Research Article

Muhammad Adeel*, Khuram Ali Khan, Đilda Pečarić, and Josip Pečarić

Levinson-type inequalities via new Green functions and Montgomery identity

<https://doi.org/10.1515/math-2020-0163>

received September 15, 2019; accepted April 15, 2020

Abstract: In this study, Levinson-type inequalities are generalized by using new Green functions and Montgomery identity for the class of k -convex functions ($k \geq 3$). Čebyšev-, Grüss- and Ostrowski-type new bounds are found for the functionals involving data points of two types. Moreover, a new functional is introduced based on f divergence and then some estimates for new functional are obtained. Some inequalities for Shannon entropies are obtained too.

Keywords: Levinson's inequality, Montgomery identity, information theory

MSC 2010: 26D20, 26D15, 94A20

1 Introduction and preliminaries

The theory of convex functions has encountered a fast advancement. This can be attributed to a few causes: first, applications of convex functions are directly involved in modern analysis; second, many important inequalities are results of applications of convex functions, and convex functions are closely related to inequalities (see [1]). Divided differences are found to be very helpful when we are dealing with functions having different degrees of smoothness. The following definition of divided difference is given in [1, p. 14].

1.1 k th-order divided difference

The k th-order divided difference of a function $f: [\zeta_1, \zeta_2] \rightarrow \mathbb{R}$ at mutually distinct points $x_0, \dots, x_k \in [\zeta_1, \zeta_2]$ is defined recursively by

$$[x_\rho; f] = f(x_\rho), \quad \rho = 0, \dots, k,$$

$$[x_0, \dots, x_k; f] = \frac{[x_1, \dots, x_k; f] - [x_0, \dots, x_{k-1}; f]}{x_k - x_0}. \quad (1)$$

It is easy to see that (1) is equivalent to

$$[x_0, \dots, x_k; f] = \sum_{\rho=0}^k \frac{f(x_\rho)}{q'(x_\rho)}, \quad \text{where } q(x) = \prod_{j=0}^k (x - x_j).$$

The following definition of a real valued convex function is characterized by the k th-order divided difference (see [1, p. 15]).

* **Corresponding author: Muhammad Adeel**, Department of Mathematics, University of Sargodha, Sargodha-40100, Pakistan; Department of Mathematics, University of Central Punjab, Lahore, Pakistan, e-mail: adeel.uosmaths@gmail.com

Khuram Ali Khan: Department of Mathematics, University of Sargodha, Sargodha-40100, Pakistan, e-mail: khuramsms@gmail.com

Đilda Pečarić: Catholic University of Croatia, Ilica 242, Zagreb, Croatia, e-mail: gildapecar@gmail.com

Josip Pečarić: Rudn University, Moscow, Russia, e-mail: pecaric@element.hr

Definition 1.1. A function $f: [\zeta_1, \zeta_2] \rightarrow \mathbb{R}$ is said to be k -convex ($k \geq 0$) if and only if for all choices of $(k+1)$ distinct points $x_0, \dots, x_k \in [\zeta_1, \zeta_2]$, $[x_0, \dots, x_k; f] \geq 0$ holds.

If this inequality is reversed, then f is said to be k -concave. If the inequality is strict, then f is said to be a strictly k -convex (k -concave) function.

Note that 0-convex functions, 1-convex functions and 2-convex functions are non-negative functions, increasing functions and simply convex functions, respectively.

1.2 Criteria for k convex functions

In [1, p. 16], criteria to examine the n -convexity of a function f are given as:

“If $f^{(k)}$ exists, then f is k -convex if and only if $f^{(k)} \geq 0$ ”.

In [2], (see also [3, p. 32, Theorem 1]) Ky Fan’s inequality is generalized by Levinson for 3-convex functions as follows:

Theorem A. Let $f: \mathcal{J} = (0, 2\alpha) \rightarrow \mathbb{R}$ be such that $f^{(3)}(t) \geq 0$. Let $x_\rho \in (0, \alpha)$ and $p_\rho > 0$. Then,

$$\frac{1}{P_n} \sum_{\rho=1}^n p_\rho f(x_\rho) - f\left(\frac{1}{P_n} \sum_{\rho=1}^n p_\rho x_\rho\right) \leq \frac{1}{P_n} \sum_{\rho=1}^n p_\rho f(2\alpha - x_\rho) - f\left(\frac{1}{P_n} \sum_{\rho=1}^n p_\rho (2\alpha - x_\rho)\right). \quad (2)$$

Functional form of (2) is defined as follows:

$$J_1(f(\cdot)) = \frac{1}{P_n} \sum_{\rho=1}^n p_\rho f(2\alpha - x_\rho) - f\left(\frac{1}{P_n} \sum_{\rho=1}^n p_\rho (2\alpha - x_\rho)\right) - \frac{1}{P_n} \sum_{\rho=1}^n p_\rho f(x_\rho) + f\left(\frac{1}{P_n} \sum_{\rho=1}^n p_\rho x_\rho\right) \geq 0. \quad (3)$$

In [4], Popoviciu noted that (2) is valid on $(0, 2\alpha)$ for 3-convex functions, while in [5] (see also [3, p. 32, Theorem 2]) Bullen provided a different proof of Popoviciu’s result and also the converse of (2).

Theorem B. (a) Let $f: \mathcal{J} = [\zeta_1, \zeta_2] \rightarrow \mathbb{R}$ be such that $f^{(3)}(t) \geq 0$ and $x_\rho, y_\rho \in [\zeta_1, \zeta_2]$ for $\rho = 1, 2, \dots, n$ such that

$$\max\{x_1 \dots x_n\} \leq \min\{y_1 \dots y_n\}, \quad x_1 + y_1 = \dots = x_n + y_n \quad (4)$$

and $p_\rho > 0$ then

$$\frac{1}{P_n} \sum_{\rho=1}^n p_\rho f(x_\rho) - f\left(\frac{1}{P_n} \sum_{\rho=1}^n p_\rho x_\rho\right) \leq \frac{1}{P_n} \sum_{\rho=1}^n p_\rho f(y_\rho) - f\left(\frac{1}{P_n} \sum_{\rho=1}^n p_\rho y_\rho\right). \quad (5)$$

(b) f is 3-convex if f is continuous, $p_\rho > 0$ and (5) holds for all x_ρ, y_ρ satisfying (4).

Functional form of (5) is defined as follows:

$$J_2(f(\cdot)) = \frac{1}{P_n} \sum_{\rho=1}^n p_\rho f(y_\rho) - f\left(\frac{1}{P_n} \sum_{\rho=1}^n p_\rho y_\rho\right) - \frac{1}{P_n} \sum_{\rho=1}^n p_\rho f(x_\rho) + f\left(\frac{1}{P_n} \sum_{\rho=1}^n p_\rho x_\rho\right) \geq 0. \quad (6)$$

Remark 1.1. It is essential to take note of that under the suppositions of Theorems A and B, if the function f is 3-convex, then $J_i(f(\cdot)) \geq 0$ for $i = 1, 2$.

In [6], (see also [3, p. 32, Theorem 4]) the following result holds.

Theorem C. Let $f: \mathcal{J} = [\zeta_1, \zeta_2] \rightarrow \mathbb{R}$ be a 3-convex function, $p_\rho > 0$, and let $x_\rho, y_\rho \in [\zeta_1, \zeta_2]$ be such that $x_\rho + y_\rho = 2\tilde{c}$, for $\rho = 1, \dots, n$, $x_\rho + x_{n-\rho+1} \leq 2\tilde{c}$ and $\frac{p_\rho x_\rho + p_{n-\rho+1} x_{n-\rho+1}}{p_\rho + p_{n-\rho+1}} \leq \tilde{c}$. Then, (5) holds.

In [7], Mercer replaced the condition of symmetric distribution of points x_ρ and y_ρ with symmetric variances of points x_ρ and y_ρ .

Theorem D. Let $f: \mathcal{I} = [\zeta_1, \zeta_2] \rightarrow \mathbb{R}$ be a 3-convex function, p_ρ are positive such that $\sum_{\rho=1}^n p_\rho = 1$. Also, let x_ρ, y_ρ satisfy $\max\{x_1, \dots, x_n\} \leq \min\{y_1, \dots, y_n\}$ and

$$\sum_{\rho=1}^n p_\rho \left(x_\rho - \sum_{\rho=1}^n p_\rho x_\rho \right)^2 = \sum_{\rho=1}^n p_\rho \left(y_\rho - \sum_{\rho=1}^n p_\rho y_\rho \right)^2, \quad (7)$$

then (5) holds.

In [8], Adeel et al. generalized Levinson's inequality for 3-convex functions; they also obtained some results for information theory. Similar results can be found in [9,10]. In [11], Witkowski showed the Levinson inequality with random variables. Furthermore, he showed that it is enough to assume that f is 3-convex and that assumption (7) can be weakened to inequality in a certain direction.

In [12], Pečarić et al. provided the probabilistic version of Levinson's inequality (2) under Mercer's assumption of equal variances for the family of 3-convex functions at a point. They showed that this is the largest family of continuous functions for which inequality (5) holds. The operator version of probabilistic Levinson's inequality was discussed in [13]. In [14], Pavić provided geometrical interpretation of different inequalities involving convex functions. Many other researchers generalized different inequalities for different classes of convex functions [15–18].

For our main results, we use the generalized Montgomery identity via Taylor's formula given in [19].

Theorem E. Let $k \in \mathbb{N}$, $f: \mathcal{I} \rightarrow \mathbb{R}$ be such that $f^{(k-1)}$ is absolutely continuous, $\mathcal{I}$ is an open interval contained in $\mathbb{R}$, $\zeta_1, \zeta_2 \in \mathcal{I}$ such that $\zeta_1 < \zeta_2$. Then, the following identity holds:

$$\begin{aligned} f(s) &= \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} f(v) dv + \sum_{l=0}^{k-2} \frac{f^{(l+1)}(\zeta_1)}{l!(l+2)} \frac{(s - \zeta_1)^{l+2}}{\zeta_2 - \zeta_1} - \sum_{l=0}^{k-2} \frac{f^{(l+1)}(\zeta_2)}{l!(l+2)} \frac{(s - \zeta_2)^{l+2}}{\zeta_2 - \zeta_1} \\ &\quad + \frac{1}{(k-1)!} \int_{\zeta_1}^{\zeta_2} R_k(s, v) f^{(k)}(v) dv, \end{aligned} \quad (8)$$

where

$$R_k(s, v) = \begin{cases} -\frac{(s-v)^k}{k(\zeta_2 - \zeta_1)} + \frac{s - \zeta_1}{\zeta_2 - \zeta_1} ((s-v)^{k-1}), & \zeta_1 \leq v \leq s; \\ -\frac{(s-v)^k}{k(\zeta_2 - \zeta_1)} + \frac{s - \zeta_2}{\zeta_2 - \zeta_1} ((s-v)^{k-1}), & s < v \leq \zeta_2. \end{cases} \quad (9)$$

When $k = 1$, identity (8) reduces to the well-known Montgomery identity (see [20]).

$$f(s) = \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} f(v) dv + \int_{\zeta_1}^{\zeta_2} P(s, v) f'(v) dv,$$

where $P(s, v)$ is the Peano kernel defined as

$$P(s, v) = \begin{cases} \frac{v - \zeta_1}{\zeta_2 - \zeta_1}, & \zeta_1 \leq v \leq s; \\ \frac{v - \zeta_2}{\zeta_2 - \zeta_1}, & s < v \leq \zeta_2. \end{cases} \quad (10)$$

The error function $e_{\mathcal{F}}(t)$ can be represented in terms of the Green functions $G_{\mathcal{F},k}(t, s)$ of the boundary value problem

$$z^{(k)}(t) = 0, \quad z^{(i)}(a_1) = 0, \quad 0 \leq i \leq p, \quad z^{(i)}(a_2) = 0, \quad p+1 \leq i \leq k-1,$$

$$e_{\mathcal{F}}(t) = \int_{\zeta_1}^{\zeta_2} G_{F,k}(t, s) f^{(k)}(s) ds, \quad t \in [\zeta_1, \zeta_2],$$

where

$$G_{F,k}(t, s) = \frac{1}{(k-1)!} \begin{cases} \sum_{i=0}^p \binom{k-1}{i} (t-a_1)^i (a_1-s)^{k-i-1}, & \zeta_1 \leq s \leq t; \\ - \sum_{i=p+1}^{k-1} \binom{k-1}{i} (t-a_1)^i (a_1-s)^{k-i-1}, & t \leq s \leq \zeta_2. \end{cases} \quad (11)$$

In [21], the following result holds.

