Research Article

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On finite dual Cayley graphs

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Abstract: A Cayley graph Γ on a group G is called a *dual Cayley graph* on G if the left regular representation of G is a subgroup of the automorphism group of Γ (note that the right regular representation of G is always an automorphism group of Γ). In this article, we study finite dual Cayley graphs regarding identification, construction, transitivity and such graphs with automorphism groups as small as possible. A few problems worth further research are also proposed.

Keywords: dual Cayley graph, identification, construction, transitivity

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1 Introduction

Graphs considered in this article are finite and undirected. For a graph Γ , we denote by $V\Gamma$ and $\text{Aut}\Gamma$ its vertex set and its full automorphism group, respectively. For a vertex α of Γ , we denote by $\Gamma(\alpha) := \{\beta \in V\Gamma \mid \beta \sim \alpha\}$ the neighbor set of α in Γ , where $\beta \sim \alpha$ means that β is adjacent to α .

Cayley graphs were introduced by Cayley [1] in 1878, which provide an important and rich source of transitive graphs, stated as follows: a graph Γ is called a *Cayley graph* on a group G if there is a subset $S \subseteq G \setminus \{1\}$, with $S = S^{-1} := \{g^{-1} \mid g \in S\}$, such that $V\Gamma = G$ and two vertices g and h are adjacent if and only if $hg^{-1} \in S$. This Cayley graph is denoted by $\text{Cay}(G, S)$. It is known that a graph Γ is isomorphic to a Cayley graph on a group G if and only if $\text{Aut}\Gamma$ contains a subgroup which is isomorphic to G and acts regularly on $V\Gamma$, see [2, Proposition 16.3].

Let $\Gamma = \text{Cay}(G, S)$ be a Cayley graph, and let

$$\begin{aligned}\hat{G} &= \{\hat{g} \mid \hat{g}: x \mapsto xg, \text{ for all } g, x \in G\}, \\ \check{G} &= \{\check{g} \mid \check{g}: x \mapsto g^{-1}x, \text{ for all } g, x \in G\},\end{aligned}$$

be the right regular representation and left regular representation of G , respectively. It is well known (also easy to prove) that $\hat{G}$ is always an automorphism group of Γ , but $\check{G}$ is not necessarily. We call $\text{Cay}(G, S)$ a *dual Cayley graph* on G if $\check{G} \leq \text{Aut}\Gamma$. It is then natural and interesting to propose the following problem.

Problem 1. Characterizing finite dual Cayley graphs.

Clearly, if G is abelian, then $\hat{G} = \check{G}$, so the family of dual Cayley graphs on abelian groups is exactly the family of Cayley graphs on abelian groups. We thus mainly focus on nontrivial dual Cayley graphs, namely, dual Cayley graphs on nonabelian groups.

The aim of this article is to investigate the identification, construction and transitivity of dual Cayley graphs and propose several problems needing further research.

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2 Properties of dual Cayley graphs

For a Cayley graph Γ on a group G , the following two notations will be used throughout the article:

- 1: the vertex of Γ corresponding to the identity element of G ;
- τ : the permutation on $V\Gamma$ via $\tau: g \rightarrow g^{-1}$ for $g \in G$.

Other notations used in this article are standard. For example, for a positive integer n , we use $\mathbb{Z}_n$ and D_{2n} to denote the cyclic group of order n and the dihedral group of order $2n$, respectively, and use A_n and S_n to denote the alternating group and the symmetric group of degree n , respectively. Given two groups N and H , by $N \times H$ we denote the direct product of N and H , by $N.H$ an extension of N by H , and if such an extension is split, then we write $N : H$ instead of $N.H$.

Let $\Gamma = \text{Cay}(G, S)$ and let

$$\text{Aut}(G, S) = \{\sigma \in \text{Aut}(G) \mid S^\sigma = S\},$$

the subgroup of $\text{Aut}(G)$ preserving S setwise. Then, $\hat{G}$ is a subgroup of $\text{Aut}\Gamma$, and $\text{Aut}(G, S)$ is a subgroup of both $\text{Aut}(G)$ and $\text{Aut}\Gamma$. Furthermore, the following nice property holds.

Lemma 2.1. [3, Lemma 2.1] *Let $\Gamma = \text{Cay}(G, S)$. Then, the normalizer $\mathbf{N}_{\text{Aut}\Gamma}(\hat{G}) = \hat{G} : \text{Aut}(G, S)$.*

The following theorem gives necessary and sufficient conditions for a Cayley graph to be a dual Cayley graph, where

$$\tilde{G} = \{\tilde{g} \mid \tilde{g}: x \mapsto g^{-1}xg, \text{ for all } g, x \in G\}$$

is the conjugate representation of G .

