Research Article

Xiaohuang Huang, Bingmao Deng, and Mingliang Fang*

Entire functions that share two pairs of small functions

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Abstract: In this paper, we study the unicity of entire functions and their derivatives and obtain the following result: let f be a non-constant entire function, let a_1, a_2, b_1 , and b_2 be four small functions of f such that $a_1 \neq b_1, a_2 \neq b_2$, and none of them is identically equal to ∞ . If f and $f^{(k)}$ share (a_1, a_2) CM and share (b_1, b_2) IM, then $(a_2 - b_2)f - (a_1 - b_1)f^{(k)} \equiv a_2b_1 - a_1b_2$. This extends the result due to Li and Yang [*Value sharing of an entire function and its derivatives*, J. Math. Soc. Japan. **51** (1999), no. 7, 781–799].

Keywords: unicity, entire functions, derivatives, small functions

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1 Introduction and main results

In this paper, we use the general notations in the Nevanlinna value distribution theory, see ([1–3]).

Let f be a meromorphic function on the whole complex plane. For $0 < r < R$, we define the following functions:

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta$$

is the average of the positive logarithmic $|f(z)|$ on the circle $|z| = r$;

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r$$

is called the counting function of poles of $f(z)$, where $n(r, f)$ denotes the number of poles of $f(z)$ on the disc $|z| \leq t$, multiple poles are counted according to their multiplicities, and $n(0, f)$ denotes the multiplicity of pole of $f(z)$ at the origin (if $f(0) \neq \infty$, then $n(0, f) = 0$). We denote $\overline{N}(r, f)$ the reduced counting function of $f(z)$ whose the multiplicity of poles only counts once.

$$T(r, f) = m(r, f) + N(r, f).$$

$T(r, f)$ is said to be the characteristic function of $f(z)$ which is obviously a non-negative function.

* **Corresponding author: Mingliang Fang**, Department of Mathematics, Hangzhou Dianzi University, Hangzhou 310012, China, e-mail: mlfang@hdu.edu.cn

Xiaohuang Huang: Department of Mathematics, Hangzhou Dianzi University, Hangzhou 310012, China, e-mail: 1838394005@qq.com

Bingmao Deng: School of Financial Mathematics & Statistics, Guangdong University of Finance, Guangzhou 510521, China, e-mail: dbmao2012@163.com

Let a be a complex number. Obviously, $\frac{1}{f(z)-a}$ is meromorphic on the disc $|z| \leq R$. Similar to the above definitions, we define the following functions.

$$m\left(r, \frac{1}{f-a}\right) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\log^+ |f(re^{i\theta}) - a|} d\theta$$

is the average of the positive logarithmic $\left|\frac{1}{f(z)-a}\right|$ on the circle $|z| = r$.

$$N\left(r, \frac{1}{f(z)-a}\right) = \int_0^r \frac{n\left(t, \frac{1}{f(z)-a}\right) - n\left(0, \frac{1}{f(z)-a}\right)}{t} dt + n(0, f) \log r,$$

where $n\left(r, \frac{1}{f(z)-a}\right)$ denotes the number of zeros of $f(z) - a$ on the disc $|z| \leq t$, multiple poles are counted according to their multiplicities. $n\left(0, \frac{1}{f(z)-a}\right)$ denotes the multiplicity of zeros of $f(z) - a$ at the origin. We denote $\bar{N}\left(r, \frac{1}{f-a}\right)$ the reduced counting function of $f(z) - a$ whose the multiplicity of zero only counts once.

$$T\left(r, \frac{1}{f(z)-a}\right) = m\left(r, \frac{1}{f(z)-a}\right) + N\left(r, \frac{1}{f(z)-a}\right).$$

$T\left(r, \frac{1}{f(z)-a}\right)$ is said to be the characteristic function of $\frac{1}{f(z)-a}$.

Obviously, $T(r, f)$ is a non-decreasing function of r , and a convex function of $\log r$.

We denote $S(r, f) = o(T(r, f))$, as $r \rightarrow \infty$ outside of a possible exceptional set of finite linear measures. We say that a meromorphic function a is called a small function of f if it satisfies $T(r, a) = S(r, f)$.

Nevanlinna established two fundamental theorems, called the first fundamental theorem and the second fundamental theorem.

The first fundamental theorem. Let f be a meromorphic function on the whole complex plane. Then,

$$T(r, f) = T\left(r, \frac{1}{f-a}\right) + O(1).$$

The second fundamental theorem. Let f be a meromorphic function on the whole complex plane and let $a_1, \dots, a_q$ ($q \geq 3$) be distinct complex values in the extended plane. Then,

$$(q-2)T(r, f) \leq \sum_{j=1}^q \bar{N}\left(r, \frac{1}{f-a_j}\right) + S(r, f).$$

Let f and g be two meromorphic functions, and let a and b be two small functions of f and g . We say that f and g share a pair of small functions (a, b) CM (resp. IM) if $f - a$ and $g - b$ have the same zeros counting multiplicities (resp. ignoring multiplicities). When $a = b$, we say that f and g share a CM (resp. IM) if $f - a$ and $g - a$ have the same zeros counting multiplicities (resp. ignoring multiplicities).