Theorem F. Let $f \in C^k[\zeta_1, \zeta_2]$, and let P_F be its “two-point right focal” interpolating polynomial. Then, for $\zeta_1 \leq a_1 < a_2 \leq \zeta_2$ and $0 \leq p \leq k-2$,

$$\begin{aligned} f(t) = P_F(t) + e_F(t) &= \sum_{i=0}^p \frac{(t-a_1)^i}{i!} f^{(i)}(a_1) + \sum_{j=0}^{k-p-2} \left(\sum_{i=0}^j \frac{(t-a_1)^{p+1+i} (a_1-a_2)^{j-i}}{(p+1+i)!(j-i)!} \right) f^{(p+1+j)}(a_2) \\ &+ \int_{\zeta_1}^{\zeta_2} G_{F,k}(t, s) f^{(k)}(s) ds, \end{aligned} \quad (12)$$

where $G_{F,k}(t, s)$ is the Green function, defined by (11).

Let $f \in C^k[\zeta_1, \zeta_2]$, and let P_F be its “two-point right focal” interpolating polynomial for $\zeta_1 \leq a_1 < a_2 \leq \zeta_2$. Then, for $k=3$ and $p=0$, (12) becomes

$$f(t) = f(a_1) + (t-a_1)f^{(1)}(a_1) + (t-a_1)(a_1-a_2)f^{(2)}(a_2) + \frac{(t-a_1)^2}{2}f^{(2)}(a_2) + \int_{\zeta_1}^{\zeta_2} G_1(t, s)f^{(3)}(s)ds, \quad (13)$$

where

$$G_1(t, s) = \begin{cases} \frac{1}{2}(a_1-s)^2, & \zeta_1 \leq s \leq t; \\ - (t-a_1)(a_1-s) - \frac{1}{2}(t-a_1)^2, & t \leq s \leq \zeta_2. \end{cases} \quad (14)$$

For $k=3$ and $p=1$, (12) becomes

$$f(t) = f(a_1) + (t-a_1)f^{(1)}(a_1) + \frac{(t-a_1)^2}{2}f^{(2)}(a_2) + \int_{\zeta_1}^{\zeta_2} G_2(t, s)f^{(3)}(s)ds, \quad (15)$$

where

$$G_2(t, s) = \begin{cases} \frac{1}{2}(a_1-s)^2 + (t-a_1)(a_1-s), & \zeta_1 \leq s \leq t; \\ - \frac{1}{2}(t-a_1)^2, & t \leq s \leq \zeta_2. \end{cases} \quad (16)$$

In [22], Mehmood et al. generalized Popoviciu-type inequalities via new Green functions and Montgomery identity. All generalizations that exist in the literature are only for one type of data points. But in this study, motivated by the above discussion Levinson-type inequalities are generalized via new Green functions and Montgomery identity involving two types of data points for higher order convex functions. Čebyšev-, Grüss- and Ostrowski-type new bounds are also found for the functionals involving data points of two types.

2 Main results

Motivated by identity (6), we construct the following identities.

2.1 Generalization of Bullen-type inequalities for higher order convex functions

First, we define the following functional:

$\mathcal{F} : \text{let } f: \mathcal{J}_1 = [\zeta_1, \zeta_2] \rightarrow \mathbb{R}$ be a function, $x_1, \dots, x_n$ and $y_1, \dots, y_m \in \mathcal{J}_1$ such that $\max\{x_1, \dots, x_n\} \leq \min\{y_1, \dots, y_m\}$ and

$$\sum_{\rho=1}^n p_{\rho} \left(x_{\rho} - \sum_{\rho=1}^n p_{\rho} x_{\rho} \right)^2 = \sum_{\varrho=1}^m q_{\varrho} \left(y_{\varrho} - \sum_{\varrho=1}^m q_{\varrho} y_{\varrho} \right)^2. \quad (17)$$

Also, let $(p_1, \dots, p_n) \in \mathbb{R}^n$ and $(q_1, \dots, q_m) \in \mathbb{R}^m$ be such that $\sum_{\rho=1}^n p_{\rho} = 1$, $\sum_{\varrho=1}^m q_{\varrho} = 1$ and $x_{\rho}, y_{\varrho}, \sum_{\rho=1}^n p_{\rho} x_{\rho}, \sum_{\varrho=1}^m q_{\varrho} y_{\varrho} \in \mathcal{J}_1$. Then,

$$\mathcal{J}(f(\cdot)) = \sum_{\varrho=1}^m q_{\varrho} f(y_{\varrho}) - f\left(\sum_{\varrho=1}^m q_{\varrho} y_{\varrho}\right) - \sum_{\rho=1}^n p_{\rho} f(x_{\rho}) + f\left(\sum_{\rho=1}^n p_{\rho} x_{\rho}\right). \quad (18)$$

Theorem 2.1. Assume $\mathcal{F}$. Let $f \in C^k[\zeta_1, \zeta_2]$ be such that $f^{(k-1)}$ is absolutely continuous. Then, we have the following new identities for $t = 1, 2$:

(i)

$$\begin{aligned} \mathcal{J}(f(\cdot)) &= \frac{1}{2} \left[\sum_{\varrho=1}^m q_{\varrho} y_{\varrho}^2 - \left(\sum_{\varrho=1}^m q_{\varrho} y_{\varrho} \right)^2 - \sum_{\rho=1}^n p_{\rho} x_{\rho}^2 + \left(\sum_{\rho=1}^n p_{\rho} x_{\rho} \right)^2 \right] f^{(2)}(\zeta_2) + 2 \left(\frac{f(\zeta_1) - f(\zeta_2)}{\zeta_2 - \zeta_1} \right) \int_{\zeta_1}^{\zeta_2} \mathcal{J}(G_t(\cdot, s)) ds \\ &\quad + \int_{\zeta_1}^{\zeta_2} \mathcal{J}(G_t(\cdot, s)) \sum_{l=3}^{k-1} \frac{l(l-1)}{(l-1)!} \left[\frac{f^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - f^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] ds \\ &\quad + \frac{1}{(k-4)!} \int_{\zeta_1}^{\zeta_2} f^{(k)}(v) \left(\int_{\zeta_1}^{\zeta_2} \mathcal{J}(G_t(\cdot, s)) \tilde{R}_{k-3}(s, v) ds \right) dv, \end{aligned} \quad (19)$$

where

$$\tilde{R}_{k-3}(s, v) = \begin{cases} 2 \frac{(s-v)^{k-3}}{(k-3)(\zeta_2 - \zeta_1)} + \frac{s - \zeta_1}{\zeta_2 - \zeta_1} ((s-v)^{k-4}), & \zeta_1 \leq v \leq s; \\ 2 \frac{(s-v)^{k-3}}{(k-3)(\zeta_2 - \zeta_1)} + \frac{s - \zeta_2}{\zeta_2 - \zeta_1} ((s-v)^{k-4}), & s < v \leq \zeta_2, \end{cases} \quad (20)$$

and

$$\mathcal{J}(G_t(\cdot, s)) = \sum_{\varrho=1}^m q_{\varrho} G_t(y_{\varrho}, s) - G_t\left(\sum_{\varrho=1}^m q_{\varrho} y_{\varrho}, s\right) - \sum_{\rho=1}^n p_{\rho} G_t(x_{\rho}, s) + G_t\left(\sum_{\rho=1}^n p_{\rho} x_{\rho}, s\right), \quad (21)$$

for $t = 1, 2$, $G_t(\cdot, s)$ is defined in (14) and (16), respectively.

(ii)

$$\begin{aligned}
\mathcal{J}(f(\cdot)) &= \frac{1}{2} \left[\sum_{\varrho=1}^m q_{\varrho} y_{\varrho}^2 - \left(\sum_{\varrho=1}^m q_{\varrho} y_{\varrho} \right)^2 - \sum_{\rho=1}^n p_{\rho} x_{\rho}^2 + \left(\sum_{\rho=1}^n p_{\rho} x_{\rho} \right)^2 \right] f^{(2)}(\zeta_2) + 2 \left(\frac{f'(\zeta_2) - f'(\zeta_1)}{\zeta_2 - \zeta_1} \right) \int_{\zeta_1}^{\zeta_2} \mathcal{J}(G_t(\cdot, s)) ds \\
&\quad + \int_{\zeta_1}^{\zeta_2} \mathcal{J}(G_t(\cdot, s)) \sum_{l=4}^{k-1} \frac{1}{(l-2)(l-4)!} \left[\frac{f^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - f^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] ds \\
&\quad + \frac{1}{(k-4)!} \int_{\zeta_1}^{\zeta_2} f^{(k)}(v) \left(\int_{\zeta_1}^{\zeta_2} \mathcal{J}(G_t(\cdot, s)) R_{k-3}(s, v) ds \right) dv,
\end{aligned} \tag{22}$$

where

$$R_{k-3}(s, v) = \begin{cases} -\frac{(s-v)^{k-3}}{(k-3)(\zeta_2 - \zeta_1)} + \frac{s - \zeta_1}{\zeta_2 - \zeta_1} ((s-v)^{k-4}), & \zeta_1 \leq v \leq s; \\ -\frac{(s-v)^{k-3}}{(k-3)(\zeta_2 - \zeta_1)} + \frac{s - \zeta_2}{\zeta_2 - \zeta_1} ((s-v)^{k-4}), & s < v \leq \zeta_2. \end{cases} \tag{23}$$

Proof.

