$

Theorem 4.1. (Gevrey ultradifferentiability of order $\beta \geq 1$)

Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$ with spectral measure $E_A(\cdot)$ and $1 \leq \beta < \infty$. Then the following statements are equivalent.

- (i) Every weak solution of equation (1.1) belongs to the β th-order Beurling-type Gevrey class $\mathcal{E}^{(\beta)}(\mathbb{R}, X)$.
- (ii) Every weak solution of equation (1.1) belongs to the β th-order Roumieu-type Gevrey class $\mathcal{E}^{(\beta)}(\mathbb{R}, X)$.
- (iii) There exist $b_+ > 0$ and $b_- > 0$ such that the set $\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta$, where

$$\mathcal{P}_{b_-, b_+}^\beta := \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leq -b_- |\operatorname{Im} \lambda|^{1/\beta} \text{ or } \operatorname{Re} \lambda \geq b_+ |\operatorname{Im} \lambda|^{1/\beta}\},$$

is bounded (see Figure 1).

Proof. We are to prove the closed chain of implications

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i),$$

the implication $(i) \Rightarrow (ii)$ following immediately from the inclusion

$$\mathcal{E}^{(\beta)}(\mathbb{R}, X) \subseteq \mathcal{E}^{(\beta)}(\mathbb{R}, X)$$

(see Remarks 2.5).

To prove the implication $(iii) \Rightarrow (i)$, suppose that there exist $b_+ > 0$ and $b_- > 0$ such that the set $\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta$ is bounded and let $y(\cdot)$ be an arbitrary weak solution of equation (1.1).

Figure 1: Gevrey ultradifferentiability of order $1 \leq \beta < \infty$.

By Theorem 2.1,

$$y(t) = e^{tA}f, \quad t \in \mathbb{R}, \quad \text{with some } f \in \bigcap_{t \in \mathbb{R}} D(e^{tA}).$$

Our purpose is to show that $y(\cdot) \in \mathcal{E}^{(\beta)}(\mathbb{R}, X)$, which, by Proposition 3.1 and (2.15), is accomplished by showing that, for each $t \in \mathbb{R}$,

$$y(t) \in \mathcal{E}^{(\beta)}(A) = \bigcap_{s>0} D(e^{s|A|^{1/\beta}}).$$

Let us proceed by proving that, for any $t \in \mathbb{R}$ and $s > 0$,

$$y(t) \in D(e^{s|A|^{1/\beta}})$$

via Proposition 2.1.

For any $s > 0$, $t \in \mathbb{R}$ and an arbitrary $g^* \in X^*$,

$$\begin{aligned} \int_{\sigma(A)} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) &= \int_{\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\ &+ \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid -1 < \operatorname{Re} \lambda < 1\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\ &+ \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\ &+ \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -1\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) < \infty. \end{aligned} \quad (4.16)$$

Indeed,

$$\int_{\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) < \infty$$

and

$$\int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid -1 < \operatorname{Re} \lambda < 1\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) < \infty$$

due to the boundedness of the sets

$$\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta \quad \text{and} \quad \{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid -1 < \operatorname{Re} \lambda < 1\},$$

the continuity of the integrated function on $\mathbb{C}$, and the finiteness of the measure $\nu(f, g^*, \cdot)$.

Further, for any $s > 0$, $t \in \mathbb{R}$ and an arbitrary $g^* \in X^*$,

$$\begin{aligned}
 & \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
 & \leq \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1\}} e^{s[|\operatorname{Re} \lambda| + |\operatorname{Im} \lambda|]^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
 & \quad \text{since, for } \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \text{ with } \operatorname{Re} \lambda \geq 1, b_+^{-\beta} \operatorname{Re} \lambda^\beta \geq |\operatorname{Im} \lambda|; \\
 & \leq \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1\}} e^{s[\operatorname{Re} \lambda + b_+^{-\beta} \operatorname{Re} \lambda^\beta]^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
 & \quad \text{since, in view of } \operatorname{Re} \lambda \geq 1 \text{ and } \beta \geq 1, \operatorname{Re} \lambda^\beta \geq \operatorname{Re} \lambda; \\
 & \leq \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1\}} e^{s(1+b_+^{-\beta})^{1/\beta} \operatorname{Re} \lambda} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
 & = \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1\}} e^{[s(1+b_+^{-\beta})^{1/\beta} + t] \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
 & \quad \text{since } f \in \bigcap_{t \in \mathbb{R}} D(e^{tA}), \text{ by Proposition 2.1;} \\
 & < \infty.
 \end{aligned} \tag{4.17}$$

Finally, for any $s > 0$, $t \in \mathbb{R}$ and an arbitrary $g^* \in X^*$,

$$\begin{aligned}
 & \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -1\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
 & \leq \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -1\}} e^{s[|\operatorname{Re} \lambda| + |\operatorname{Im} \lambda|]^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
 & \quad \text{since, for } \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \text{ with } \operatorname{Re} \lambda \leq -1, b_-^{-\beta} (-\operatorname{Re} \lambda)^\beta \geq |\operatorname{Im} \lambda|; \\
 & \leq \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -1\}} e^{s[-\operatorname{Re} \lambda + b_-^{-\beta} (-\operatorname{Re} \lambda)^\beta]^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
 & \quad \text{since, in view of } -\operatorname{Re} \lambda \geq 1 \text{ and } \beta \geq 1, (-\operatorname{Re} \lambda)^\beta \geq -\operatorname{Re} \lambda; \\
 & \leq \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -1\}} e^{s(1+b_-^{-\beta})^{1/\beta} (-\operatorname{Re} \lambda)} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
 & = \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -1\}} e^{[t - s(1+b_-^{-\beta})^{1/\beta}] \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
 & \quad \text{since } f \in \bigcap_{t \in \mathbb{R}} D(e^{tA}), \text{ by Proposition 2.1;} \\
 & < \infty.
 \end{aligned} \tag{4.18}$$

Also, for any $s > 0$, $t \in \mathbb{R}$ and an arbitrary $n \in \mathbb{N}$,

$$\begin{aligned}
 & \sup_{\{g^* \in X^* \mid \|g^*\| = 1\}} \int_{\{\lambda \in \sigma(A) \mid e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
 & \leq \sup_{\{g^* \in X^* \mid \|g^*\| = 1\}} \int_{\{\lambda \in \sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta \mid e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda)
 \end{aligned}$$

$$\begin{aligned}
& + \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid -1 < \operatorname{Re} \lambda < 1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
& + \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
& + \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \rightarrow 0, \quad n \rightarrow \infty.
\end{aligned} \tag{4.19}$$

Indeed, since, due to the boundedness of the sets

$$\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta \quad \text{and} \quad \{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid -1 < \operatorname{Re} \lambda < 1\}$$

and the continuity of the integrated function on $\mathbb{C}$, the sets

$$\{\lambda \in \sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta \mid e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}$$

and

$$\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid -1 < \operatorname{Re} \lambda < 1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}$$

are *empty* for all sufficiently large $n \in \mathbb{N}$, we immediately infer that, for any $s > 0$ and $t \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta \mid e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) = 0$$

and

$$\lim_{n \rightarrow \infty} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid -1 < \operatorname{Re} \lambda < 1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) = 0.$$