In [3], Nevanlinna proved the following famous five-value theorem:

Theorem A. Let f and g be two non-constant meromorphic functions, and let a_j ($j = 1, 2, 3, 4, 5$) be five distinct values in the extended complex plane. If f and g share a_j ($j = 1, 2, 3, 4, 5$) IM, then $f \equiv g$.

Brosch [4], Czubiak and Gundersen [5], Gundersen [6], Steinmetz [7], and Gundersen et al. [8] studied shared pairs of values. For five shared pairs of values, Gundersen et al. [8] proved the next result, which is the best possible:

Theorem B. If f and g are two non-constant meromorphic functions that share two pairs of values CM and share three other pairs of values IM, then f is a Möbius transformation of g .

Li and Qiao [9] proved that Theorem A is still valid for five distinct small functions, and they proved the following:

Theorem C. *Let f and g be two non-constant meromorphic functions, and let a_j ($j = 1, 2, 3, 4, 5$) (one of them can be ∞) be five distinct small functions of f and g . If f and g share a_j ($j = 1, 2, 3, 4, 5$) IM, then $f \equiv g$.*

Zhang and Yang [10] and Nguyen and Si [11] studied shared pairs of small functions.

Rubel and Yang [12] investigated the uniqueness of an entire function and its derivative and proved.

Theorem D. *Let f be a non-constant entire function, and let a, b be two finite distinct complex values. If f and f' share a, b CM, then $f \equiv f'$.*

Zheng and Wang [13] improved Theorem D and proved the following: if f is a non-constant entire function, and f and $f^{(k)}$ share a, b CM, where a and b are two small functions of f , then $f \equiv f^{(k)}$.

Li and Yang [14] improved the result of Zheng and Wang [13] and proved the following result:

Theorem E. *Let f be a non-constant entire function, and let a, b be two distinct small functions of f . If f and $f^{(k)}$ share a CM, and share b IM, then $f \equiv f^{(k)}$.*

In this paper, we study shared pair of small functions and extend Theorem E as follows.

Theorem 1. *Let f be a non-constant entire function, let a_1, a_2, b_1 , and b_2 be four small functions of f such that $a_1 \neq b_1$ and $a_2 \neq b_2$, and none of them is identically equal to ∞ . If f and $f^{(k)}$ share (a_1, a_2) CM, and share (b_1, b_2) IM, then $(a_2 - b_2)f - (a_1 - b_1)f^{(k)} \equiv a_2b_1 - a_1b_2$.*

Remark 1. If $a_1 \equiv a_2, b_1 \equiv b_2$, then by Theorem 1, we obtain Theorem F.

The following example shows that the conclusion of Theorem 1 is not valid for meromorphic functions.

Example 1. [15] Let $f = h + \frac{h-g}{q-1}$, where

$$g = -\left(\frac{1}{3e^{2z}} + \frac{1}{2e^z}\right), \quad h = \frac{1}{2e^z} - \frac{1}{3e^{2z}}, \quad q = e^{-e^z},$$

and let $a = h', b = g'$. By simple calculation, we obtain

$$\begin{aligned} T(r, a) &= S(r, f), \quad T(r, b) = S(r, f), \\ f' - a &= e^{2z}(f - a)(f - h), \quad f' - b = e^{2z}(f - b)(f - g). \end{aligned}$$

So f and f' share (a, a) CM and (b, b) IM. However, $f \neq f'$.

The following example shows that there exist a transcendental entire function f and two pairs of small functions of f satisfying Theorem 1.

Example 2. [16] Suppose $f = e^{3z} + 10z, a_1 = 11z, a_2 = 9z, b_1 = 10z, b_2 = 0$. Then it is an easy work to obtain

$$T(r, a_i) = S(r, f), \quad T(r, b_i) = S(r, f)$$

for $i = 1, 2$, and

$$\begin{aligned} f - a_1 &= e^{3z} - z, \quad f'' - b_1 = 9(e^{3z} - z), \\ f - a_2 &= e^{3z}, \quad f'' - b_2 = 9e^{3z}. \end{aligned}$$

Thus, we see that f and f'' share (a_1, b_1) CM and (a_2, b_2) IM. Furthermore, $(a_2 - b_2)f - (a_1 - b_1)f'' \equiv a_2b_1 - a_1b_2$.

In 2020, Sahoo-Halder [16] proved.

Theorem F. Let f be a non-constant entire function of finite order, let $f \in S_k(a_1, a_2, b_1, b_2)$, and let a_1, a_2, b_1 and b_2 be four small functions of f such that $a_1 \neq b_1$ and $a_2 \neq b_2$, none of them is identically equal to ∞ , and $a_i - b_i$ have at least one zero for $i = 1, 2$. If f and $f^{(k)}$ share (a_1, a_2) and (b_1, b_2) IM, then $(a_2 - b_2)f - (a_1 - b_1)f^{(k)} \equiv a_2b_1 - a_1b_2$.

The authors raised two questions as follows.

Question 1

The condition of finite order of Theorem F and Corollary 1 in [16] can be removed in any way?

Question 2

How far do the conclusions in Theorem F and Corollary 1 in [16] hold for a non-constant entire function?

In Theorem 1, we answer Question 1 in a setting with stronger conditions, and we answer that for any positive integers k Theorem F and Corollary 1 in [16] hold.

