Entire functions that share two pairs of small functions

Abstract: In this paper, we study the unicity of entire functions and their derivatives and obtain the following result: let \( f \) be a non-constant entire function, let \( a_1, a_2, b_1, \) and \( b_2 \) be four small functions of \( f \) such that \( a_1 \neq b_1, a_2 \neq b_2, \) and none of them is identically equal to \( \infty. \) If \( f \) and \( f^{(k)} \) share \( (a_1, a_2) \) CM and share \( (b_1, b_2) \) IM, then \((a_2 - b_2)f - (a_1 - b_1)f^{(k)} \equiv a_2b_1 - a_1b_2. \) This extends the result due to Li and Yang [Value sharing of an entire function and its derivatives, J. Math. Soc. Japan. 51 (1999), no. 7, 781–799].

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1 Introduction and main results

In this paper, we use the general notations in the Nevanlinna value distribution theory, see ([1–3]).

Let \( f \) be a meromorphic function on the whole complex plane. For \( 0 < r < R, \) we define the following functions:

\[
m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta
\]

is the average of the positive logarithmic \(|f(z)|\) on the circle \(|z| = r;\)

\[
N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} \, dt + n(0, f) \log r
\]

is called the counting function of poles of \( f(z), \) where \( n(t, f) \) denotes the number of poles of \( f(z) \) on the disc \(|z| \leq t, \) multiple poles are counted according to their multiplicities, and \( n(0, f) \) denotes the multiplicity of pole of \( f(z) \) at the origin (if \( f(0) \neq \infty, \) then \( n(0, f) = 0). \) We denote \( N(r, f) \) the reduced counting function of \( f(z) \) whose the multiplicity of poles only counts once.

\[
T(r, f) = m(r, f) + N(r, f).
\]

\( T(r, f) \) is said to be the characteristic function of \( f(z) \) which is obviously a non-negative function.
Let \( a \) be a complex number. Obviously, \( \frac{1}{f(z) - a} \) is meromorphic on the disc \( |z| \leq R \). Similar to the above definitions, we define the following functions.

\[
m(r, \frac{1}{f(z) - a}) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta}) - a| \, d\theta
\]

is the average of the positive logarithmic \( \frac{1}{f(z) - a} \) on the circle \( |z| = r \).

\[
N(r, \frac{1}{f(z) - a}) = \int_0^r n(t, \frac{1}{f(z) - a}) - n(0, \frac{1}{f(z) - a}) \frac{dt}{t} + n(0, f) \log r,
\]

where \( n(r, \frac{1}{f(z) - a}) \) denotes the number of zeros of \( f(z) - a \) on the disc \( |z| \leq t \), multiple poles are counted according to their multiplicities. \( n(0, \frac{1}{f(z) - a}) \) denotes the multiplicity of zeros of \( f(z) - a \) at the origin. We denote \( \frac{N(r, \frac{1}{f(z) - a})}{\pi} \) the reduced counting function of \( f(z) - a \) whose the multiplicity of zero only counts once.

\[
T(r, \frac{1}{f(z) - a}) = m(r, \frac{1}{f(z) - a}) + N(r, \frac{1}{f(z) - a}).
\]

\( T(r, \frac{1}{f(z) - a}) \) is said to be the characteristic function of \( \frac{1}{f(z) - a} \).

Obviously, \( T(r, f) \) is a non-decreasing function of \( r \), and a convex function of \( \log r \).

We denote \( S(r, f) = o(T(r, f)) \), as \( r \to \infty \) outside of a possible exceptional set of finite linear measures. We say that a meromorphic function \( a \) is called a small function of \( f \) if it satisfies \( T(r, a) = S(r, f) \).

Nevanlinna established two fundamental theorems, called the first fundamental theorem and the second fundamental theorem.

**The first fundamental theorem.** Let \( f \) be a meromorphic function on the whole complex plane. Then,

\[
T(r, f) = T(r, \frac{1}{f(z) - a}) + O(1).
\]

**The second fundamental theorem.** Let \( f \) be a meromorphic function on the whole complex plane and let \( a_1, \ldots, a_q \) (\( q \geq 3 \)) be distinct complex values in the extended plane. Then,

\[
(q - 2) T(r, f) \leq \sum_{j=1}^q N(r, \frac{1}{f(z) - a_j}) + S(r, f).
\]

Let \( f \) and \( g \) be two meromorphic functions, and let \( a \) and \( b \) be two small functions of \( f \) and \( g \). We say that \( f \) and \( g \) share a pair of small functions \( (a, b) \) CM (resp. IM) if \( f - a \) and \( g - b \) have the same zeros counting multiplicities (resp. ignoring multiplicities). When \( a = b \), we say that \( f \) and \( g \) share a CM (resp. IM) if \( f - a \) and \( g - a \) have the same zeros counting multiplicities (resp. ignoring multiplicities).

In [3], Nevanlinna proved the following famous five-value theorem:

**Theorem A.** Let \( f \) and \( g \) be two non-constant meromorphic functions, and let \( a_j \) (\( j = 1, 2, 3, 4, 5 \)) be five distinct values in the extended complex plane. If \( f \) and \( g \) share \( a_j \) (\( j = 1, 2, 3, 4, 5 \)) IM, then \( f \equiv g \).