Further, for any $s > 0$, $t \in \mathbb{R}$ and an arbitrary $n \in \mathbb{N}$,

$$\begin{aligned}
& \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
& \qquad \qquad \qquad \text{as in (4.17);} \\
& \leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}} e^{[s(1+b_+^{-\beta})^{1/\beta} + t] \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
& \qquad \qquad \qquad \text{since } f \in \bigcap_{t \in \mathbb{R}} D(e^{tA}), \text{ by (2.9);} \\
& \leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} 4M \|E_A(\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}) e^{[s(1+b_+^{-\beta})^{1/\beta} + t]Af}\| \|g^*\| \\
& \leq 4M \|E_A(\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}) e^{[s(1+b_+^{-\beta})^{1/\beta} + t]Af}\| \\
& \qquad \qquad \qquad \text{by the strong continuity of the s.m.;} \\
& \rightarrow 4M \|E_A(\emptyset) e^{[s(1+b_+^{-\beta})^{1/\beta} + t]Af}\| = 0, \quad n \rightarrow \infty.
\end{aligned}$$

Finally, for any $s > 0$, $t \in \mathbb{R}$ and an arbitrary $n \in \mathbb{N}$,

$$\begin{aligned}
& \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
& \qquad \qquad \qquad \text{as in (4.18);}
\end{aligned}$$

$$\begin{aligned}
&\leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}} e^{[t-s(1+b_-^{-\beta})^{1/\beta}] \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
&\quad \text{since } f \in \bigcap_{t \in \mathbb{R}} D(e^{tA}), \text{ by (2.9);} \\
&\leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} 4M \|E_A(\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}) e^{[t-s(1+b_-^{-\beta})^{1/\beta}] A} f\| \|g^*\| \\
&\leq 4M \|E_A(\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}) e^{[t-s(1+b_-^{-\beta})^{1/\beta}] A} f\| \\
&\quad \text{by the strong continuity of the s.m.;} \\
&\rightarrow 4M \|E_A(\emptyset) e^{[t-s(1+b_-^{-\beta})^{1/\beta}] A} f\| = 0, \quad n \rightarrow \infty.
\end{aligned}$$

By Proposition 2.1 and the properties of the operational calculus (see [11, Theorem XVIII.2.11 (f)]), (4.16) and (4.19) jointly imply that, for any $s > 0$ and $t \in \mathbb{R}$,

$$f \in D(e^{s|A|^{1/\beta}} e^{tA}),$$

which, in view of (2.15), further implies that, for each $t \in \mathbb{R}$,

$$y(t) = e^{tA} f \in \bigcap_{s>0} D(e^{s|A|^{1/\beta}}) = \mathcal{E}^{(\beta)}(A).$$

Whence, by Proposition 3.1, we infer that

$$y(\cdot) \in \mathcal{E}^{(\beta)}(\mathbb{R}, X),$$

which completes the proof for the implication (iii) $\Rightarrow$ (i).

Let us prove the remaining implication (ii) $\Rightarrow$ (iii) by *contrapositive* assuming that, for any $b_+ > 0$ and $b_- > 0$, the set $\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta$ is *unbounded*. In particular, this means that, for any $n \in \mathbb{N}$, unbounded is the set

$$\sigma(A) \setminus \mathcal{P}_{n^{-1}, n^{-2}}^\beta = \{\lambda \in \sigma(A) \mid -n^{-2} |\operatorname{Im} \lambda|^{1/\beta} < \operatorname{Re} \lambda < n^{-2} |\operatorname{Im} \lambda|^{1/\beta}\}.$$

Hence, we can choose a sequence $\{\lambda_n\}_{n=1}^\infty$ of points in the complex plane as follows:

$$\begin{aligned}
&\lambda_n \in \sigma(A), \quad n \in \mathbb{N}, \\
&-n^{-2} |\operatorname{Im} \lambda_n|^{1/\beta} < \operatorname{Re} \lambda_n < n^{-2} |\operatorname{Im} \lambda_n|^{1/\beta}, \quad n \in \mathbb{N}, \\
&\lambda_0 := 0, |\lambda_n| > \max[n, |\lambda_{n-1}|], \quad n \in \mathbb{N}.
\end{aligned}$$

The latter implies, in particular, that the points λ_n , $n \in \mathbb{N}$, are *distinct* ($\lambda_i \neq \lambda_j$, $i \neq j$).

Since, for each $n \in \mathbb{N}$, the set

$$\{\lambda \in \mathbb{C} \mid -n^{-2} |\operatorname{Im} \lambda|^{1/\beta} < \operatorname{Re} \lambda < n^{-2} |\operatorname{Im} \lambda|^{1/\beta}, |\lambda| > \max[n, |\lambda_{n-1}|]\}$$

is *open* in $\mathbb{C}$, along with the point λ_n , it contains an *open disk*

$$\Delta_n := \{\lambda \in \mathbb{C} \mid |\lambda - \lambda_n| < \varepsilon_n\}$$

centered at λ_n of some radius $\varepsilon_n > 0$, i.e., for each $\lambda \in \Delta_n$,

$$-n^{-2} |\operatorname{Im} \lambda|^{1/\beta} < \operatorname{Re} \lambda < n^{-2} |\operatorname{Im} \lambda|^{1/\beta} \quad \text{and} \quad |\lambda| > \max[n, |\lambda_{n-1}|]. \quad (4.20)$$

Furthermore, we can regard the radii of the disks to be small enough so that

$$\begin{aligned}
&0 < \varepsilon_n < \frac{1}{n}, \quad n \in \mathbb{N}, \quad \text{and} \\
&\Delta_i \cap \Delta_j = \emptyset, \quad i \neq j \quad (\text{i.e., the disks are pairwise disjoint}).
\end{aligned} \quad (4.21)$$

Whence, by the properties of the s.m.,

$$E_A(\Delta_i) E_A(\Delta_j) = 0, \quad i \neq j,$$

where 0 stands for the *zero operator* on X .

Observe also that the subspaces $E_A(\Delta_n)X$, $n \in \mathbb{N}$, are *nontrivial* since

$$\Delta_n \cap \sigma(A) \neq \emptyset, \quad n \in \mathbb{N},$$

with Δ_n being an *open set* in $\mathbb{C}$.

In view of the pairwise disjointness of the disks Δ_n , $n \in \mathbb{N}$, we can choose a unit vector $e_n \in E_A(\Delta_n)X$ for each $n \in \mathbb{N}$ obtaining a sequence $\{e_n\}_{n=1}^\infty$ in X such that

$$\|e_n\| = 1, \quad n \in \mathbb{N}, \quad \text{and} \quad E_A(\Delta_i)e_j = \delta_{ij}e_j, \quad i, j \in \mathbb{N}, \quad (4.22)$$

where δ_{ij} is the *Kronecker delta*.