2 Some lemmas

Lemma 2.1. [1–3] Let f be a non-constant meromorphic function, and let k be a positive integer. Then

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f).$$

Lemma 2.2. [2] Let f be a meromorphic function on the complex plane, let n be a positive integer, and let $\psi(f) = a_0f + a_1f' + \dots + a_nf^{(n)}$, where $a_0, a_1, \dots, a_n (\neq 0)$ are small functions of f . Then

$$m\left(r, \frac{\psi(f)}{f}\right) = S(r, f), \quad T(r, \psi) \leq T(r, f) + k\overline{N}(r, f) + S(r, f).$$

Lemma 2.3. [1–3] Let f_1, f_2 be two non-constant meromorphic functions, then

$$N(r, f_1f_2) - N\left(r, \frac{1}{f_1f_2}\right) = N(r, f_1) + N(r, f_2) - N\left(r, \frac{1}{f_1}\right) - N\left(r, \frac{1}{f_2}\right).$$

Lemma 2.4. Let f be a transcendental meromorphic function, let a_1, a_2, b_1 , and b_2 be four small functions of f such that $a_1 \neq b_1$ and $a_2 \neq b_2$, and let

$$L(f) = \begin{vmatrix} f - a_1 & a_1 - b_1 \\ f' - a_1' & a_1' - b_1' \end{vmatrix}, \quad L(f^{(k)}) = \begin{vmatrix} f^{(k)} - a_2 & a_2 - b_2 \\ f^{(k+1)} - a_2' & a_2' - b_2' \end{vmatrix}.$$

If f and $f^{(k)}$ share (a_1, a_2) CM, share (b_1, b_2) IM, then $L(f) \neq 0, L(f^{(k)}) \neq 0$.

Proof. Suppose that $L(f) \equiv 0$. Then we get $\frac{f' - a_1'}{f - a_1} \equiv \frac{a_1' - b_1'}{a_1 - b_1}$. It follows that $f - a_1 = c(a_1 - b_1)$, where c is a nonzero constant. So $T(r, f) = T(r, c(a_1 - b_1) + a_1) = S(r, f)$, a contradiction. Hence, $L(f) \neq 0$.

Since f is an entire function, f and $f^{(k)}$ share (a_1, a_2) CM, share (b_1, b_2) IM, we have

$$\begin{aligned} T(r, f) &\leq \overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - b_1}\right) + S(r, f) = \overline{N}\left(r, \frac{1}{f^{(k)} - a_2}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - b_2}\right) + S(r, f) \\ &\leq 2T(r, f^{(k)}) + S(r, f). \end{aligned}$$

Hence, a_2, b_2 are small functions of $f^{(k)}$. Using the same argument as proving $L(f) \neq 0$, it is easy to obtain $L(f^{(k)}) \neq 0$. Thus, the lemma is proved. $\square$

Lemma 2.5. Let k_j ($j = 1, 2, \dots, q$) be positive integers, let a_1, b_1 be two distinct small functions of f and let $d_j = a_1 - l_j(a_1 - b_1)$ ($j = 1, 2, \dots, q$). Then,

$$m\left(r, \frac{L(f)}{f - a_1}\right) = S(r, f), \quad m\left(r, \frac{L(f)}{f - b_1}\right) = S(r, f)$$

and

$$m\left(r, \frac{L(f)f}{(f - d_1)(f - d_2)\cdots(f - d_m)}\right) = S(r, f), \quad (2.1)$$

where $L(f)$ is defined as Lemma 2.4, and $2 \leq m \leq q$.

Proof. Since $L(f) = (a'_1 - b'_1)(f - a_1) - (a_1 - b_1)(f' - a'_1)$, by Lemma 2.1, we have

$$m\left(r, \frac{L(f)}{f - a_1}\right) \leq m(r, a'_1 - b'_1) + m\left(r, \frac{(a_1 - b_1)(f' - a'_1)}{f - a_1}\right) = S(r, f).$$

Similarly, we have

$$m\left(r, \frac{L(f)}{f - b_1}\right) = S(r, f).$$

Obviously,

$$L(f) = \begin{vmatrix} f - a_1 + l_j(a_1 - b_1) & a_1 - b_1 \\ f' - a'_1 + l_j(a'_1 - b'_1) & a'_1 - b'_1 \end{vmatrix} = \begin{vmatrix} f - d_j & a_1 - b_1 \\ f' - d'_j & a'_1 - b'_1 \end{vmatrix},$$

$$\frac{L(f)f}{(f - d_1)(f - d_2)\cdots(f - d_q)} = \sum_{i=1}^q \frac{c_i L(f)}{f - d_i},$$

where c_i ($i = 1, 2, \dots, q$) are small functions of f .

Thus, by Lemma 2.2 and above two formulas, we obtain (2.1). $\square$

Lemma 2.6. Let f and g be two non-constant entire functions and let a_1, a_2, b_1 , and b_2 be four small functions of f and g such that $a_1 \neq b_1, a_2 \neq b_2$, and none of them is identically equal to ∞ . Suppose that

$$H = \frac{L(f)}{(f - a_1)(f - b_1)} - \frac{L(g)}{(g - a_2)(g - b_2)} \equiv 0,$$

where

$$L(f) = (a'_1 - b'_1)(f - a_1) - (a_1 - b_1)(f' - a'_1),$$

$$L(g) = (a'_2 - b'_2)(g - a_2) - (a_2 - b_2)(g' - a'_2).$$

If f and g share (a_1, a_2) CM and share (b_1, b_2) IM, then either

$$2T(r, f) \leq \overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - b_1}\right) + S(r, f) \quad (2.2)$$

or

$$(a_2 - b_2)f - (a_1 - b_1)g + a_1 b_2 - a_2 b_1 \equiv 0. \quad (2.3)$$

Proof. Since $H \equiv 0$, it is easy to get

$$\frac{g - b_2}{g - a_2} = c \frac{f - b_1}{f - a_1}, \quad (2.4)$$

where c is a nonzero constant.

