Research Article

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On non-normal cyclic subgroups of prime order or order 4 of finite groups

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Abstract: In this paper, we call a finite group G an NLM -group (NCM -group, respectively) if every non-normal cyclic subgroup of prime order or order 4 (prime power order, respectively) in G is contained in a non-normal maximal subgroup of G . Using the property of NLM -groups and NCM -groups, we give a new necessary and sufficient condition for G to be a solvable T -group (normality is a transitive relation), some sufficient conditions for G to be supersolvable, and the classification of those groups whose all proper subgroups are NLM -groups.

Keywords: normal subgroups, minimal subgroups, maximal subgroups, supersolvable groups, T -groups

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1 Introduction

All groups considered in this paper are finite and our notation is standard. Normal subgroups play a crucial role in investigating group structures. Dually, the influence of non-normal subgroups on the structure of groups is also enormous. There are many good results in characterizing the structure of finite groups by applying some given properties of non-normal subgroups, for instance, [1–8]. Specifically, Kaplan [8] introduced the following definition in investigating solvable T -groups (normality is a transitive relation in G , or, equivalently, a group in which every subnormal subgroup is normal) although non-normal subgroups do not seem to be directly related to solvable T -groups.

Definition 1.1. [8, Definition 1.1] Let G be a group. We say that G has the NNM property – shortly: G is an NNM -group – if each non-normal subgroup of G is contained in a non-normal maximal subgroup of G .

Kaplan [8] proved that a group G is a solvable T -group if and only if all subgroups of G are NNM -groups and gave some sufficient conditions for G to be supersolvable. It is natural to consider the structure of a group G in which only a part of non-normal subgroups is contained in a non-normal maximal subgroup of G . We give the following definitions which are weaker than NNM property.

Definition 1.2. Let G be a group. We say that G has the NCM property – shortly: G is an NCM -group – if each non-normal cyclic subgroup of prime power order in G is contained in a non-normal maximal subgroup of G .

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Definition 1.3. Let G be a group. We say that G has the *NLM* property – shortly: G is an *NLM*-group – if each non-normal cyclic subgroup of prime order or order 4 in G is contained in a non-normal maximal subgroup of G .

The aim of this paper is threefold. First, we give a new characterization of solvable T -groups in terms of *NCM* property. We generalize Theorem 1 in [8].

Theorem A. A group G is a solvable T -group if and only if every subgroup of G is an *NCM*-group.

Next, we present a sufficient condition, which generalizes Theorem 7 in [8], for supersolvability of a group G .

Theorem B. Let G be a group such that all its non-nilpotent subgroups are *NLM*-groups. Then G is supersolvable.

Finally, we obtain a classification of those groups whose all proper subgroups are *NLM*-groups.

Theorem C. Let G be a non-nilpotent minimal non-*NLM*-group (non-*NLM*-group all of whose proper subgroups are *NLM*-groups). Then G must be one of the following types:

- (1) G is supersolvable.
- (2) $G = P \rtimes Q$, where P is isomorphic to the quaternion group Q_8 , $Q = \langle z \rangle$ is cyclic of order $3^r > 1$, z induces an automorphism in P such that $P/\Phi(P)$ is a faithful and irreducible Q -module, and z centralizes $\Phi(P)$.
- (3) $G = P \rtimes Q$, where $Q = \langle z \rangle$ is cyclic of order $q^r > 1$, $q \nmid p - 1$, and P is an irreducible Q -module over the field of p elements with kernel $\langle z^q \rangle$ in Q .
- (4) $G = P \rtimes Q$, where $P = \langle a_0, a_1, \dots, a_{q-1} \rangle$ is an elementary abelian p -group of order p^q , $Q = \langle z \rangle$ is cyclic of order q^r , q^f is the highest power of q dividing $p - 1$ and $r > f \geq 1$. Define $a_j^z = a_{j+1}$ for $0 \leq j < q - 1$ and $a_{q-1}^z = a_0^i$, where i is a primitive q^f th root of unity modulo p .

2 Preliminary results

In this section, we collect some necessary results used frequently in the proof of theorems. For the sake of clearness for proof, an easy lemma with no proof is as follows.