Theorem 2.2. *Let $\Gamma = \text{Cay}(G, S)$. Then, the following statements are equivalent:*

- (1) Γ is a dual Cayley graph on G .
- (2) $\tilde{G} \leq \text{Aut}\Gamma$.
- (3) $\tau \in \text{Aut}\Gamma$.

Proof. Let $g, h, x \in G$.

(1) $\Rightarrow$ (2). Since Γ is a dual Cayley graph on G , $\tilde{G} \leq \text{Aut}\Gamma$, then as $\tilde{g} = \tilde{g}\hat{g}$, we obtain $\tilde{G} \leq \tilde{G}\hat{G} \leq \text{Aut}\Gamma$.

(2) $\Rightarrow$ (3). Since $\tilde{h} \in \tilde{G} \leq \text{Aut}\Gamma$ fixes **1** and $\Gamma(\mathbf{1}) = S$, $\tilde{h}$ fixes S setwise. Then, we have the following equivalences.

$$g \sim h \text{ in } \Gamma \Leftrightarrow hg^{-1} \in S \Leftrightarrow gh^{-1} \in S^{-1} = S \Leftrightarrow h^{-1}g = (gh^{-1})^{\tilde{h}} \in S \Leftrightarrow g^\tau \sim h^\tau \text{ in } \Gamma.$$

Therefore, $\tau \in \text{Aut}\Gamma$.

(3) $\Rightarrow$ (1). Since $x^{\tilde{g}} = g^{-1}x = (x^{-1}g)^{-1} = x^{\tau\hat{g}\tau}$, we have $\tilde{g} = \tau\hat{g}\tau \in \text{Aut}\Gamma$. Hence, $\tilde{G} \leq \text{Aut}\Gamma$, and Γ is a dual Cayley graph on G . $\square$

Denote by $Z(G)$ the center of a group G .

Lemma 2.3.

- (i) $\hat{G} \cap \tilde{G} \cong Z(G)$; in particular, $\hat{G} \neq \tilde{G}$ if and only if G is a nonabelian group.
- (ii) $\hat{G}^\tau = \tilde{G}$ and $\tilde{G}^\tau = \hat{G}$.
- (iii) $\langle \text{Aut}(G), \tau \rangle = \begin{cases} \text{Aut}(G) & \text{if } G \text{ is abelian;} \\ \text{Aut}(G) \times \langle \tau \rangle & \text{if } G \text{ is nonabelian.} \end{cases}$

Proof.

- (i) If $\hat{g} = \tilde{h}$ for some $g, h \in G$, then $xg = x^{\hat{g}} = x^{\tilde{h}} = h^{-1}x$ for each $x \in G$, and by choosing $x = 1$ we obtain $h^{-1} = g$. So $xg = gx$, namely, $g \in Z(G)$. Consequently, $\hat{G} \cap \tilde{G} = \{\hat{g} \mid g \in Z(G)\} \cong Z(G)$.

- (ii) Obviously, $o(\tau) = 1$ if G is an elementary abelian 2-group, and $o(\tau) = 2$ otherwise. For each $g \in G$, as discussed above, $\hat{g}^\tau = \tau \hat{g} \tau = \check{g}$, thus $\hat{G}^\tau = \check{G}$. One similarly shows $\check{G}^\tau = \hat{G}$.
- (iii) Let $\sigma \in \text{Aut}(G)$. For each $x \in G$, we have $x^{\sigma\tau} = (\sigma(x))^{-1} = \sigma(x^{-1}) = x^{\tau\sigma}$, hence $\sigma\tau = \tau\sigma$. Now note that $o(\tau) \leq 2$, and $\tau \in \text{Aut}(G)$ if and only if G is abelian, part (iii) follows. $\square$

For an element $g \in G$, denote by g^G the conjugate class of g in G . The next lemma reveals the structure of dual Cayley graphs.

Lemma 2.4. *Let $\Gamma = \text{Cay}(G, S)$ be a dual Cayley graph on a group G . Then,*

$$\Gamma = \text{Cay}(G, \{s_1, s_1^{-1}\}^G \cup \dots \cup \{s_m, s_m^{-1}\}^G), \quad \text{for some } s_1, \dots, s_m \in G \setminus \{1\},$$

or equivalently, $S = S^{-1}$ is a union of some G -conjugate classes.

Conversely, each Cayley graph constructed as above is a dual Cayley graph.