Brosch [4], Czubiak and Gundersen [5], Gundersen [6], Steinmetz [7], and Gundersen et al. [8] studied shared pairs of values. For five shared pairs of values, Gundersen et al. [8] proved the next result, which is the best possible:

**Theorem B.** If \( f \) and \( g \) are two non-constant meromorphic functions that share two pairs of values CM and share three other pairs of values IM, then \( f \) is a Möbius transformation of \( g \).
Li and Qiao [9] proved that Theorem A is still valid for five distinct small functions, and they proved the following:

**Theorem C.** Let \( f \) and \( g \) be two non-constant meromorphic functions, and let \( a_j \) (j = 1, 2, 3, 4, 5) (one of them can be \( \infty \)) be five distinct small functions of \( f \) and \( g \). If \( f \) and \( g \) share \( a_j \) (j = 1, 2, 3, 4, 5) IM, then \( f \equiv g \).


**Theorem D.** Let \( f \) be a non-constant entire function, and let \( a, b \) be two finite distinct complex values. If \( f \) and \( f' \) share \( a, b \) CM, then \( f \equiv f' \).

Zheng and Wang [13] improved Theorem D and proved the following: if \( f \) is a non-constant entire function, and \( f \) and \( f^{(k)} \) share \( a, b \) CM, where \( a \) and \( b \) are two small functions of \( f \), then \( f \equiv f^{(k)} \).

Li and Yang [14] improved the result of Zheng and Wang [13] and proved the following result:

**Theorem E.** Let \( f \) be a non-constant entire function, and let \( a, b \) be two distinct small functions of \( f \). If \( f \) and \( f^{(k)} \) share \( a \) CM, and share \( b \) IM, then \( f \equiv f^{(k)} \).

In this paper, we study shared pair of small functions and extend Theorem E as follows.

**Theorem 1.** Let \( f \) be a non-constant entire function, let \( a_1, a_2, b_1, b_2 \) be four small functions of \( f \) such that \( a_1 \neq b_1 \) and \( a_2 \neq b_2 \), and none of them is identically equal to \( \infty \). If \( f \) and \( f^{(k)} \) share \( (a_1, a_2) \) CM, and share \( (b_1, b_2) \) IM, then \( (a_2 - b_2) f - (a_1 - b_1) f^{(k)} \equiv a_2 b_1 - a_1 b_2 \).

**Remark 1.** If \( a_1 \equiv a_2, b_1 \equiv b_2 \), then by Theorem 1, we obtain Theorem F.

The following example shows that the conclusion of Theorem 1 is not valid for meromorphic functions.

**Example 1.** [15] Let \( f = h + \frac{h - g}{q - 1} \), where
\[
g = -\left(\frac{1}{3e^{2z}} + \frac{1}{2e^z}\right), \quad h = \frac{1}{2e^z} - \frac{1}{3e^{2z}}, \quad q = e^{-e^z},
\]
and let \( a = h', b = g' \). By simple calculation, we obtain
\[
T(r, a) = S(r, f), \quad T(r, b) = S(r, f), \quad f' - a = e^{2z}(f - a)(f - h), \quad f' - b = e^{2z}(f - b)(f - g).
\]
So \( f \) and \( f' \) share \( (a, a) \) CM and \( (b, b) \) IM. However, \( f \neq f' \).

The following example shows that there exist a transcendental entire function \( f \) and two pairs of small functions of \( f \) satisfying Theorem 1.

**Example 2.** [16] Suppose \( f = e^{3z} + 10z, a_1 = 11z, a_2 = 9z, b_1 = 10z, b_2 = 0 \). Then it is an easy work to obtain
\[
T(r, a_i) = S(r, f), \quad T(r, b_i) = S(r, f)
\]
for \( i = 1, 2 \), and
\[
f - a_1 = e^{3z} - z, \quad f'' - b_1 = 9(e^{3z} - z),
\]
\[
f - a_2 = e^{3z}, \quad f'' - b_2 = 9e^{3z}.
\]
Thus, we see that \( f \) and \( f'' \) share \( (a_1, b_1) \) CM and \( (a_2, b_2) \) IM. Furthermore, \( (a_2 - b_2) f - (a_1 - b_1) f'' \equiv a_2 b_1 - a_1 b_2 \).
In 2020, Sahoo-Halder [16] proved.

**Theorem F.** Let $f$ be a non-constant entire function of finite order, let $f \in S_k(a_1, a_2, b_1, b_2)$, and let $a_1, a_2, b_1$ and $b_2$ be four small functions of $f$ such that $a_1 \neq b_1$ and $a_2 \neq b_2$, none of them is identically equal to $\infty$, and $a_i - b_i$ have at least one zero for $i = 1, 2$. If $f$ and $f^{(k)}$ share $(a_1, a_2)$ and $(b_1, b_2)$ IM, then $(a_2 - b_2)f - (a_1 - b_1)f^{(k)} = a_2b_1 - a_1b_2$.

The authors raised two questions as follows.

**Question 1**
The condition of finite order of Theorem F and Corollary 1 in [16] can be removed in any way?

**Question 2**
How far do the conclusions in Theorem F and Corollary 1 in [16] hold for a non-constant entire function?

In Theorem 1, we answer Question 1 in a setting with stronger conditions, and we answer that for any positive integers $k$ Theorem F and Corollary 1 in [16] hold.