Lemma 2.1. Let $G = P \rtimes Q$, where P and Q are Sylow subgroups of G , P is normal in G and Q is abelian. Then all maximal subgroups of G are of type either PQ_1^g or P_1Q^g , where Q_1 is a maximal subgroup of Q and P_1 is such that P/P_1 is non-trivial and G -simple, where g varies in G and PQ_1^g is normal in G .

The main properties of minimal non-supersolvable groups appear in Doerk's paper [9].

Lemma 2.2. Let G be a minimal non-supersolvable group. Then

- (1) G has a unique normal Sylow p -subgroup P for some prime p .
- (2) $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.
- (3) If $p \neq 2$, then the exponent of P is p .
- (4) If P is non-abelian, then $\Phi(P) = Z(P) = P'$.
- (5) If P is non-abelian and $p = 2$, then the exponent of P is 4.
- (6) If P is abelian, then the exponent of P is p .

Lemma 2.3. [9, Satz 4] If a group G has four maximal supersolvable subgroups of pairwise coprime indices, then G is supersolvable.

Lemma 2.4. [10, Theorem 1] *A group G is a minimal non- T -group (non- T -group all of whose proper subgroups are T -groups) if and only if it is one of the following seven types.*

- (1) *The generalized quaternion group of order 16.*
- (2) *$G = \langle x, y | x^{p^m} = 1 = y^{p^n}, y^{-1}xy = x^{1+p^{m-1}} \rangle$, where p is any prime, $m > 1$ and $n > 0$.*
- (3) *$G = \langle x, y, z | z = [x, y], 1 = x^{p^m} = y^{p^n} = z^p = [x, z] = [y, z] \rangle$, where p is any prime, $m > 0$ and $n > 0$.*
- (4) *$G = P \rtimes Q$, where P is the quaternion group of order 8, $Q = \langle y \rangle$ is cyclic of order $3^r > 1$, y induces an automorphism permuting cyclically the three maximal subgroups of P .*
- (5) *$G = P \rtimes Q$, where $P = \langle a, b \rangle$ is an elementary abelian p -group of order p^2 , and $Q = \langle y \rangle$ is cyclic of order q^r . Define $a^y = a^i$, $b^y = b^j$, $p \equiv 1 \pmod{q^f}$, and $r \geq f > 0$, where i is the least positive primitive q^f th root of unity modulo p , $j = 1 + kq^{f-1}$, with $0 < k < q$.*
- (6) *$G = P \rtimes Q$, where $Q = \langle y \rangle$ is cyclic of order $q^r > 1$, with $q \nmid p - 1$, and P is an irreducible Q -module over the field of p elements with kernel $\langle y^q \rangle$ in Q .*
- (7) *$G = P \rtimes Q$, where $P = \langle a_0, a_1, \dots, a_{q-1} \rangle$ is an elementary abelian p -group of order p^q , $Q = \langle y \rangle$ is cyclic of order q^r , q^f is the highest power of q dividing $p - 1$ and $r > f \geq 1$. Define $a_j^y = a_{j+1}$ for $0 \leq j < q - 1$ and $a_{q-1}^y = a_0^i$, where i is a primitive q^f th root of unity modulo p .*

3 A necessary and sufficient condition for solvable T -groups

In this section, we will give the proof of Theorem A and its corollary.

Proof of Theorem A. It is obvious that a nilpotent group is an NCM -group if and only if it is a Dedekind group, i.e., its all subgroups are normal.

By [8, Theorem 1], we only need to prove the sufficient condition for the case that G is non-nilpotent.

Assume that G is a counterexample of minimal order and every subgroup of G is an NCM -group. Surely, the assumption is inherited by subgroups. Consequently, every proper subgroup of G is a solvable T -group by the minimality of G .

Let G be of type (4) in Lemma 2.4. By Lemma 2.1, G has two kinds of maximal subgroups $\Phi(P)\langle y \rangle^g$ and $P\langle y^3 \rangle$, where $g \in G$, $\Phi(P)\langle y \rangle^g \not\trianglelefteq G$, and $P\langle y^3 \rangle \trianglelefteq G$. Since y induces an automorphism permuting cyclically the three maximal subgroups of the quaternion group, three maximal subgroups of Q_8 are not normal in G . However, they are not contained in $\Phi(P)\langle y \rangle^g$, a contradiction.

Let G be of type (5) in Lemma 2.4. Then non-normal subgroup $\langle ab \rangle$ of G is only contained in $P\langle y^q \rangle$ which is normal in G , a contradiction.