If $c = 1$, then by (2.4), we get (2.3).

If $c \neq 1$, then by (2.4), we have

$$\frac{a_2 - b_2}{g - a_2} \equiv \frac{(c - 1)f - (cb_1 - a_1)}{f - a_1} \quad (2.5)$$

and

$$T(r, f) = T(r, g) + S(r, f) + S(r, g).$$

Obviously, $\frac{cb_1 - a_1}{c - 1} \neq a_1$, $\frac{cb_1 - a_1}{c - 1} \neq b_1$. It follows from (2.5) and the fact that f and g share (a_1, a_2) CM that

$N\left(r, \frac{1}{f - \frac{cb_1 - a_1}{c - 1}}\right) = N\left(r, \frac{1}{a_2 - b_2}\right) = S(r, f)$. Thus by Nevanlinna's second fundamental theorem, we have

$$\begin{aligned} 2T(r, f) &\leq \bar{N}\left(r, \frac{1}{f - a_1}\right) + \bar{N}\left(r, \frac{1}{f - b_1}\right) + \bar{N}\left(r, \frac{1}{f - \frac{cb_1 - a_1}{c - 1}}\right) + S(r, f) \\ &\leq \bar{N}\left(r, \frac{1}{f - a_1}\right) + \bar{N}\left(r, \frac{1}{f - b_1}\right) + S(r, f). \end{aligned}$$

It follows that $2T(r, f) \leq \bar{N}\left(r, \frac{1}{f - a_1}\right) + \bar{N}\left(r, \frac{1}{f - b_1}\right) + S(r, f)$. □

3 Proof of Theorem 1

We prove Theorem 1 by contradiction. Suppose that $(a_2 - b_2)f - (a_1 - b_1)f^{(k)} + a_1b_2 - a_2b_1 \neq 0$. Set

$$\frac{f - a_1}{f^{(k)} - a_2} = H. \quad (3.1)$$

Since f is a transcendental entire function, and f and $f^{(k)}$ share (a_1, a_2) CM, then by (3.1), we have

$$N(r, H) = S(r, f), \quad N\left(r, \frac{1}{H}\right) = S(r, f). \quad (3.2)$$

From the fact that f and $f^{(k)}$ share (a_1, a_2) CM and share (b_1, b_2) IM, and f is a transcendental entire function, by Nevanlinna's second fundamental theorem and Lemma 2.1, we obtain

$$\begin{aligned} T(r, f) &\leq \bar{N}\left(r, \frac{1}{f - a_1}\right) + \bar{N}\left(r, \frac{1}{f - b_1}\right) + \bar{N}(r, f) + S(r, f) \\ &= \bar{N}\left(r, \frac{1}{f^{(k)} - a_2}\right) + \bar{N}\left(r, \frac{1}{f^{(k)} - b_2}\right) + S(r, f) \\ &\leq N\left(r, \frac{1}{(a_2 - b_2)f - (a_1 - b_1)f^{(k)} + a_1b_2 - a_2b_1}\right) + S(r, f) \\ &\leq T(r, (a_2 - b_2)f - (a_1 - b_1)f^{(k)} + a_1b_2 - a_2b_1) + S(r, f) \\ &\leq m(r, (a_2 - b_2)f - (a_1 - b_1)f^{(k)} + a_1b_2 - a_2b_1) + S(r, f) \\ &\leq m(r, f) + m\left(r, \frac{(a_2 - b_2)f - (a_1 - b_1)f^{(k)}}{f}\right) + S(r, f) \\ &\leq T(r, f) + S(r, f). \end{aligned}$$

Thus, we have

$$T(r, f) = \bar{N}\left(r, \frac{1}{f - a_1}\right) + \bar{N}\left(r, \frac{1}{f - b_1}\right) + S(r, f) \quad (3.3)$$

and

$$\begin{aligned} T(r, f) &= T(r, (a_2 - b_2)f - (a_1 - b_1)f^{(k)} + a_1b_2 - a_2b_1) + S(r, f) \\ &= N\left(r, \frac{1}{(a_2 - b_2)f - (a_1 - b_1)f^{(k)} + a_1b_2 - a_2b_1}\right) + S(r, f). \end{aligned} \quad (3.4)$$

Obviously,

$$T(r, H) = m\left(r, \frac{1}{H}\right) + S(r, f) = m\left(r, \frac{f^{(k)} - a_2}{f - a_1}\right) + S(r, f) \leq m\left(r, \frac{1}{f - a_1}\right) + S(r, f). \quad (3.5)$$