## 2 Some lemmas

**Lemma 2.1.** [1–3] Let $f$ be a non-constant meromorphic function, and let $k$ be a positive integer. Then

$$m\left( r, \frac{f^{(k)}}{f} \right) = S(r, f).$$

**Lemma 2.2.** [2] Let $f$ be a meromorphic function on the complex plane, let $n$ be a positive integer, and let $\psi(f) = a_0f + a_1f^2 + \cdots + a_nf^{[n]}$, where $a_0, a_1, \ldots, a_n \neq 0$ are small functions of $f$. Then

$$m\left( r, \frac{\psi(f)}{f} \right) = S(r, f), \quad T(r, \psi) \leq T(r, f) + kN(r, f) + S(r, f).$$

**Lemma 2.3.** [1–3] Let $f_1, f_2$ be two non-constant meromorphic functions, then

$$N(r, f_1f_2) - N\left( r, \frac{1}{f_1f_2} \right) = N(r, f_1) + N(r, f_2) - N\left( r, \frac{1}{f_1} \right) - N\left( r, \frac{1}{f_2} \right).$$

**Lemma 2.4.** Let $f$ be a transcendental meromorphic function, let $a_1, a_2, b_1,$ and $b_2$ be four small functions of $f$ such that $a_1 \neq b_1$ and $a_2 \neq b_2$, and let

$$L(f) = \begin{vmatrix} f - a_1 & a_1 - b_1 \\ f' - a_1' & a_1' - b_1' \end{vmatrix}, \quad L(f^{(k)}) = \begin{vmatrix} f^{(k)} - a_2 & a_2 - b_2 \\ f^{(k)} - a_2' & a_2' - b_2' \end{vmatrix}.$$

If $f$ and $f^{(k)}$ share $(a_1, a_2)$ CM, share $(b_1, b_2)$ IM, then $L(f) \neq 0$, $L(f^{(k)}) \neq 0$.

**Proof.** Suppose that $L(f) \equiv 0$. Then we get $f - a_1 \equiv \frac{a_1 - b_1}{a_1 - b_1}$. It follows that $f - b_1 = c(a_1 - b_1)$, where $c$ is a nonzero constant. So $T(r, f) = T(r, c(a_1 - b_1) + a_1) = S(r, f)$, a contradiction. Hence, $L(f) \neq 0$.

Since $f$ is an entire function, $f$ and $f^{(k)}$ share $(a_1, a_2)$ CM, share $(b_1, b_2)$ IM, we have

$$T(r, f) \leq N\left( r, \frac{1}{f - a_1} \right) + N\left( r, \frac{1}{f - b_1} \right) + S(r, f) = N\left( r, \frac{1}{f^{(k)} - a_2} \right) + N\left( r, \frac{1}{f^{(k)} - b_2} \right) + S(r, f)$$

$$\leq 2T(r, f^{(k)}) + S(r, f).$$
Hence, $a_2, b_2$ are small functions of $f^{(k)}$. Using the same argument as proving $L(f) \neq 0$, it is easy to obtain $L(f^{(k)}) \neq 0$. Thus, the lemma is proved.

**Lemma 2.5.** Let $k_j$ ($j = 1, 2, \ldots, q$) be positive integers, let $a_1, b_1$ be two distinct small functions of $f$ and let $d_j = a_i - l_j(a_i - b_1)$ ($j = 1, 2, \ldots, q$). Then,

$$m(r, \frac{L(f)}{f - a_i}) = S(r, f), \quad m(r, \frac{L(f)}{f - b_i}) = S(r, f)$$

and

$$m(r, \frac{L(f)f}{(f - d_1)(f - d_2)\cdots(f - d_m)}) = S(r, f), \quad (2.1)$$

where $L(f)$ is defined as Lemma 2.4, and $2 \leq m \leq q$.

**Proof.** Since $L(f) = (a'_1 - b'_1)(f - a_i) - (a_i - b_1)(f' - a'_1)$, by Lemma 2.1, we have

$$m(r, \frac{L(f)}{f - a_i}) \leq m(r, a'_1 - b'_1) + m(r, \frac{(a_i - b_1)(f' - a'_1)}{f - a_i}) = S(r, f).$$

Similarly, we have

$$m(r, \frac{L(f)}{f - b_i}) = S(r, f).$$

Obviously,

$$L(f) = \begin{vmatrix} f - a_i + l_j(a_i - b_1) & a_1 - b_1 \\ f' - a'_1 + l_j(a'_1 - b'_1) & a'_1 - b'_1 \end{vmatrix} = \begin{vmatrix} f - d_j & a_i - b_1 \\ f' - d'_j & a'_1 - b'_1 \end{vmatrix},$$

$$\frac{L(f)f}{(f - d_1)(f - d_2)\cdots(f - d_m)} = \sum_{i=1}^{q} c_i L(f),$$

where $c_i$ ($i = 1, 2, \ldots, q$) are small functions of $f$.

Thus, by Lemma 2.2 and above two formulas, we obtain (2.1).

