Research Article

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Range-kernel weak orthogonality of some elementary operators

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Abstract: We study the range-kernel weak orthogonality of certain elementary operators induced by non-normal operators, with respect to usual operator norm and the Von Neumann-Schatten p -norm ($1 \leq p < \infty$).

Keywords: range-kernel orthogonality, elementary operator, Schatten p -classes, quasinormal operator, sub-normal operator, k -quasihyponormal operator

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1 Introduction

Let $B(\mathcal{H})$ be the algebra of all bounded linear operators acting on a complex separable Hilbert space $\mathcal{H}$. Given $A, B \in B(\mathcal{H})$, we define the generalized derivation $\delta_{A,B} : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ by $\delta_{A,B}(X) = AX - XB$.

Let $X \in B(\mathcal{H})$ be a compact operator, and let $s_1(X) \geq s_2(X) \geq \dots \geq 0$ denote the eigenvalues of $|X| = (X^*X)^{\frac{1}{2}}$ arranged in their decreasing order. The operator X is said to belong to the Schatten p -class $C_p(\mathcal{H})$, if

$$\|X\|_p = \left[\sum_{i=1}^{\infty} s_i(X)^p \right]^{\frac{1}{p}} = \text{tr}(|X|^p)^{\frac{1}{p}} < +\infty, \quad (1)$$

where tr denotes the trace functional. In case $p = \infty$, we denote by $C_{\infty}(\mathcal{H})$, the ideal of compact operators equipped with the norm $\|X\|_{\infty} = s_1(X)$. For $p = 1$, $C_1(\mathcal{H})$ is called the trace class, and for $p = 2$, $C_2(\mathcal{H})$ is called the Hilbert-Schmidt class and the case $p = \infty$ corresponds to the class. For more details, the reader is referred to [1]. In the sequel, we will use the following further notations and definitions. The closure of the range of an operator $T \in B(\mathcal{H})$ will be denoted by $\overline{\text{ran } T}$ and $\ker T$ denotes the kernel of T . The restriction of T to an invariant subspace $\mathcal{M}$ will be denoted by $T|_{\mathcal{M}}$, and the commutator $AB - BA$ of the operators A, B will be denoted by $[A, B]$. We recall the definition of Birkhoff-James's orthogonality in Banach spaces [2,3].

Definition 1. If $\mathcal{X}$ is a complex Banach space, then for any elements $x, y \in \mathcal{X}$, we say that x is orthogonal to y , noted by $x \perp y$, iff for all $\alpha, \beta \in \mathbb{C}$ there holds

$$\|\alpha y + \beta x\| \geq \|\beta x\|. \quad (2)$$

for all $\alpha, \beta \in \mathbb{C}$ or $\mathbb{R}$.

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If $\mathcal{M}$ and $\mathcal{N}$ are linear subspaces in $\mathcal{X}$, we say that $\mathcal{M}$ is orthogonal to $\mathcal{N}$ if $x \perp_{B-J} y$ for all $x \in \mathcal{M}$ and all $y \in \mathcal{N}$. The orthogonality in this sense is asymmetric.

Let $*$: $\mathcal{R} \rightarrow B(\mathcal{X})$ be an involution defined on a linear subspace $\mathcal{R}$ of $B(\mathcal{X})$ onto the algebra of all bounded linear operators acting on the Banach space $\mathcal{X}$ and $\mathcal{R}^* = \mathcal{R}$. According to the definition given by Harte [4], E is called the Fuglede operator if $\ker E \subseteq \ker E^*$.

The elementary operator is an operator E defined on Banach $(\mathcal{A}, \mathcal{B})$ -bimodule $\mathcal{M}$ with its representation $E(x) = \sum_{i=1}^n a_i x b_i$, where $a = (a_i)_i \in \mathcal{A}^n$, $b = (b_i)_i \in \mathcal{B}^n$ are n -tuples of algebra elements. The length of E is defined to be the smallest number of multiplication terms required for any representation $\sum_j a_j x b_j$ for E .

In this note, we consider $\mathcal{A} = \mathcal{B} = B(\mathcal{H})$ and $\mathcal{R} = B(\mathcal{H})$ or $\mathcal{R} = C_p : (1 \leq p < \infty)$ and the length of E will be less or equal to 2, i.e., if $A = (A_1, A_2)$, $B = (B_1, B_2)$ are 2-tuples of operators in $B(\mathcal{H})^2$, then the elementary operator induced by A and B is defined by $E(X) = A_1 X B_1 - A_2 X B_2$ for all $X \in \mathcal{R}$. We will denote by $\tilde{E}$ the formal adjoint of E defined by $\tilde{E}(X) = \sum_{i \in J} A_i^* X B_i^*$ for all $X \in \mathcal{R}$. Note also that $\tilde{E}|_{C_2} = (E|_{C_2})^*$ and $E^*(X) = \sum_{i \in J} B_i X A_i$ on any separable ideal of compact operator, where E^* is the operator adjoint of E in the sense of duality.

J. Anderson [5] proved that if A and B are normal operators, then

$$\text{for all } X, S \in B(\mathcal{H}) : S \in \ker \delta_{A,B} \Rightarrow \|\delta_{A,B}(X) + S\| \geq \|S\|. \quad (3)$$

This means that the kernel of $\delta_{A,B}$ is orthogonal to its range.

F. Kittaneh [6] extended this result to an u.i. ideal norm $\mathcal{J}$ in $B(\mathcal{H})$, by proving that the range of $\delta_{A,B}|_{\mathcal{J}}$ is orthogonal to $\ker \delta_{A,B} \cap \mathcal{J}$.

A detailed study of range-kernel orthogonality for generalized derivation $\delta_{A,B}$ has received much attention in recent years and has been carried out in a large number of studies [3,5,7–13] and are based on the following result.

Theorem 2. *Let A, B be operators in $B(\mathcal{H})$. If $\delta_{A,B}$ is Fuglede, then the range of $\delta_{A,B}$ (resp. the range of $\delta_{A,B}|_{C_p}$) is orthogonal to the kernel of $\delta_{A,B}$ (resp. the kernel of $\delta_{A,B}|_{C_p}$) for all $1 \leq p \leq \infty$.*

D. Keckic [14] and A. Turnšek [15] extended Theorem 2 to the elementary operator E defined by $E(X) = AXB - CXD$, where (A, C) and (B, D) are 2-tuples of commuting normal operators. Duggal [16] generalized the famous theorem to the case (A, C) and (B, D) are 2-tuples of commuting operators, where A, B are normal and C, D^* are hyponormal.

In this paper, our goal is to extend the previous theorem to non-normal operators including quasi-normal, subnormal, and k -quasihyponormal operators.

In the following, we recall some definitions about the range-kernel weak orthogonality.

Definition 3. [4] If $E : \mathcal{X} \rightarrow \mathcal{Y}$ and $T : \mathcal{Y} \rightarrow \mathcal{Z}$ are bounded linear operators between Banach spaces and $0 < k \leq 1$,

$$[s \in \ker T \Rightarrow \text{dist}(s, \text{ran } E) \geq \|s\|] \Leftrightarrow T \angle_k E \Leftrightarrow \ker T \angle_k \text{ran } E. \quad (4)$$

We say that T is weakly orthogonal to E , written $T \angle E$, or equivalently

$$\ker T \angle \text{ran } E \Leftrightarrow \exists k : 0 < k \leq 1 : T \angle_k E. \quad (5)$$

For $0 < k \leq 1$, we say that (E, T) has a $\frac{1}{k}$ -gap if $T \angle_k E$.