As is easily seen, (4.22) implies that the vectors e_n , $n \in \mathbb{N}$, are *linearly independent*. Furthermore, there exists an $\varepsilon > 0$ such that

$$d_n := \text{dist}(e_n, \text{span}(\{e_i \mid i \in \mathbb{N}, i \neq n\})) \geq \varepsilon, \quad n \in \mathbb{N}. \quad (4.23)$$

Indeed, otherwise there exists a subsequence $\{d_{n(k)}\}_{k=1}^\infty$ such that

$$d_{n(k)} \rightarrow 0, \quad k \rightarrow \infty.$$

Then, by selecting a vector

$$f_{n(k)} \in \text{span}(\{e_i \mid i \in \mathbb{N}, i \neq n(k)\}), \quad k \in \mathbb{N},$$

such that

$$\|e_{n(k)} - f_{n(k)}\| < d_{n(k)} + 1/k, \quad k \in \mathbb{N},$$

we arrive at

$$\begin{aligned} 1 &= \|e_{n(k)}\| \\ &\quad \text{since, by (4.22), } E_A(D_{n(k)})e_{n(k)} = e_{n(k)} \text{ and } E_A(D_{n(k)})f_{n(k)} = 0; \\ &= \|E_A(\Delta_{n(k)})(e_{n(k)} - f_{n(k)})\| \leq \|E_A(\Delta_{n(k)})\|\|e_{n(k)} - f_{n(k)}\| \quad \text{by (2.8);} \\ &\leq M\|e_{n(k)} - f_{n(k)}\| \leq M[d_{n(k)} + 1/k] \rightarrow 0, \quad k \rightarrow \infty, \end{aligned}$$

which is a *contradiction* proving (4.23).

As follows from the *Hahn-Banach theorem*, for any $n \in \mathbb{N}$, there is an $e_n^* \in X^*$ such that

$$\|e_n^*\| = 1, \quad n \in \mathbb{N}, \quad \text{and} \quad \langle e_i, e_j^* \rangle = \delta_{ij}d_i, \quad i, j \in \mathbb{N}. \quad (4.24)$$

Let us consider separately the two possibilities concerning the sequence of the real parts $\{\text{Re } \lambda_n\}_{n=1}^\infty$: its being *bounded* or *unbounded*.

First, suppose that the sequence $\{\text{Re } \lambda_n\}_{n=1}^\infty$ is *bounded*, i.e., there is such an $\omega > 0$ that

$$|\text{Re } \lambda_n| \leq \omega, \quad n \in \mathbb{N}, \quad (4.25)$$

and consider the element

$$f := \sum_{k=1}^\infty k^{-2}e_k \in X,$$

which is well defined since $\{k^{-2}\}_{k=1}^\infty \in l_1$ (l_1 is the space of absolutely summable sequences) and $\|e_k\| = 1$, $k \in \mathbb{N}$ (see (4.22)).

In view of (4.22), by the properties of the *s.m.*,

$$E_A(\cup_{k=1}^\infty \Delta_k)f = f \quad \text{and} \quad E_A(\Delta_k)f = k^{-2}e_k, \quad k \in \mathbb{N}. \quad (4.26)$$

For any $t \geq 0$ and an arbitrary $g^* \in X^*$,

$$\begin{aligned} &\int_{\sigma(A)} e^{t \text{Re } \lambda} d\nu(f, g^*, \lambda) && \text{by (4.26);} \\ &= \int_{\sigma(A)} e^{t \text{Re } \lambda} d\nu(E_A(\cup_{k=1}^\infty \Delta_k)f, g^*, \lambda) && \text{by (2.12);} \end{aligned} \quad (4.27)$$

$$\begin{aligned}
&= \sum_{k=1}^{\infty} \int_{\sigma(A) \cap \Delta_k} e^{t \operatorname{Re} \lambda} d\nu(E_A(\Delta_k)f, g^*, \lambda) && \text{by (4.26);} \\
&= \sum_{k=1}^{\infty} k^{-2} \int_{\sigma(A) \cap \Delta_k} e^{t \operatorname{Re} \lambda} d\nu(e_k, g^*, \lambda)
\end{aligned}$$

since, for $\lambda \in D_k$, by (4.25) and (4.21),

$$\begin{aligned}
&\operatorname{Re} \lambda = \operatorname{Re} \lambda_k + (\operatorname{Re} \lambda - \operatorname{Re} \lambda_k) \leq \operatorname{Re} \lambda_k + |\lambda - \lambda_k| \leq \omega + \varepsilon_k \leq \omega + 1; \\
&\leq e^{t(\omega+1)} \sum_{k=1}^{\infty} k^{-2} \int_{\sigma(A) \cap \Delta_k} 1 \, d\nu(e_k, g^*, \lambda) = e^{t(\omega+1)} \sum_{k=1}^{\infty} k^{-2} \nu(e_k, g^*, \Delta_k)
\end{aligned}$$

by (2.10);

$$\leq e^{t(\omega+1)} \sum_{k=1}^{\infty} k^{-2} 4M \|e_k\| \|g^*\| = 4Me^{t(\omega+1)} \|g^*\| \sum_{k=1}^{\infty} k^{-2} < \infty.$$

Also, for any $t < 0$ and an arbitrary $g^* \in X^*$,

$$\begin{aligned}
&\int_{\sigma(A)} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) && \text{by (4.26);} \\
&= \int_{\sigma(A)} e^{t \operatorname{Re} \lambda} d\nu(E_A(\cup_{k=1}^{\infty} \Delta_k)f, g^*, \lambda) && \text{by (2.12);} \\
&= \sum_{k=1}^{\infty} \int_{\sigma(A) \cap \Delta_k} e^{t \operatorname{Re} \lambda} d\nu(E_A(\Delta_k)f, g^*, \lambda) && \text{by (4.26);} \\
&= \sum_{k=1}^{\infty} k^{-2} \int_{\sigma(A) \cap \Delta_k} e^{t \operatorname{Re} \lambda} d\nu(e_k, g^*, \lambda) && (4.28)
\end{aligned}$$

since, for $\lambda \in D_k$, by (4.25) and (4.21),

$$\begin{aligned}
&\operatorname{Re} \lambda = \operatorname{Re} \lambda_k - (\operatorname{Re} \lambda_k - \operatorname{Re} \lambda) \geq \operatorname{Re} \lambda_k - |\operatorname{Re} \lambda_k - \operatorname{Re} \lambda| \geq -\omega - \varepsilon_k \geq -\omega - 1; \\
&\leq e^{-t(\omega+1)} \sum_{k=1}^{\infty} k^{-2} \int_{\sigma(A) \cap \Delta_k} 1 \, d\nu(e_k, g^*, \lambda) = e^{-t(\omega+1)} \sum_{k=1}^{\infty} k^{-2} \nu(e_k, g^*, \Delta_k)
\end{aligned}$$

by (2.10);

$$\leq e^{-t(\omega+1)} \sum_{k=1}^{\infty} k^{-2} 4M \|e_k\| \|g^*\| = 4Me^{-t(\omega+1)} \|g^*\| \sum_{k=1}^{\infty} k^{-2} < \infty.$$

Similar to (4.27) for any $t \geq 0$ and an arbitrary $n \in \mathbb{N}$,

$$\begin{aligned}
&\sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
&\leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} e^{t(\omega+1)} \sum_{k=1}^{\infty} k^{-2} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\} \cap \Delta_k} 1 \, d\nu(e_k, g^*, \lambda) && \text{by (4.26);} \\
&= e^{t(\omega+1)} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\} \cap \Delta_k} 1 \, d\nu(E_A(\Delta_k)f, g^*, \lambda) && (4.29) \\
&= e^{t(\omega+1)} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} 1 \, d\nu(E_A(\cup_{k=1}^{\infty} \Delta_k)f, g^*, \lambda) && \text{by (2.12);} \\
&= e^{t(\omega+1)} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} 1 \, d\nu(f, g^*, \lambda) && \text{by (4.26);} \\
&= e^{t(\omega+1)} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} 1 \, d\nu(f, g^*, \lambda) && \text{by (2.9);}
\end{aligned}$$

$$\begin{aligned}
&\leq e^{t(\omega+1)} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} 4M \|E_A(\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\})f\| \|g^*\| \\
&\leq 4Me^{t(\omega+1)} \|E_A(\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\})f\| \\
&\quad \text{by the strong continuity of the s. m. ;} \\
&\rightarrow 4Me^{t(\omega+1)} \|E_A(\emptyset)f\| = 0, \quad n \rightarrow \infty.
\end{aligned}$$