It follows from (3.1) and (3.4) that

$$\begin{aligned} m\left(r, \frac{1}{f - a_1}\right) &= m\left(r, -\frac{(a_1 - b_1)H^{-1} + b_2 - a_2}{(a_2 - b_2)f - (a_1 - b_1)f^{(k)} + a_1b_2 - a_2b_1}\right) \\ &\leq m\left(r, \frac{1}{(a_2 - b_2)f - (a_1 - b_1)f^{(k)} + a_1b_2 - a_2b_1}\right) + m\left(r, \frac{a_1 - b_1}{H} + b_2 - a_2\right) + S(r, f) \\ &\leq T(r, H) + S(r, f). \end{aligned} \quad (3.6)$$

By (3.5) and (3.6), we get

$$T(r, H) = m\left(r, \frac{1}{f - a_1}\right) + S(r, f). \quad (3.7)$$

On the other hand, (3.1) can be rewritten as

$$\frac{(a_2 - b_2)f - (a_1 - b_1)f^{(k)} + a_1b_2 - a_2b_1}{f - a_1} = -[(a_1 - b_1)H^{-1} + b_2 - a_2], \quad (3.8)$$

which implies

$$\bar{N}\left(r, \frac{1}{f - b_1}\right) \leq \bar{N}\left(r, \frac{1}{(a_1 - b_1)H^{-1} + b_2 - a_2}\right) \leq T(r, H) + S(r, f). \quad (3.9)$$

Thus, by (3.3), (3.7), and (3.9)

$$\begin{aligned} m\left(r, \frac{1}{f - a_1}\right) + N\left(r, \frac{1}{f - a_1}\right) &= \bar{N}\left(r, \frac{1}{f - a_1}\right) + \bar{N}\left(r, \frac{1}{f - b_1}\right) + S(r, f) \\ &\leq \bar{N}\left(r, \frac{1}{f - a_1}\right) + \bar{N}\left(r, \frac{1}{(a_1 - b_1)H^{-1} + b_2 - a_2}\right) + S(r, f) \\ &\leq \bar{N}\left(r, \frac{1}{f - a_1}\right) + m\left(r, \frac{1}{f - a_1}\right) + S(r, f). \end{aligned}$$

Hence, by the above formula, (3.7), (3.9), we obtain

$$N\left(r, \frac{1}{f - a_1}\right) = \bar{N}\left(r, \frac{1}{f - a_1}\right) + S(r, f), \quad (3.10)$$

$$\bar{N}\left(r, \frac{1}{f - b_1}\right) = T(r, H) + S(r, f). \quad (3.11)$$

Set

$$\varphi = \frac{L(f)((a_2 - b_2)f - (a_1 - b_1)f^{(k)} + a_1b_2 - a_2b_1)}{(f - a_1)(f - b_1)} \quad (3.12)$$

and

$$\psi = \frac{L(f^{(k)})((a_2 - b_2)f - (a_1 - b_1)f^{(k)} + a_1b_2 - a_2b_1)}{(f^{(k)} - a_2)(f^{(k)} - b_2)}. \quad (3.13)$$

Obviously, $\varphi \neq 0$. By (3.12), it is easy to know that $N(r, \varphi) = S(r, f)$. Thus, by Lemmas 2.1, 2.4, and 2.5, we have

$$\begin{aligned} T(r, \varphi) &= m(r, \varphi) + N(r, \varphi) \\ &= m\left(r, \frac{L(f)((a_2 - b_2)f - (a_1 - b_1)f^{(k)} + a_1b_2 - a_2b_1)}{(f - a_1)(f - b_1)}\right) + S(r, f) \\ &\leq m\left(r, \frac{L(f)f}{(f - a_1)(f - b_1)}\right) + m\left(r, \frac{(a_2 - b_2)f - (a_1 - b_1)f^{(k)}}{f}\right) \\ &\quad + m\left(r, \frac{L(f)(a_1b_2 - a_2b_1)}{(f - a_1)(f - b_1)}\right) + S(r, f) \leq S(r, f), \end{aligned}$$

that is

$$T(r, \varphi) = S(r, f). \quad (3.14)$$

Let $d_1 = a_1 - k(a_1 - b_1)$ ($k \neq 0, 1$) and $d_2 = a_2 - k(a_2 - b_2)$. It follows from Nevanlinna's second fundamental theorem and (3.3) that

$$\begin{aligned} 2T(r, f) &\leq \bar{N}\left(r, \frac{1}{f - a_1}\right) + \bar{N}\left(r, \frac{1}{f - b_1}\right) + \bar{N}\left(r, \frac{1}{f - d_1}\right) + S(r, f) \\ &\leq T(r, f) + \bar{N}\left(r, \frac{1}{f - d_1}\right) + S(r, f) \\ &\leq 2T(r, f) - m\left(r, \frac{1}{f - d_1}\right) + S(r, f), \end{aligned}$$

which implies

$$m\left(r, \frac{1}{f - d_1}\right) = S(r, f). \quad (3.15)$$

Rewrite (3.13) as

$$\psi = \left[\frac{a_2 - d_2}{a_2 - b_2} \frac{L(f^{(k)})}{f^{(k)} - a_2} - \frac{b_2 - d_2}{a_2 - b_2} \frac{L(f^{(k)})}{f^{(k)} - b_2} \right] \left[\frac{(a_2 - b_2)(f - d_1)}{f^{(k)} - d_2} - a_1 + b_1 \right] \quad (3.16)$$

and set

$$\phi = \frac{L(f)}{(f - a_1)(f - b_1)} - \frac{L(f^{(k)})}{(f^{(k)} - a_2)(f^{(k)} - b_2)}. \quad (3.17)$$

Next, we consider two cases.