(i) Applying (18) to identities (13) and (15) along with their respective new Green functions, by means of simple calculations and following the properties of $\mathcal{J}(f(\cdot))$, we get

$$\mathcal{J}(f(\cdot)) = \frac{1}{2} \left[\sum_{\varrho=1}^m q_{\varrho} y_{\varrho}^2 - \left(\sum_{\varrho=1}^m q_{\varrho} y_{\varrho} \right)^2 - \sum_{\rho=1}^n p_{\rho} x_{\rho}^2 + \left(\sum_{\rho=1}^n p_{\rho} x_{\rho} \right)^2 \right] f^{(2)}(\zeta_2) + \int_{\zeta_1}^{\zeta_2} \mathcal{J}(G_t(\cdot, s)) f^{(3)}(s) ds. \tag{24}$$

Differentiating (8) with respect to “s”, we have

$$f^{(3)}(s) = 2 \left(\frac{f''(\zeta_1) - f''(\zeta_2)}{\zeta_2 - \zeta_1} \right) + \sum_{l=3}^{k-1} \frac{l(l-1)}{(l-1)!} \left(\frac{f^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - f^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right) + \frac{1}{(k-4)!} \int_{\zeta_1}^{\zeta_2} \tilde{R}_{k-3}(s, v) f^{(k)}(v) dv. \tag{25}$$

Using (25) in (24) and following the properties of $\mathcal{J}(f(\cdot))$, we get

$$\begin{aligned}
\mathcal{J}(f(\cdot)) &= \frac{1}{2} \left[\sum_{\varrho=1}^m q_{\varrho} y_{\varrho}^2 - \left(\sum_{\varrho=1}^m q_{\varrho} y_{\varrho} \right)^2 - \sum_{\rho=1}^n p_{\rho} x_{\rho}^2 + \left(\sum_{\rho=1}^n p_{\rho} x_{\rho} \right)^2 \right] f^{(2)}(\zeta_2) + \int_{\zeta_1}^{\zeta_2} \mathcal{J}(G_t(\cdot, s)) \left[2 \left(\frac{f''(\zeta_1) - f''(\zeta_2)}{\zeta_2 - \zeta_1} \right) \right. \\
&\quad \left. + \sum_{l=3}^{k-1} \frac{l(l-1)}{(l-1)!} \left(\frac{f^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - f^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right) + \frac{1}{(k-4)!} \int_{\zeta_1}^{\zeta_2} \tilde{R}_{k-3}(s, v) f^{(k)}(v) dv \right] ds. \\
\mathcal{J}(f(\cdot)) &= \frac{1}{2} \left[\sum_{\varrho=1}^m q_{\varrho} y_{\varrho}^2 - \left(\sum_{\varrho=1}^m q_{\varrho} y_{\varrho} \right)^2 - \sum_{\rho=1}^n p_{\rho} x_{\rho}^2 + \left(\sum_{\rho=1}^n p_{\rho} x_{\rho} \right)^2 \right] f^{(2)}(\zeta_2) + 2 \left(\frac{f''(\zeta_1) - f''(\zeta_2)}{\zeta_2 - \zeta_1} \right) \int_{\zeta_1}^{\zeta_2} \mathcal{J}(G_t(\cdot, s)) ds \\
&\quad + \int_{\zeta_1}^{\zeta_2} \mathcal{J}(G_t(\cdot, s)) \left[\sum_{l=3}^{k-1} \frac{l(l-1)}{(l-1)!} \left(\frac{f^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - f^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right) \right] ds \\
&\quad + \frac{1}{(k-4)!} \int_{\zeta_1}^{\zeta_2} \mathcal{J}(G_t(\cdot, s)) \left(\int_{\zeta_1}^{\zeta_2} \tilde{R}_{k-3}(s, v) f^{(k)}(v) dv \right) ds.
\end{aligned}$$

Execute Fubini's theorem in the last term to get (19) for $t = 1, 2$.

(ii) Using formula (8) on the function $f^{(3)}$, replacing k by $k - 3$ and rearranging the indices, we have

$$f^{(3)}(s) = 2 \left(\frac{f''(\zeta_2) - f''(\zeta_1)}{\zeta_2 - \zeta_1} \right) + \sum_{l=4}^{k-1} \frac{l}{(l-2)(l-4)!} \left(\frac{f^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - f^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right) + \frac{1}{(k-4)!} \int_{\zeta_1}^{\zeta_2} R_{k-3}(s, v) f^{(k)}(v) dv. \quad (26)$$

Using (26) in (24) and Fubini's theorem we conclude (22) for $t = 1, 2$. $\square$

As an application of new identities (19) and (22), the next result gives generalization of Bullen-type inequalities involving new Green functions for k -convex ($k \geq 3$) functions.

Theorem 2.2. Assuming the conditions of Theorem 2.1 with f as the k -convex function, then for $t = 1, 2$ we have the following two results:

If

$$\int_{\zeta_1}^{\zeta_2} \check{J}(G_t(\cdot, s)) \check{R}_{k-3}(s, v) ds \geq 0, \quad v \in \mathcal{I}_1, \quad (27)$$

then

$$\begin{aligned} \check{J}(f(\cdot)) &\geq \frac{1}{2} \left[\sum_{q=1}^m q_{\varrho} y_{\varrho}^2 - \left(\sum_{q=1}^m q_{\varrho} y_{\varrho} \right)^2 - \sum_{\rho=1}^n p_{\rho} x_{\rho}^2 + \left(\sum_{\rho=1}^n p_{\rho} x_{\rho} \right)^2 \right] f^{(2)}(\zeta_2) + 2 \left(\frac{f'(\zeta_1) - f'(\zeta_2)}{\zeta_2 - \zeta_1} \right) \int_{\zeta_1}^{\zeta_2} \check{J}(G_t(\cdot, s)) ds \\ &\quad + \int_{\zeta_1}^{\zeta_2} \check{J}(G_t(\cdot, s)) \sum_{l=3}^{k-1} \frac{l(l-1)}{(l-1)!} \left[\frac{f^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - f^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] ds, \end{aligned} \quad (28)$$

and if

$$\int_{\zeta_1}^{\zeta_2} \check{J}(G_t(\cdot, s)) R_{k-3}(s, v) ds \geq 0, \quad v \in \mathcal{I}_1, \quad (29)$$

then

$$\begin{aligned} \check{J}(f(\cdot)) &\geq \frac{1}{2} \left[\sum_{q=1}^m q_{\varrho} y_{\varrho}^2 - \left(\sum_{q=1}^m q_{\varrho} y_{\varrho} \right)^2 - \sum_{\rho=1}^n p_{\rho} x_{\rho}^2 + \left(\sum_{\rho=1}^n p_{\rho} x_{\rho} \right)^2 \right] f^{(2)}(\zeta_2) + 2 \left(\frac{f''(\zeta_2) - f''(\zeta_1)}{\zeta_2 - \zeta_1} \right) \int_{\zeta_1}^{\zeta_2} \check{J}(G_t(\cdot, s)) ds \\ &\quad + \int_{\zeta_1}^{\zeta_2} \check{J}(G_t(\cdot, s)) \sum_{l=4}^{k-1} \frac{1}{(l-2)(l-4)!} \left[\frac{f^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - f^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] ds. \end{aligned} \quad (30)$$

Proof. As function f is k -convex ($k \geq 3$) and is k -times differentiable, so

$$f^{(k)}(s) \geq 0 \quad \forall s \in \mathcal{I}_1,$$

then by using (27) in (19) and (29) in (22), we get (28) and (30), respectively. $\square$

Remark 2.1. In Theorem 2.2, inequalities (28) and (30) hold in the reverse direction if inequalities (27) and (29) are reversed.

If we put $m = n$, $p_\rho = q_\rho$ and by using positive weights in (18), then $\mathcal{J}(\cdot)$ converted to the functional $J_2(\cdot)$ defined in (6) also in this case, (19), (21), (22), (27), (28), (29) and (30) become

$$\begin{aligned} J_2(f(\cdot)) &= 2 \left(\frac{f''(\zeta_1) - f''(\zeta_2)}{\zeta_2 - \zeta_1} \right) \int_{\zeta_1}^{\zeta_2} J_2(G_t(\cdot, s)) ds \\ &\quad + \int_{\zeta_1}^{\zeta_2} J_2(G_t(\cdot, s)) \sum_{l=3}^{k-1} \frac{l(l-1)}{(l-1)!} \left[\frac{f^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - f^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] ds \\ &\quad + \frac{1}{(k-4)!} \int_{\zeta_1}^{\zeta_2} f^{(k)}(v) \left(\int_{\zeta_1}^{\zeta_2} J_2(G_t(\cdot, s)) \tilde{R}_{k-3}(s, v) ds \right) dv, \end{aligned} \quad (31)$$

$$J_2(G_t(\cdot, s)) = \sum_{\rho=1}^n p_\rho G_t(y_\rho, s) - G_t \left(\sum_{\rho=1}^n p_\rho y_\rho, s \right) - \sum_{\rho=1}^n p_\rho G_t(x_\rho, s) + G_t \left(\sum_{\rho=1}^n p_\rho x_\rho, s \right), \quad (32)$$

$$\begin{aligned} J_2(f(\cdot)) &= 2 \left(\frac{f''(\zeta_2) - f''(\zeta_1)}{\zeta_2 - \zeta_1} \right) \int_{\zeta_1}^{\zeta_2} J_2(G_t(\cdot, s)) ds \\ &\quad + \int_{\zeta_1}^{\zeta_2} J_2(G_t(\cdot, s)) \sum_{l=4}^{k-1} \frac{1}{(l-2)(l-4)!} \left[\frac{f^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - f^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] ds \\ &\quad + \frac{1}{(k-4)!} \int_{\zeta_1}^{\zeta_2} f^{(k)}(v) \left(\int_{\zeta_1}^{\zeta_2} J_2(G_t(\cdot, s)) R_{k-3}(s, v) ds \right) dv, \end{aligned} \quad (33)$$

$$\int_{\zeta_1}^{\zeta_2} J_2(G_t(\cdot, s)) \tilde{R}_{k-3}(s, v) ds \geq 0, \quad v \in \mathcal{I}_1, \quad (34)$$

$$\begin{aligned} J_2(f(\cdot)) &\geq 2 \left(\frac{f''(\zeta_1) - f''(\zeta_2)}{\zeta_2 - \zeta_1} \right) \int_{\zeta_1}^{\zeta_2} J_2(G_t(\cdot, s)) ds \\ &\quad + \int_{\zeta_1}^{\zeta_2} J_2(G_t(\cdot, s)) \sum_{l=3}^{k-1} \frac{l(l-1)}{(l-1)!} \left[\frac{f^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - f^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] ds, \end{aligned} \quad (35)$$

$$\int_{\zeta_1}^{\zeta_2} J_2(G_t(\cdot, s)) R_{k-3}(s, v) ds \geq 0, \quad v \in \mathcal{I}_1 \quad (36)$$

and

$$\begin{aligned} J_2(f(\cdot)) &\geq 2 \left(\frac{f''(\zeta_2) - f''(\zeta_1)}{\zeta_2 - \zeta_1} \right) \int_{\zeta_1}^{\zeta_2} J_2(G_t(\cdot, s)) ds \\ &\quad + \int_{\zeta_1}^{\zeta_2} J_2(G_t(\cdot, s)) \sum_{l=4}^{k-1} \frac{1}{(l-2)(l-4)!} \left[\frac{f^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - f^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] ds. \end{aligned} \quad (37)$$

Theorem 2.3. Let $f: \mathcal{I}_1 = [\zeta_1, \zeta_2] \rightarrow \mathbb{R}$ be a k -convex ($k \geq 3$) function. Also, let $f^{(k-1)}$ be an absolutely continuous function and $(p_1, \dots, p_n)$ are positive real numbers such that $\sum_{\rho=1}^n p_\rho = P_n$. Then, for the functional $J_2(\cdot)$ defined in (6), we have the following:

- (i) For $t = 1, 2$, inequalities (35) and (37) hold provided that k is odd.
- (ii) For fixed $t = 1, 2$, let inequality (35) be satisfied and

$$(f''(\zeta_1) - f''(\zeta_2)) + \sum_{l=3}^{k-1} \frac{l(l-1)}{(l-1)!} \left[\frac{f^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - f^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] \geq 0 \quad \forall s \in \mathcal{I}_1, \quad (38)$$

or (37) be satisfied and

$$\sum_{l=4}^{k-1} \frac{1}{(l-2)(l-4)!} \left[\frac{f^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - f^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] \geq 0. \quad (39)$$

Then, we have

$$J_2(f(\cdot)) \geq 0. \quad (40)$$

Proof. It is clear that Green functions $G_t(\cdot, s)$ defined in (14) and (16) are 3-convex functions. Since weights p_ρ are positive, applying Theorem B and by using Remark 1.1, we have $J_2(G_t(\cdot, s)) \geq 0$ for fixed $t = 1, 2$.