Let G be of type (6) in Lemma 2.4. In fact, G is minimal non-nilpotent and every subgroup of P is not normal in G . However, G has two kinds of maximal subgroups $\langle y \rangle^g$ and $P\langle y^q \rangle \trianglelefteq G$, where $g \in G$. This induces a contradiction.

Let G be of type (7) in Lemma 2.4. Similar arguments as above, every subgroup of P is not normal in G and only contained in $P\langle y^q \rangle \trianglelefteq G$, a contradiction. $\square$

By [8, Theorem 1] and Theorem A, the following result is obvious.

Corollary 3.1. *Let G be a group. Then the following conditions are equivalent:*

- (1) *G is a solvable T -group.*
- (2) *Every subgroup of G is an NNM -group.*
- (3) *Every subgroup of G is an NCM -group.*

However, a group G in which all subgroups are NLM -groups is not necessarily a solvable T -group.

Example 3.2. Let $G = \langle x, y | x^{p^2} = 1 = y^{p^3}, y^{-1}xy = x^{1+p} \rangle$, where p is any prime. Then G is minimal non-abelian and not a T -group.

4 Sufficient conditions for a group G to be supersolvable

We consider the structure of a group G by means of the NLM property. The following result plays a key role in proving later.

Lemma 4.1. *Let $G = P \rtimes Q$ be a supersolvable NLM -group, where P and Q are Sylow subgroups of G , P is elementary abelian and Q is cyclic non-normal in G . If all subgroups of G are NLM -groups, then G is a T -group.*

Proof. Clearly, we only need to consider the case $|P| \geq p^2$ for a prime p . Since G is supersolvable and P is elementary abelian, P can be regarded as a completely reducible Q -module over the field of p elements, say $P = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_m \rangle$, $Q = \langle y \rangle$ with $x_i^y = x_i^{\alpha_i}$ for some $1 \leq \alpha_i < p$, $1 \leq i \leq m$. According to the structure of T -groups [11], if G is not a T -group, then there exist i, j such that $\alpha_i \neq \alpha_j$. Take the subgroup $H = \langle x_i, x_j, y \rangle$. The non-normal maximal subgroups of H are $\langle x_i \rangle \langle y \rangle^h$ and $\langle x_j \rangle \langle y \rangle^h$, where h varies in H . The subgroup $U = \langle x_i x_j \rangle$ has order p , but is neither normal in H nor contained in a non-normal maximal subgroup of H . Hence, H is not an NLM -group. This contradiction induces that G is a T -group. $\square$

Proof of Theorem B. Assume that G is a counterexample of minimal order. Obviously, the assumption is inherited by subgroups. By the minimality of G , every proper subgroup of G is supersolvable. By Lemma 2.3, $\pi(G)$ has cardinal 2 or 3.

Now we only need to consider the following two cases.

Case 1. Let $G = PQ$, where $P \in \text{Syl}_p(G)$, $Q \in \text{Syl}_q(G)$, and $P \trianglelefteq G$, $Q \not\trianglelefteq G$.

By Lemma 2.1, all maximal subgroups of G are either of the type PQ_1 or of the type $\Phi(P)Q^g$, where $g \in G$, and Q_1 is a maximal subgroup of Q . By the assumption that all non-nilpotent subgroups of G are NLM -groups and Lemma 2.2(3)(5)(6), the exponent of P is p or 4 if $p = 2$, and P has a cyclic subgroup N of order p or 4, which is not normal in G and $N \not\leq \Phi(P)Q^g$. However, N is only contained in PQ_1 which is normal in G , a contradiction to the fact that G is an NLM -group as it is non-nilpotent. So G is supersolvable.

Case 2. Let $G = PQR$, where $P \in \text{Syl}_p(G)$, $Q \in \text{Syl}_q(G)$, $R \in \text{Syl}_r(G)$, $P \trianglelefteq G$, $Q \not\trianglelefteq G$, and $R \not\trianglelefteq G$.

By the minimality of G , we have easily that p is the maximal prime divisor of $\pi(G)$ and let $p > q > r$. Lemma 2.2(3) implies that the exponent of P is p .

(1) Assume that P is abelian.