Proof. Let $\Gamma = \text{Cay}(G, S)$ be a dual Cayley graph. Then, $\langle \tilde{G}, \tau \rangle \leq \text{Aut} \Gamma$ by Theorem 2.2 and fixes **1**, hence it preserves $S = \Gamma(\mathbf{1})$ setwise. Noting that each orbit of $\tilde{G} \times \langle \tau \rangle$ on S has the form $\{s, s^{-1}\}^G$ with $s \in G \setminus \{1\}$, we obtain $S = \{s_1, s_1^{-1}\}^G \cup \dots \cup \{s_m, s_m^{-1}\}^G$ for some $s_1, \dots, s_m \in G \setminus \{1\}$.

Conversely, assume $\Gamma = \text{Cay}(G, S)$ with $S = \{s_1, s_1^{-1}\}^G \cup \dots \cup \{s_m, s_m^{-1}\}^G$. Since $\tilde{G} \leq \text{Aut}(G)$ fixes S setwise, we have $\tilde{G} \leq \text{Aut}(G, S) \leq \text{Aut} \Gamma$, hence Γ is a dual Cayley graph by Theorem 2.2. $\square$

There are some typical symmetric graphs: K_n (the complete graph with n vertices), $K_{n,n}$ (the complete bipartite graph with $2n$ vertices) and $K_{n,n} - nK_2$ (the graph deleted a 1-match from $K_{n,n}$). It is known that all of them are Cayley graphs, but the next lemma shows not all of them are nontrivial dual Cayley graphs.

Lemma 2.5.

- (1) *The complete graphs K_n with $n \geq 6$ not a prime are nontrivial dual Cayley graphs.*
- (2) *The complete bipartite graphs $K_{n,n}$ with $n \geq 3$ are nontrivial dual Cayley graphs.*
- (3) *The graphs $K_{p,p} - pK_2$ with p an odd prime are not nontrivial dual Cayley graphs.*

Proof. Write $\Gamma = K_n$, $\Sigma = K_{n,n}$ and $\Delta = K_{p,p} - pK_2$.

- (1) Let G be a nonabelian group of order n . Then, $\Gamma \cong \text{Cay}(G, G \setminus \{1\})$. Since $G \setminus \{1\} = (G \setminus \{1\})^{-1}$ is a union of some G -conjugate classes, by Lemma 2.4, K_n is a nontrivial dual Cayley graph on G .
- (2) Let $H = \langle a, b | a^n = b^2 = 1, a^b = a^{-1} \rangle \cong D_{2n}$ be a dihedral group of order $2n$. It is easy to show that $\Sigma \cong \text{Cay}(H, \{b, ab, \dots, a^{n-1}b\})$. Since $b^{a^i} = a^{-i}(ba^ib)b = a^{-2i}b$, and $(ab)^{a^i} = ab^{a^i} = a^{-2i+1}b$, we conclude that $\{b, ab, \dots, a^{n-1}b\} = b^G \cup (ab)^G$. It then follows from Lemma 2.4 that $K_{n,n}$ is a nontrivial dual Cayley graph on H .
- (3) Suppose for a contrary that $\Delta = \text{Cay}(R, S)$ is a dual Cayley graph on a nonabelian group R . Since Δ is of valency $p - 1$ and order $2p$, we have $|S| = p - 1$ and $|R| = 2p$. Since R is nonabelian of order twice a prime, it is easy to show $R \cong \langle x, y | x^p = y^2 = 1, x^y = x^{-1} \rangle \cong D_{2p}$. If some $x^iy \in S$, by Lemma 2.4, $(x^iy)^R \subseteq S$; however, as p is an odd prime, a simple computation shows $(x^iy)^R = \{x, xy, \dots, x^{p-1}y\}$, contradicting $|S| = p - 1$. Hence, $S \subseteq \langle x \rangle$. It follows that $\langle S \rangle \leq \langle x \rangle < R$, and so Δ is disconnected, also a contradiction. $\square$

For a graph Γ , if a subgroup $X \leq \text{Aut} \Gamma$ acts transitively on the edge set or arc set of Γ , then Γ is called *X-edge-transitive* or *X-arc-transitive*, respectively. Both edge-transitive graphs and arc-transitive graphs are the main research objects in the field of algebraic graph theory. An arc-transitive graph is obviously edge-transitive. It is known that an edge-transitive Cayley graph $\text{Cay}(G, S)$ is arc-transitive if and only if there is an automorphism of the graph which maps an element $s \in S$ to its inverse (thus edge-transitive Cayley graphs $\text{Cay}(G, S)$ on abelian groups G are arc-transitive because $\tau \in \text{Aut}(G, S) \leq \text{Aut} \Gamma$ maps s to s^{-1}). A group X is a central product of two subgroups M and N , denoted by $X = M \circ N$, if $X = MN$, M and N commute each other, and $M \cap N$ is the center of X , see [4, p. 141].