- (i) $\tilde{R}_k(s, v) \geq 0$ and $R_k(s, v) \geq 0$ for $k = 5, 7, \dots$, so (33) and (35) hold. As f is k -convex, hence by following Theorem 2.2 we get (34) and (36).
- (ii) Using (38) in (35) and (39) in (37) we get (40) for fixed $t = 1, 2$.

Next, we have a generalized form of Levinson-type inequality for $2n$ points given in [6] (see also [3]). $\square$

Theorem 2.4. Let $f \in C^k[\zeta_1, \zeta_2]$ be such that $f^{(k-1)}$ is absolutely continuous and $(p_1, \dots, p_n)$ be the positive real numbers such that $\sum_{\rho=1}^n p_\rho = 1$. Also, let $x_\rho + y_\rho = 2\tilde{c}$ and $x_\rho + x_{n-\rho+1} \leq 2\tilde{c}$, $\frac{p_\rho x_\rho + p_{n-\rho+1} x_{n-\rho+1}}{p_\rho + p_{n-\rho+1}} \leq \tilde{c}$. Then, for the functional $J_2(\cdot)$ defined in (6), we have the following:

- (i) For $t = 1, 2$, inequalities (35) and (37) hold provided that k is odd.
- (ii) For fixed $t = 1, 2$, let inequality (35) be satisfied and inequality (38) is valid or (37) be satisfied and inequality (38) is valid then we have inequality (40).

Proof. Proof is similar to that of Theorem 2.3. $\square$

In [7], Mercer made a significant improvement by replacing the condition $x_1 + y_1 = \dots = x_n + y_n$ of symmetric distribution with the weaker one that the variances of the two sequences are equal.

In the next result, Levinson-type inequality is given (for positive weights) under Mercer's condition given in (7).

Corollary 2.1. Let $f: \mathcal{I}_1 = [\zeta_1, \zeta_2] \rightarrow \mathbb{R}$ be such that $f^{(k-1)}$ is absolutely continuous, x_ρ, y_ρ satisfy $\max\{x_1, \dots, x_n\} \leq \min\{y_1, \dots, y_n\}$ and (7). Also, let $(p_1, \dots, p_n) \in \mathbb{R}^n$ such that $\sum_{\rho=1}^n p_\rho = 1$. Then, (31) and (33) are valid.

Proof. Proof is similar to that of Theorem 2.1 and by using the conditions given in the statement we get (31) and (33). $\square$

Theorem 2.5. Let $f: \mathcal{I}_1 = [\zeta_1, \zeta_2] \rightarrow \mathbb{R}$ be a k -convex ($k > 3$) function. Also, let $f^{(k-1)}$ be an absolutely continuous function and $(p_1, \dots, p_n)$ are positive real numbers such that $\sum_{\rho=1}^n p_\rho = 1$, $\max\{x_1, \dots, x_n\} \leq \min\{y_1, \dots, y_n\}$ and

$$\sum_{\rho=1}^n p_\rho \left(x_\rho - \sum_{\rho=1}^n p_\rho x_\rho \right)^2 = \sum_{\rho=1}^n p_\rho \left(y_\rho - \sum_{\rho=1}^n p_\rho y_\rho \right)^2. \quad (41)$$

Then,

- (i) For $t = 1, 2$, inequalities (35) and (37) hold provided that k is odd.
(ii) For fixed $t = 1, 2$, let inequality (35) be satisfied and

$$(f'(\zeta_1) - f'(\zeta_2)) + \sum_{l=3}^{k-1} \frac{l(l-1)}{(l-1)!} \left[\frac{f^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - f^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] \geq 0 \quad \forall s \in \mathcal{I}_1, \quad (42)$$

or (37) be satisfied and

$$\sum_{l=4}^{k-1} \frac{1}{(l-2)(l-4)!} \left[\frac{f^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - f^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] \geq 0. \quad (43)$$

Then, we have

$$J_2(f(\cdot)) \geq 0. \quad (44)$$

Proof. It is clear that Green functions $G_t(\cdot, s)$ defined in (14) and (16) are 3-convex functions. Since weights are positive, applying Theorem D, we have $J_2(G_t(\cdot, s)) \geq 0$ for fixed $t = 1, 2$.

- (i) $\tilde{R}_k(s, v) \geq 0$ and $R_k(s, v) \geq 0$ for $k = 5, 7, \dots$, so (34) and (36) hold. As f is k -convex, hence by following Theorem 2.2 we get (35) and (37).
(ii) Using (42) in (35) and (43) in (37) we get (44) for fixed $t = 1, 2$. $\square$

2.2 Levinson-type inequality for k -convex ($k \geq 3$) functions

Motivated by identity (3), we construct the following identities.

First, we defined the following functional:

$\mathcal{H}$: let $f: \mathcal{I}_2 = [0, 2a] \rightarrow \mathbb{R}$ be a function, $x_1, \dots, x_n \in (0, a)$, $(p_1, \dots, p_n) \in \mathbb{R}^n$, $(q_1, \dots, q_m) \in \mathbb{R}^m$ are real numbers such that $\sum_{\rho=1}^n p_\rho = 1$ and $\sum_{\varrho=1}^m q_\varrho = 1$. Also, let x_ρ , $\sum_{\varrho=1}^m q_\varrho(2a - x_\varrho)$ and $\sum_{\rho=1}^n p_\rho \in \mathcal{I}_2$. Then,

$$\tilde{J}(f(\cdot)) = \sum_{\varrho=1}^m q_\varrho f(2a - x_\varrho) - f\left(\sum_{\varrho=1}^m q_\varrho(2a - x_\varrho)\right) - \sum_{\rho=1}^n p_\rho f(x_\rho) + f\left(\sum_{\rho=1}^n p_\rho x_\rho\right). \quad (45)$$

Theorem 2.6. Assume $\mathcal{H}$. Let $f \in C^k[0, 2a]$ ($k \geq 3$) be such that $f^{(k-1)}$ is absolutely continuous. Then, we have the following new identities for $t = 1, 2$ and $0 \leq \zeta_1 < \zeta_2 \leq 2a$:

(i)

$$\begin{aligned} \tilde{J}(f(\cdot)) &= \frac{1}{2} \left[\sum_{\varrho=1}^m q_\varrho (2a - x_\varrho)^2 - \left(\sum_{\varrho=1}^m q_\varrho (2a - x_\varrho) \right)^2 - \sum_{\rho=1}^n p_\rho x_\rho^2 + \left(\sum_{\rho=1}^n p_\rho x_\rho \right)^2 \right] f^{(2)}(\zeta_2) + 2 \left(\frac{f'(\zeta_1) - f'(\zeta_2)}{\zeta_2 - \zeta_1} \right) \\ &\quad \times \int_{\zeta_1}^{\zeta_2} \tilde{J}(G_t(\cdot, s)) ds + \int_{\zeta_1}^{\zeta_2} \tilde{J}(G_t(\cdot, s)) \times \sum_{l=3}^{k-1} \frac{l(l-1)}{(l-1)!} \left[\frac{f^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - f^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] ds \\ &\quad + \frac{1}{(k-4)!} \int_{\zeta_1}^{\zeta_2} f^{(k)}(v) \left(\int_{\zeta_1}^{\zeta_2} \tilde{J}(G_t(\cdot, s)) \tilde{R}_{k-3}(s, v) ds \right) dv, \end{aligned} \quad (46)$$

where

$$\tilde{R}_{k-3}(s, v) = \begin{cases} 2 \frac{(s-v)^{k-3}}{(k-3)(\zeta_2 - \zeta_1)} + \frac{s - \zeta_1}{\zeta_2 - \zeta_1} ((s-v)^{k-4}), & \zeta_1 \leq v \leq s; \\ 2 \frac{(s-v)^{k-3}}{(k-3)(\zeta_2 - \zeta_1)} + \frac{s - \zeta_2}{\zeta_2 - \zeta_1} ((s-v)^{k-4}), & s < v \leq \zeta_2, \end{cases}$$

and

$$\tilde{J}(G_t(\cdot, s)) = \sum_{\varrho=1}^m q_{\varrho} G_t(2a - x_{\varrho}, s) - G_t \left(\sum_{\varrho=1}^m q_{\varrho} (2a - x_{\varrho}, s) \right) - \sum_{\rho=1}^n p_{\rho} G_t(x_{\rho}, s) + G_t \left(\sum_{\rho=1}^n p_{\rho} x_{\rho}, s \right), \quad (47)$$

for $t = 1, 2$, $G_t(\cdot, s)$ is defined in (14) and (16), respectively.

(ii)

$$\begin{aligned} \tilde{J}(\mathfrak{f}(\cdot)) &= \frac{1}{2} \left[\sum_{\varrho=1}^m q_{\varrho} (2a - x_{\varrho})^2 - \left(\sum_{\varrho=1}^m q_{\varrho} (2a - x_{\varrho}) \right)^2 - \sum_{\rho=1}^n p_{\rho} x_{\rho}^2 + \left(\sum_{\rho=1}^n p_{\rho} x_{\rho} \right)^2 \right] f^{(2)}(\zeta_2) \\ &\quad + 2 \left(\frac{\mathfrak{f}''(\zeta_2) - \mathfrak{f}''(\zeta_1)}{\zeta_2 - \zeta_1} \right) \int_{\zeta_1}^{\zeta_2} \tilde{J}(G_t(\cdot, s)) ds \\ &\quad + \int_{\zeta_1}^{\zeta_2} \tilde{J}(G_t(\cdot, s)) \sum_{l=4}^{k-1} \frac{1}{(l-2)(l-4)!} \left[\frac{\mathfrak{f}^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - \mathfrak{f}^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] ds \\ &\quad + \frac{1}{(k-4)!} \int_{\zeta_1}^{\zeta_2} \mathfrak{f}^{(k)}(v) \left(\int_{\zeta_1}^{\zeta_2} \tilde{J}(G_t(\cdot, s)) R_{k-3}(s, v) ds \right) dv. \end{aligned} \quad (48)$$

Proof. Replace $\mathfrak{I}_1$ with $\mathfrak{I}_2$ and y_{ϱ} with $(2a - x_{\varrho})$ in Theorem 2.1, we get the required result. $\square$

As an application of the above obtained identities (45) and (47), the next Theorem gives generalization of Levinson's inequality (for real weights) for k -convex ($k \geq 3$) functions involving new Green functions.