Case 1. $\phi \equiv 0$. Integrating both sides of (3.17) which implies

$$\frac{f - a_1}{f - b_1} = C \frac{f^{(k)} - a_2}{f^{(k)} - b_2}, \quad (3.18)$$

where C is a nonzero constant.

Then, by Lemma 2.6, we get

$$2T(r, f) \leq \bar{N}\left(r, \frac{1}{f - a_1}\right) + \bar{N}\left(r, \frac{1}{f - b_1}\right) + S(r, f), \quad (3.19)$$

which contradicts with (3.3).

Case 2. $\phi \neq 0$. It is easy to obtain that $m(r, \phi) = S(r, f)$, and all poles of ϕ must come from the zeros of $f - b_1$. Hence, we have

$$T(r, \phi) = m(r, \phi) + N(r, \phi) \leq \bar{N}\left(r, \frac{1}{f - b_1}\right) + S(r, f). \quad (3.20)$$

It follows from (3.4), (3.17), and (3.20) that

$$\begin{aligned} m(r, f) &= m(r, (a_2 - b_2)f - (a_1 - a_1)f^{(k)} + a_1b_2 - a_2b_1) + S(r, f) \\ &= m\left(r, \frac{\phi((a_2 - b_2)f - (a_1 - a_1)f^{(k)} + a_1b_2 - a_2b_1)}{\phi}\right) + S(r, f) \\ &= m\left(r, \frac{\varphi - \psi}{\phi}\right) + S(r, f) \\ &\leq T\left(r, \frac{\phi}{\varphi - \psi}\right) + S(r, f) \\ &\leq T(r, \varphi - \psi) + T(r, \phi) + S(r, f) \\ &\leq T(r, \psi) + T(r, \phi) + S(r, f) \\ &\leq T(r, \psi) + \bar{N}\left(r, \frac{1}{f - b_1}\right) + S(r, f). \end{aligned} \quad (3.21)$$

On the other hand, by Lemma 2.4, (3.1), (3.8), and (3.11), we have

$$\begin{aligned} T(r, \psi) &= T\left(r, \frac{L(f^{(k)})((a_2 - b_2)f - (a_1 - a_1)f^{(k)} + a_1b_2 - a_2b_1)}{(f^{(k)} - a_2)(f^{(k)} - b_2)}\right) \\ &= m\left(r, \frac{L(f^{(k)})((a_2 - b_2)f - (a_1 - a_1)f^{(k)} + a_1b_2 - a_2b_1)}{(f^{(k)} - a_2)(f^{(k)} - b_2)}\right) + S(r, f) \\ &\leq m\left(r, \frac{L(f^{(k)})}{f^{(k)} - b_2}\right) + m\left(r, \frac{(a_2 - b_2)f - (a_1 - a_1)f^{(k)} + a_1b_2 - a_2b_1}{f^{(k)} - a_2}\right) + S(r, f) \\ &\leq m\left(r, \frac{(a_2 - b_2)f - (a_1 - a_1)f^{(k)} + a_1b_2 - a_2b_1}{f - a_1} \frac{f - a_1}{f^{(k)} - a_2}\right) + S(r, f) \\ &\leq m(r, (a_2 - b_2)H + b_1 - a_1) + S(r, f) \\ &\leq m\left(r, \frac{1}{f - a_1}\right) + S(r, f) \\ &= \bar{N}\left(r, \frac{1}{f - b_1}\right) + S(r, f). \end{aligned} \quad (3.22)$$

Thus, by (3.21) and (3.22), we obtain

$$T(r, f) \leq 2\bar{N}\left(r, \frac{1}{f - b_1}\right) + S(r, f). \quad (3.23)$$

If $a_1^{(k)} \equiv a_2$, then by (3.1), (3.2), and Lemma 2.1, we get

$$T(r, H) = m\left(r, \frac{1}{H}\right) + S(r, f) = m\left(r, \frac{f^{(k)} - a_1^{(k)}}{f - a_1}\right) + S(r, f) \leq m\left(r, \frac{f^{(k)} - a_1^{(k)}}{f - a_1}\right) + S(r, f) = S(r, f). \quad (3.24)$$

It follows from (3.11), (3.23), and (3.24) that $T(r, f) = S(r, f)$, a contradiction.

If $a_1^{(k)} \equiv b_2$, then by (3.11), (3.23), and

$$\begin{aligned} T(r, f) &\leq m\left(r, \frac{1}{f - a_1}\right) + \bar{N}\left(r, \frac{1}{f - b_1}\right) + S(r, f) \\ &\leq m\left(r, \frac{1}{f^{(k)} - b_2}\right) + N\left(r, \frac{1}{f^{(k)} - b_2}\right) + S(r, f) \\ &\leq T(r, f^{(k)}) + S(r, f), \end{aligned}$$

which implies

$$T(r, f) \leq T(r, f^{(k)}) + S(r, f). \quad (3.25)$$

On the other hand, it follows from Lemma 2.2 that

$$T(r, f^{(k)}) \leq T(r, f) + S(r, f). \quad (3.26)$$

Thus,

$$T(r, f) = T(r, f^{(k)}) + S(r, f). \quad (3.27)$$

Then, by Nevanlinna's second fundamental theorem, Lemma 2.1, (3.3), and (3.27), we have