**Lemma 2.6.** Let $f$ and $g$ be two non-constant entire functions and let $a_1, a_2, b_1, b_2$ be four small functions of $f$ and $g$ such that $a_1 \neq b_1, a_2 \neq b_2$, and none of them is identically equal to $\infty$. Suppose that

$$H = \frac{L(f)}{(f - a_1)(f - b_1)} - \frac{L(g)}{(g - a_2)(g - b_2)} \equiv 0,$$

where

$$L(f) = (a'_1 - b'_1)(f - a_i) - (a_i - b_1)(f' - a'_1),$$

$$L(g) = (a'_2 - b'_2)(g - a_2) - (a_2 - b_2)(g'_2 - a_2).$$

If $f$ and $g$ share $(a_1, a_2)$ CM and share $(b_1, b_2)$ IM, then either

$$2T(r, f) \leq N\left(r, \frac{1}{f - a_1}\right) + N\left(r, \frac{1}{f - b_1}\right) + S(r, f) \quad (2.2)$$

or

$$(a_2 - b_2)f - (a_1 - b_1)g + a_1b_2 - a_2b_1 \equiv 0. \quad (2.3)$$
\textbf{Proof.} Since $H \equiv 0$, it is easy to get
\[
\frac{g - b_2}{g - a_2} = c \frac{f - b_1}{f - a_1},
\]  
where $c$ is a nonzero constant.

If $c = 1$, then by (2.4), we get (2.3).

If $c \neq 1$, then by (2.4), we have
\[
\frac{a_2 - b_2}{g - a_2} \equiv \frac{(c - 1)f - (cb_1 - a_1)}{f - a_1}
\]  
and
\[
T(r, f) = T(r, g) + S(r, f) + S(r, g).
\]
Obviously, $\frac{cb_1 - a_1}{c - 1} \neq a_1$, $\frac{cb_1 - a_1}{c - 1} \neq b_1$. It follows from (2.5) and the fact that $f$ and $g$ share $(a_1, a_2)$ CM that
\[
N \left( r, \frac{1}{f - a_1} \right) = N \left( r, \frac{1}{a_1 - b_1} \right) = S(r, f).
\]
Thus by Nevanlinna’s second fundamental theorem, we have
\[
2T(r, f) \leq \overline{N} \left( r, \frac{1}{f - a_1} \right) + \overline{N} \left( r, \frac{1}{f - b_1} \right) + \overline{N} \left( r, \frac{1}{f - \frac{c b_1 - a_1}{c - 1}} \right) + S(r, f)
\]
\[
\leq \overline{N} \left( r, \frac{1}{f - a_1} \right) + \overline{N} \left( r, \frac{1}{f - b_1} \right) + S(r, f).
\]
It follows that $2T(r, f) \leq \overline{N} \left( r, \frac{1}{f - a_1} \right) + \overline{N} \left( r, \frac{1}{f - b_1} \right) + S(r, f)$. \hfill \Box

\section{3 Proof of Theorem 1}

We prove Theorem 1 by contradiction. Suppose that $(a_2 - b_2)f - (a_1 - b_1)f^{(k)} + a_1 b_2 - a_2 b_1 \neq 0$. Set
\[
\frac{f - a_1}{f^{(k)} - a_2} = H.
\]
Since $f$ is a transcendental entire function, and $f$ and $f^{(k)}$ share $(a_1, a_2)$ CM, then by (3.1), we have
\[
N(r, H) = S(r, f), \quad N \left( r, \frac{1}{H} \right) = S(r, f).
\]

From the fact that $f$ and $f^{(k)}$ share $(a_1, a_2)$ CM and share $(b_1, b_2)$ IM, and $f$ is a transcendental entire function, by Nevanlinna’s second fundamental theorem and Lemma 2.1, we obtain
\[
T(r, f) \leq \overline{N} \left( r, \frac{1}{f - a_1} \right) + \overline{N} \left( r, \frac{1}{f - b_1} \right) + \overline{N}(r, f) + S(r, f)
\]
\[
= \overline{N} \left( r, \frac{1}{f^{(k)} - a_2} \right) + \overline{N} \left( r, \frac{1}{f^{(k)} - b_2} \right) + S(r, f)
\]
\[
\leq \overline{N} \left( r, \frac{1}{(a_2 - b_2)f - (a_1 - b_1)f^{(k)} + a_1 b_2 - a_2 b_1} \right) + S(r, f)
\]
\[
\leq T(r, (a_2 - b_2)f - (a_1 - b_1)f^{(k)} + a_1 b_2 - a_2 b_1) + S(r, f)
\]
\[
\leq m(r, (a_2 - b_2)f - (a_1 - b_1)f^{(k)} + a_1 b_2 - a_2 b_1) + S(r, f)
\]
\[
\leq m(r, f) + m \left( r, \frac{(a_2 - b_2)f - (a_1 - b_1)f^{(k)}}{f} \right) + S(r, f)
\]
\[
\leq T(r, f) + S(r, f).
\]
Thus, we have

\[
T(r, f) = N \left( r, \frac{1}{f - a_1} \right) + N \left( r, \frac{1}{f - b_1} \right) + S(r, f)
\]  