Similar to (4.28) for any $t < 0$ and an arbitrary $n \in \mathbb{N}$,

$$\begin{aligned}
&\sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
&\leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} e^{-t(\omega+1)} \sum_{k=1}^{\infty} k^{-2} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\} \cap \Delta_k} 1 d\nu(e_k, g^*, \lambda) \\
&\quad \text{by (4.26);} \\
&= e^{-t(\omega+1)} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\} \cap \Delta_k} 1 d\nu(E_A(\Delta_k)f, g^*, \lambda) \\
&\quad \text{by (2.12);} \\
&= e^{-t(\omega+1)} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} 1 d\nu(E_A(\cup_{k=1}^{\infty} \Delta_k)f, g^*, \lambda) \\
&\quad \text{by (4.26);} \\
&= e^{-t(\omega+1)} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} 1 d\nu(f, g^*, \lambda) \\
&\quad \text{by (2.9);} \\
&\leq e^{-t(\omega+1)} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} 4M \|E_A(\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\})f\| \|g^*\| \\
&\leq 4Me^{-t(\omega+1)} \|E_A(\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\})f\| \\
&\quad \text{by the strong continuity of the s. m. ;} \\
&\rightarrow 4Me^{-t(\omega+1)} \|E_A(\emptyset)f\| = 0, \quad n \rightarrow \infty.
\end{aligned} \tag{4.30}$$

By Proposition 2.1, (4.27), (4.28), (4.29), and (4.30) jointly imply that

$$f \in \bigcap_{t \in \mathbb{R}} D(e^{tA}),$$

and hence, by Theorem 2.1,

$$y(t) := e^{tA}f, \quad t \in \mathbb{R},$$

is a weak solution of equation (1.1).

Let

$$h^* := \sum_{k=1}^{\infty} k^{-2} e_k^* \in X^*, \tag{4.31}$$

the functional being well defined since $\{k^{-2}\}_{k=1}^{\infty} \in l_1$ and $\|e_k^*\| = 1$, $k \in \mathbb{N}$ (see (4.24)).

In view of (4.24) and (4.23), we have:

$$\langle e_n, h^* \rangle = \langle e_k, k^{-2} e_k^* \rangle = d_k k^{-2} \geq \varepsilon k^{-2}, \quad k \in \mathbb{N}. \tag{4.32}$$

For any $s > 0$,

$$\begin{aligned}
&\int_{\sigma(A)} e^{s|\lambda|^{1/\beta}} d\nu(f, h^*, \lambda) \\
&= \sum_{k=1}^{\infty} k^{-2} \int_{\sigma(A) \cap \Delta_k} e^{s|\lambda|^{1/\beta}} d\nu(e_k, h^*, \lambda) \quad \text{since, for } \lambda \in D_k, \text{ by (4.20), } |\lambda| \geq k;
\end{aligned} \quad \text{by (2.12) as in (4.27);}$$

$$\begin{aligned}
&\geq \sum_{k=1}^{\infty} k^{-2} e^{sk^{1/\beta}} \int_{\sigma(A) \cap \Delta_k} 1 \, dv(e_k, h^*, \lambda) = \sum_{k=1}^{\infty} k^{-2} e^{sk^{1/\beta}} v(e_k, h^*, \Delta_k) \\
&\geq \sum_{k=1}^{\infty} k^{-2} e^{sk^{1/\beta}} |\langle E_A(\Delta_k) e_k, h^* \rangle| \quad \text{by (4.22) and (4.32);} \\
&\geq \sum_{k=1}^{\infty} \varepsilon k^{-4} e^{sk^{1/\beta}} = \infty.
\end{aligned}$$

Whence, by Proposition 2.1 and (2.15), we infer that

$$y(0) = f \notin \bigcup_{s>0} D(e^{s|A|^{1/\beta}}) = \mathcal{E}^{\{\beta\}}(A),$$

which, by Proposition 3.1, implies that the weak solution $y(t) = e^{tA}f$, $t \in \mathbb{R}$, of equation (1.1) does not belong to the Roumieu-type Gevrey class $\mathcal{E}^{\{\beta\}}(\mathbb{R}, X)$ and completes our consideration of the case of the sequence's $\{\operatorname{Re} \lambda_n\}_{n=1}^{\infty}$ being *bounded*.

Now, suppose that the sequence $\{\operatorname{Re} \lambda_n\}_{n=1}^{\infty}$ is *unbounded*. Therefore, there is a subsequence $\{\operatorname{Re} \lambda_{n(k)}\}_{k=1}^{\infty}$ such that

$$\operatorname{Re} \lambda_{n(k)} \rightarrow \infty \quad \text{or} \quad \operatorname{Re} \lambda_{n(k)} \rightarrow -\infty, \quad k \rightarrow \infty.$$

Let us consider separately each of the two cases.

First, suppose that

$$\operatorname{Re} \lambda_{n(k)} \rightarrow \infty, \quad k \rightarrow \infty.$$

Then, without loss of generality, we can regard that

$$\operatorname{Re} \lambda_{n(k)} \geq k, \quad k \in \mathbb{N}. \quad (4.33)$$

Consider the elements

$$f := \sum_{k=1}^{\infty} e^{-n(k) \operatorname{Re} \lambda_{n(k)}} e_{n(k)} \in X \quad \text{and} \quad h := \sum_{k=1}^{\infty} e^{-\frac{n(k)}{2} \operatorname{Re} \lambda_{n(k)}} e_{n(k)} \in X,$$

well defined since, by (4.33),

$$\left\{ e^{-n(k) \operatorname{Re} \lambda_{n(k)}} \right\}_{k=1}^{\infty}, \left\{ e^{-\frac{n(k)}{2} \operatorname{Re} \lambda_{n(k)}} \right\}_{k=1}^{\infty} \in l_1$$

and $\|e_{n(k)}\| = 1$, $k \in \mathbb{N}$ (see (4.22)).