If $T = E$ and $k \neq 1$, we shall call E w -orthogonal ($E \angle E$), consequently we get a $\frac{1}{k}$ -gap between the subspaces $\ker E$ and $\text{ran } E$, which corresponds to the “range-kernel weak-orthogonality” for an operator E . If $k = 1$, we shall say that T is orthogonal to E , written $T \perp E$, also if $\mathcal{X} = (\mathcal{Y}) = \mathcal{Z}$ and $T = E$ we get a 1-gap between the subspaces $\ker E$ and $\text{ran } E$.

T is said to be a quasi-normal if $[T, T^*T] = 0$, subnormal if there exist a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a normal operator $N \in B(\mathcal{K})$ such that $N|_{\mathcal{H}} = T$. Also, T is called hyponormal if $[T^*, T]$ is a positive operator. Furthermore, we have the following proper inclusion

$$T \text{ normal} \Rightarrow T \text{ quasi-normal} \Rightarrow T \text{ subnormal} \Rightarrow T \text{ hyponormal}.$$

$A \in B(\mathcal{X})$ is said the Fuglede operator [13] if $\ker A \subseteq \ker A^*$.

Recall that an n -tuple $A = (A_1, A_2, \dots, A_n) \in B(\mathcal{H})^n$ is said commuting (resp. doubly commuting) if $[A_i, A_j] = 0$ (resp. $[A_i, A_j] = 0$ and $[A_i^*, A_j] = 0$) for all $i, j = 1, \dots, n, i \neq j$. The n -tuple A is said to be normal if A is commuting and each A_i ($i = 1, \dots, n$) is normal, and A is subnormal if A is the restriction of a normal n -tuple to a common invariant subspace. Clearly, every normal n -tuple is subnormal n -tuple. Any other notation or definition will be explained as and when required.

2 Preliminaries

We summarize the results given by D. Keckic [14], A. Turnšek [15], and B. P. Duggal [16] in the following theorem.

Theorem 4. [14–16] Let A, B be normal operators, C, D^* be hyponormal operators in $B(\mathcal{H})$ such that $[A, C] = [B, D] = 0$ and $\mathcal{J} = B(\mathcal{H})$ or $\mathcal{J} = C_p : 1 \leq p \leq \infty$, then

(i) If $\ker A \cap \ker C = \{0\} = \ker B^* \cap \ker D^*$, then for all $X \in B(\mathcal{H})$ such that $\tilde{E}(X), E(X) \in \mathcal{J}$

$$S \in \ker E \cap \mathcal{J} \Rightarrow \min\{\|E(X) + S\|_{\mathcal{J}}, \|\tilde{E}(X) + S\|_{\mathcal{J}}\} \geq \|S\|_{\mathcal{J}}.$$

(ii) If $\ker A \cap \ker C \neq \{0\}$ or $\ker B^* \cap \ker D^* \neq \{0\}$, then there exists k verifying $0 < k < 1$ such that

$$\forall X \in B(\mathcal{H}), \quad \forall S \in \ker E \cap \mathcal{J} : \|E(X) + S\|_{\mathcal{J}} \geq k\|S\|_{\mathcal{J}}, \quad \text{where } E(X) \in \mathcal{J}.$$

(iii) If $\mathcal{J} = C_2(\mathcal{H})$ with its inner product $\langle X, Y \rangle = \text{tr}(Y^*X)$ and E is defined on $C_2(\mathcal{H})$, then for all $S \in \ker E$, we get

$$\|E(X) + S\|_2^2 = \|E(X)\|_2^2 + \|S\|_2^2; \quad \|\tilde{E}(X) + S\|_2^2 = \|\tilde{E}(X)\|_2^2 + \|S\|_2^2.$$

We recall some useful results which are important in the sequel.

Lemma 5. [13] Let A, B be commuting operators in $B(\mathcal{H})$ with no trivial kernel.

1. Let ξ be the elementary Fuglede operator defined by $\xi(X) = AXA^* - BXB^*$, then

(i) if $\ker A \cap \ker B = \{0\}$, then $\ker A$ reduces A and $\ker B$ reduces B .

(ii) if $[A, B^*] = 0$ and $\ker A \cap \ker B \neq \{0\}$, then $\ker A \cap \ker B$ reduces A and B .

2. Let $A \in B(\mathcal{H})$, $B \in B(\mathcal{K})$, and $E(X) = AXB$; $X \in B(\mathcal{K}, \mathcal{H})$, then E is the Fuglede operator if and only if $\ker A$ reduces A and $\ker B^*$ reduces B^* .

Lemma 6. [9] Let T be an operator represented by block matrix as $T = (T_{i,j})_{i,j=1}^n$.

(i) If $T \in B(\mathcal{H})$, then $\frac{1}{n^2} \sum_{i,j} \|T_{i,j}\|^2 \leq \|T\|^2 \leq \sum_{i,j} \|T_{i,j}\|^2$

(ii) If $T \in C_p(T)$; $1 \leq p < \infty$, then

$$\begin{aligned} \frac{1}{n^{p-2}} \|T\|_p^p &\leq \sum_{i,j} \|T_{i,j}\|_p^p \leq \|T\|_p^p \quad \text{for all } 2 \leq p < \infty, \\ \|T\|_p^p &\leq \sum_{i,j} \|T_{i,j}\|_p^p \leq \frac{1}{n^{p-2}} \|T\|_p^p \quad \text{for all } 1 \leq p \leq 2. \end{aligned}$$

3 Main results

Proposition 7. Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and $A \in B(\mathcal{H})$, $B \in B(\mathcal{K})$, and $E \in B(B(\mathcal{K}, \mathcal{H}))$ such that $E(X) = AXB$. If E is the Fuglede operator, then E is w -orthogonal, $E \perp \tilde{E}$, and the inequality

$$\min\{\|E(X) + S\|_{\mathcal{J}}, \|\tilde{E}(X) + S\|_{\mathcal{J}}\} \geq \|S\|_{\mathcal{J}} \quad (6)$$

holds if

- $\mathcal{J} = B(\mathcal{K}, \mathcal{H})$ with $k = 1/2$ or
- $\mathcal{J} = C_p(\mathcal{K}, \mathcal{H})$ with $k = \frac{1}{2^{\frac{2-p}{p}}}$, if $1 \leq p \leq 2$ and $k = \frac{1}{2^{\frac{p-2}{p}}}$, if $2 \leq p < \infty$.

Proof. If A and B^* are injective operators, then E is injective. So there is nothing to prove.

Suppose that A or B^* is the non-injective operator. From Lemma 5(2) and with respect to the decompositions:

$$\mathcal{H} = (\ker A)^\perp \oplus \ker A, \quad \mathcal{K} = (\ker B^*)^\perp \oplus \ker B^*$$

we obtain $A = A_1 \oplus A$; $B = B_1 \oplus 0$. Let $X = (X_{i,j})_{i,j=1,2} : \mathcal{K} \rightarrow \mathcal{H}$. Then

$$E(X) = A_1 X_{11} B_1 \oplus 0.$$

From the injectivity of A_1 and B_1^* it yields that any $S = (S_{ij})_{i,j=1,2}$ in $\ker E$ can be written as

$$S = \begin{bmatrix} 0 & S_{12} \\ S_{21} & S_{22} \end{bmatrix},$$

where the operators S_{12} , S_{21} , and S_{22} are arbitrary.