Theorem 2.7. Assuming the conditions of Theorem 2.6 with $\mathfrak{f}$ as the k -convex function, then for $t = 1, 2$ we have the following two results:

If

$$\int_{\zeta_1}^{\zeta_2} \tilde{J}(G_t(\cdot, s)) \tilde{R}_{k-3}(s, v) ds \geq 0, \quad v \in \mathfrak{I}_2, \quad (49)$$

then

$$\begin{aligned} \tilde{J}(\mathfrak{f}(\cdot)) &\geq \frac{1}{2} \left[\sum_{\varrho=1}^m q_{\varrho} (2a - x_{\varrho})^2 - \left(\sum_{\varrho=1}^m q_{\varrho} (2a - x_{\varrho}) \right)^2 - \sum_{\rho=1}^n p_{\rho} x_{\rho}^2 + \left(\sum_{\rho=1}^n p_{\rho} x_{\rho} \right)^2 \right] f^{(2)}(\zeta_2) \\ &\quad + 2 \left(\frac{\mathfrak{f}''(\zeta_1) - \mathfrak{f}''(\zeta_2)}{\zeta_2 - \zeta_1} \right) \int_{\zeta_1}^{\zeta_2} \tilde{J}(G_t(\cdot, s)) ds \\ &\quad + \int_{\zeta_1}^{\zeta_2} \tilde{J}(G_t(\cdot, s)) \sum_{l=3}^{k-1} \frac{l(l-1)}{(l-1)!} \left[\frac{\mathfrak{f}^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - \mathfrak{f}^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] ds, \end{aligned} \quad (50)$$

and if

$$\int_{\zeta_1}^{\zeta_2} \tilde{J}(G_t(\cdot, s)) R_{k-3}(s, v) ds \geq 0, \quad v \in \mathfrak{I}_1, \quad (51)$$

then

$$\begin{aligned} \tilde{J}(\mathfrak{f}(\cdot)) &\geq \frac{1}{2} \left[\sum_{\varrho=1}^m q_{\varrho} (2a - x_{\varrho})^2 - \left(\sum_{\varrho=1}^m q_{\varrho} (2a - x_{\varrho}) \right)^2 - \sum_{\rho=1}^n p_{\rho} x_{\rho}^2 + \left(\sum_{\rho=1}^n p_{\rho} x_{\rho} \right)^2 \right] f^{(2)}(\zeta_2) \\ &\quad + 2 \left(\frac{\mathfrak{f}''(\zeta_2) - \mathfrak{f}''(\zeta_1)}{\zeta_2 - \zeta_1} \right) \int_{\zeta_1}^{\zeta_2} \tilde{J}(G_t(\cdot, s)) ds \\ &\quad + \int_{\zeta_1}^{\zeta_2} \tilde{J}(G_t(\cdot, s)) \sum_{l=4}^{k-1} \frac{1}{(l-2)(l-4)!} \left[\frac{\mathfrak{f}^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - \mathfrak{f}^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] ds. \end{aligned} \quad (52)$$

Proof. Proof is similar to that of Theorem 2.2.

If we put $m = n$, $p_{\rho} = q_{\varrho}$ and use positive weights in (44), then $\tilde{J}(\cdot)$ converted to the functional $J_1(\cdot)$ defined in (2), also in this case (46), (47), (48), (49), (50), (51) and (52) become

$$\begin{aligned} J_1(\mathfrak{f}(\cdot)) &= 2 \left(\frac{\mathfrak{f}''(\zeta_1) - \mathfrak{f}''(\zeta_2)}{\zeta_2 - \zeta_1} \right) \int_{\zeta_1}^{\zeta_2} J_1(G_t(\cdot, s)) ds \\ &\quad + \int_{\zeta_1}^{\zeta_2} J_1(G_t(\cdot, s)) \sum_{l=3}^{k-1} \frac{l(l-1)}{(l-1)!} \left[\frac{\mathfrak{f}^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - \mathfrak{f}^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] ds \\ &\quad + \frac{1}{(k-4)!} \int_{\zeta_1}^{\zeta_2} \mathfrak{f}^{(k)}(v) \left(\int_{\zeta_1}^{\zeta_2} J_1(G_t(\cdot, s)) \tilde{R}_{k-3}(s, v) ds \right) dv, \end{aligned} \quad (53)$$

$$J_1(G_t(\cdot, s)) = \sum_{\rho=1}^n p_{\rho} G_t(2a - x_{\rho}, s) - G_t \left(\sum_{\rho=1}^n p_{\rho} 2a - x_{\rho}, s \right) - \sum_{\rho=1}^n p_{\rho} G_t(x_{\rho}, s) + G_t \left(\sum_{\rho=1}^n p_{\rho} x_{\rho}, s \right), \quad (54)$$

$$\begin{aligned} J_1(\mathfrak{f}(\cdot)) &= 2 \left(\frac{\mathfrak{f}''(\zeta_2) - \mathfrak{f}''(\zeta_1)}{\zeta_2 - \zeta_1} \right) \int_{\zeta_1}^{\zeta_2} J_1(G_t(\cdot, s)) ds \\ &\quad + \int_{\zeta_1}^{\zeta_2} J_1(G_t(\cdot, s)) \sum_{l=4}^{k-1} \frac{1}{(l-2)(l-4)!} \left[\frac{\mathfrak{f}^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - \mathfrak{f}^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] ds \\ &\quad + \frac{1}{(k-4)!} \int_{\zeta_1}^{\zeta_2} \mathfrak{f}^{(k)}(v) \left(\int_{\zeta_1}^{\zeta_2} J_1(G_t(\cdot, s)) R_{k-3}(s, v) ds \right) dv, \end{aligned} \quad (55)$$

$$\int_{\zeta_1}^{\zeta_2} J_1(G_t(\cdot, s)) \tilde{R}_{k-3}(s, v) ds \geq 0, \quad v \in \mathcal{J}_1 \quad (56)$$

$$\begin{aligned} J_1(\mathfrak{f}(\cdot)) &\geq 2 \left(\frac{\mathfrak{f}''(\zeta_1) - \mathfrak{f}''(\zeta_2)}{\zeta_2 - \zeta_1} \right) \int_{\zeta_1}^{\zeta_2} J_1(G_t(\cdot, s)) ds \\ &\quad + \int_{\zeta_1}^{\zeta_2} J_1(G_t(\cdot, s)) \sum_{l=3}^{k-1} \frac{l(l-1)}{(l-1)!} \left[\frac{\mathfrak{f}^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - \mathfrak{f}^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] ds, \end{aligned} \quad (57)$$

$$\int_{\zeta_1}^{\zeta_2} J_1(G_t(\cdot, s)) R_{k-3}(s, v) ds \geq 0, \quad v \in \mathcal{I}_1 \quad (58)$$

and

$$\begin{aligned} J_1(f(\cdot)) &\geq 2 \left(\frac{f''(\zeta_2) - f''(\zeta_1)}{\zeta_2 - \zeta_1} \right) \int_{\zeta_1}^{\zeta_2} J_1(G_t(\cdot, s)) ds \\ &\quad + \int_{\zeta_1}^{\zeta_2} J_1(G_t(\cdot, s)) \sum_{l=4}^{k-1} \frac{1}{(l-2)(l-4)!} \left[\frac{f^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - f^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] ds. \end{aligned} \quad (59)$$

□

Theorem 2.8. Let $f \in C^k[\zeta_1, \zeta_2]$ be such that $f^{(k-1)}$ is absolutely continuous. Also, let $(p_1, \dots, p_n)$ be positive real numbers such that $\sum_{\rho=1}^n p_\rho = 1$. Then, for the functional $J_1(\cdot)$ defined in (2), we have the following:

- (i) For $t = 1, 2$, inequalities (57) and (59) hold, provided that n is odd.
- (ii) For fixed $k = 1, 2$, let inequality (57) be satisfied and

$$(f''(\zeta_1) - f''(\zeta_2)) + \sum_{l=3}^{k-1} \frac{l(l-1)}{(l-1)!} \left[\frac{f^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - f^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] \geq 0 \quad \forall s \in \mathcal{I}_2, \quad (60)$$

or (59) be satisfied and

$$\sum_{l=4}^{k-1} \frac{1}{(l-2)(l-4)!} \left[\frac{f^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - f^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right]. \quad (61)$$

Then,

$$J_1(f(\cdot)) \geq 0.$$

Proof. Proof is similar to that of Theorem 2.3. □

3 Bounds for identities related to generalization of Levinson's inequality

For two Lebesgue integrable functions $f_1, f_2: [\zeta_1, \zeta_2] \rightarrow \mathbb{R}$, we consider the Čebyšev functional

$$\Theta(f_1, f_2) = \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} f_1(t) f_2(t) dt - \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} f_1(t) dt \cdot \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} f_2(t) dt, \quad (62)$$

where the integrals are assumed to exist.

Theorem 3.1. [23] Let $f_1: [\zeta_1, \zeta_2] \rightarrow \mathbb{R}$ be a Lebesgue integrable function and $f_2: [\zeta_1, \zeta_2] \rightarrow \mathbb{R}$ be an absolutely continuous function with $(\cdot, -\zeta_1)(\cdot, -\zeta_2)[f_2']^2 \in L[\zeta_1, \zeta_2]$. Then, we have the inequality

$$|\Theta(f_1, f_2)| \leq \frac{1}{\sqrt{2}} [\Theta(f_1, f_1)]^{\frac{1}{2}} \frac{1}{\sqrt{\zeta_2 - \zeta_1}} \left(\int_{\zeta_1}^{\zeta_2} (x - \zeta_1)(\zeta_2 - x) [f_2'(x)]^2 dx \right)^{\frac{1}{2}}. \quad (63)$$

The constant $\frac{1}{\sqrt{2}}$ is the best possible value.

Theorem 3.2. [23] Let $f_1: [\zeta_1, \zeta_2] \rightarrow \mathbb{R}$ be an absolutely continuous with $f_1' \in L_\infty[\zeta_1, \zeta_2]$ and let $f_2: [\zeta_1, \zeta_2] \rightarrow \mathbb{R}$ be monotonic non-decreasing on $[\zeta_1, \zeta_2]$. Then, we have the inequality

$$|\Theta(f_1, f_2)| \leq \frac{1}{2(\zeta_2 - \zeta_1)} \|f'_1\|_\infty \int_{\zeta_1}^{\zeta_2} (x - \zeta_1)(\zeta_2 - x) df_2(x). \quad (64)$$

The constant $\frac{1}{2}$ is the best possible value.

Now we consider Theorems 3.1 and 3.2 to generalize results given in the previous section. In order to avoid many notions let us denote

$$\widetilde{\mathcal{R}}_t(v) = \int_{\zeta_1}^{\zeta_2} \widetilde{J}(G_t(\cdot, s)) \widetilde{R}_{k-3}(s, v) ds, \quad v \in [\zeta_1, \zeta_2], \quad (65)$$

and

$$\mathcal{R}_t(v) = \int_{\zeta_1}^{\zeta_2} J(G_t(\cdot, s)) R_{k-3}(s, v) ds, \quad v \in [\zeta_1, \zeta_2]. \quad (66)$$

First we intend to give the Ostrowski-type inequalities related to generalizations of the Bullen inequality.

Theorem 3.3. Assuming the conditions of Theorem 2.1, moreover, assume (r, s) is a pair of conjugate exponents that is $1 \leq r, s \leq \infty$ such that $\frac{1}{r} + \frac{1}{s} = 1$. Let $|f^{(k)}|^r: [\zeta_1, \zeta_2] \rightarrow \mathbb{R}$ be the Riemann integrable function. Then,

$$\begin{aligned} & \left| \widetilde{J}(f(\cdot)) - \frac{1}{2} \left[\sum_{q=1}^m q_{\mathbb{Q}} y_{\mathbb{Q}}^2 - \left(\sum_{q=1}^m q_{\mathbb{Q}} y_{\mathbb{Q}} \right)^2 - \sum_{\rho=1}^n p_{\rho} x_{\rho}^2 + \left(\sum_{\rho=1}^n p_{\rho} x_{\rho} \right)^2 \right] f^{(2)}(\zeta_2) - 2 \left(\frac{f''(\zeta_1) - f''(\zeta_2)}{\zeta_2 - \zeta_1} \right) \int_{\zeta_1}^{\zeta_2} \widetilde{J}(G_t(\cdot, s)) ds \right. \\ & \quad \left. - \int_{\zeta_1}^{\zeta_2} \widetilde{J}(G_t(\cdot, s)) \sum_{l=3}^{k-1} \frac{l(l-1)}{(l-1)!} \left[\frac{f^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - f^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] ds \right| \\ & \leq \frac{1}{(k-4)!} \|f^{(k)}\|_r \left(\int_{\zeta_1}^{\zeta_2} \left| \int_{\zeta_1}^{\zeta_2} \widetilde{J}(G_t(\cdot, s)) \widetilde{R}_{k-3}(s, v) ds \right|^s dv \right)^{\frac{1}{s}}, \end{aligned} \quad (67)$$

and

$$\begin{aligned} & \left| \widetilde{J}(f(\cdot)) - \frac{1}{2} \left[\sum_{q=1}^m q_{\mathbb{Q}} y_{\mathbb{Q}}^2 - \left(\sum_{q=1}^m q_{\mathbb{Q}} y_{\mathbb{Q}} \right)^2 - \sum_{\rho=1}^n p_{\rho} x_{\rho}^2 + \left(\sum_{\rho=1}^n p_{\rho} x_{\rho} \right)^2 \right] f^{(2)}(\zeta_2) - 2 \left(\frac{f''(\zeta_2) - f''(\zeta_1)}{\zeta_2 - \zeta_1} \right) \int_{\zeta_1}^{\zeta_2} \widetilde{J}(G_t(\cdot, s)) ds \right. \\ & \quad \left. - \int_{\zeta_1}^{\zeta_2} \widetilde{J}(G_t(\cdot, s)) \sum_{l=3}^{k-1} \frac{l(l-1)}{(l-1)!} \left[\frac{f^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - f^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] ds \right| \\ & \leq \frac{1}{(k-4)!} \|f^{(k)}\|_r \left(\int_{\zeta_1}^{\zeta_2} \left| \int_{\zeta_1}^{\zeta_2} \widetilde{J}(G_t(\cdot, s)) R_{k-3}(s, v) ds \right|^s dv \right)^{\frac{1}{s}}. \end{aligned} \quad (68)$$

The constants on the right-hand side (R.H.S.) of (67) and (68) are sharp for $1 < r \leq \infty$ and best possible $r = 1$.