By (4.22),

$$E_A(\bigcup_{k=1}^{\infty} \Delta_{n(k)})f = f \quad \text{and} \quad E_A(\Delta_{n(k)})f = e^{-n(k) \operatorname{Re} \lambda_{n(k)}} e_{n(k)}, \quad k \in \mathbb{N}, \quad (4.34)$$

and

$$E_A(\bigcup_{k=1}^{\infty} \Delta_{n(k)})h = h \quad \text{and} \quad E_A(\Delta_{n(k)})h = e^{-\frac{n(k)}{2} \operatorname{Re} \lambda_{n(k)}} e_{n(k)}, \quad k \in \mathbb{N}. \quad (4.35)$$

For any $t \geq 0$ and an arbitrary $g^* \in X^*$,

$$\begin{aligned}
&\int_{\sigma(A)} e^{t \operatorname{Re} \lambda} dv(f, g^*, \lambda) \quad \text{by (2.12) as in (4.27);} \\
&= \sum_{k=1}^{\infty} e^{-n(k) \operatorname{Re} \lambda_{n(k)}} \int_{\sigma(A) \cap \Delta_{n(k)}} e^{t \operatorname{Re} \lambda} dv(e_{n(k)}, g^*, \lambda) \\
&\quad \text{since, for } \lambda \in \Delta_{n(k)}, \text{ by (4.21),} \\
&\leq \sum_{k=1}^{\infty} e^{-n(k) \operatorname{Re} \lambda_{n(k)}} e^{t(\operatorname{Re} \lambda_{n(k)}+1)} \int_{\sigma(A) \cap \Delta_{n(k)}} 1 \, dv(e_{n(k)}, g^*, \lambda)
\end{aligned} \quad (4.36)$$

$$\operatorname{Re} \lambda = \operatorname{Re} \lambda_{n(k)} + (\operatorname{Re} \lambda - \operatorname{Re} \lambda_{n(k)}) \leq \operatorname{Re} \lambda_{n(k)} + |\lambda - \lambda_{n(k)}| \leq \operatorname{Re} \lambda_{n(k)} + 1;$$

$$\begin{aligned}
&= e^t \sum_{k=1}^{\infty} e^{-[n(k)-t] \operatorname{Re} \lambda_{n(k)}} \nu(e_{n(k)}, g^*, \Delta_{n(k)}) && \text{by (2.10);} \\
&\leq e^t \sum_{k=1}^{\infty} e^{-[n(k)-t] \operatorname{Re} \lambda_{n(k)}} 4M \|e_{n(k)}\| \|g^*\| = 4Me^t \|g^*\| \sum_{k=1}^{\infty} e^{-[n(k)-t] \operatorname{Re} \lambda_{n(k)}} \\
&< \infty.
\end{aligned}$$

Indeed, for all $k \in \mathbb{N}$ sufficiently large so that

$$n(k) \geq t + 1,$$

in view of (4.33),

$$e^{-[n(k)-t] \operatorname{Re} \lambda_{n(k)}} \leq e^{-k}.$$

For any $t < 0$ and an arbitrary $g^* \in X^*$,

$$\begin{aligned}
&\int_{\sigma(A)} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) && \text{by (2.12) as in (4.27);} \\
&= \sum_{k=1}^{\infty} e^{-n(k) \operatorname{Re} \lambda_{n(k)}} \int_{\sigma(A) \cap \Delta_{n(k)}} e^{t \operatorname{Re} \lambda} d\nu(e_{n(k)}, g^*, \lambda) \\
&&& \text{since, for } \lambda \in \Delta_{n(k)}, \text{ by (4.21),} \\
&\quad \operatorname{Re} \lambda = \operatorname{Re} \lambda_{n(k)} - (\operatorname{Re} \lambda_{n(k)} - \operatorname{Re} \lambda) \geq \operatorname{Re} \lambda_{n(k)} - |\operatorname{Re} \lambda_{n(k)} - \operatorname{Re} \lambda| \geq \operatorname{Re} \lambda_{n(k)} - 1; \\
&\leq \sum_{k=1}^{\infty} e^{-n(k) \operatorname{Re} \lambda_{n(k)}} e^{t(\operatorname{Re} \lambda_{n(k)} - 1)} \int_{\sigma(A) \cap \Delta_{n(k)}} 1 d\nu(e_{n(k)}, g^*, \lambda) && (4.37) \\
&= e^{-t} \sum_{k=1}^{\infty} e^{-[n(k)-t] \operatorname{Re} \lambda_{n(k)}} \nu(e_{n(k)}, g^*, \Delta_{n(k)}) && \text{by (2.10);} \\
&\leq e^{-t} \sum_{k=1}^{\infty} e^{-[n(k)-t] \operatorname{Re} \lambda_{n(k)}} 4M \|e_{n(k)}\| \|g^*\| = 4Me^{-t} \|g^*\| \sum_{k=1}^{\infty} e^{-[n(k)-t] \operatorname{Re} \lambda_{n(k)}} \\
&< \infty.
\end{aligned}$$

Indeed, for all $k \in \mathbb{N}$, in view of $t < 0$,

$$n(k) - t \geq n(k) \geq 1,$$

and hence, in view of (4.33),

$$e^{-[n(k)-t] \operatorname{Re} \lambda_{n(k)}} \leq e^{-k}.$$

Similarly to (4.36), for any $t \geq 0$ and an arbitrary $n \in \mathbb{N}$,

$$\begin{aligned}
&\sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
&\leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} e^t \sum_{k=1}^{\infty} e^{-[n(k)-t] \operatorname{Re} \lambda_{n(k)}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\} \cap \Delta_{n(k)}} 1 d\nu(e_{n(k)}, g^*, \lambda) \\
&= e^t \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} e^{-\left[\frac{n(k)}{2}-t\right] \operatorname{Re} \lambda_{n(k)}} e^{-\frac{n(k)}{2} \operatorname{Re} \lambda_{n(k)}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\} \cap \Delta_{n(k)}} 1 d\nu(e_{n(k)}, g^*, \lambda) && (4.38) \\
&\quad \text{since, by (4.33), there is an } L > 0 \text{ such that } e^{-\left[\frac{n(k)}{2}-t\right] \operatorname{Re} \lambda_{n(k)}} \leq L, k \in \mathbb{N}; \\
&\leq Le^t \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} e^{-\frac{n(k)}{2} \operatorname{Re} \lambda_{n(k)}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\} \cap \Delta_{n(k)}} 1 d\nu(e_{n(k)}, g^*, \lambda) \\
&&& \text{by (4.35);}
\end{aligned}$$

$$\begin{aligned}
&= Le^t \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} 1 \, dv(E_A(\cup_{k=1}^{\infty} \Delta_{n(k)})h, g^*, \lambda) && \text{by (2.12);} \\
&= Le^t \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} 1 \, dv(h, g^*, \lambda) && \text{by (2.12);} \\
& && \text{by (4.35);} \\
&= Le^t \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} 1 \, dv(h, g^*, \lambda) && \text{by (2.9);} \\
&\leq Le^t \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} 4M \|E_A(\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\})h\| \|g^*\| \\
&\leq 4LMe^t \|E_A(\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\})h\| \\
& && \text{by the strong continuity of the s.m.;} \\
&\rightarrow 4LMe^t \|E_A(\emptyset)h\| = 0, \quad n \rightarrow \infty.
\end{aligned}$$

Similarly to (4.37), for any $t < 0$ and an arbitrary $n \in \mathbb{N}$,

$$\begin{aligned}
&\sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} e^{t \operatorname{Re} \lambda} dv(f, g^*, \lambda) \\
&\leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} e^{-t} \sum_{k=1}^{\infty} e^{-[n(k)-t] \operatorname{Re} \lambda_{n(k)}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\} \cap \Delta_{n(k)}} 1 \, dv(e_{n(k)}, g^*, \lambda) \\
&= e^{-t} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} e^{-\left[\frac{n(k)}{2}-t\right] \operatorname{Re} \lambda_{n(k)}} e^{-\frac{n(k)}{2} \operatorname{Re} \lambda_{n(k)}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\} \cap \Delta_{n(k)}} 1 \, dv(e_{n(k)}, g^*, \lambda) \\
& && \text{since, by (4.33), there is an } L > 0 \text{ such that } e^{-\left[\frac{n(k)}{2}-t\right] \operatorname{Re} \lambda_{n(k)}} \leq L, \quad k \in \mathbb{N}; \\
&\leq Le^{-t} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} e^{-\frac{n(k)}{2} \operatorname{Re} \lambda_{n(k)}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\} \cap \Delta_{n(k)}} 1 dv(e_{n(k)}, g^*, \lambda) && \text{by (4.35);} \\
&= Le^{-t} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\} \cap \Delta_{n(k)}} 1 \, dv(E_A(\Delta_{n(k)})h, g^*, \lambda) && (4.39) \\
& && \text{by (2.12);} \\
&= Le^{-t} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} 1 \, dv(E_A(\cup_{k=1}^{\infty} \Delta_{n(k)})h, g^*, \lambda) && \text{by (4.35);} \\
&= Le^{-t} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} 1 \, dv(h, g^*, \lambda) && \text{by (2.9);} \\
&\leq Le^{-t} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} 4M \|E_A(\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\})h\| \|g^*\| \\
&\leq 4LMe^{-t} \|E_A(\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\})h\| \\
& && \text{by the strong continuity of the s.m.;} \\
&\rightarrow 4LMe^{-t} \|E_A(\emptyset)h\| = 0, \quad n \rightarrow \infty.
\end{aligned}$$