Hence, for all $S \in \ker E$ and all $X \in B(\mathcal{K}, \mathcal{H})$, by Lemma 5(2), we have

$$\min\{\|E(X) + S\|, \|\tilde{E}(X) + S\|\} = \left\| \begin{bmatrix} A_1 X_{11} B_1 & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \right\| \geq \frac{1}{2} (\|S_{12}\|^2 + \|S_{21}\|^2 + \|S_{22}\|^2)^{1/2} \geq \frac{1}{2} \|S\|.$$

Also, for all $S \in \ker E \cap C_p(\mathcal{K}, \mathcal{H})$ and all $X \in B(\mathcal{K}, \mathcal{H})$ such that $E(X) \in C_p(\mathcal{K}, \mathcal{H})$, we get

(i) if $1 \leq p \leq 2$, then

$$\min\{\|E(X) + S\|_p, \|\tilde{E}(X) + S\|_p\} = \left\| \begin{bmatrix} A_1 X_{11} B_1 & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \right\|_p \geq \frac{1}{2^{\frac{2-p}{p}}} (\|S_{12}\|_p^p + \|S_{21}\|_p^p + \|S_{22}\|_p^p)^{1/p} \geq \frac{1}{2^{\frac{2-p}{p}}} \|S\|_p.$$

(ii) if $2 \leq p < \infty$, then

$$\min\{\|E(X) + S\|_p, \|\tilde{E}(X) + S\|_p\} = \left\| \begin{bmatrix} A_1 X_{11} B_1 & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \right\|_p \geq \frac{1}{2^{\frac{p-2}{p}}} (\|S_{12}\|_p^p + \|S_{21}\|_p^p + \|S_{22}\|_p^p)^{1/p} \geq \frac{1}{2^{\frac{p-2}{p}}} \|S\|_p.$$

□

Using Lemma 5(2), we get a simple form of the previous result as follows.

Corollary 8. If $\ker A$ reduces A and $\ker B^*$ reduces B^* , then E is w -orthogonal, $E \perp \tilde{E}$ and satisfies the relation (6).

In the sequel ξ denotes the elementary operator defined by

$$\xi(X) = AXA^* - BXB^*$$

from $B(\mathcal{H})$ to $\mathcal{J}$, where A and B are operators in $B(\mathcal{H})$ and $\mathcal{J} = B(\mathcal{H})$ or $\mathcal{J} = C_p(\mathcal{H})$; $1 \leq p < \infty$.

Lemma 9. Let Δ be the elementary operator defined on $B(\mathcal{H})$ by $\Delta(X) = AXB - X$, where $A, B \in B(\mathcal{H})$. If Δ is a Fuglede operator, then Δ is orthogonal and $\Delta \perp \tilde{\Delta}$.

Proof. The proof is the same as the one in Theorem 4. □

Proposition 10. Let A, B be doubly commuting operators in $B(\mathcal{H})$ and

$$\xi : B(\mathcal{H}) \rightarrow C_p(\mathcal{H}); \quad (1 \leq p \leq \infty, \quad p \neq 2).$$

If $\ker A$ reduces A , $\ker B$ reduces B and ξ orthogonal, then $\ker A \cap \ker B = \{0\}$.

Proof. We consider the following three cases:

- (i) Let us suppose that $\mathcal{N} = \ker A \cap \ker B \neq \{0\}$.

If $\ker A \neq \ker B$ and $\ker B \not\subseteq \ker A$, then with respect to the decomposition

$$\mathcal{H} = (\ker B)^\perp \oplus (\ker B \ominus \mathcal{N}) \oplus \mathcal{N}$$

and from the hypothesis it yields

$$A = A_1 \oplus A_2 \oplus 0, \quad B = B_1 \oplus 0, \quad \text{and } S = (S_{ij})_{i,j=1,\dots,3},$$

where A_2 is an injective operator. Hence,

$$S \in \ker \xi \Rightarrow A_1 S_{11} A_1^* = B_1 S_{11} B_1^*; \quad A_1 S_{12} A_2^* = A_2 S_{21} A_1^* = 0; \quad S_{22} = 0$$

and the other entries are arbitrary. Choosing X and S as follows:

$$X = 0 \oplus (e \otimes e) \oplus 0, \quad S = 0 \oplus \begin{bmatrix} 0 & R \\ R^* & C \end{bmatrix},$$

where e is a non-zero vector in $\mathcal{H}$, R is an operator of rank one, and C is a self-adjoint operator of rank one. Then

$$\xi(X) + S = 0 \oplus \begin{bmatrix} A_2 e \otimes A_2 e & R \\ R^* & C \end{bmatrix}.$$

Applying Lemma (2.4) [15], we get

$$\|\xi(X) + S\|_p < \left\| \begin{bmatrix} 0 & R \\ R^* & C \end{bmatrix} \right\|_p = \|S\|_p \quad (p \neq 2).$$

- (ii) If $\ker A \neq \ker B$ and $\ker A \not\subseteq \ker B$, we proceed similarly as in the first case, it suffices to replace A by B and B by A in the preceding argument.
- (iii) If $\ker A = \ker B$, then with respect to the decomposition

$$\mathcal{H} = (\ker B)^\perp \oplus \ker B,$$

we get

$$A = A_1 \oplus 0, \quad B = B_1 \oplus 0.$$

Letting $S = (S_{ij})_{i,j=1,\dots,3}$. Then

$$S \in \ker \xi \Rightarrow A_1 S_{11} A_1^* = B_1 S_{11} B_1^*,$$

and the other entries are arbitrary. Choosing X and S as

$$X = (e \otimes e) \oplus 0, \quad S = \begin{bmatrix} 0 & R \\ R^* & C \end{bmatrix},$$

where e and R are as in (i). Then

$$\xi(X) + S = \begin{bmatrix} A_1 e \otimes A_1 e - B_1 e \otimes B_1 e & R \\ R^* & C \end{bmatrix}.$$

If $A_1 e \otimes A_1 e = B_1 e \otimes B_1 e$ for all $e \in \mathcal{H}$, then it follows from this fact and the injectivity of A_1 and B_1 that $A_1 = \alpha B_1$ with $|\alpha| = 1$ and hence $\xi = 0$, which is a contradiction with the assumption $\xi \neq 0$. We use Lemma (2.4) [15] to complete the proof for $p \notin \{1, 2\}$.

In the case $p = 1$, let us assume that $A_1 e \otimes A_1 e - B_1 e \otimes B_1 e$ is an operator of rank 2 and has eigenvalues λ_1, λ_2 with $|\lambda_1| = |\lambda_2|$ for all $e \in \mathcal{H}$, then we can check that

$$|\lambda_1| = |\lambda_2| \Leftrightarrow \|A_1 e\| = \|B_1 e\| \quad \text{and} \quad \langle A_1 e, B_1 e \rangle = 0 \quad \forall e \in \mathcal{H}.$$

If $\ker B^* \subseteq \ker B$, then by the injectivity of B_1^* , it follows that $A_1 = 0$, which is a contradiction since $A \neq 0$.