Proof. Fix $t = 1, 2$. Rearrange identity (19) in such a way

$$\left| \mathcal{J}(f(\cdot)) - \frac{1}{2} \left[\sum_{\varrho=1}^m q_{\varrho} y_{\varrho}^2 - \left(\sum_{\varrho=1}^m q_{\varrho} y_{\varrho} \right)^2 - \sum_{\rho=1}^n p_{\rho} x_{\rho}^2 + \left(\sum_{\rho=1}^n p_{\rho} x_{\rho} \right)^2 \right] f^{(2)}(\zeta_2) - 2 \left(\frac{f''(\zeta_1) - f''(\zeta_2)}{\zeta_2 - \zeta_1} \right) \int_{\zeta_1}^{\zeta_2} \mathcal{J}(G_t(\cdot, s)) ds \right. \\ \left. - \int_{\zeta_1}^{\zeta_2} \mathcal{J}(G_t(\cdot, s)) \sum_{l=3}^{k-1} \frac{l(l-1)}{(l-1)!} \left[\frac{f^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - f^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] ds \right| \leq \frac{1}{(k-4)!} \left| \int_{\zeta_1}^{\zeta_2} \widetilde{\mathcal{R}}_t(v) f^{(k)}(v) dv \right|. \quad (69)$$

Employing the classical Holder's inequality to R.H.S. of (68) yields (66). The proof for sharpness is similar to Theorem 3.5 given in [13].

The proof of (67) is similar to that of (68). $\square$

The Grüss-type inequalities can be obtained by using Theorem 3.2.

Theorem 3.4. Assume $\mathcal{F}$. Let $f \in C^k[\zeta_1, \zeta_2]$ and $f^{(k)}$ be an absolutely continuous with $(\cdot - \zeta_1)(\zeta_2 - \cdot)[f^{(k+1)}]^2 \in L[\zeta_1, \zeta_2]$. Also, $\widetilde{\mathcal{R}}_t(v)$ and $\mathcal{R}_t(v)$ are defined in (65) and (66), respectively. Then, we have

$$\mathcal{J}(f(\cdot)) - \frac{1}{2} \left[\sum_{\varrho=1}^m q_{\varrho} y_{\varrho}^2 - \left(\sum_{\varrho=1}^m q_{\varrho} y_{\varrho} \right)^2 - \sum_{\rho=1}^n p_{\rho} x_{\rho}^2 + \left(\sum_{\rho=1}^n p_{\rho} x_{\rho} \right)^2 \right] f^{(2)}(\zeta_2) - 2 \left(\frac{f''(\zeta_1) - f''(\zeta_2)}{\zeta_2 - \zeta_1} \right) \int_{\zeta_1}^{\zeta_2} \mathcal{J}(G_t(\cdot, s)) ds \\ - \int_{\zeta_1}^{\zeta_2} \mathcal{J}(G_t(\cdot, s)) \sum_{l=3}^{k-1} \frac{l(l-1)}{(l-1)!} \left[\frac{f^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - f^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] ds \\ - \frac{f^{(k-1)}(\zeta_2) - f^{(k-1)}(\zeta_1)}{(\zeta_2 - \zeta_1)(k-4)!} \int_{\zeta_1}^{\zeta_2} \widetilde{\mathcal{R}}_t(v) dv = \mathfrak{R}_p(\zeta_1, \zeta_2, \widetilde{\mathcal{R}}_t, f^{(k)}), \quad (70)$$

and

$$\mathcal{J}(f(\cdot)) - \frac{1}{2} \left[\sum_{\varrho=1}^m q_{\varrho} y_{\varrho}^2 - \left(\sum_{\varrho=1}^m q_{\varrho} y_{\varrho} \right)^2 - \sum_{\rho=1}^n p_{\rho} x_{\rho}^2 + \left(\sum_{\rho=1}^n p_{\rho} x_{\rho} \right)^2 \right] f^{(2)}(\zeta_2) - 2 \left(\frac{f''(\zeta_2) - f''(\zeta_1)}{\zeta_2 - \zeta_1} \right) \int_{\zeta_1}^{\zeta_2} \mathcal{J}(G_t(\cdot, s)) ds \\ - \int_{\zeta_1}^{\zeta_2} \mathcal{J}(G_t(\cdot, s)) \times \sum_{l=3}^{k-1} \frac{l(l-1)}{(l-1)!} \left[\frac{f^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - f^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] ds \\ - \frac{f^{(k-1)}(\zeta_2) - f^{(k-1)}(\zeta_1)}{(\zeta_2 - \zeta_1)(k-4)!} \int_{\zeta_1}^{\zeta_2} \mathcal{R}_t(v) dv = \mathfrak{R}_p(\zeta_1, \zeta_2, \mathcal{R}_t, f^{(k)}), \quad (71)$$

where the remainder $\mathfrak{R}_p(\zeta_1, \zeta_2, \widetilde{\mathcal{R}}_t, f^{(k)})$ satisfies the bound

$$|\mathfrak{R}_p(\zeta_1, \zeta_2, \widetilde{\mathcal{R}}_t, f^{(k)})| \leq \frac{(\zeta_2 - \zeta_1)^{\frac{1}{2}}}{\sqrt{2}(k-4)!} [\Theta(\widetilde{\mathcal{R}}_t(v), \widetilde{\mathcal{R}}_t(v))]^{\frac{1}{2}} \left| \int_{\zeta_1}^{\zeta_2} (v - \zeta_1)(\zeta_2 - v) [f^{(k+1)}(v)]^2 dv \right|^{\frac{1}{2}}, \quad (72)$$

and

$$|\mathfrak{R}_p(\zeta_1, \zeta_2, \mathcal{R}_t, f^{(k)})| \leq \frac{(\zeta_2 - \zeta_1)^{\frac{1}{2}}}{\sqrt{2}(k-4)!} [\Theta(\mathcal{R}_t(v), \mathcal{R}_t(v))]^{\frac{1}{2}} \left| \int_{\zeta_1}^{\zeta_2} (v - \zeta_1)(\zeta_2 - v) [f^{(k+1)}(v)]^2 dv \right|^{\frac{1}{2}}, \quad (73)$$

respectively.

Proof. Fix $t = 1, 2$. Using the Čebyšev functional for $f_1 = \widetilde{\mathcal{R}}_t$ and $f_2 = f^{(k)}$, by comparing (70) with (19), we have

$$\mathfrak{R}_p(\zeta_1, \zeta_2, \widetilde{\mathcal{R}}_t, f^{(k)}) = \frac{\zeta_2 - \zeta_1}{(k-4)!} \Theta(\widetilde{\mathcal{R}}_t, f^{(k)}).$$

Now applying Theorem 3.1 for the corresponding functions, we get the required bound.

Similarly, by comparing (71) with identity (22), we get the respective bound. $\square$

Theorem 3.5. Assume $\mathcal{F}$. Let $f \in C^k[\zeta_1, \zeta_2]$ and $f^{(k)}$ be an absolutely continuous and $f^{(k+1)} \geq 0$ on $[\zeta_1, \zeta_2]$ with $\widetilde{\mathcal{R}}_t(v)$ and $\mathcal{R}_t(v)$ defined in (65) and (66), respectively. Then, in representation (70) the remainder $\mathfrak{R}_p(\zeta_1, \zeta_2, \widetilde{\mathcal{R}}_t, f^{(k)})$ satisfies the estimation

$$|\mathfrak{R}(\zeta_1, \zeta_2, \widetilde{\mathcal{R}}_t, f^{(k)})| \leq \frac{(\zeta_2 - \zeta_1) \|\widetilde{\mathcal{R}}_t'\|_{\infty}}{(k-4)!} \left[\frac{f^{(k-1)}(\zeta_2) + f^{(k-1)}(\zeta_1)}{2} - \frac{f^{(k-2)}(\zeta_2) - f^{(k-2)}(\zeta_1)}{\zeta_2 - \zeta_1} \right], \quad (74)$$

whereas in representation (71) the remainder $\mathfrak{R}_p(\zeta_1, \zeta_2, \mathcal{R}_t, f^{(k)})$ satisfies the estimation

$$|\mathfrak{R}(\zeta_1, \zeta_2, \mathcal{R}_t, f^{(k)})| \leq \frac{(\zeta_2 - \zeta_1) \|\widetilde{\mathcal{R}}_t'\|_{\infty}}{(k-4)!} \left[\frac{f^{(k-1)}(\zeta_2) + f^{(k-1)}(\zeta_1)}{2} - \frac{f^{(k-2)}(\zeta_2) - f^{(k-2)}(\zeta_1)}{\zeta_2 - \zeta_1} \right]. \quad (75)$$

Proof. Fix $t = 1, 2$. We have established

$$\mathfrak{R}_p(\zeta_1, \zeta_2, \widetilde{\mathcal{R}}_t, f^{(k)}) = \frac{\zeta_2 - \zeta_1}{(k-4)!} \Theta(\widetilde{\mathcal{R}}_t, f^{(k)}).$$

Now setting $f_1 \mapsto \widetilde{\mathcal{R}}_t$ and $f_2 \mapsto f^{(k)}$ in Theorem 3.2, we get

$$|\mathfrak{R}(\zeta_1, \zeta_2, \mathcal{R}_t, f^{(k)})| = \frac{1}{(k-4)!} |\Theta(\widetilde{\mathcal{R}}_t, f^{(k)})| \leq \frac{\|\mathcal{R}_t'\|_{\infty}}{2(\zeta_2 - \zeta_1)(k-4)!} \int_{\zeta_1}^{\zeta_2} (v - \zeta_1)(\zeta_2 - v) f^{(k+1)}(v) dv. \quad (76)$$

Simplifying the integral on R.H.S. of (75) we get the required results. $\square$

Remark 3.1. Similar work can be done for Levinson's inequality (2) (one type of data points) for k -convex functions.

Remark 3.2. We can give related mean value theorems by using non-negative functionals (19), (22), (46) and (47), and we can construct the new families of k -exponentially convex functions and Cauchy means related to these functionals as given in Section 4 of [24].