By Proposition 2.1, (4.36), (4.37), (4.38), and (4.39) jointly imply that

$$f \in \bigcap_{t \in \mathbb{R}} D(e^{tA}),$$

and hence, by Theorem 2.1,

$$y(t) := e^{tA}f, \quad t \in \mathbb{R},$$

is a weak solution of equation (1.1).

Since, for any $\lambda \in \Delta_{n(k)}$, $k \in \mathbb{N}$, by (4.21), (4.33),

$$\begin{aligned} \operatorname{Re} \lambda &= \operatorname{Re} \lambda_{n(k)} - (\operatorname{Re} \lambda_{n(k)} - \operatorname{Re} \lambda) \geq \operatorname{Re} \lambda_{n(k)} - |\operatorname{Re} \lambda_{n(k)} - \operatorname{Re} \lambda| \\ &\geq \operatorname{Re} \lambda_{n(k)} - \varepsilon_{n(k)} \geq \operatorname{Re} \lambda_{n(k)} - 1/n(k) \geq k - 1 \geq 0 \end{aligned}$$

and, by (4.20),

$$\operatorname{Re} \lambda < n(k)^{-2} |\operatorname{Im} \lambda|^{1/\beta},$$

we infer that, for any $\lambda \in \Delta_{n(k)}$, $k \in \mathbb{N}$,

$$|\lambda| \geq |\operatorname{Im} \lambda| \geq [n(k)^2 \operatorname{Re} \lambda]^\beta \geq [n(k)^2 (\operatorname{Re} \lambda_{n(k)} - 1/n(k))]^\beta.$$

Using this estimate, for an arbitrary $s > 0$ and the functional $h^* \in X^*$ defined by (4.31), we have:

$$\begin{aligned} &\int_{\sigma(A)} e^{s|\lambda|^{1/\beta}} d\nu(f, h^*, \lambda) && \text{by (2.12) as in (4.27);} \\ &= \sum_{k=1}^{\infty} e^{-n(k) \operatorname{Re} \lambda_{n(k)}} \int_{\sigma(A) \cap \Delta_{n(k)}} e^{s|\lambda|^{1/\beta}} d\nu(e_{n(k)}, h^*, \lambda) \\ &\geq \sum_{k=1}^{\infty} e^{-n(k) \operatorname{Re} \lambda_{n(k)}} e^{sn(k)^2 (\operatorname{Re} \lambda_{n(k)} - 1/n(k))} \nu(e_{n(k)}, h^*, \Delta_{n(k)}) \\ &\geq \sum_{k=1}^{\infty} e^{-n(k) \operatorname{Re} \lambda_{n(k)}} e^{sn(k)^2 (\operatorname{Re} \lambda_{n(k)} - 1/n(k))} |\langle E_A(\Delta_{n(k)}) e_{n(k)}, h^* \rangle| \\ & && \text{by (4.22) and (4.32);} \\ &\geq \sum_{k=1}^{\infty} \varepsilon e^{(sn(k)-1)n(k) \operatorname{Re} \lambda_{n(k)} - sn(k)} n(k)^{-2} = \infty. \end{aligned} \tag{4.40}$$

Indeed, for all $k \in \mathbb{N}$ sufficiently large so that

$$sn(k) \geq s + 2,$$

in view of (4.33),

$$e^{(sn(k)-1)n(k) \operatorname{Re} \lambda_{n(k)} - sn(k)} n(k)^{-2} \geq e^{(s+1)n(k) - sn(k)} n(k)^{-2} = e^{n(k)} n(k)^{-2} \rightarrow \infty, \quad k \rightarrow \infty.$$

Whence, by Proposition 2.1 and (2.15), we infer that

$$y(0) = f \notin \bigcup_{s>0} D(e^{s|A|^{1/\beta}}) = \mathcal{E}^{[\beta]}(A),$$

which, by Proposition 3.1, implies that the weak solution $y(t) = e^{tA}f$, $t \in \mathbb{R}$, of equation (1.1) does not belong to the Roumieu-type Gevrey class $\mathcal{E}^{[\beta]}(\mathbb{R}, X)$.

The remaining case of

$$\operatorname{Re} \lambda_{n(k)} \rightarrow -\infty, \quad k \rightarrow \infty,$$

is symmetric to the case of

$$\operatorname{Re} \lambda_{n(k)} \rightarrow \infty, \quad k \rightarrow \infty,$$

and is considered in absolutely the same manner, which furnishes a weak solution $y(\cdot)$ of equation (1.1) such that

$$y(0) = f \notin \bigcup_{s>0} D(e^{s|A|^{1/\beta}}) = \mathcal{E}^{[\beta]}(A),$$

and hence, by Proposition 3.1, not belonging to the Roumieu-type Gevrey class $\mathcal{E}^{[\beta]}(\mathbb{R}, X)$.

With every possibility concerning $\{\operatorname{Re} \lambda_n\}_{n=1}^{\infty}$ considered, the proof by contrapositive of the “only if” part is complete and so is the proof of the entire statement. $\square$

Remark 4.1. Due to the *scalar type spectrality* of the operator A , Theorem 4.1 is stated exclusively in terms of the location of its *spectrum* in the complex plane, and hence, is an intrinsically qualitative statement (cf. [1,2,4]).

From Theorem 4.1 and [4, Theorem 3.1], the latter characterizing the Roumieu-type strong Gevrey ultradifferentiability of all weak solutions of equation (1.2) of order $\beta \geq 1$ on $(0, \infty)$, we derive:

Corollary 4.1. *Let A be a scalar type spectral operator in a complex Banach space and $1 \leq \beta < \infty$. All weak solutions of equation (1.2) are β th-order Roumie-type Gevrey ultradifferentiable on $(0, \infty)$ iff all weak solutions of equation (1.1) are β th-order Beurling-type Gevrey ultradifferentiable on $\mathbb{R}$.*

For $\beta = 1$, we obtain the following important particular case.

Corollary 4.2. (Characterization of the entireness of weak solutions)

Let A be a scalar type spectral operator in a complex Banach space. Every weak solution of equation (1.1) is an entire vector function iff there exist $b_+ > 0$ and $b_- > 0$ such that the set $\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^1$, where

$$\mathcal{P}_{b_-, b_+}^1 := \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leq -b_- |\operatorname{Im} \lambda| \quad \text{or} \quad \operatorname{Re} \lambda \geq b_+ |\operatorname{Im} \lambda|\},$$

is bounded (see Figure 2).