If $\ker B^* \not\subseteq \ker B$, with respect to the decomposition

$$\mathcal{H} = (\ker B_1^*)^\perp \oplus \ker B_1^* \oplus \ker B$$

we get

$$A = A_1 \oplus A_2 \oplus 0, \quad B = \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} \oplus 0, \quad \text{and} \quad S = (S_{ij})_{i,j=1,\dots,3}.$$

Since A_1 is injective and $S \in \ker \xi$, we obtain

$$A_1 S_{11} A_1^* = B_1 S_{11} B_1^*; \quad S_{12} = S_{21} = S_{22} = 0$$

and the other entries are arbitrary.

We rewrite S on the following decomposition

$$\mathcal{H} = (\ker B_1^*)^\perp \oplus \ker B \oplus \ker B_1^*$$

and choose $S_{11} = S_{23} = S_{32} = 0$, $S_{13} = R$, $S_{31} = R^*$, $S_{34} = C$, and $X = e \otimes e \oplus 0$ (R, C, e are as in (i)). Then

$$\xi(X) + S = \begin{bmatrix} A_1 e \otimes A_1 e - B_1 e \otimes B_1 e & R \\ R^* & C \end{bmatrix} \oplus 0.$$

If $A_1 e \otimes A_1 e - B_1 e \otimes B_1 e$ is an operator of rank two and has eigenvalues λ_1, λ_2 with $|\lambda_1| = |\lambda_2|$ for all $e \in \mathcal{H}$, then from the previous argument we conclude that $A_1 = 0$. On the other hand, if $B_2 \neq 0$, then from $[A, B] = 0$ it follows that $A_2 = 0$, also a contradiction with the fact that $A \neq 0$.

If $B_2 = 0$ ($\ker B^*$ reduces B^*), then $B = B_1 \oplus 0$, $A = 0 \oplus A_2 \oplus 0$ (A_2 is injective), and $S = (S_{ij})_{i,j=1,\dots,3}$. Hence, it follows from $ASA^* = BSB^*$ that

$$S = \begin{bmatrix} 0 & S_{12} & S_{13} \\ S_{21} & 0 & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

and the other entries are arbitrary.

To conclude the proof we can argue similarly as in the first case (i). □

Corollary 11. Let $(A_1, A_2), (B_1, B_2)$ be 2-tuples of doubly commuting operators in $B(\mathcal{H})$ and $E : B(\mathcal{H}) \rightarrow C_p(\mathcal{H})$; ($1 \leq p \leq \infty$, $p \neq 2$) be the elementary operator defined by $E(X) = A_1 X B_1 - A_2 X B_2$ such that $\ker A_1, \ker A_2, \ker B_1^*$, and $\ker B_2^*$ reduce A_1, A_2, B_1^* , and B_2^* , respectively. If E is orthogonal, then

$$\ker A_1 \cap \ker A_2 = \{0\} = \ker B_1^* \cap \ker B_2^*.$$

Proof. Consider the space $\mathcal{H} \oplus \mathcal{H}$ and the following decompositions

$$A = A_1 \oplus B_1^*, \quad B = A_2 \oplus B_2^*, \quad \text{and} \quad Y = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}.$$

Then, for all $X \in B(\mathcal{H})$, we have $AYA^* - BYB^* = A_1 X B_1 - A_2 X B_2$, $\ker A = \ker A_1 \cap \ker B_1^*$, and $\ker B = \ker A_2 \cap \ker B_2^*$. Hence,

$$\ker A_1 \cap \ker A_2 = \{0\} = \ker B_1^* \cap \ker B_2^* \Leftrightarrow \ker A \cap \ker B = \{0\}.$$

So to achieve the proof, it suffices to apply the previous proposition. □

Theorem 12. Let A_1, A_2, N_1 , and N_2 be operators in $B(\mathcal{H})$ such that (N_1, N_2) is 2-tuple normal and $[A_1, N_1] = [A_2, N_2] = 0$. Let $E(X) = A_1XA_2 - N_1XN_2$ such that $E(X), \tilde{E}(X) \in \mathcal{J}$. If E is the Fuglede operator, then E is w -orthogonal and $E \perp \tilde{E}$. Furthermore,

(i) If $\mathcal{J} = C_p(\mathcal{H}) : (1 \leq p < \infty, p \neq 2)$, then

$$E \text{ is orthogonal if and only if } \ker A_1 \cap \ker N_1 = \{0\} = \ker A_2^* \cap \ker N_2^*;$$

(ii) If $\ker A_1 \cap \ker N_1 = \{0\} = \ker A_2^* \cap \ker N_2^*$, then E is orthogonal and $E \perp \tilde{E}$;

(iii) If $\ker A_1 \cap \ker N_1 \neq \{0\}$ or $\ker A_2^* \cap \ker N_2^* \neq 0$, then for all $X \in B(\mathcal{H})$ and all $S \in \ker E \cap \mathcal{J}$,

$$\min\{\|E(X) + S\|_{\mathcal{J}}, \|\tilde{E}(X) + S\|_{\mathcal{J}}\} \geq k\|S\|_{\mathcal{J}},$$

where

$$- \mathcal{J} = B(\mathcal{H}) : k = \frac{1}{2^n};$$

$$- \mathcal{J} = C_p(\mathcal{H}) : k = \frac{1}{2^{\frac{4-2p}{p}}}, \text{ if } 1 \leq p \leq 2 \text{ and } k = \frac{1}{2^{\frac{2p-4}{p}}}, \text{ if } 2 \leq p < \infty.$$

Proof.

(i) The implication $(\Rightarrow)$ follows from Corollary 8.

Let $\xi : B(\mathcal{H}) \rightarrow \mathcal{J}$ be the elementary operator defined by $\xi(X) = AXA^* - NXN^*$, where $A, N \in B(\mathcal{H})$, N is normal with $[A, N] = 0$. Assume that ξ is the Fuglede operator.

(ii) Suppose that N is invertible and set $D = N^{-1}A$. ξ is Fuglede implies that ξ_D is Fuglede, where ξ_D is the elementary operator induced by D . By Lemma 2.4 [16], we have

$$\|\xi(X) + S\|_{\mathcal{J}} = \|AXA^* - NXN^* + S\|_{\mathcal{J}} = \|D(NXN^*)D^* - NXN^* + S\|_{\mathcal{J}} \geq \|S\|_{\mathcal{J}}.$$

Similarly, it can be shown that $\|\tilde{\xi}(X) + S\|_{\mathcal{J}} \geq \|S\|_{\mathcal{J}}$ for any operator $S \in (\ker \xi) \cap \mathcal{J}$.

(iii) Suppose that N is injective, set $\Delta_n = \{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{n}; n \in \mathbb{N}\}$ and $\mu(\Delta_n)$ denotes the corresponding spectral projection, where $P_n = I - \mu(\Delta_n)$ converges strongly to I .