Since the exponent of P is p , we have that P is an elementary abelian group and it is a minimal normal subgroup of G . If $P\langle x \rangle$ is nilpotent for any $x \in Q$, then Q is also normal in G , which contradicts the fact that minimal non-supersolvable has a unique normal Sylow subgroup. So there exists some element y such that $P\langle y \rangle$ is a non-nilpotent NLM -group, and $P\langle y \rangle$ is a T -group by Lemma 4.1. Thus, every subgroup of P is normal in PQ . Similarly, every subgroup of P is normal in PR , and is normal in G as well, a contradiction.

(2) Assume that P is non-abelian.

By Lemma 2.2(4), $\Phi(P) = Z(P) = P'$. Note that the exponent of P is p , we have easily that $G/\Phi(P)$ is minimal non-supersolvable and it satisfies the condition of the theorem, and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$. By induction, similar arguments to (1), $G/\Phi(P)$ is supersolvable, and so $G/\Phi(G)$ is supersolvable. Therefore, G is supersolvable, which is the final contradiction. $\square$

The following two results are obvious by Theorem B.

Corollary 4.2. *Let G be a group such that all its non-nilpotent subgroups are NCM -groups. Then G is supersolvable.*

Corollary 4.3. [8, Theorem 7] *Let G be a group such that all its non-nilpotent subgroups are NNM -groups. Then G is supersolvable.*

5 Non-*NLM*-groups whose proper subgroups are *NLM*-groups

The final aim is to give the structure of minimal non-*NLM*-groups. Buckley [12] called a group G a *PN*-group if every minimal subgroup of G is normal in G and gave some characterizations of *PN*-groups. Sastry [13] investigated the structure of minimal non-*PN*-groups (non-*PN*-groups whose proper subgroups are *PN*-groups). Guo et al. [14] called a group G a *PN**-group if all minimal subgroups and cyclic subgroups of order 4 of G are normal in G , and classified completely minimal non-*PN**-groups (non-*PN**-groups whose proper subgroups are *PN**-groups). Obviously, a nilpotent group is an *NLM*-group if and only if it is a *PN**-group. By combining [13, Theorem] and [14, Theorem 2.1], nilpotent minimal non-*NLM*-groups are as follows.

Theorem D. *Let G be a p -group. Then G is a minimal non-*NLM*-group if and only if it is one of the following types:*

- (1) $G = \langle a, b | a^{p^n} = 1, b^p = 1, [a, b] = a^{p^{n-1}} \rangle$, p an odd prime.
- (2) $G = \langle a, b, c | a^{p^n} = b^p = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle$, p an odd prime.
- (3) $G = \langle a, b, c | a^{2^n} = b^{2^m} = c^2 = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle$, where $n \geq m$ and $1 \leq m \leq 2$ (if $n = 2, m = 1$, then G is isomorphic to the dihedral group D_8).
- (4) $G = \langle a, b | a^{2^n} = 1, b^{2^m} = 1, [a, b] = a^{2^{n-1}} \rangle$, $n \geq 2, m \geq 2$, and at least one of n and m is 2.
- (5) $G = \langle a, b | a^8 = 1, b^2 = a^4, [a, b] = a^{-2} \rangle$.

A classification of non-nilpotent minimal non-*NLM*-groups is given in Theorem C, which will be proved as follows.

Proof of Theorem C. Suppose that G is a minimal non-*NLM*-group, then by Theorem B, every proper subgroup of G is supersolvable. Therefore, G is either supersolvable as type (1) or minimal non-supersolvable. In the following, we assume that G is minimal non-supersolvable.

By Lemma 2.2(1)(3)(5), G has a unique normal Sylow p -subgroup P , and the exponent of P is p or 4 if P is a non-abelian 2-group. Since all proper subgroups of G are *NLM*-groups, P is either elementary abelian or isomorphic to the quaternion group Q_8 .

First, we examine the classification of minimal non-supersolvable groups in [15, Theorem 9]. Groups of types 5, 7, 9, and 12 in [15, Theorem 9] all contain an extraspecial subgroup P of order p^3 , where $p > 2$. Note that P is not an *NLM*-group. Those groups are not satisfied with minimal non-*NLM*-groups. Let P be Q_8 as type 3 in [15, Theorem 9]. Note that $G/C_G(P) \leq S_4$, where S_4 is the symmetric group of degree 4. It makes Q is a Sylow 3-subgroup of G . Furthermore, G is also a minimal non-nilpotent group, thus G is of type (2).