Figure 2: The case of $\beta = 1$.

Remark 4.2. As is established in [1], all weak solutions of equation (1.1) with a scalar type spectral operator A in a complex Banach space can be *strongly infinite differentiable*, while the operator A is *unbounded*. Corollary 4.2 further implies that all weak solutions of equation (1.1) with a scalar type spectral operator A in a complex Banach space can be *entire* vector functions without A being bounded, e.g., when A is an unbounded *self-adjoint* operator in a complex Hilbert space (cf. Corollary 7.3, [36, Corollary 4.1], and [36, Corollary 5.1]). This fact contrasts the situation when a closed densely defined linear operator A in a complex Banach space generates a strongly continuous group $\{T(t)\}_{t \in \mathbb{R}}$ of bounded linear operators, i.e., the associated ACP is *well-posed* (see Remark 2.1), in which case even the (left or right) strong differentiability of all weak solutions of equation (2.1) at 0 immediately implies *boundedness* for A (cf., e.g., [7]).

5 Inherent smoothness improvement effects

Theorem 4.1 implies, in particular, that

if, for some $1 \leq \beta < \infty$, every weak solution of equation (1.1) with a scalar type spectral operator A in a complex Banach space X belongs to the β th-order Roumieu-type Gevrey class $\mathcal{E}^{(\beta)}(\mathbb{R}, X)$, then all of them belong to the narrower β th-order Beurling-type Gevrey class $\mathcal{E}^{\{\beta\}}(\mathbb{R}, X)$.

Such a jump-like inherent smoothness improvement effect replicates the situation of the strong Gevrey ultradifferentiability of weak solutions of equation (1.2) on $[0, \infty)$ [2, Theorem 4.1].

Notably, for $\beta = 1$, we have:

if every weak solution of equation (1.1) with a scalar type spectral operator A in a complex Banach space X is analytically continuable into a complex neighborhood of $\mathbb{R}$ (each one into its own), then all of them are entire vector functions,

which can be further strengthened as follows.

Proposition 5.1. (Smoothness improvement effect)

Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$. If every weak solution of equation (1.1) is analytically continuable into a complex neighborhood of 0 (each one into its own), then all of them are entire vector functions.

Proof. Let us first show that if a weak solution $y(\cdot)$ of equation (1.1) is analytically continuable into a complex neighborhood of 0, then $y(0)$ is an *analytic vector* of the operator A , i.e.,

$$y(0) \in \mathcal{E}^{\{1\}}(A).$$

Let a weak solution $y(\cdot)$ of equation (1.1) be analytically continuable into a complex neighborhood of 0. This implies that there is a $\delta > 0$ such that

$$y(t) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} t^n, \quad t \in [-\delta, \delta].$$

The power series converging at $t = \delta$, there is a $c > 0$ such that

$$\left\| \frac{y^{(n)}(0)}{n!} \delta^n \right\| \leq c, \quad n \in \mathbb{Z}_+.$$

Whence, considering that, by Proposition 3.1 with $I = [-\delta, \delta]$,

$$y(0) \in C^\infty(A) \quad \text{and} \quad y^{(n)}(0) = A^n y(0), \quad n \in \mathbb{Z}_+,$$

we infer that

$$\|A^n y(0)\| = \|y^{(n)}(0)\| \leq c[\delta^{-1}]^n n!, \quad n \in \mathbb{Z}_+,$$

which implies

$$y(0) \in \mathcal{E}^{\{1\}}(A).$$

Now, let us prove the statement *by contrapositive* assuming that there is a weak solution of equation (1.1), which is not an entire vector function. This, by Theorem 4.1 with $\beta = 1$, implies that there is a weak solution $y(\cdot)$ of equation (1.1), which is not analytically continuable into a complex neighborhood of $\mathbb{R}$. Then, by Proposition 3.1, for some $t_0 \in \mathbb{R}$,

$$y(t_0) \notin \mathcal{E}^{\{1\}}(A).$$

Therefore, for the weak solution

$$y_{t_0}(t) := y(t + t_0), \quad t \in \mathbb{R},$$

of equation (1.1),

$$y_{t_0}(0) = y(t_0) \notin \mathcal{E}^{\{1\}}(A),$$

which, as is shown above, implies that $y_{t_0}(\cdot)$ is not analytically continuable into a complex neighborhood of 0, and hence, completes the proof by contrapositive. $\square$

Proposition 5.1 replicates the similar situation for weak solutions of equation (1.2) [2, Proposition 5.1].

6 Gevrey ultradifferentiability of orders less than one

While Corollary 4.2 implies that all weak solutions of equation (1.1) with a scalar type spectral operator A in a complex Banach space can be *entire* vector functions without the operator's A being bounded (see Remark 4.2), the following analogue of [3, Theorem 4.1] shows that A cannot remain unbounded, if each weak solution $y(\cdot)$ of (1.1), in addition to being entire, satisfies the growth estimate

$$\|y(z)\| \leq M e^{\gamma|z|^{1/(1-\beta)}}, \quad z \in \mathbb{C},$$

with some $0 \leq \beta < 1$, $\gamma > 0$, and $M > 0$ depending on $y(\cdot)$ (see (2.14)).

Theorem 6.1. (Gevrey ultradifferentiability of orders less than one)

Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$ with spectral measure $E_A(\cdot)$. If every weak solution of equation (1.1) belongs to the β th-order Roumieu-type Gevrey class $\mathcal{E}^{[\beta]}(\mathbb{R}, X)$ with some $\beta \in [0, 1)$ (each one to its own), then the operator A is bounded, and hence, all weak solutions of equation (1.1) are necessarily entire vector functions of exponential type.

Proof. For the scalar type spectral operators

$$A_- := AE_A(\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda < 0\})$$

and

$$A_+ := AE_A(\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda \geq 0\}),$$

by the properties of the operational calculus (see [11, Theorem XVIII.2.11]),

$$A = A_- + A_+.$$

Let

$$A_1 := A|_{E_A(\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda < 0\})X}$$

and

$$A_2 := A|_{E_A(\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda \geq 0\})X},$$

where $\cdot|$ is the *restriction* of an operator (left) to a subspace (right).

Since, by the properties of the operational calculus, the complementary subspaces

$$E_A(\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda \geq 0\})X \quad \text{and} \quad E_A(\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda < 0\})X$$

reduce the operators A , A_- , and A_+ ,

$$\begin{aligned} \sigma(A) &= \sigma(A_1) \cup \sigma(A_2), \\ \sigma(A_1) &\subseteq \sigma(A_-) \subseteq \sigma(A_1) \cup \{0\}, \\ \sigma(A_2) &\subseteq \sigma(A_+) \subseteq \sigma(A_2) \cup \{0\} \end{aligned}$$

(cf. [37]), with $\sigma(A_i) = \emptyset$, $i = 1, 2$, provided the underlying subspace is $\{0\}$.