From $[A, N] = 0$ and by Fuglede-Putnam's theorem it follows that $[A, N^*] = 0$ and therefore $P_n\mathcal{H}$ reduces both A and N . Let

$$\mathcal{H} = P_n\mathcal{H} \oplus (I - P_n)\mathcal{H},$$

then $A = A_{1n} \oplus A_{2n}$; $N = N_{1n} \oplus N_{2n}$ and $P_n = I \oplus 0$, where N_{1n} is invertible. Hence,

$$P_n(\xi(X) + S)P_n = [A_{1n}X_{11}A_{1n}^* - N_{1n}X_{11}N_{1n}^* + S_{11}] \oplus 0$$

for all $X = (X_{ij})_{i,j=1,2} \in B(\mathcal{H})$ and $S = (S_{ij})_{i,j=1,2} \in \ker \xi$ implying

$$\|\xi(X) + S\|_{\mathcal{J}} \geq \|P_n(\xi(X) + S)P_n\|_{\mathcal{J}} = \|A_{1n}X_{11}A_{1n}^* - N_{1n}X_{11}N_{1n}^* + S_{11}\|_{\mathcal{J}} \geq \|S_{11}\|_{\mathcal{J}}.$$

Since $P_n(\xi^*(X) + S)P_n = [A_{1n}^*X_{11}A_{1n} - N_{1n}^*X_{11}N_{1n} + S_{11}] \oplus 0$, then

$$\|\xi^*(X) + S\|_{\mathcal{J}} \geq \|P_n(\xi^*(X) + S)P_n\|_{\mathcal{J}} = \|A_{1n}^*X_{11}A_{1n} - N_{1n}^*X_{11}N_{1n} + S_{11}\|_{\mathcal{J}} \geq \|S_{11}\|_{\mathcal{J}}.$$

On the other hand, for all positive integer n , we have

$$\|S_{11}\|_{\mathcal{J}} = \|P_nSP_n\| \leq \min\{\|\xi(X) + S\|_{\mathcal{J}}, \|\xi^*(X) + S\|_{\mathcal{J}}\} < \infty$$

and thus $\sup_n \|P_nSP_n\| < \infty$ for all $S \in \ker \xi \cap \mathcal{J}$ and $X \in B(\mathcal{H})$. It follows by Lemma 3 [14] that

$$\min\{\|\xi(X) + S\|_{\mathcal{J}}, \|\tilde{\xi}(X) + S\|_{\mathcal{J}}\} \geq \|S\|_{\mathcal{J}}.$$

(iii) Suppose that N is an arbitrary normal operator.

If $\ker A \cap \ker N = \{0\}$, then $\mathcal{H}$ may be decomposed as

$$\mathcal{H} = [(\ker N)^{\perp} \ominus \ker A] \oplus \ker A \oplus \ker N.$$

Since ξ is Fuglede and by Lemma 5(1.(i)), we have $\ker A$ reduces A , $A = A_{11} \oplus 0 \oplus A_{22}$ and $N = N_{11} \oplus N_{22} \oplus 0$. For $X = (X_{ij})_{i,j=1,2,3} \in B(\mathcal{H})$, set $\xi_1(X_{11}) = A_{11}X_{11}A_{11}^* - N_{11}X_{11}N_{11}^*$ and $S = (S_{ij})_{i,j=1,2,3} \in \ker \xi$, then

$$\begin{aligned}\xi_1(S_{11}) &= 0 \text{ and } A_{11}S_{13}A_{22}^* = A_{22}S_{31}A_{11}^* = A_{22}S_{33}A_{22}^* = 0, \\ N_{11}S_{12}N_{22}^* &= N_{22}S_{21}N_{11}^* = N_{22}S_{22}N_{22}^* = 0.\end{aligned}$$

Since ξ is Fuglede, then $S \in \ker \tilde{\xi}$ and

$$\begin{aligned}\xi_1^*(S_{11}) &= 0 \text{ and } A_{11}^*S_{13}A_{22} = A_{22}^*S_{31}A_{11} = A_{22}^*S_{33}A_{22} = 0, \\ N_{11}^*S_{12}N_{22} &= N_{22}^*S_{21}N_{11} = N_{22}^*S_{22}N_{22} = 0.\end{aligned}\tag{7}$$

Since N_{11} , N_{22} , A_{11} , and A_{22} are injective, we get from (ii) that ξ_1 is orthogonal, $\xi_1 \perp \tilde{\xi}_1$ and $S_{13} = S_{31} = S_{33} = S_{12} = S_{22} = S_{21} = 0$ and any operator $S \in \ker \xi$ has the form

$$S = \begin{bmatrix} S_{11} & 0 & 0 \\ 0 & 0 & S_{23} \\ 0 & S_{32} & 0 \end{bmatrix},$$

where S_{23} and S_{32} are arbitrary.

Let $S_{23} = U_{23}|S_{23}|$, $S_{32} = U_{32}|S_{32}|$ be the polar decompositions of S_{23} and S_{32} , respectively, let V be the operator defined by

$$V = I \oplus \begin{bmatrix} 0 & U_{32}^* \\ U_{23}^* & 0 \end{bmatrix}.$$

Then

$$\|\xi(X) + S\|_{\mathcal{J}} \geq \|V(\xi(X) + S)\|_{\mathcal{J}} = \left\| \begin{bmatrix} \xi_1(X_{11}) + S_{11} & * & * \\ * & |S_{32}| & * \\ * & * & |S_{23}| \end{bmatrix} \right\|.$$

Applying Lemma 6, we get

– if $\mathcal{J} = B(\mathcal{H})$:

$$\|\xi(X) + S\| \geq \max\{\|\xi_1(X_{11}) + S_{11}\|, \|S_{32}\|, \|S_{23}\|\} \geq \max\{\|S_{11}\|, \|S_{32}\|, \|S_{23}\|\} = \|S\|.$$

– if $\mathcal{J} = C_p(\mathcal{H})$; ($1 \leq p < \infty$):

$$\|\xi(X) + S\|_p \geq (\|\xi_1(X_{11}) + S_{11}\|_p^p + \|S_{32}\|_p^p + \|S_{23}\|_p^p)^{1/p} \geq (\|S_{11}\|_p^p + \|S_{32}\|_p^p + \|S_{23}\|_p^p)^{1/p} = \|S\|_p.$$

By the same method, we have that $\xi \perp \tilde{\xi}$.

If $M = \ker A \cap \ker N \neq \{0\}$ and $\mathcal{H}$ is decomposed as

$$\mathcal{H} = (\ker N)^\perp \oplus [\ker N \ominus M] \oplus M,$$

then by Lemma 5(ii) and the fact that ξ is Fuglede, it follows that M reduces A , and hence

$$A = A_{11} \oplus A_{22} \oplus 0, \quad N = N_{11} \oplus 0.$$

For $X = (X_{ij})_{i,j=1,2,3}$, we set $\xi_1(X) = A_{11}X_{11}A_{11}^* - N_{11}X_{11}N_{11}^*$ and let $S = (S_{ij})_{i,j=1,2,3} \in \ker \xi$. From the injectivity of A_{11} and A_{22} , we obtain

$$\xi_1(S_{11}) = 0 \text{ and } S_{12} = S_{21} = S_{22} = 0.$$

By simple computation, we get

$$\|\xi(X) + S\|_{\mathcal{J}} = \left\| \begin{bmatrix} \xi_1(X_{11}) + S_{11} & * & S_{13} \\ * & * & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \right\|_{\mathcal{J}}.$$

Applying Lemma 6 yields

– for $\mathcal{J} = B(\mathcal{H})$:

$$\|\xi(X) + S\|^2 \geq \frac{1}{2^2} (\|\xi_1(X_{11} + S_{11})\|^2 + \|S_{13}\|^2 + \|S_{23}\|^2 + \|S_{32}\|^2 + \|S_{13}\|^2 + \|S_{33}\|^2) \geq \frac{1}{2^4} \|S\|^2.$$