In general, an edge-transitive Cayley graph is not necessarily arc-transitive. However, for dual Cayley graphs, edge-transitivity and arc-transitivity are equivalent, which was first observed in [5, Corollary 2.5], but the statement there is not precise.

Lemma 2.6. *Let $\Gamma = \text{Cay}(G, S)$ be a dual Cayley graph on a nonabelian group G . Then, the following statements hold.*

- (i) *If Γ is X -vertex-transitive and X -edge-transitive, then Γ is $\langle X, \tau \rangle$ -arc-transitive. In particular, each edge-transitive dual Cayley graph is arc-transitive.*
- (ii) $\langle \hat{G}, \check{G}, \tau \rangle = \langle \hat{G} \circ \check{G} \rangle : \langle \tau \rangle \leq \text{Aut} \Gamma$.

Proof.

- (i) Since $\Gamma = \text{Cay}(G, S)$ is a dual Cayley graph, by Theorem 2.2, $\tau \in \text{Aut} \Gamma$, and as τ maps s to s^{-1} for $s \in S$, we conclude that Γ is $\langle X, \tau \rangle$ -arc-transitive.
- (ii) By Theorem 2.2, $\langle \hat{G}, \check{G}, \tau \rangle \leq \text{Aut} \Gamma$. Since $\check{G}$ and $\hat{G}$ centralize each other, we have $\langle \hat{G}, \check{G} \rangle = \hat{G}\check{G} = \hat{G} \circ \check{G}$. For $x, g, h \in G$, since $x^{\tau^{-1}\check{g}\check{h}\tau} = (h^{-1}x^{-1}g)^{\tau} = g^{-1}xh = x^{\check{g}\check{h}}$, we have $(\check{g}\check{h})^{\tau} = \check{g}\hat{h} = \hat{g}\check{h} \in \hat{G}\check{G}$, namely, τ normalizes $\hat{G}\check{G}$. Thus to finish the proof, we only need to prove $\tau \notin \hat{G}\check{G}$. If not, then $\tau = \check{g}\hat{h}$ for some $g, h \in G$. Hence for each $x \in G$, we have $x^{-1} = x^{\tau} = x^{\check{g}\hat{h}} = h^{-1}xg$. Choosing $x = 1$, we obtain $g = h$ and $x^{\tau} = g^{-1}xg$, namely, τ is an inner automorphism of G induced by g , which is a contradiction as G is a nonabelian group.

From Lemma 2.6, we have the next corollary. Recall that a Cayley graph Γ on a group G is called a normal Cayley graph if $\hat{G} \triangleleft \text{Aut} \Gamma$, see [6].

Corollary 2.7. *The dual Cayley graphs on nonabelian groups G are never normal Cayley graphs on G .*

Proof. Suppose for a contradiction that Γ is a dual Cayley graph and is normal on a nonabelian group G . Then, $\hat{G} \triangleleft \text{Aut} \Gamma$ and by Theorem 2.2, $\tau \in \text{Aut} \Gamma$; however, as G is nonabelian, by Lemma 2.3, $\hat{G}^{\tau} = \check{G} \neq \hat{G}$, yielding a contradiction. $\square$

A Cayley graph $\text{Cay}(G, S)$ is called a *graphical regular representation* (GRR) of a group G if $\text{Aut}(\text{Cay}(G, S)) = \hat{G}$ (i.e., the graph $\text{Cay}(G, S)$ has its full automorphism group as small as possible). There are many studies regarding graphical regular representations in the literature, which culminates in [7,8] with the classification of finite groups admitting GRRs. Motivated by Lemma 2.6(ii), we call that a dual Cayley graph $\text{Cay}(G, S)$ with G nonabelian is a dual graphical representation with smallest automorphism group (DGRSA) of G if $\text{Aut}(\text{Cay}(G, S)) = \langle \hat{G} \circ \check{G} \rangle : \langle \tau \rangle$. It is then natural to ask the following basic “DGRSA” problem.

Problem 2.

- (1) Which nonabelian groups (especially nonabelian simple groups) admit DGRSAs?
- (2) Which nonabelian groups (especially nonabelian simple groups) admit arc-transitive DGRSAs?

The following observation limits the structure of DGRSAs that are arc-transitive.