Therefore, we have the following inclusions:

$$\sigma(A) \subseteq \sigma(A_-) \cup \sigma(A_+) \subseteq \sigma(A) \cup \{0\}. \quad (6.41)$$

By the properties of the operational calculus (see [11, Theorem XVIII.2.11]),

$$\begin{aligned} e^{tA} &= \int_{\sigma(A)} e^{t\chi_{\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda \geq 0\}}(\lambda)} dE_A(\lambda) = \int_{\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda \geq 0\}} e^{t\lambda} dE_A(\lambda) + \int_{\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda < 0\}} 1 dE_A(\lambda) \\ &= e^{tA} E_A(\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda \geq 0\}) + E_A(\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda < 0\}), \quad t \in \mathbb{R}, \end{aligned} \quad (6.42)$$

($\chi_\delta(\cdot)$ is the *characteristic function* of a set $\delta \subseteq \mathbb{C}$).

Let

$$f \in \bigcap_{t \geq 0} D(e^{tA_+})$$

be arbitrary. Then, by (6.42),

$$E_A(\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda \geq 0\})f \in \bigcap_{t \geq 0} D(e^{tA}).$$

Since, for arbitrary $t \in \mathbb{R}$ and any Borel set $\delta \subseteq \mathbb{C}$,

$$\int_{\delta} e^{t\lambda} d\nu(E_A(\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda \geq 0\})f, g^*, \lambda) = \int_{\delta \cap \{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda \geq 0\}} e^{t\lambda} d\nu(f, g^*, \lambda)$$

(see, e.g., [3, Preliminaries]), by Proposition 2.1, we infer that

$$E_A(\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda \geq 0\})f \in \bigcap_{t < 0} D(e^{tA}).$$

Hence, for any $f \in \bigcap_{t \geq 0} D(e^{tA_+})$,

$$E_A(\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda \geq 0\})f \in \bigcap_{t \in \mathbb{R}} D(e^{tA}).$$

This, by [21, Theorem 4.2], implies that every weak solution

$$e^{tA_+}f = e^{tA}E_A(\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda \geq 0\})f + E_A(\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda < 0\})f, \quad t \geq 0,$$

where

$$f \in \bigcap_{t \geq 0} D(e^{tA_+})$$

is arbitrary, of the equation

$$y'(t) = A_+y(t), \quad t \geq 0,$$

along with the weak solution

$$e^{tA}E_A(\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda \geq 0\})f, \quad t \in \mathbb{R},$$

of equation (1.1) and the constant vector function

$$E_A(\{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda < 0\})f, \quad t \in \mathbb{R},$$

belongs to the β th-order Roumieu-type Gevrey class $\mathcal{E}^{(\beta)}([0, \infty), X)$ with some $\beta \in [0, 1)$ (each one to its own), which, by [3, Theorem 4.1], implies that the operator A_+ is *bounded*, and hence, the spectrum $\sigma(A_+)$ is a *bounded set* in $\mathbb{C}$ (see, e.g., [18,38]).

Using similar reasoning for the scalar type spectral operator $-A_-$ and the evolution equation

$$y'(t) = -A_-y(t), \quad t \geq 0,$$

(see Remarks 2.1), one can show that the spectrum of the operator $-A_-$, and hence, of the operator A_- is a *bounded set* in $\mathbb{C}$.

Thus, from inclusion (6.41), we infer that $\sigma(A)$ is a *bounded set* in $\mathbb{C}$, which, by [11, Theorem XVIII.2.11 (c)], means that the operator A is *bounded* and completes the proof implying that every weak solution $y(\cdot)$ of equation (1.1) is an entire vector function of the form

$$y(z) = e^{zA}f = \sum_{n=0}^{\infty} \frac{z^n}{n!} A^n f, \quad z \in \mathbb{C}, \text{ with some } f \in X,$$

and hence, satisfying the growth condition

$$\|y(z)\| \leq \|f\| e^{\|A\||z|}, \quad z \in \mathbb{C},$$

is of *exponential type* (see Preliminaries). □

7 The case of a normal operator

As an important particular case of Theorem 4.1, we obtain:

Corollary 7.1. (Gevrey ultradifferentiability of order $\beta \geq 1$)

Let A be a normal operator in a complex Hilbert space X and $1 \leq \beta < \infty$. Then the following statements are equivalent.

- (i) Every weak solution of equation (1.1) belongs to the β th-order Beurling-type Gevrey class $\mathcal{E}^{(\beta)}(\mathbb{R}, X)$.
- (ii) Every weak solution of equation (1.1) belongs to the β th-order Roumieu-type Gevrey class $\mathcal{E}^{[\beta]}(\mathbb{R}, X)$.
- (iii) There exist $b_+ > 0$ and $b_- > 0$ such that the set $\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta$, where

$$\mathcal{P}_{b_-, b_+}^\beta := \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leq -b_- |\operatorname{Im} \lambda|^{1/\beta} \text{ or } \operatorname{Re} \lambda \geq b_+ |\operatorname{Im} \lambda|^{1/\beta}\},$$

is bounded (see Figure 1).

Remark 7.1. Corollary 7.1 develops the results of paper [36], where similar consideration is given to the strong Gevrey ultradifferentiability of the weak solutions of equation (1.2) with a normal operator in a complex Hilbert space on $[0, \infty)$ and $(0, \infty)$.

For $\beta = 1$, we obtain the following important particular case.

Corollary 7.2. (Characterization of the entireness of weak solutions)

Let A be a normal operator in a complex Hilbert space. Every weak solution of equation (1.1) is an entire vector function iff there exist $b_+ > 0$ and $b_- > 0$ such that the set $\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta$, where

$$\mathcal{P}_{b_-, b_+}^\beta := \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leq -b_- |\operatorname{Im} \lambda| \text{ or } \operatorname{Re} \lambda \geq b_+ |\operatorname{Im} \lambda|\},$$

is bounded (see Figure 2).

Considering that, for a self-adjoint operator A in a complex Hilbert space X ,

$$\sigma(A) \subseteq \mathbb{R}$$

(see, e.g., [12, 13]), by Corollary 7.2, we can strengthen [1, Corollary 18] as follows.

Corollary 7.3. (The case of a self-adjoint operator)

Every weak solution of equation (1.1) with a self-adjoint operator A in a complex Hilbert space is an entire vector function.

Cf. [36, Corollary 4.1] and [36, Corollary 5.1].

From Corollary 4.1, we immediately obtain:

Corollary 7.4. Let A be a normal operator in a complex Hilbert space and $1 \leq \beta < \infty$. All weak solutions of equation (1.2) are β th-order Roumieu-type Gevrey ultradifferentiable on $(0, \infty)$ iff all weak solutions of equation (1.1) are β th-order Beurling-type Gevrey ultradifferentiable on $\mathbb{R}$.

Cf. [36, Theorem 4.2].

For a normal operator in a complex Hilbert space, Proposition 5.1 and Theorem 6.1 acquire the following form, respectively.

Corollary 7.5. (Smoothness improvement effect)

Let A be a normal operator in a complex Hilbert space. If every weak solution of equation (1.1) is analytically continuable into a complex neighborhood of 0 (each one into its own), then all of them are entire vector functions.

Corollary 7.6. (Gevrey ultradifferentiability of orders less than one)

Let A be a normal operator in a complex Hilbert space X . If every weak solution of equation (1.1) belongs to the β th-order Roumieu type Gevrey class $\mathcal{E}^{\{\beta\}}(\mathbb{R}, X)$ with $0 \leq \beta < 1$ (each one to its own), then the operator A is bounded, and hence, all weak solutions of equation (1.1) are necessarily entire vector functions of exponential type.

Observe that Corollary 7.5 replicates the similar situation for weak solutions of equation (1.2) with a normal operator in a complex Hilbert space [36, Proposition 5.1].

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