– for $\mathcal{J} = C_p(\mathcal{H})$; $1 \leq p \leq 2$:

$$\|\xi(X) + S\|_p^p \geq \frac{1}{2^{2-p}} (\|\xi_1(X_{11} + S_{11})\|_p^p + \|S_{13}\|_p^p + \|S_{23}\|_p^p + \|S_{32}\|_p^p + \|S_{13}\|_p^p + \|S_{33}\|_p^p) \geq \frac{1}{2^{4-2p}} \|S\|_p^p.$$

– for $\mathcal{J} = C_p(\mathcal{H})$; $2 \leq p < \infty$:

$$\|\xi(X) + S\|_p^p \geq \frac{1}{2^{2-p}} (\|\xi_1(X_{11} + S_{11})\|_p^p + \|S_{13}\|_p^p + \|S_{23}\|_p^p + \|S_{32}\|_p^p + \|S_{13}\|_p^p + \|S_{33}\|_p^p) \geq \frac{1}{2^{2p-4}} \|S\|_p^p.$$

Similarly,

– for $\mathcal{J} = B(\mathcal{H})$:

$$\|\tilde{\xi}(X) + S\| \geq \frac{1}{2^2} \|S\|.$$

– for $\mathcal{J} = C_p(\mathcal{H})$; $1 \leq p \leq 2$:

$$\|\tilde{\xi}(X) + S\|_p \geq \frac{1}{2^{\frac{4-2p}{p}}} \|S\|_p.$$

– for $\mathcal{J} = C_p(\mathcal{H})$; $2 \leq p < \infty$:

$$\|\tilde{\xi}(X) + S\|_p \geq \frac{1}{2^{\frac{2p-4}{p}}} \|S\|_p.$$

Let us now finish the proof for the elementary operator $E(X) = A_1XA_2 - N_1XN_2$. Consider the space $\mathcal{H} \oplus \mathcal{H}$ and define the following operators on $B(\mathcal{H} \oplus \mathcal{H})$ as

$$N = N_1 \oplus N_2^*, \quad A = A_1 \oplus A_2^*, \quad \text{and} \quad Z = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}.$$

It is clear that N is normal operator and $[A_1, N_1] = [A_2, N_2] = 0$ imply $[A, N] = 0$ and $E(Z) = AZA^* - NZN^*$. Applying the preceding result, the proof is complete. $\square$

Let $A, B \in B(\mathcal{H})$ and Ω be a set in $B(\mathcal{H})^2$ defined by $(A, B) \in \Omega$ if and only if $\Delta(X) = ANXB - X$ is the Fuglede operator for any normal N, M satisfying $[N, A] = [M, B] = 0$.

It follows from the definition that Δ is Fuglede and $\Omega \neq \emptyset$ since $(I, I) \in \Omega$.

It is shown in [13] that if $(A, B) \in \Omega$, then the elementary operator E defined by $E(X) = NXM - AXB$ is Fuglede for any normal operators N and M in $B(\mathcal{H})$. So as a consequence of the previous theorem, we get the following corollaries.

Corollary 13. Let $A, B \in B(\mathcal{H})$ and N, M be arbitrary normal operators in $B(\mathcal{H})$ such that $[N, A] = [M, B] = 0$ and E be the elementary operator defined by $E(X) = NXM - AXB$ for all $X \in B(\mathcal{H})$.

If $(A, B) \in \Omega$, then E is w -orthogonal and $E \perp \tilde{E}$. Furthermore, E and $\tilde{E}$ verify assertions (i), (ii), and (iii) in Theorem 12.

Lemma 14. [13] Let $A, B \in B(\mathcal{H})$ and N, M be normal operators in $B(\mathcal{H})$ such that $[N, A] = [M, B] = 0$, then $(A, B) \in \Omega$ in each of the following cases:

- (i) A and B^* are hyponormal operators;
- (ii) A is k -quasihyponormal and B^* is injective k -quasihyponormal operator.

Corollary 15. If $(A_1, A_2), (B_1, B_2)$ are 2-tuples of commuting operators in $B(\mathcal{H})$ such that A_1, B_1 are normal operators and $E(X) = A_1XB_1 - A_2XB_2$, then E is w -orthogonal and $E \perp \tilde{E}$. Furthermore, E and $\tilde{E}$ satisfy assertions (i), (ii), and (iii) cited in Theorem 12, in each of the following cases:

- (i) A_1 and B_1^* are hyponormal operators;
- (ii) A_1 is k -quasihyponormal and B_1^* is injective k -quasihyponormal operator.

In the next theorem, we give a positive answer to a question raised by P. B. Duggal [16]: Is Theorem 2 still true if the hypothesis is related to A and B^* being subnormal?

Theorem 16. If A and B^* are 2-tuples of commuting subnormal operators in $[B(\mathcal{H})]^2$ such that $A = (A_1, A_2)$, $B = (B_1, B_2)$, and $E(X) = A_1XB_1 - A_2XB_2$, then E is w -orthogonal and $E \perp \tilde{E}$.

Proof. From the definition of subnormality of 2-tuple operator, we have A is the restriction of a 2-tuple normal $N = (N_1, N_2)$ to a common invariant subspace $\mathcal{H}$ and B^* is the restriction of a 2-tuple normal $M = (M_1, M_2)$ to a common invariant subspace $\mathcal{H}$ equivalently to $A_i = N_i|_{\mathcal{H}}; B_i^* = M_i|_{\mathcal{H}}, i = 1, 2$ with N_i, M_i are normal operators on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and $[N_1, N_2] = [M_1, M_2] = 0$. If $S \in \ker E$, then for all $X \in B(\mathcal{H})$, we have

$$N_1\tilde{X}M_1^* - N_2\tilde{X}M_2^* + \tilde{S} = \begin{bmatrix} E(X) + S & 0 \\ 0 & 0 \end{bmatrix},$$

where $\tilde{X} = X \oplus 0$ and $\tilde{S} = S \oplus 0$. Hence,

$$\|N_1\tilde{X}M_1^* - N_2\tilde{X}M_2^* + \tilde{S}\|_{\mathcal{J}} = \|E(X) + S\|_{\mathcal{J}}.$$

Since $N_i, M_i : i = 1, 2$ are normal, we get the w -orthogonality of E . With similar argument, $E \perp \tilde{E}$ follows. $\square$

Corollary 17. If (A_1, A_2) and $B = (B_1, B_2)$ are 2-tuples of commuting operators in $[B(\mathcal{H})]^2$ such that $[A_i, A_1^*A_1 + A_2^*A_2] = [B_i^*, B_1B_1^* + B_2B_2^*] = 0; i = 1, 2$ and $E(X) = A_1XB_1 - A_2XB_2$, then E is w -orthogonal and $E \perp \tilde{E}$.