Lemma 2.8. *Let $\Gamma = \text{Cay}(G, S)$ be a DGRSA of a nonabelian group G , and suppose Γ is arc-transitive. Then, $S = \{g, g^{-1}\}^G$ for some $g \in G \setminus \{1\}$.*

Proof. By assumption, $\text{Aut} \Gamma = \langle \check{G} \circ \hat{G} \rangle : \langle \tau \rangle$, hence $(\text{Aut} \Gamma)_1 = \check{G} \times \langle \tau \rangle$ by Lemma 2.3. By the arc-transitivity, the vertex stabilizer $(\text{Aut} \Gamma)_1$ is transitive on $\Gamma(1) = S$, thus $S = g^{(\text{Aut} \Gamma)_1} = g^{\check{G} \times \langle \tau \rangle} = \{g, g^{-1}\}^G$ for some $g \in G \setminus \{1\}$. $\square$

The next lemma shows that, for dihedral groups, only D_6 admits a unique arc-transitive DGRSA.

Lemma 2.9. *Let Γ be a connected arc-transitive DGRSA of a dihedral group $G \cong D_{2n}$. Then, $n = 3$ and $\Gamma \cong K_{3,3}$.*

Proof. Let $G = \langle a, b | a^n = b^2 = 1, a^b = a^{-1} \rangle$ and let $\Gamma = \text{Cay}(G, S)$. By Lemma 2.8, $S = \{g, g^{-1}\}^G$ for some $g \in G$, and by the connectivity, $\langle g^G \rangle = G$. Set $g = a^i b^j$ with $0 \leq i \leq n-1$ and $0 \leq j \leq 1$. If $j = 0$, then $g \in \langle a \rangle$, and as $\langle a \rangle$ is a characteristic subgroup of G , we deduce $\langle S \rangle \leq \langle a \rangle < G$, a contradiction. Therefore, $g = a^i b$ is of order 2 and $S = g^G$.

Assume n is even. Since $b^a = a^{-1} a^b b = a^{-2} b$, we easily deduce $S = \{b, a^2 b, \dots, a^{n-2} b\}$ if g is conjugate to b in G , or $S = \{ab, a^3 b, \dots, a^{n-1} b\}$ if g is not conjugate to b in G . For the former case, $\langle S \rangle = \langle a^2, b \rangle = \langle a^2 \rangle : \langle b \rangle \cong D_n$, a contradiction. For the latter case, as $(a^2)^{ab} = (a^2)^b = a^{-2} \in \langle a^2 \rangle$, one easily obtains $\langle S \rangle = \langle ab, a^2 \rangle = \langle a^2 \rangle : \langle ab \rangle \cong D_n$, also a contradiction.

Therefore, n is odd. Then, it is easy to see that each involution of G is conjugate to b , and a direct computation shows that $S = b^G = \{b, ab, \dots, a^{n-1} b\}$ is of size n . Note that Γ is a bipartite graph with bipartitions $\langle a \rangle$ and S , and we further conclude that $\Gamma \cong K_{n,n}$ and $\text{Aut} \Gamma = (S_n \times S_n) : \mathbb{Z}_2$. On the other hand, since Γ is a DGRSA of D_{2n} , we have $\text{Aut} \Gamma = (\hat{G} \circ \check{G}) : \langle \tau \rangle \cong (\hat{D}_{2n} \times \check{D}_{2n}) : \langle \tau \rangle$ as n is odd. It follows $(n!)^2 \cdot 2 = (2n)^2 \cdot 2$, which implies that $n = 3$ and $\Gamma \cong K_{3,3}$ is an arc-transitive DGRSA of D_6 . $\square$

We remark that, quite different from Lemma 2.9, Theorem 3.3 shows that each alternating simple group A_n has an arc-transitive DGRSA, and Theorem 3.5 shows that $\text{PSL}(2, p)$ with $p \geq 5$ a prime has an arc-transitive DGRSA if and only if $p \equiv 1 \pmod{4}$.

3 Dual Cayley graphs on nonabelian simple groups

For a positive integer s , an s -arc of Γ is a sequence $\alpha_0, \alpha_1, \dots, \alpha_s$ of $s+1$ vertices such that $\alpha_{i-1} \sim \alpha_i$ for $1 \leq i \leq s$ and $\alpha_{i-1} \neq \alpha_{i+1}$ for $1 \leq i \leq s-1$. Then, Γ is called (X, s) -arc-transitive if it is transitive on the set of s -arcs of Γ . An s -transitive graph with $s \geq 2$ is $(s-1)$ -arc-transitive. If a graph Γ is s -arc-transitive but not $(s+1)$ -arc-transitive, then Γ is simply called s -transitive. A remarkable result of Tutte (1949) shows that cubic graphs are at most 5-arc-transitive, and Tutte's result was generalized by Weiss [9], which says that graphs except cycles are at most 7-arc-transitive. It is known that the complete graph K_n with $n \geq 4$ is 2-transitive, and the complete bipartite graph is 3-transitive.