(3.3)

and

\[
T(r, f) = T(r, (a_2 - b_2) f - (a_1 - b_1) f^{(k)} + a_1 b_2 - a_2 b_1) + S(r, f)
\]

\[
= N \left( r, \frac{1}{(a_2 - b_2) f - (a_1 - b_1) f^{(k)} + a_1 b_2 - a_2 b_1} \right) + S(r, f).
\]  

(3.4)

Obviously,

\[
T(r, H) = m \left( r, \frac{1}{H} \right) + S(r, f) = m \left( r, \frac{f^{(k)} - a_2}{f - a_1} \right) + S(r, f) \leq m \left( r, \frac{1}{f - a_1} \right) + S(r, f). \]  

(3.5)

It follows from (3.1) and (3.4) that

\[
m \left( r, \frac{1}{f - a_1} \right) = m \left( r, \frac{(a_1 - b_1) H^{-1} + b_2 - a_2}{(a_2 - b_2) f - (a_1 - b_1) f^{(k)} + a_1 b_2 - a_2 b_1} \right) \\
\leq m \left( r, \frac{1}{(a_2 - b_2) f - (a_1 - b_1) f^{(k)} + a_1 b_2 - a_2 b_1} \right) + m \left( r, \frac{a_1 - b_1}{H} + b_2 - a_2 \right) + S(r, f) \\
\leq T(r, H) + S(r, f).
\]  

(3.6)

By (3.5) and (3.6), we get

\[
T(r, H) = m \left( r, \frac{1}{f - a_1} \right) + S(r, f).
\]  

(3.7)

On the other hand, (3.1) can be rewritten as

\[
\frac{(a_2 - b_2) f - (a_1 - b_1) f^{(k)} + a_1 b_2 - a_2 b_1}{f - a_1} = -[(a_1 - b_1) H^{-1} + b_2 - a_2],\]

(3.8)

which implies

\[
N \left( r, \frac{1}{f - b_1} \right) \leq N \left( r, \frac{1}{(a_1 - b_1) H^{-1} + b_2 - a_2} \right) \leq T(r, H) + S(r, f).
\]  

(3.9)

Thus, by (3.3), (3.7), and (3.9)

\[
m \left( r, \frac{1}{f - a_1} \right) + N \left( r, \frac{1}{f - a_1} \right) = N \left( r, \frac{1}{f - a_1} \right) + N \left( r, \frac{1}{f - b_1} \right) + S(r, f) \\
\leq N \left( r, \frac{1}{f - a_1} \right) + N \left( r, \frac{1}{(a_1 - b_1) H^{-1} + b_2 - a_2} \right) + S(r, f) \\
\leq N \left( r, \frac{1}{f - a_1} \right) + m \left( r, \frac{1}{f - a_1} \right) + S(r, f).
\]  

Hence, by the above formula, (3.7), (3.9), we obtain

\[
N \left( r, \frac{1}{f - a_1} \right) = N \left( r, \frac{1}{f - a_1} \right) + S(r, f), \]  

(3.10)

\[
N \left( r, \frac{1}{f - b_1} \right) = T(r, H) + S(r, f).
\]  

(3.11)
Set
\[ \varphi = \frac{L(f)((a_2 - b_2)f - (a_1 - b_1)f^{(k)}) + a_1b_2 - a_2b_1}{(f - a_1)(f - b_1)} \]  
and
\[ \psi = \frac{L(f^{(k)})(a_2 - b_2)f - (a_1 - b_1)f^{(k)} + a_1b_2 - a_2b_1}{(f^{(k)} - a_2)(f^{(k)} - b_2)}. \]

Obviously, \( \varphi \neq 0 \). By (3.12), it is easy to know that \( N(r, \varphi) = S(r, f) \). Thus, by Lemmas 2.1, 2.4, and 2.5, we have
\[ T(r, \varphi) = m(r, \varphi) + N(r, \varphi) \]
\[ = m\left( r, \frac{L(f)((a_2 - b_2)f - (a_1 - b_1)f^{(k)}) + a_1b_2 - a_2b_1}{(f - a_1)(f - b_1)} \right) + S(r, f) \]
\[ \leq m\left( r, \frac{L(f)}{(f - a_1)(f - b_1)} \right) + m\left( r, \frac{(a_2 - b_2)f - (a_1 - b_1)f^{(k)}}{f} \right) \]
\[ + m\left( r, \frac{L(f)(a_2b_2 - a_2b_1)}{(f - a_1)(f - b_1)} \right) + S(r, f) \leq S(r, f), \]
that is
\[ T(r, \varphi) = S(r, f). \]

Let \( d_1 = a_1 - k(a_1 - b_1) \) \((k \neq 0, 1)\) and \( d_2 = a_2 - k(a_2 - b_2) \). It follows from Nevanlinna’s second fundamental theorem and (3.3) that
\[ 2T(r, f) \leq N\left( r, \frac{1}{f - a_1} \right) + N\left( r, \frac{1}{f - b_1} \right) + N\left( r, \frac{1}{f - d_1} \right) + S(r, f) \]
\[ \leq T(r, f) + N\left( r, \frac{1}{f - d_1} \right) + S(r, f) \]
\[ \leq 2T(r, f) - m\left( r, \frac{1}{f - d_1} \right) + S(r, f), \]
which implies
\[ m\left( r, \frac{1}{f - d_1} \right) = S(r, f). \]

Rewrite (3.13) as
\[ \psi = \left[ \frac{a_2 - d_2}{a_2 - b_2} \frac{L(f^{(k)})}{f^{(k)} - a_2} - \frac{b_2 - d_2}{a_2 - b_2} \frac{L(f^{(k)})}{f^{(k)} - b_2} \right] \frac{(a_2 - b_2)(f - d_1)}{f^{(k)} - d_2} - a_1 + b_1 \]
and set
\[ \phi = \frac{L(f)}{(f - a_1)(f - b_1)} - \frac{L(f^{(k)})}{(f^{(k)} - a_2)(f^{(k)} - b_2)}. \]

Next, we consider two cases.