$$\begin{aligned} 2T(r, f) &\leq 2T(r, f^{(k)}) + S(r, f) \\ &\leq \bar{N}\left(r, \frac{1}{f^{(k)} - a_2}\right) + \bar{N}\left(r, \frac{1}{f^{(k)} - b_2}\right) + \bar{N}\left(r, \frac{1}{f^{(k)} - d_2}\right) + S(r, f) \\ &\leq \bar{N}\left(r, \frac{1}{f - a_1}\right) + \bar{N}\left(r, \frac{1}{f - b_1}\right) + T\left(r, \frac{1}{f^{(k)} - d_2}\right) - m\left(r, \frac{1}{f^{(k)} - d_2}\right) + S(r, f) \\ &\leq T(r, f) + T(r, f^{(k)}) - m\left(r, \frac{1}{f^{(k)} - d_2}\right) + S(r, f) \\ &\leq 2T(r, f) - m\left(r, \frac{1}{f^{(k)} - d_2}\right) + S(r, f). \end{aligned}$$

Thus,

$$m\left(r, \frac{1}{f^{(k)} - d_2}\right) = S(r, f). \quad (3.28)$$

It is easy to see that $N(r, \psi) = S(r, f)$. By Nevanlinna's first fundamental theorem, Lemmas 2.1 and 2.3, (3.15), (3.27), (3.28), and the fact that f is a transcendental entire function, we obtain

$$\begin{aligned} m\left(r, \frac{f - d_1}{f^{(k)} - d_2}\right) &= T\left(r, \frac{f^{(k)} - d_2}{f - d_1}\right) - N\left(r, \frac{f - d_1}{f^{(k)} - d_2}\right) + O(1) \\ &\leq m\left(r, \frac{f^{(k)} - d_1^{(k)}}{f - d_1}\right) + m\left(r, \frac{d_1^{(k)} - d_2}{f - d_1}\right) + N\left(r, \frac{f^{(k)} - d_2}{f - d_1}\right) - N\left(r, \frac{f - d_1}{f^{(k)} - d_2}\right) + O(1) \\ &\leq N\left(r, \frac{1}{f - d_1}\right) - N\left(r, \frac{1}{f^{(k)} - d_2}\right) + S(r, f) \\ &\leq T(r, f) - T(r, f^{(k)}) + S(r, f) = S(r, f). \end{aligned}$$

It follows from Lemma 2.5 and (3.16) that

$$\begin{aligned} T(r, \psi) &= m(r, \psi) + N(r, \psi) \\ &= m\left(r, \frac{a_2 - d_2}{a_2 - b_2} \frac{L(f^{(k)})}{f^{(k)} - a_2}\right) + m\left(r, \frac{b_2 - d_2}{a_2 - b_2} \frac{L(f^{(k)})}{f^{(k)} - b_2}\right) + S(r, f) \\ &\quad + m\left(r, \frac{(a_2 - b_2)(f - d_1)}{f^{(k)} - d_2} - a_1 + b_1\right) + S(r, f) = S(r, f). \end{aligned} \quad (3.29)$$

By (3.3), (3.10), (3.21), and (3.29), we get

$$N\left(r, \frac{1}{f-a_1}\right) = \bar{N}\left(r, \frac{1}{f-a_1}\right) + S(r, f) = S(r, f). \quad (3.30)$$

Moreover, by (3.3), (3.27), and (3.30), we have

$$m\left(r, \frac{1}{(f-a_1)^{(k)}}\right) = S(r, f), \quad (3.31)$$

which implies

$$\bar{N}\left(r, \frac{1}{(f-a_1)^{(k)}}\right) = m\left(r, \frac{1}{f-a_1}\right) + S(r, f) \leq m\left(r, \frac{1}{(f-a_1)^{(k)}}\right) + S(r, f) \leq S(r, f). \quad (3.32)$$

Then by (3.3), (3.30), and (3.32), we obtain $T(r, f) = S(r, f)$, a contradiction.

Hence, we have $a_1^{(k)} \neq a_2$ and $a_1^{(k)} \neq b_2$. By (3.7), (3.11), and (3.23), we obtain

$$\begin{aligned} T(r, f) &\leq 2m\left(r, \frac{1}{f-a_1}\right) + S(r, f) \\ &\leq 2m\left(r, \frac{1}{f^{(k)}-a_1^{(k)}}\right) + S(r, f) \\ &= 2T(r, f^{(k)}) - 2N\left(r, \frac{1}{f^{(k)}-a_1^{(k)}}\right) + S(r, f) \\ &\leq \bar{N}\left(r, \frac{1}{f^{(k)}-a_2}\right) + \bar{N}\left(r, \frac{1}{f^{(k)}-b_2}\right) + \bar{N}\left(r, \frac{1}{f^{(k)}-a_1^{(k)}}\right) - 2N\left(r, \frac{1}{f^{(k)}-a_1^{(k)}}\right) + S(r, f) \\ &\leq T(r, f) - N\left(r, \frac{1}{f^{(k)}-a_1^{(k)}}\right) + S(r, f), \end{aligned}$$

which implies that

$$N\left(r, \frac{1}{f^{(k)}-a_1^{(k)}}\right) = S(r, f). \quad (3.33)$$

In the following, we will prove $T(r, H) = S(r, f)$, then combining (3.11) with (3.23), we get $T(r, f) = S(r, f)$, which is impossible.