The following theorem classifies arc-transitive nontrivial dual Cayley graphs on nonabelian simple groups. Note that according to the structure and the action of the socle, the well-known O'Nan–Scott theorem divided the finite primitive permutation groups into eight types, see Praeger [10] or [11].

Theorem 3.1. *Let Γ be an arc-transitive dual Cayley graph on a nonabelian simple group T . Then, Γ is connected and s -transitive with $s = 1$ or 2. Furthermore, either*

- (i) $s = 2$ and Γ is a complete graph; or
- (ii) $s = 1$, and there is a group H with $\text{Inn}(T) \leq H \leq \text{Aut}(T)$ and an element $g \in T \setminus \{1\}$, such that $\text{Aut} \Gamma = (\hat{T} : H) : \langle \tau \rangle$ and

$$\Gamma = \text{Cay}(T, \{g, g^{-1}\}^H).$$

Proof. Suppose $\Gamma = \text{Cay}(T, S)$ and set $A = \text{Aut} \Gamma$. Since T is nonabelian simple, $\check{T} \circ \hat{T} = \check{T} \times \hat{T}$, and by Lemma 2.6, $A \geq (\check{T} \times \hat{T}) : \langle \tau \rangle$. It follows that $A_1 \geq \check{T}$ and $g^T \subseteq S$ for $g \in S$. Noting that $\langle g^T \rangle$ is a normal subgroup of T , we conclude that $\langle S \rangle = T$, so Γ is connected.

If Γ is a complete graph, then Γ is 2-transitive, as in part (i).

Now suppose Γ is not a complete graph. Then, $\text{Aut} \Gamma$ is not the alternating or symmetric group on VT . Note that $(\check{T} \times \hat{T}) : \langle \tau \rangle \cong T : \mathbb{Z}_2$ is a primitive group of simple diagonal type on $VT = T$, by [10, Proposition 8.1], its overgroup A is also a primitive group of simple diagonal type with unique minimal normal subgroup T^2 .

Hence, $A = (\tilde{T} \times \hat{T}).(o \times \langle \tau \rangle)$ with $o \leq \text{Out}(T)$. Consequently, $A_1 = \tilde{T}.o \times \langle \tau \rangle = (\text{Inn}(T).o) \times \langle \tau \rangle$ because $\langle \text{Aut}(T), \tau \rangle = \text{Aut}(T) \times \langle \tau \rangle$ by Lemma 2.3. Set $H = \text{Inn}(T).o$. By the arc-transitivity of Γ , $A_1 = H \times \langle \tau \rangle$ is transitive on $\Gamma(1) = S$, hence $S = \{g, g^{-1}\}^H$ for some $g \in T \setminus \{1\}$. Furthermore, if Γ is 2-arc-transitive, as A is primitive on VT , by [12, Theorem 2], A should be an affine group, an almost simple group, or is of twisted wreath product type or of product action type, which is a contradiction as A is of simple diagonal type. Therefore, Γ is 1-transitive. $\square$

Theorem 3.1 Particularly says that dual Cayley graphs on nonabelian simple groups are at most 2-transitive, and Lemma 2.9 shows that there is a 3-transitive nontrivial dual Cayley graph on soluble group. We have not found examples with higher transitivity on insoluble and nonabelian soluble groups.

Problem 3.

- (1) Are there 3-transitive dual Cayley graphs on insoluble (not nonabelian simple) groups?
- (2) Are there 4-transitive dual Cayley graphs on nonabelian soluble groups?

We finally answer Problem 2(2) for alternating groups and $\text{PSL}(2, p)$ with p a prime. Denote by $C_G(g)$ the centralizer of an element g in a group G .

Lemma 3.2. *Let g be an element of A_n . Then, $C_{S_n}(g) = C_{A_n}(g)$ if and only if g may be written as a product of disjoint cycles with different odd lengths.*

Proof. (Sufficiency) Suppose $g = (\alpha_1 \dots \alpha_{m_1})(\beta_1 \dots \beta_{m_2}) \dots (\gamma_1 \dots \gamma_{m_s})$ is a product of disjoint cycles, where $m_1, \dots, m_s$ are different odd positive integers. If $h \in C_{S_n}(g)$, then $h^{-1}gh = g$, and one may deduce

$$h = (\alpha_1 \dots \alpha_{m_1})^{k_1} (\beta_1 \dots \beta_{m_2})^{k_2} \dots (\gamma_1 \dots \gamma_{m_s})^{k_m}$$

for some positive integers $k_1, \dots, k_m$. Note that a cycle of odd length is an even permutation, so is h . Hence, $C_{S_n}(g) = C_{A_n}(g)$.