Proof. By assumption, A and B^* are spherically quasi-normal commuting 2-tuples (see definition [17]) and also by [17], A and B^* are subnormal 2-tuples. So the desired result follows from Theorem 16. $\square$

Theorem 18. Let $(A_1, A_2), (B_1, B_2)$ be 2-tuples of doubly commuting operators in $B(\mathcal{H})$ and E be the elementary operator defined by $E(X) = A_1XB_1 - A_2XB_2$ such that $E(X), \tilde{E}(X) \in \mathcal{J}, A_1, B_1^*$ are quasi-normal operators and A_2, B_2^* are k -quasihyponormal operators ($k \geq 1$) with $\ker A_2 \subseteq \ker A_2^*$ and $\ker B_2^* \subseteq \ker B_2$. Then E is w -orthogonal and $E \perp \tilde{E}$. Furthermore,

1. If $\mathcal{J} = C_p(\mathcal{H}); (1 \leq p < \infty, p \neq 2)$, then

$$E \text{ is orthogonal if and only if } \ker A_1 \cap \ker N_1 = \ker A_2^* \cap \ker N_2^* = \{0\};$$

2. If $\ker A_1 \cap \ker N_1 = \{0\} = \ker A_2^* \cap \ker N_2^*$, then E is orthogonal and $E \perp \tilde{E}$;
3. If $\ker A_1 \cap \ker A_2 \neq 0$ or $\ker B_1^* \cap \ker B_2^* \neq \{0\}$, then for all $X \in B(\mathcal{H})$ and all $S \in \ker E \cap \mathcal{J}$,

$$\min\{\|E(X) + S\|_{\mathcal{J}}, \|\tilde{E}(X) + S\|_{\mathcal{J}}\} \geq k\|S\|_{\mathcal{J}}.$$

- For $\mathcal{J} = B(\mathcal{H}), k = \frac{1}{6}$;
- For $\mathcal{J} = C_p(\mathcal{H}), k = \frac{1}{6^{\frac{1}{2-p}}}, \text{ if } 1 \leq p \leq 2$;
- For $\mathcal{J} = C_p(\mathcal{H}), k = \frac{1}{6^{\frac{1}{p-2}}}, \text{ if } 2 \leq p < \infty$.

Proof. Consider the following decompositions

$$\mathcal{H} = \mathcal{H}_1 = (\ker A_2)^\perp \oplus \ker A_2, \quad \mathcal{H} = \mathcal{H}_2 = (\ker B_2^*)^\perp \oplus \ker B_2^*.$$

Then

$$A_2 = C_1 \oplus 0, \quad B_2 = C_2 \oplus 0,$$

where C_1, C_2^* are injective k -quasihyponormal and by hypothesis, we get

$$A_1 = T_1 \oplus T_2, \quad B_1 = R_1 \oplus R_2$$

with T_1, T_2, R_1^* , and R_2^* are quasinormal operators and $(T_1, C_1), (R_1, C_2)$ are 2-tuples of doubly commuting operators.

Since $\ker T_1$ reduces T_1 (resp. $\ker R_1^*$ reduces R_1^*) and by commutativity, we have that

$$T_1 = A_{11} \oplus 0, \quad R_1 = B_{11} \oplus 0, \quad C_1 = A_{21} \oplus A_{22}, \quad C_2 = B_{21} \oplus B_{22}$$

with respect to the following decompositions

$$(\ker A_2)^\perp = (\ker T_1)^\perp \oplus \ker T_1, \quad (\ker B_2^*)^\perp = (\ker R_1^*)^\perp \oplus \ker R_1^*,$$

where A_{11} and B_{11}^* are injective quasinormal operators, A_{21}, A_{22}, B_{21}^* , and B_{22}^* are injective k -quasihyponormal. We set $A_{11} = U|A_{11}|$ and $B_{11} = V|B_{11}| = |B_{11}^*|V$. Then it follows from the injectivity of A_{11} and B_{11}^* that U and V^* are isometry operators and by $[T_1, C_1] = [T_1^*, C_1] = 0$, we get that

$$[A_{11}, A_{21}] = [A_{11}^*, A_{21}] = 0. \quad (8)$$

Then $[A_{11}^* A_{11}, A_{21}] = [A_{11} A_{11}^*, A_{21}] = 0$ and $[|A_{11}|, A_{21}] = [|A_{11}^*|, A_{21}] = 0$. Hence, $[U, A_{21}] = [U^*, A_{21}] = 0$. Similarly, we obtain that $[V, B_{21}] = [V^*, B_{21}] = 0$, and $(UA_{21}^*)^*$ is k -quasihyponormal. Indeed,

$$\begin{aligned} (A_{21}^* U)^* k [U^* A_{21}, A_{21}^* U] A_{21}^{*k} U^k &= (A_{21})^{k+1} U^{*k} [U^*, U] U^k A_{21}^{*k+1} + U^{*k} U A_{21}^k [A_{21}, A_{21}^*] A_{21}^{*k} U^k \\ &= U^* U^{*k} A_{21}^k [A_{21}, A_{21}^*] A_{21}^{*k} U^k \leq 0. \end{aligned}$$

Similarly, we get $(V^* B_{21})^*$ is an injective k -quasihyponormal.

Let $X, S \in B(\mathcal{H}) : X = (X_{ij})_{i,j=1,2,3}$ and $S = (S_{ij})_{i,j=1,2,3}$. If $S \in \ker E$, then

$$A_{11} S_{11} B_{11} = A_{21} S_{11} B_{21}, \quad (9)$$

$$A_{11} S_{13} R_2 = T_2 S_{31} B_{11} = T_2 S_{33} R_2 = 0, \quad (10)$$

$$A_{21} S_{12} B_{22} = A_{22} S_{21} B_{21} = A_{22} S_{22} B_{22} = 0. \quad (11)$$

And

$$U^*(A_{11} X_{11} B_{11} - A_{21} X_{11} B_{21} + S_{11}) V^* = |A_{11}| X_{11} |B_{11}^*| - U^* A_{21} X_{11} B_{21} V^* + U^* S_{11} V^*.$$

We derive from 8 that

$$|A_{11}| U S_{11} V^* |B_{11}| = U^* A_{21} U^* S_{11} V^* B_{21} V^*.$$

Applying Corollary 8,

$$\|A_{11} X_{11} B_{11} - A_{21} X_{11} B_{21} + S_{11}\| \geq \|U^*(A_{11} X_{11} B_{11} - A_{21} X_{11} B_{21} + S_{11}) V^*\| \geq \|U^* S_{11} V^*\|.$$

from the injectivity of A_{11} and its polar decomposition, we have

$$(\ker T_1)^\perp = (\ker A_{11})^\perp = (\ker U)^\perp; \quad \overline{\ker A_{11}} = \text{ran } U$$

and $U : (\ker U)^\perp \rightarrow \text{ran } U$ is unitary. Taking the following decompositions yields

$$(\ker U)^\perp = \text{ran } U \oplus (\text{ran } U)^\perp; \quad (\ker V^*)^\perp = \overline{\text{ran } V^*} \oplus (\text{ran } V^*)^\perp.$$

Then

$$\begin{aligned} A_{11} &= \begin{bmatrix} A_{11}^1 & \gamma \\ 0 & 0 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} B_{11}^1 & 0 \\ \zeta & 0 \end{bmatrix}, \quad S_{11} = \begin{bmatrix} S_{11}^1 & S_{11}^2 \\ S_{11}^3 & S_{11}^4 \end{bmatrix}, \\ U &= \begin{bmatrix} U_1 \\ 0 \end{bmatrix}: (\ker U)^\perp \rightarrow \operatorname{ran} U \oplus (\operatorname{ran} U)^\perp, \\ V &= [V_1 \ 0]: \overline{\operatorname{ran} V^*} \oplus (\operatorname{ran} V^*)^\perp \rightarrow (\ker V^*)^\perp. \end{aligned}$$

From the commutativity, we obtain

$$A_{21} = A_{21}^1 \oplus A_{21}^2; \quad B_{21} = B_{21}^1 \oplus B_{21}^2.$$

By simple computation, we get

$$U^* S_{11} V^* = U_1^* S_{11}^1 V_1^*, \quad A_{21}^1 S_{11}^2 B_{21}^2 = A_{21}^2 S_{11}^3 B_{21}^1 = A_{21}^2 S_{11}^4 B_{21}^2 = 0.$$

From the injectivity of A_{21}^i and B_{21}^i : $i = 1, 2$, we derive that

$$S_{11}^2 = S_{11}^3 = S_{11}^4 = 0.$$

The injectivity of A_{21} , B_{22} , A_{22} , B_{21} , A_{11} , and B_{11}^* in the equalities 10 and 11 implies that

$$S_{12} = S_{21} = S_{22} = 0, \quad S_{13} R_2 = T_2 S_{31} = T_2 S_{33} R_2 = 0.$$

Setting $E_{11}(X_{11}) = A_{11} X_{11} B_{11} - A_{21} X_{11} B_{21}$.