**Case 1.** \( \phi \equiv 0 \). Integrating both sides of (3.17) which implies
\[ \frac{f - a_1}{f - b_1} = C \frac{f^{(k)} - a_2}{f^{(k)} - b_2}, \]
where \( C \) is a nonzero constant.
Then, by Lemma 2.6, we get

\[ 2T(r,f) \leq \overline{N}\left(r, \frac{1}{f-a_1}\right) + \overline{N}\left(r, \frac{1}{f-b_1}\right) + S(r,f), \tag{3.19} \]

which contradicts with (3.3).

**Case 2.** \( \phi \neq 0 \). It is easy to obtain that \( m(r, \phi) = S(r,f) \), and all poles of \( \phi \) must come from the zeros of \( f - b_1 \). Hence, we have

\[ T(r, \phi) = m(r, \phi) + N(r, \phi) \leq \overline{N}\left(r, \frac{1}{f-b_1}\right) + S(r,f). \tag{3.20} \]

It follows from (3.4), (3.17), and (3.20) that

\[
m(r,f) = m(r,(a_2-b_2)f - (a_1-a_2)f^{(k)} + a_1b_2 - a_2b_1) + S(r,f) \\
= m\left(r, \frac{\phi((a_2-b_2)f - (a_1-a_2)f^{(k)} + a_1b_2 - a_2b_1)}{\phi}\right) + S(r,f) \\
= m\left(r, \frac{\varphi - \psi}{\phi}\right) + S(r,f) \\
\leq T\left(r, \frac{\varphi - \psi}{\varphi - \psi}\right) + S(r,f) \\
\leq T(r, \varphi - \psi) + T(r, \phi) + S(r,f) \\
\leq T(r, \psi) + T(r, \phi) + S(r,f) \\
\leq T(r, \psi) + \overline{N}\left(r, \frac{1}{f-b_1}\right) + S(r,f). \tag{3.21} \]

On the other hand, by Lemma 2.4, (3.1), (3.8), and (3.11), we have

\[
T(r, \psi) = T\left(r, \frac{L(f^{(k)})((a_2-b_2)f - (a_1-a_2)f^{(k)} + a_1b_2 - a_2b_1)}{f^{(k)} - a_2}(f^{(k)} - b_2)\right) \\
= m\left(r, \frac{L(f^{(k)})((a_2-b_2)f - (a_1-a_2)f^{(k)} + a_1b_2 - a_2b_1)}{f^{(k)} - a_2}(f^{(k)} - b_2)\right) + S(r,f) \\
\leq m\left(r, \frac{L(f^{(k)})}{f^{(k)} - b_2}\right) + m\left(r, \frac{(a_2-b_2)f - (a_1-a_2)f^{(k)} + a_1b_2 - a_2b_1}{f^{(k)} - a_2}\right) + S(r,f) \\
\leq m\left(r, \frac{(a_2-b_2)f - (a_1-a_2)f^{(k)} + a_1b_2 - a_2b_1}{f^{(k)} - a_2}\right) + S(r,f) \\
\leq m\left(r, \frac{1}{f - a_1}\right) + S(r,f) \\
= \overline{N}\left(r, \frac{1}{f-b_1}\right) + S(r,f). \tag{3.22} \]

Thus, by (3.21) and (3.22), we obtain

\[ T(r,f) \leq 2\overline{N}\left(r, \frac{1}{f-b_1}\right) + S(r,f). \tag{3.23} \]

If \( d_1^{(k)} \equiv a_2 \), then by (3.1), (3.2), and Lemma 2.1, we get

\[ T(r,H) = m\left(r, \frac{1}{H}\right) + S(r,f) = m\left(r, \frac{f^{(k)} - a^{(k)}_1}{f-a_1}\right) + S(r,f) \leq m\left(r, \frac{f^{(k)} - a^{(k)}_1}{f-a_1}\right) + S(r,f) = S(r,f). \tag{3.24} \]

It follows from (3.11), (3.23), and (3.24) that \( T(r,f) = S(r,f) \), a contradiction.
If \( a_1^{(k)} = b_2 \), then by (3.11), (3.23), and
\[
T(r, f) \leq m\left( r, \frac{1}{f - a_1} \right) + N\left( r, \frac{1}{f - b_1} \right) + S(r, f)
\]
\[
\leq m\left( r, \frac{1}{f^{(k)} - b_2} \right) + N\left( r, \frac{1}{f^{(k)} - b_2} \right) + S(r, f)
\]
\[
\leq T(r, f^{(k)}) + S(r, f),
\]
which implies
\[
T(r, f) \leq T(r, f^{(k)}) + S(r, f). \tag{3.25}
\]