(Necessity) Assume on the contrary that $C_{S_n}(g) = C_{A_n}(g)$ but g is not a product of disjoint odd cycles with different lengths. Write $g = (a_1 \dots a_{n_1})(b_1 \dots b_{n_2}) \dots (c_1 \dots c_{n_t})$, a product of disjoint cycles. Then, either at least one of $n_1, \dots, n_t$, say n_1 , is an even integer, or there are at least two equal odd integers in $n_1, \dots, n_t$, say n_1 and n_2 . For the former case, $(a_1 \dots a_{n_1})$ is an odd permutation and centralizes h , and for the latter case, $(a_1 b_1) \dots (a_{n_1} b_{n_1})$ is an odd permutation and centralizes g , hence we always have $C_{S_n}(g) > C_{A_n}(g)$, which is a contradiction. $\square$

Theorem 3.3. *Each alternating simple group A_n with $n \geq 5$ has an arc-transitive DGRSA.*

Proof. Let $g = (123) \in A_n$ and let

$$\Gamma = \text{Cay}(A_n, \{g, g^{-1}\}^{A_n}).$$

Set $T = A_n$ and $S = \{g, g^{-1}\}^{A_n}$. Clearly, $\langle \hat{T}, \tilde{T} \rangle = \hat{T} \times \tilde{T}$. By Lemma 2.4, Γ is a dual Cayley graph on T . Since $n \geq 5$, and $g = (123)$ and $g^{-1} = (132)$ are conjugate in S_3 , it is easy to see that g and g^{-1} are conjugate in A_n ; we further deduce $S = g^{A_n}$ and Γ is $\hat{T} \times \tilde{T}$ -arc-transitive. Also, since each element in S is of order 3, $\{g, g^{-1}\}^{A_n} \neq A_n \setminus \{1\}$, so Γ is not a complete graph. It then follows from Theorem 3.1 that $\text{Aut} \Gamma = (\hat{A}_n : H) : \langle \tau \rangle$ where $\text{Inn}(T) \leq H \leq \text{Aut}(T)$, hence $(\text{Aut} \Gamma)_1 = H \times \langle \tau \rangle$ by Lemma 2.3. By the arc-transitivity, $S = g^{H \times \langle \tau \rangle} = \{g, g^{-1}\}^H = g^H$.

Assume first $n \neq 6$. Then, $H \leq \text{Aut}(A_n) = S_n$. If $H = S_n$, then $g^{A_n} = g^{S_n}$, so $|A_n : C_{A_n}(g)| = |g^{A_n}| = |g^{S_n}| = |S_n : C_{S_n}(g)|$, implying $|C_{S_n}(g)| = 2 |C_{A_n}(g)|$; however, as $g = (123)$ is a 3-cycle, Lemma 3.2 implies $C_{S_n}(g) = C_{A_n}(g)$, yielding a contradiction. Hence, $H = \text{Inn}(T) = \tilde{T}$ and $\text{Aut}(\Gamma) = (\hat{T} : \tilde{T}) : \langle \tau \rangle = (\tilde{T} \times \hat{T}) : \langle \tau \rangle$, namely, Γ is a DGRSA of A_n .

Assume now $n = 6$. Then, $\text{Inn}(A_6) \leq H \leq \text{Aut}(A_6) \cong A_6 : \mathbb{Z}_2^2$, by Atlas [13, Page 4], we have $C_H(g) = C_{A_6}(g) \cong \mathbb{Z}_3^2$, so $|H|/9 = |g^H| = |g^{A_6}| = |A_6|/9$. It follows $H = \text{Inn}(A_6) = \tilde{A}_6$ and $\text{Aut}(\Gamma) = (\tilde{A}_6 \times \hat{A}_6) : \langle \tau \rangle$, hence Γ is a DGRSA of A_6 . $\square$

The maximal subgroups of $\text{PSL}(2, q)$ and $\text{PGL}(2, q)$ with q a prime power are known, refer to [14, p. 417] and [15, Theorem 2]. Then, one has the following.