Rewrite (3.1) as

$$f - a_1 = H(f^{(k)} - a_1^{(k)}) + H(a_1^{(k)} - a_2). \quad (3.34)$$

Set $g = f^{(k)} - a_1^{(k)}$. Differentiating (3.34) k times, we have

$$g = H^{(k)}g + kH^{(k-1)}g' + \cdots + kH'g^{(k-1)} + Hg^{(k)} + B^{(k)}, \quad (3.35)$$

where $B = H(a_1^{(k)} - a_2)$. It is easy to see that $g \neq 0$. Then, we rewrite (3.35) as

$$1 - \frac{B^{(k)}}{g} = DH, \quad (3.36)$$

where

$$D = \frac{H^{(k)}}{H} + \frac{kH^{(k-1)}g'}{Hg} + \cdots + \frac{kH'g^{(k-1)}}{Hg} + \frac{g^k}{g}. \quad (3.37)$$

Note that $N\left(r, \frac{1}{f^{(k)}-a_1^{(k)}}\right) = N\left(r, \frac{1}{g}\right) = S(r, f)$. Then it follows from (3.2) that

$$\begin{aligned}
T(r, D) &\leq \sum_{i=1}^k \left(T\left(r, \frac{H^{(i)}}{H}\right) + T\left(r, \frac{g^{(i)}}{g}\right) \right) + S(r, f) \\
&\leq \sum_{i=1}^k \left(m\left(r, \frac{H^{(i)}}{H}\right) + N\left(r, \frac{H^{(i)}}{H}\right) + m\left(r, \frac{g^{(i)}}{g}\right) + N\left(r, \frac{g^{(i)}}{g}\right) \right) + S(r, f) = S(r, H) + S(r, f).
\end{aligned} \tag{3.38}$$

By (3.34) and Lemma 2.2, we get

$$T(r, H) \leq T(r, f) + T(r, f^{(k)}) + S(r, f) \leq 2T(r, f) + S(r, f). \tag{3.39}$$

Then it follows from (3.38) that $T(r, D) = S(r, f)$. Next, we discuss two subcases.

Subcase 2.1. $H^{-1} - D \neq 0$. Rewrite (3.36) as

$$gH(H^{-1} - D) = B^{(k)}. \tag{3.40}$$

We claim that $D \equiv 0$. Otherwise, it follows from (3.40) that $N\left(r, \frac{1}{H^{-1}-D}\right) = S(r, f)$. Then applying Nevanlinna's second fundamental theorem to H , we can obtain

$$\begin{aligned}
T(r, H) &= T(r, H^{-1}) + O(1) \\
&\leq \bar{N}(r, H^{-1}) + \bar{N}\left(r, \frac{1}{H^{-1}}\right) + \bar{N}\left(r, \frac{1}{H^{-1}-D}\right) + S(r, H) \\
&\leq S(r, H) \leq S(r, f).
\end{aligned} \tag{3.41}$$

It follows from (3.11) and (3.23) that $T(r, f) = S(r, f)$, a contradiction. Thus, $D \equiv 0$. Then by (3.40), we get

$$g = B^{(k)}. \tag{3.42}$$

Integrating (3.42), we get

$$f = H(a_1^{(k)} - a_2) + P(z) + a_1, \tag{3.43}$$

where $P(z)$ is a polynomial of degree at most $k - 1$. By (3.43), we have

$$T(r, f) = T(r, H) + S(r, f). \tag{3.44}$$

It follows from (3.3), (3.11), and (3.44) that

$$\bar{N}\left(r, \frac{1}{f^{(k)} - a_2}\right) = \bar{N}\left(r, \frac{1}{f - a_1}\right) = S(r, f). \tag{3.45}$$

Then combining (3.33) with (3.45), and applying Nevanlinna's second fundamental theorem to $f^{(k)}$, we get

$$T(r, f^{(k)}) \leq \bar{N}\left(r, \frac{1}{f^{(k)} - a_2}\right) + \bar{N}\left(r, \frac{1}{f^{(k)} - a_1^{(k)}}\right) = S(r, f). \tag{3.46}$$

Because f and $f^{(k)}$ share (a_1, a_2) CM and (b_1, b_2) IM, then (3.3) and (3.46) imply

$$\begin{aligned}
T(r, f) &= \bar{N}\left(r, \frac{1}{f - a_1}\right) + \bar{N}\left(r, \frac{1}{f - b_1}\right) + S(r, f) \\
&= \bar{N}\left(r, \frac{1}{f^{(k)} - a_2}\right) + \bar{N}\left(r, \frac{1}{f^{(k)} - b_2}\right) + S(r, f) \\
&\leq 2T(r, f)^{(k)} + S(r, f) = S(r, f).
\end{aligned} \tag{3.47}$$

It is impossible.

Subcase 2.2 $H^{-1} - D \equiv 0$. Then by (3.38), we have $T(r, H) = S(r, f)$. It follows from (3.11) and (3.23) that $T(r, f) = S(r, f)$, but it is impossible.

This completes the proof of Theorem 1.

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