On the other hand, it follows from Lemma 2.2 that
\[
T(r, f^{(k)}) \leq T(r, f) + S(r, f). \tag{3.26}
\]

Thus,
\[
T(r, f) = T(r, f^{(k)}) + S(r, f). \tag{3.27}
\]

Then, by Nevanlinna’s second fundamental theorem, Lemma 2.1, (3.3), and (3.27), we have
\[
2T(r, f) \leq 2T(r, f^{(k)}) + S(r, f)
\]
\[
\leq N\left( r, \frac{1}{f^{(k)} - a_2} \right) + N\left( r, \frac{1}{f^{(k)} - b_2} \right) + S(r, f)
\]
\[
\leq N\left( r, \frac{1}{f - a_1} \right) + N\left( r, \frac{1}{f - b_1} \right) + T\left( r, \frac{1}{f^{(k)} - d_2} \right) - m\left( r, \frac{1}{f^{(k)} - d_2} \right) + S(r, f)
\]
\[
\leq T(r, f) + T(r, f^{(k)}) - m\left( r, \frac{1}{f^{(k)} - d_2} \right) + S(r, f)
\]
\[
\leq 2T(r, f) - m\left( r, \frac{1}{f^{(k)} - d_2} \right) + S(r, f).
\]

Thus,
\[
m\left( r, \frac{1}{f^{(k)} - d_2} \right) = S(r, f). \tag{3.28}
\]

It is easy to see that \( N(r, \psi) = S(r, f) \). By Nevanlinna’s first fundamental theorem, Lemmas 2.1 and 2.3, (3.15), (3.27), (3.28), and the fact that \( f \) is a transcendental entire function, we obtain
\[
m\left( r, \frac{f - d_1}{f^{(k)} - d_2} \right) = T\left( r, \frac{f^{(k)} - d_2}{f^{(k)} - d_1} \right) - N\left( r, \frac{f - d_1}{f^{(k)} - d_2} \right) + O(1)
\]
\[
\leq m\left( r, \frac{f^{(k)} - d_1}{f^{(k)} - d_2} \right) + m\left( r, \frac{d_1}{f - d_1} \right) + N\left( r, \frac{f^{(k)} - d_1}{f - d_1} \right) - N\left( r, \frac{f - d_1}{f^{(k)} - d_2} \right) + O(1)
\]
\[
\leq N\left( r, \frac{1}{f - d_1} \right) - N\left( r, \frac{1}{f^{(k)} - d_2} \right) + S(r, f)
\]
\[
\leq T(r, f) - T(r, f^{(k)}) + S(r, f) = S(r, f).
\]

It follows from Lemma 2.5 and (3.16) that
\[
T(r, \psi) = m(r, \psi) + N(r, \psi)
\]
\[
= m\left( r, \frac{a_2 - d_2}{a_2 - b_2 f^{(k)} - a_2} \right) + m\left( r, \frac{b_2 - d_2}{a_2 - b_2 f^{(k)} - b_2} \right) + S(r, f)
\]
\[
+ m\left( r, \frac{(a_2 - b_2)(f - d_1)}{f^{(k)} - d_2} - a_1 + b_1 \right) + S(r, f) = S(r, f). \tag{3.29}
\]
By (3.3), (3.10), (3.21), and (3.29), we get
\[ N(r, \frac{1}{f-a_1}) = \overline{N}(r, \frac{1}{f-a_1}) + S(r, f) = S(r, f). \] (3.30)

Moreover, by (3.3), (3.27), and (3.30), we have
\[ m\left(r, \frac{1}{(f-a_1)^{k}}\right) = S(r, f), \] (3.31)
which implies
\[ \overline{N}\left(r, \frac{1}{(f-a_1)^{k}}\right) = m\left(r, \frac{1}{f-a_1}\right) + S(r, f) \leq m\left(r, \frac{1}{(f-a_1)^{k}}\right) + S(r, f) \leq S(r, f). \] (3.32)

Then by (3.3), (3.30), and (3.32), we obtain \( T(r, f) = S(r, f) \), a contradiction.

Hence, we have \( a_1^{(k)} \neq a_2 \) and \( a_1^{(k)} \neq b_2 \). By (3.7), (3.11), and (3.23), we obtain
\[ T(r, f) \leq 2m\left(r, \frac{1}{f-a_1}\right) + S(r, f) \]
\[ \leq 2m\left(r, \frac{1}{f^{(k)}-a_1^{(k)}}\right) + S(r, f) \]
\[ = 2T(r, f^{(k)}) - 2N\left(r, \frac{1}{f^{(k)}-a_1^{(k)}}\right) + S(r, f) \]
\[ \leq \overline{N}\left(r, \frac{1}{f^{(k)}-a_2}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}-b_2}\right) - 2N\left(r, \frac{1}{f^{(k)}-a_1^{(k)}}\right) + S(r, f) \]
\[ \leq T(r, f) - N\left(r, \frac{1}{f^{(k)}-a_1^{(k)}}\right) + S(r, f), \]
which implies that
\[ N\left(r, \frac{1}{f^{(k)}-a_1^{(k)}}\right) = S(r, f). \] (3.33)

In the following, we will prove \( T(r, H) = S(r, f) \), then combing (3.11) with (3.23), we get \( T(r, f) = S(r, f) \), which is impossible.