4 Application to information theory

The idea of Shannon entropy is the central job of information speculation now and again implied as a measure of uncertainty. The entropy of random variable is described with respect to probability distribution and can be shown to be a decent measure of random. The assignment of Shannon entropy is to assess the typical least number of bits expected to encode a progression of pictures subject to the letters, including the size and the repetition of the symbols.

Divergences between probability distributions have been familiar with the measure of the difference between them. An assortment of sorts of divergences exists, for example, the f -divergences (especially, Kullback–Leibler divergences, Hellinger distance and total variation distance), Rényi divergences, Jensen–Shannon divergences, etc. (see [28,29]). There are a lot of papers overseeing inequalities and entropies, see, e.g., [8,25–27] and references therein. The Jensen inequality is an essential job in a bit of these inequalities. Regardless, Jensen's inequality manages one kind of data points and Levinson's inequality deals two types of data points.

4.1 Csiszár divergence

In [30,31], Csiszár provided the following definition.

Definition 4.1. Let f be a convex function from $\mathbb{R}^+$ to $\mathbb{R}^+$. Let $\tilde{\mathbf{r}}, \tilde{\mathbf{k}} \in \mathbb{R}_+^n$ be such that $\sum_{v=1}^n r_v = 1$ and $\sum_{v=1}^n k_v = 1$. Then, the f -divergence functional is defined by

$$I_f(\tilde{\mathbf{r}}, \tilde{\mathbf{k}}) := \sum_{v=1}^n k_v f\left(\frac{r_v}{k_v}\right).$$

By defining the following:

$$f(0) := \lim_{x \rightarrow 0^+} f(x); \quad of\left(\frac{0}{0}\right) := 0; \quad of\left(\frac{a}{x}\right) := \lim_{x \rightarrow 0^+} x f\left(\frac{a}{x}\right), \quad a > 0,$$

he stated that nonnegative probability distributions can also be used.

Using the definition of the f -divergence functional, Horváth et al. [32] provided the following functional.

Definition 4.2. Let I be an interval contained in $\mathbb{R}$ and $f: I \rightarrow \mathbb{R}$ be a function. Also, let $\tilde{\mathbf{r}} = (r_1, \dots, r_n) \in \mathbb{R}^n$ and $\tilde{\mathbf{k}} = (k_1, \dots, k_n) \in (0, \infty)^n$ be such that

$$\frac{r_v}{k_v} \in I, \quad v = 1, \dots, n.$$

Then,

$$\hat{I}_f(\tilde{\mathbf{r}}, \tilde{\mathbf{k}}) := \sum_{v=1}^n k_v f\left(\frac{r_v}{k_v}\right). \quad (77)$$

We apply Theorem 2.2 for k -convex functions to $\hat{I}_f(\tilde{\mathbf{r}}, \tilde{\mathbf{k}})$.

Theorem 4.1. Let $\tilde{\mathbf{r}} = (r_1, \dots, r_n) \in \mathbb{R}^n$, $\tilde{\mathbf{w}} = (w_1, \dots, w_m) \in \mathbb{R}^m$, $\tilde{\mathbf{k}} = (k_1, \dots, k_n) \in (0, \infty)^n$ and $\tilde{\mathbf{t}} = (t_1, \dots, t_m) \in (0, \infty)^m$ such that

$$\frac{r_v}{k_v} \in I, \quad v = 1, \dots, n,$$

and

$$\frac{w_u}{t_u} \in I, \quad u = 1, \dots, m.$$

Also, let $f \in C^k[\zeta_1, \zeta_2]$ be such that f is the k -convex function. Then,

(i) For $k = 5, 7, \dots$

$$\begin{aligned} J_{cis}(f(\cdot)) &\geq \frac{1}{2} \left[\frac{1}{\sum_{u=1}^m t_u} \sum_{u=1}^m \frac{(w_u)^2}{t_u} - \left(\sum_{u=1}^m \frac{w_u}{\sum_{u=1}^m t_u} \right)^2 - \frac{1}{\sum_{v=1}^n k_v} \sum_{v=1}^n \frac{(r_v)^2}{k_v} + \left(\sum_{v=1}^n \frac{r_v}{\sum_{v=1}^n k_v} \right)^2 \right] f^{(2)}(\zeta_2) \\ &\quad + 2 \left(\frac{f'(\zeta_1) - f'(\zeta_2)}{\zeta_2 - \zeta_1} \right) \int_{\zeta_1}^{\zeta_2} J(G_t(\cdot, s)) ds \\ &\quad + \int_{\zeta_1}^{\zeta_2} J(G_t(\cdot, s)) \sum_{l=3}^{k-1} \frac{l(l-1)}{(l-1)!} \left[\frac{f^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - f^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] ds \end{aligned} \quad (78)$$

and

$$\begin{aligned}
J_{\text{cis}}(f(\cdot)) &\geq \frac{1}{2} \left[\frac{1}{\sum_{u=1}^m t_u} \sum_{u=1}^m \frac{(w_u)^2}{t_u} - \left(\sum_{u=1}^m \frac{w_u}{\sum_{u=1}^m t_u} \right)^2 - \frac{1}{\sum_{v=1}^n k_v} \sum_{v=1}^n \frac{(r_v)^2}{k_v} + \left(\sum_{v=1}^n \frac{r_v}{\sum_{v=1}^n k_v} \right)^2 \right] f^{(2)}(\zeta_2) \\
&\quad + 2 \left(\frac{f''(\zeta_2) - f''(\zeta_1)}{\zeta_2 - \zeta_1} \right) \int_{\zeta_1}^{\zeta_2} J(G_t(\cdot, s)) ds \\
&\quad + \int_{\zeta_1}^{\zeta_2} J(G_t(\cdot, s)) \sum_{l=4}^{k-1} \frac{1}{(l-2)(l-4)!} \left[\frac{f^{(l)}(\zeta_1)(s - \zeta_1)^{l-2} - f^{(l)}(\zeta_2)(s - \zeta_2)^{l-2}}{\zeta_2 - \zeta_1} \right] ds,
\end{aligned} \tag{79}$$

where

$$J_{\text{cis}}(f(\cdot)) = \frac{1}{\sum_{u=1}^m t_u} \hat{I}_f(\tilde{\mathbf{w}}, \tilde{\mathbf{t}}) - \left(\sum_{u=1}^m \frac{w_u}{\sum_{u=1}^m t_u} \right) - \frac{1}{\sum_{v=1}^n k_v} \hat{I}_f(\tilde{\mathbf{r}}, \tilde{\mathbf{k}}) + \left(\sum_{v=1}^n \frac{r_v}{\sum_{v=1}^n k_v} \right) \tag{80}$$

and

$$J(G_t(\cdot, s)) = \sum_{\rho=1}^m \frac{t_\rho}{\sum_{u=1}^m t_u} G_t\left(\frac{w_\rho}{t_\rho}, s\right) - G_t\left(\sum_{u=1}^m \frac{w_u}{\sum_{u=1}^m t_u}, s\right) - \sum_{\rho=1}^n \frac{k_\rho}{\sum_{v=1}^n k_v} G_t\left(\frac{r_\rho}{k_\rho}, s\right) + G_t\left(\sum_{v=1}^n \frac{r_v}{\sum_{v=1}^n k_v}, s\right). \tag{81}$$

(ii) For $k = 4, 6, \dots$, (78) and (79) hold in the reverse direction.

Proof. It is clear that Green functions $G_t(\cdot, s)$ defined in (14) and (16) are 3-convex functions, therefore, by using Remark 1.1, $J(G_t(\cdot, s)) \geq 0$ for fixed $t = 1, 2$.

(i) From (20) and (23), $\tilde{R}_k(s, v) \geq 0$ and $R_k(s, v) \geq 0$ for $k = 5, 7, \dots$, so (27) and (29) hold. Hence, using $p_\rho = \frac{k_v}{\sum_{v=1}^n k_v}$, $x_\rho = \frac{r_v}{k_v}$, $q_\rho = \frac{t_u}{\sum_{u=1}^m t_u}$, $y_\rho = \frac{w_u}{t_u}$ in Theorem 2.2, (28) and (30) become (78) and (79), respectively, where $\hat{I}_f(\tilde{\mathbf{r}}, \tilde{\mathbf{k}})$ is defined in (77) and

$$\hat{I}_f(\tilde{\mathbf{w}}, \tilde{\mathbf{t}}) := \sum_{u=1}^m t_u f\left(\frac{w_u}{t_u}\right). \tag{82}$$

(ii) From (20) and (23), $\tilde{R}_k(s, v) \leq 0$ and $R_k(s, v) \leq 0$ for $k = 4, 6, \dots$, so inequalities in (27) and (29) are reversed. Therefore, by using Remark 2.1, (28) and (30) hold in the reverse direction. Using the same substitutions as in (i), we get inequalities (78) and (79) in the reverse direction. $\square$

4.2 Shannon entropy

Definition 4.3. (see [32]). The Shannon entropy of the positive probability distribution $\tilde{\mathbf{k}} = (k_1, \dots, k_n)$ is defined by

$$S := - \sum_{v=1}^n k_v \log(k_v). \tag{83}$$

Corollary 4.1. Let $\tilde{\mathbf{k}} = (k_1, \dots, k_n)$ and $\tilde{\mathbf{t}} = (t_1, \dots, t_m)$ be positive probability distributions. Also, let $\tilde{\mathbf{r}} = (r_1, \dots, r_n) \in (0, \infty)^n$ and $\tilde{\mathbf{w}} = (w_1, \dots, w_m) \in (0, \infty)^m$.

(i) If base of log is greater than 1 and $k = \text{odd}(k = 3, 5, \dots)$, then

$$\begin{aligned}
J_S(\cdot) &\geq - \frac{1}{2(\zeta_2)^2} \left[\frac{1}{\sum_{u=1}^m t_u} \sum_{u=1}^m \frac{(w_u)^2}{t_u} - \left(\sum_{u=1}^m \frac{w_u}{\sum_{u=1}^m t_u} \right)^2 - \frac{1}{\sum_{v=1}^n k_v} \sum_{v=1}^n \frac{(r_v)^2}{k_v} + \left(\sum_{v=1}^n \frac{r_v}{\sum_{v=1}^n k_v} \right)^2 \right] \\
&\quad - \frac{2(\zeta_1 + \zeta_2)}{(\zeta_1 \zeta_2)^2} \int_{\zeta_1}^{\zeta_2} J(G_t(\cdot, s)) ds + \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} J(G_t(\cdot, s)) \sum_{l=3}^{k-1} (-1)^{l-1} (l-1) \left[\frac{(s - \zeta_1)^{l-2}}{(\zeta_1)^l} - \frac{(s - \zeta_2)^{l-2}}{(\zeta_2)^l} \right] ds
\end{aligned} \tag{84}$$

and

$$J_s(\cdot) \geq -\frac{1}{2(\zeta_2)^2} \left[\frac{1}{\sum_{u=1}^m t_u} \sum_{u=1}^m \frac{(w_u)^2}{t_u} - \left(\sum_{u=1}^m \frac{w_u}{\sum_{u=1}^m t_u} \right)^2 - \frac{1}{\sum_{v=1}^n k_v} \sum_{v=1}^n \frac{(r_v)^2}{k_v} + \left(\sum_{v=1}^n \frac{r_v}{\sum_{v=1}^n k_v} \right)^2 \right] \\ - \frac{(\zeta_1 + \zeta_2)}{(\zeta_1 \zeta_2)^2} \int_{\zeta_1}^{\zeta_2} J(G_t(\cdot, s)) ds + \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} J(G_t(\cdot, s)) \sum_{l=4}^{k-1} (-1)^{l-1} (l-1)(l-3) \left[\frac{(s-\zeta_1)^{l-2}}{(\zeta_1)^l} - \frac{(s-\zeta_2)^{l-2}}{(\zeta_2)^l} \right] ds, \quad (85)$$

where

$$J_s(\cdot) = \sum_{u=1}^m t_u \log(w_u) + \tilde{S} - \log \left(\sum_{u=1}^m w_u \right) - \sum_{v=1}^n k_v \log(r_v) + S + \log \left(\sum_{v=1}^n r_v \right) \quad (86)$$

and $J(G_t(\cdot, s))$ is defined in (81).