(i) If $\ker A_1 \cap \ker A_2 = \{0\} = \ker B_1^* \cap \ker B_2^*$, then $S_{13} = S_{33} = S_{31} = 0$ and therefore any operator $S \in \ker E$ has the form

$$S = \begin{bmatrix} S_{11} & 0 & 0 \\ 0 & 0 & S_{23} \\ 0 & S_{32} & 0 \end{bmatrix},$$

where S_{23} and S_{32} are arbitrary with

$$\|S_{11}\|_{\mathcal{J}} = \|S_{11}^1\|_{\mathcal{J}} = \|U_1^* S_{11}^1 V_1^*\|_{\mathcal{J}} = \|U^* S_{11} V^*\|_{\mathcal{J}}$$

and

$$\|E(X) + S\|_{\mathcal{J}} = \left\| \begin{bmatrix} E_{11}(X_{11}) + S_{11} & * & * \\ * & * & S_{23} \\ * & S_{32} & * \end{bmatrix} \right\|_{\mathcal{J}}.$$

Let $S_{23} = U_{23}|S_{23}|$, $S_{32} = S_{32}|S_{32}|$ be the polar decomposition of S_{23} and S_{32} , respectively, and set the operator

$$V = I \oplus \begin{bmatrix} 0 & U_{32}^* \\ U_{23}^* & 0 \end{bmatrix}. \text{ Then}$$

$$\|E(X) + S\|_{\mathcal{J}} \geq \|V(E(X) + S)\|_{\mathcal{J}} \geq \left\| \begin{bmatrix} E_{11}(X_{11}) + S_{11} & * & * \\ * & |S_{32}| & * \\ * & * & |S_{23}| \end{bmatrix} \right\|_{\mathcal{J}}.$$

Applying Lemma 5, we get

– $\mathcal{J} = B(\mathcal{H})$:

$$\|E(X) + S\| \geq \max\{\|E(X_{11}) + S_{11}\|, \|S_{32}\|, \|S_{23}\|\} \geq \max\{\|S_{11}\|, \|S_{32}\|, \|S_{23}\|\} = \|S\|.$$

– $\mathcal{J} = C_p(\mathcal{H})$: ($1 \leq p < \infty$)

$$\|E(X) + S\|_p \geq (\|E(X_{11}) + S_{11}\|_p^p + \|S_{32}\|_p^p + \|S_{23}\|_p^p)^{1/p} \geq (\|S_{11}\|_p^p + \|S_{32}\|_p^p + \|S_{23}\|_p^p)^{1/p} = \|S\|_p^p.$$

(ii) If $\ker A_1 \cap \ker A_2 \neq \{0\}$ or $\ker B_1^* \cap \ker B_2^* \neq \{0\}$, then any operator $S \in \ker E$ has the form

$$S = \begin{bmatrix} S_{11} & 0 & S_{13} \\ 0 & 0 & S_{23} \\ S_{13} & S_{32} & S_{33} \end{bmatrix},$$

where S_{23} and S_{32} are arbitrary. By simple calculation, we have

$$\|E(X) + S\|_{\mathcal{J}} = \left\| \begin{bmatrix} E_{11}(X_{11}) + S_{11} & * & A_{11}X_{13}R_2 + S_{13} \\ * & * & S_{23} \\ T_2X_{31}B_{11} + S_{31} & S_{32} & T_2X_{33}R_2 + S_{33} \end{bmatrix} \right\|_{\mathcal{J}}.$$

It is well known that the kernel of a quasinormal operator is a reduced subspace, then by application of Corollary 8, we obtain

$$\begin{aligned} \|A_{11}X_{13}R_2 + S_{13}\|_{\mathcal{J}} &\geq k\|S_{13}\|_{\mathcal{J}} \\ \|T_2X_{31}B_{11} + S_{31}\|_{\mathcal{J}} &\geq k\|S_{31}\|_{\mathcal{J}} \\ \|T_2X_{33}R_2 + S_{33}\|_{\mathcal{J}} &\geq k\|S_{33}\|_{\mathcal{J}}. \end{aligned}$$

Therefore, by Lemma 6, we get

– $\mathcal{J} = B(\mathcal{H})$;

$$\|E(X) + S\|^2 \geq \frac{1}{3^2} \left(\|E_{11}(X_{11}) + S_{11}\|^2 + \frac{1}{2^2} [\|S_{32}\|^2 + \|S_{23}\|^2 + \|S_{13}\|^2 + \|S_{31}\|^2 + \|S_{33}\|^2] \right) \geq \frac{1}{6^2} \|S\|^2.$$

– $\mathcal{J} = C_p(\mathcal{H})$: ($2 \leq p < \infty$)

$$\begin{aligned} \|E(X) + S\|_p^p &\geq \frac{1}{3^{p-2}} \|S_{11}\|_p^p + \frac{1}{2^{p-2}} [\|S_{32}\|_p^p + \|S_{23}\|_p^p + \|S_{13}\|_p^p + \|S_{31}\|_p^p + \|S_{33}\|_p^p] \\ &\geq \frac{1}{2^{p-2}} \frac{1}{3^{p-2}} \|S\|_p^p = \frac{1}{6^{p-2}} \|S\|_p^p. \end{aligned}$$

– $\mathcal{J} = C_p(\mathcal{H})$: ($1 \leq p \leq 2$):

$$\begin{aligned} \|E(X) + S\|_p^p &\geq \frac{1}{3^{2-p}} \|S_{11}\|_p^p + \frac{1}{2^{2-p}} [\|S_{32}\|_p^p + \|S_{23}\|_p^p + \|S_{13}\|_p^p + \|S_{31}\|_p^p + \|S_{33}\|_p^p] \\ &\geq \frac{1}{2^{2-p}} \frac{1}{3^{2-p}} \|S\|_p^p = \frac{1}{6^{p-2}} \|S\|_p^p. \end{aligned}$$

□

4 Conclusion

D. Keckic [14] and A. Turnšek [15] extended Theorem 2 to the elementary operator E defined by $E(X) = AXB - CXD$, where (A, C) and (B, D) are 2-tuples of commuting normal operators. Duggal [16] generalized the famous theorem to the case (A, C) and (B, D) are 2-tuples of commuting operators, where A, B are normal and C, D^* are hyponormal.

In this paper, Theorem 2 was extended to non-normal operators including quasinormal, subnormal, and k -quasihyponormal operators. The main results are Theorems 12 and 18, both of considerable value in the relevant area of research, also the paper includes new ideas, along with a few new tools and techniques, and likely to attract considerable attention from researchers in operator theory and Banach space theory.

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