Lemma 3.4. *Let $T = \text{PSL}(2, p)$ with $p \geq 5$ a prime, and let $g \in T$. Then, the following statements hold.*

- (a) *Elements of order p of T form two conjugate classes of T and are conjugate in $\text{Aut}(T)$.*
- (b) *Either $o(g) = p$ or $o(g)$ divides $(p - 1)/2$ or $(p + 1)/2$.*
- (c) *If $o(g) \neq p$, then g and g^{-1} are conjugate in T .*
- (d) *If $o(g) = p$, then $C_T(g) = C_{\text{Aut}(T)}(g) \cong \mathbb{Z}_p$, and g is conjugate to g^{-1} in T if and only if $p \equiv 1 \pmod{4}$.*

Theorem 3.5. *Let $T = \text{PSL}(2, p)$ with $p \geq 5$ a prime. Then, T has an arc-transitive DGRSA if and only if $p \equiv 1 \pmod{4}$.*

Proof. (Sufficiency) Let

$$\Gamma = \text{Cay}(T, \{g, g^{-1}\}^T) \text{ with } g \in T \text{ of order } p.$$

By Lemma 2.4, Γ is a dual Cayley graph on T . Set $S = \{g, g^{-1}\}^T$. Since elements in S have the same order, $S \neq T \setminus \{1\}$, and hence Γ is not a complete graph.

Since $p \equiv 1 \pmod{4}$, by Lemma 3.4(d), g is conjugate to g^{-1} in T , so $S = g^T$ and Γ is $\hat{T} \times \tilde{T}$ -arc-transitive. Since Γ is not a complete graph, by Theorem 3.1, $\text{Aut}\Gamma = (\hat{T} : H) : \langle \tau \rangle$, where $\tilde{T} \leq H \leq \text{Aut}(T) \cong \text{PGL}(2, p)$, hence $(\text{Aut}\Gamma)_1 = H \times \langle \tau \rangle$ by Lemma 2.3(iii). By the arc-transitivity, $S = g^{H \times \langle \tau \rangle} = \{g, g^{-1}\}^H = g^H$. If $H = \text{Aut}(T)$, then $|T : C_T(g)| = |g^T| = |S| = |g^{\text{Aut}(T)}| = |\text{Aut}(T) : C_{\text{Aut}(T)}(g)|$, it follows $|C_{\text{Aut}(T)}(g)| = 2 |C_T(g)|$, contradicting to Lemma 3.4(d). Hence $H = \tilde{T}$, $\text{Aut}\Gamma = (\hat{T} : \tilde{T}) : \langle \tau \rangle$ and Γ is an arc-transitive DGRSA of T .

(Necessity) Suppose for a contradiction that $p \equiv 3 \pmod{4}$ and there exists an arc-transitive DGRSA Σ of T . By Lemma 2.8, we have

$$\Sigma = \text{Cay}(T, \{h, h^{-1}\}^T) \text{ for some } h \in T \setminus \{1\},$$

and by Lemma 3.4(b), either $o(h) = p$ or $o(h)$ divides $(p - 1)/2$ or $(p + 1)/2$. Write $S' = \{h, h^{-1}\}^T$.

Assume first $o(h) = p$. Since $p \equiv 3 \pmod{4}$, by Lemma 3.4(d), S' consists of two conjugate classes of elements of order p of T , then by Lemma 3.4(a), we conclude $S' = h^{\text{Aut}(T)}$. It then follows from Lemma 2.1 that $\text{Aut}(T) = \text{Aut}(T, S') \leq \text{Aut}\Sigma$. Hence, $\text{Aut}\Sigma \geq (\hat{T} : \text{Aut}(T)) : \langle \tau \rangle$, a contradiction.

Assume now $o(h)$ divides $(p - 1)/2$ or $(p + 1)/2$. Then, $S' = h^T$ by Lemma 3.4(c). Moreover, by [14, p. 417] and [15, Theorem 2], we have: if $o(h) \mid \frac{p-1}{2}$, then $o(h)$ is odd, and $C_{\text{Aut}(T)}(h) \cong \mathbb{Z}_{p-1}$; if $o(h) = 2$, then $C_T(h) \cong D_{p+1}$ and $C_{\text{Aut}(T)}(h) \cong D_{2(p+1)}$; if $o(h) \mid \frac{p+1}{2}$ with $o(h) \neq 2$, then $C_T(h) \cong \mathbb{Z}_{\frac{p+1}{2}}$ and $C_{\text{Aut}(T)}(h) \cong \mathbb{Z}_{p+1}$. Thus, we always have $|h^T| = |T : C_T(h)| = |\text{Aut}(T) : C_{\text{Aut}(T)}(h)| = |h^{\text{Aut}(T)}|$, hence $S' = h^{\text{Aut}(T)}$. Now, Lemma 2.1 implies $\text{Aut}(T) = \text{Aut}(T, S') \leq \text{Aut}\Sigma$, and hence $\text{Aut}\Sigma \geq (\hat{T} : \text{Aut}(T)) : \langle \tau \rangle$, also a contradiction. $\square$

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