(ii) If base of log is less than 1 or k is even ($k = 4, 6, \dots$), then inequalities (84) and (85) are reversed.

Proof. (i) The function $f \rightarrow \log(x)$ is k -convex for $k = 3, 5, \dots$ and base of log is greater than 1. Therefore, using $f = \log(x)$ in Theorem 4.1 (i), we get (84) and (85), where S is defined in (83) and

$$\tilde{S} = - \sum_{u=1}^m t_u \log(t_u).$$

(ii) The function $f \rightarrow \log(x)$ is k -concave for $k = 4, 6, \dots$, therefore using $f = \log(x)$ in Theorem 4.1 (ii), we get (84) and (85) in the reverse direction. $\square$

Corollary 4.2. Let $\tilde{r} = (r_1, \dots, r_n)$ and $\tilde{w} = (w_1, \dots, w_m)$ be positive probability distributions. Also, let $\tilde{k} = (k_1, \dots, k_n) \in (0, \infty)^n$ and $\tilde{t} = (t_1, \dots, t_m) \in (0, \infty)^m$. If base of log is greater than 1 and k is even ($k \geq 4$), then

$$J_s(\cdot) \leq -\frac{1}{2\zeta_2} \left[\frac{1}{\sum_{u=1}^m t_u} \sum_{u=1}^m \frac{(w_u)^2}{t_u} - \left(\sum_{u=1}^m \frac{w_u}{\sum_{u=1}^m t_u} \right)^2 - \frac{1}{\sum_{v=1}^n k_v} \sum_{v=1}^n \frac{(r_v)^2}{k_v} + \left(\sum_{v=1}^n \frac{r_v}{\sum_{v=1}^n k_v} \right)^2 \right] \\ + \left(\frac{-2}{\zeta_1 \zeta_2} \right) \int_{\zeta_1}^{\zeta_2} J(G_t(\cdot, s)) ds + \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} J(G_t(\cdot, s)) \sum_{l=3}^{k-1} (-1)^l l \left[\frac{(s-\zeta_1)^{l-2}}{(\zeta_1)^{l-1}} - \frac{(s-\zeta_2)^{l-2}}{(\zeta_2)^{l-1}} \right] ds \quad (87)$$

and

$$J_s(\cdot) \leq -\frac{1}{2(\zeta_2)^2} \left[\frac{1}{\sum_{u=1}^m t_u} \sum_{u=1}^m \frac{(w_u)^2}{t_u} - \left(\sum_{u=1}^m \frac{w_u}{\sum_{u=1}^m t_u} \right)^2 - \frac{1}{\sum_{v=1}^n k_v} \sum_{v=1}^n \frac{(r_v)^2}{k_v} + \left(\sum_{v=1}^n \frac{r_v}{\sum_{v=1}^n k_v} \right)^2 \right] \\ + \left(\frac{2}{\zeta_1 \zeta_2} \right) \int_{\zeta_1}^{\zeta_2} J(G_t(\cdot, s)) ds + \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} J(G_t(\cdot, s)) \sum_{l=4}^{k-1} (-1)^l (l-3) \left[\frac{(s-\zeta_1)^{l-2}}{(\zeta_1)^{l-1}} - \frac{(s-\zeta_2)^{l-2}}{(\zeta_2)^{l-1}} \right] ds, \quad (88)$$

where

$$J_s(\cdot) = \frac{1}{\sum_{u=1}^m t_u} \left(\tilde{S} + \sum_{u=1}^m w_u \log(t_u) \right) - \frac{1}{\sum_{u=1}^m t_u} \log \left(\sum_{u=1}^m t_u \right) - \frac{1}{\sum_{v=1}^n k_v} \left(S + \sum_{v=1}^n r_v \log(k_v) \right) \\ + \frac{1}{\sum_{v=1}^n k_v} \log \left(\sum_{v=1}^n k_v \right) \quad (89)$$

and $J(G_t(\cdot, s))$ is defined in (81).

Proof. The function $f \rightarrow -x \log(x)$ is k -convex ($k = 4, 6, \dots$) and base of $\log$ is greater than 1. Therefore, using $f = -x \log(x)$ in Theorem 4.1 (ii), we get (87) and (88), where

$$\tilde{\mathfrak{S}} = - \sum_{u=1}^m w_u \log(w_u)$$

and

$$\mathfrak{S} = - \sum_{v=1}^n r_v \log(r_v). \quad \square$$

Remark 4.1. If base is less than 1 or $k = \text{odd}$ ($k \geq 3$), then inequalities (87) and (88) are also valid.

5 Conclusion

This study is concerned with generalization of the Levinson-type inequalities (for real weights) for two types of data points implicating higher order convex functions. New Green functions and Montgomery identity are used for the class of k -convex functions, where $k \geq 3$. As applications of new obtained results, Čebyšev-, Grüss- and Ostrowski-type new bounds are found. Moreover, the main results are applied to information theory via f -divergence and Shannon entropy. In future work, the main results can apply for other types of divergences and distances such as Rényi divergence, Rényi entropy and Zipf-Mandelbrot law.

Acknowledgments: The authors wish to thank the anonymous referees for their very careful reading of the manuscript and fruitful comments and suggestions. The research of Josip Pečarić is supported by the Ministry of Education and Science of the Russian Federation (Agreement number 02.a03.21.0008).

References

- [1] J. Pečarić, F. Proschan, and Y. L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, New York, 1992.
- [2] N. Levinson, *Generalization of an inequality of Kay Fan*, J. Math. Anal. Appl. **6** (1969), 133–134.
- [3] D. S. Mitrinović, J. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, vol. 61, Kluwer Academic Publishers, Dordrecht, 1991.
- [4] T. Popoviciu, *Sur une inegalite de N. Levinson*, Mathematica **6** (1969), 301–306.
- [5] P. S. Bullen, *An inequality of N. Levinson*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **412/460** (1973), 109–112.
- [6] J. Pečarić, *On an inequality on N. Levinson*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **678/715** (1980), 71–74.
- [7] A. McD. Mercer, *A variant of Jensen's inequality*, J. Ineq. Pure Appl. Math. **4** (2003), no. 4, 73.
- [8] M. Adeel, K. A. Khan, Đ. Pečarić, and J. Pečarić, *Generalization of the Levinson inequality with applications to information theory*, J. Inequal. Appl. **2019** (2019), 230.
- [9] M. Adeel, K. A. Khan, Đ. Pečarić, and J. Pečarić, *Levinson type inequalities for higher order convex functions via Abel-Gontscharoff interpolation*, Adv. Differ. Equ. **2019** (2019), 430.
- [10] M. Adeel, K. A. Khan, Đ. Pečarić, and J. Pečarić, *Estimation of f -divergence and Shannon entropy by Levinson type inequalities via new Green's functions and Lidstone polynomial*, Adv. Differ. Equ. **2020** (2020), 27.
- [11] A. Witkowski, *On Levinson's inequality*, Ann. Univ. Paed. Cracov. Stud. Math. **12** (2013), 59–67.
- [12] J. Pečarić, M. Praljak, and A. Witkowski, *Generalized Levinson's inequality and exponential convexity*, Opuscula Math. **35** (2015), 397–410.
- [13] J. Pečarić, M. Praljak, and A. Witkowski, *Linear operators inequality for n -convex functions at a point*, Math. Inequal. Appl. **18** (2015), 1201–1217.
- [14] Z. Pavić, *Figurative Interpolations of Basic Inequalities For Convex Functions*, Turkish J. Ineq. **2** (2018), no. 1, 21–29.
- [15] M. E. Özdemir, Ç. Yildiz, and A.O. Akdemir, *On some inequalities for s -convex functions and applications*, J. Inequal. Appl. **2013** (2013), no. 1, 333.

- [16] E. Set, S. S. Karatas, and M. A. Khan, *Hermite-Hadamard type inequalities obtained via fractional integral for differentiable m -convex and (α, m) -convex function*. Int. J. Anal. **2016** (2016), 4765691, DOI: 10.1155/2016/4765691.
- [17] M. W. Almoari and M. M. Almahameed, *Ostrowski's type inequalities for functions whose first derivatives in absolute value are MN-convex*, Turkish J. Ineq. **1** (2017), no. 1, 53–77.
- [18] E. Set, İ. İşcan, and H. H. Kara, *Hermite-Hadamard-Fejér type inequalities for s -convex function in the second sense via fractional integrals*, Filomat **30** (2016), no. 12, 3131–3138.
- [19] A. Aljinović, J. Pečarić, and A. Vukelić, *On some Ostrowski type inequalities via Montgomery identity and Taylor's formula II*, Tamkang J. Math. **36** (2005), no. 4, 279–301.
- [20] D. S. Mitrinović, J. Pečarić, and A. M. Fink, *Inequalities for Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht, 1994.
- [21] G. A. Gazić, V. Culjak, J. Pečarić, and A. Vukelić, *Generalization of Jensen's inequality by Lidstone's polynomial and related results*, Math. Inequal. Appl. **164** (2013), 1243–1267.
- [22] N. Mehmood, R. P. Agarwal, S. I. Butt, and J. Pečarić, *New generalizations of Popoviciu-type inequalities via new Green's functions and Montgomery identity*, J. Inequal. Appl. **2017** (2017), 108.
- [23] P. Cerone and S. S. Dragomir, *Some new Ostrowski-type bounds for the Čebyšev functional and applications*, J. Math. Inequal. **8** (2014), no. 1, 159–170.
- [24] S. I. Butt, K. A. Khan, and J. Pečarić, *Generalization of Popoviciu inequality for higher order convex function via Taylor's polynomial*, Acta Univ. Apulensis Math. Inform. **42** (2015), 181–200.
- [25] K. A. Khan, T. Niaz, Đ. Pečarić, and J. Pečarić, *Refinement of Jensen's inequality and estimation of f - and Rényi divergence via Montgomery identity*, J. Inequal. Appl. **2018** (2018), 318.
- [26] A. L. Gibbs, *On choosing and bounding probability metrics*, Int. Stat. Rev. **70** (2002), 419–435.
- [27] I. Sason and S. Verdú, *f -divergence inequalities*, IEEE Trans. Inf. Theory **62** (2016), 5973–6006.
- [28] F. Liese and I. Vajda, *Convex Statistical Distances*, Teubner-Texte Zur Mathematik, Teubner, Leipzig, vol. 95, 1987.
- [29] I. Vajda, *Theory of Statistical Inference and Information*, Kluwer, Dordrecht, 1989.
- [30] I. Csizsár, *Information measures: a critical survey*, in: Tans. 7th Prague Conf. on Info. Th., Statist. Decis. Funct., Random Process and 8th European Meeting of Statist., vol. B, pp. 73–86, Academia, Prague, 1978.
- [31] I. Csizsár, *Information-type measures of difference of probability distributions and indirect observations*, Stud. Sci. Math. Hungar. **2** (1967), 299–318.
- [32] L. Horváth, Đ. Pečarić, and J. Pečarić, *Estimations of f - and Rényi divergences by using a cyclic refinement of the Jensen's inequality*, Bull. Malays. Math. Sci. Soc. **42** (2019), 933–946.