Rewrite (3.1) as
\[ f - a_1 = H(f^{(k)} - a_1^{(k)}) + H(a_1^{(k)} - a_2). \] (3.34)

Set \( g = f^{(k)} - a_1^{(k)} \). Differentiating (3.34) \( k \) times, we have
\[ g = H^{(k)}g + kH^{(k-1)}g' + \cdots + kH'g^{(k-1)} + Hg^{(k)} + B^{(k)}, \] (3.35)
where \( B = H(a_1^{(k)} - a_2) \). It is easy to see that \( g \neq 0 \). Then, we rewrite (3.35) as
\[ 1 - \frac{B^{(k)}}{g} = DH, \] (3.36)
where
\[ D = \frac{H^{(k)}}{H} + \frac{kH^{(k-1)}g'}{Hg} + \cdots + \frac{kH'g^{(k-1)}}{Hg} + \frac{g^{(k)}}{g}. \] (3.37)

Note that \( N\left(r, \frac{1}{f^{(k)}-a_1^{(k)}}\right) = N\left(r, \frac{1}{g}\right) = S(r, f) \). Then it follows from (3.2) that
\[ T(r, D) \leq \sum_{i=1}^{k} \left( T \left( r, \frac{H^{(i)}}{H} \right) + T \left( r, \frac{g^{(i)}}{g} \right) \right) + S(r, f) \]

\[ \leq \sum_{i=1}^{k} \left( m \left( r, \frac{H^{(i)}}{H} \right) + N \left( r, \frac{H^{(i)}}{H} \right) + m \left( r, \frac{g^{(i)}}{g} \right) + N \left( r, \frac{g^{(i)}}{g} \right) \right) + S(r, f) = S(r, H) + S(r, f). \] (3.38)

By (3.34) and Lemma 2.2, we get

\[ T(r, H) \leq T(r, f) + T(r, f^{(k)}) + S(r, f) \leq 2T(r, f) + S(r, f). \] (3.39)

Then it follows from (3.38) that \( T(r, D) = S(r, f) \). Next, we discuss two subcases.

**Subcase 2.1.** \( H^{-1} - D \neq 0 \). Rewrite (3.36) as

\[ gH(H^{-1} - D) = B^{(k)}. \] (3.40)

We claim that \( D \equiv 0 \). Otherwise, it follows from (3.40) that \( N \left( r, \frac{1}{H^{-1} - D} \right) = S(r, f) \). Then applying Nevanlinna's second fundamental theorem to \( H \), we obtain

\[ T(r, H) = T(r, H^{-1}) + O(1) \]

\[ \leq \overline{N}(r, H^{-1}) + \overline{N} \left( r, \frac{1}{H^{-1}} \right) + \overline{N} \left( r, \frac{1}{H^{-1} - D} \right) + S(r, H) \]

\[ \leq S(r, H) \leq S(r, f). \] (3.41)

It follows from (3.11) and (3.23) that \( T(r, f) = S(r, f) \), a contradiction. Thus, \( D \equiv 0 \). Then by (3.40), we get

\[ g = B^{(k)}. \] (3.42)

Integrating (3.42), we get

\[ f = H(a^{(k)}_1 - a_2) + P(z) + a_i, \] (3.43)

where \( P(z) \) is a polynomial of degree at most \( k - 1 \). By (3.43), we have

\[ T(r, f) = T(r, H) + S(r, f). \] (3.44)

It follows from (3.3), (3.11), and (3.44) that

\[ \overline{N} \left( r, \frac{1}{f^{(k)} - a_2} \right) = \overline{N} \left( r, \frac{1}{f - a_i} \right) = S(r, f). \] (3.45)

Then combining (3.33) with (3.45), and applying Nevanlinna's second fundamental theorem to \( f^{(k)} \), we get

\[ T(r, f^{(k)}) \leq \overline{N} \left( r, \frac{1}{f^{(k)} - a_2} \right) + \overline{N} \left( r, \frac{1}{f^{(k)} - a_1^{(k)}} \right) = S(r, f). \] (3.46)

Because \( f \) and \( f^{(k)} \) share \( (a_i, a_2) \) CM and \( (b_1, b_2) \) IM, then (3.3) and (3.46) imply

\[ T(r, f) = \overline{N} \left( r, \frac{1}{f - a_i} \right) + \overline{N} \left( r, \frac{1}{f - b_1} \right) + S(r, f) \]

\[ = \overline{N} \left( r, \frac{1}{f^{(k)} - a_2} \right) + \overline{N} \left( r, \frac{1}{f^{(k)} - b_2} \right) + S(r, f) \]

\[ \leq 2T(r, f^{(k)}) + S(r, f) = S(r, f). \] (3.47)

It is impossible.

**Subcase 2.2** \( H^{-1} - D \equiv 0 \). Then by (3.38), we have \( T(r, H) = S(r, f) \). It follows from (3.11) and (3.23) that \( T(r, f) = S(r, f) \), but it is impossible.

This completes the proof of Theorem 1.
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