Next, we assume that G is isomorphic to one of types 2, 4 in [15, Theorem 9]. It is easy to see that all subgroups of G are $P\langle z^q \rangle^g$ and $\langle z \rangle^g$, where $g \in G$. In type 2, G is also a minimal non-nilpotent group, G is of type (3). In type 4, $G = P \rtimes Q$, where $P = \langle a_0, a_1, \dots, a_{q-1} \rangle$ is elementary abelian, $Q = \langle z \rangle$ is of order q^r , $a_j^z = a_{j+1}$ for $0 \leq j < q-1$ and $a_{q-1}^z = a_0^i$, q^f is the highest power of q dividing $p-1$ and $r > f \geq 1$, and i is a primitive q^f th root of unity modulo p . Since $a_j^{z^q} = a_j^i$ for $0 \leq j \leq q-1$, z^q induces a power automorphism with fixed-point-free on P . Certainly, all subgroups of $P\langle z^q \rangle^g$ are *NLM*-groups. Thus, G is of type (4).

Then, we assume that G is isomorphic to one of types 6, 8, and 10 in [15, Theorem 9]. For arbitrary element x of E , by the assumption that $P\langle x \rangle$ is an *NLM*-group and Lemma 4.1, $P\langle x \rangle$ is a *T*-group. The arbitrariness of x leads to the normality of every subgroup of P in G , a contradiction.

Finally, let G be type 11 in [15, Theorem 9]. Namely, $G = PQR$, where $P \in \text{Syl}_p(G)$, $P \trianglelefteq G$, $Q \in \text{Syl}_q(G)$, $R \in \text{Syl}_r(G)$, R is a cyclic subgroup of order r^{s+t} , with r a prime number and s and t integers such that $s \geq 1$ and $t \geq 0$, normalizing Q , $Q/\Phi(Q)$ is an irreducible R -module over the field of q elements, with kernel the subgroup D of order r^t of R , and P is an irreducible QR -module over the field of p elements, where $q|p-1$, $r^s|p-1$, and $r|q-1$. In this case, $\Phi(G)_p$, the Hall p' -subgroup of $\Phi(G)$, coincides with $\Phi(Q) \times D$ and centralizes P . By the same arguments as above, PR is a *T*-group. Therefore, every subgroup of P is normal in PR . However, the supersolvability of PQ implies that there exists a non-trivial subgroup of P which is

normal in PQ . This contradicts the fact P is an irreducible QR -module over the field of p elements. Therefore, G does not coincide with a minimal non- NLM -group. $\square$

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References

- [1] R. Brandl, *Conjugacy classes of non-normal subgroups of finite p -groups*, Israel J. Math. **195** (2013), no. 1, 473–479, DOI: <https://doi.org/10.1007/s11856-012-0156-3>.
- [2] R. Brandl, *Groups with few non-normal subgroups*, Comm. Algebra **23** (1995), no. 6, 2091–2098, DOI: <https://doi.org/10.1080/00927879508825330>.
- [3] R. Brandl, *Non-soluble groups with few conjugacy classes of non-normal subgroups*, Beitr. Algebra Geom. **54** (2013), 493–501, DOI: <https://doi.org/10.1007/s13366-012-0087-5>.
- [4] L. Li, *The number of conjugacy classes of nonnormal subgroups of finite p -groups (II)*, Comm. Algebra **48** (2020), no. 3, 1099–1113, DOI: <https://doi.org/10.1080/00927872.2019.1677688>.
- [5] L. Li and H. Qu, *The number of conjugacy classes of nonnormal subgroups of finite p -groups*, J. Algebra **466** (2016), no. 15, 44–62, DOI: <https://doi.org/10.1016/j.jalgebra.2016.06.027>.
- [6] L. Ma, W. Meng, and W. Ma, *Finite groups whose all second maximal subgroups are cyclic*, Open Math. **15** (2017), no. 1, 611–615, DOI: <https://doi.org/10.1515/math-2017-0054>.
- [7] H. Mousavi, *On finite groups with few non-normal subgroups*, Comm. Algebra **27** (1999), no. 7, 3143–3151, DOI: <https://doi.org/10.1080/00927879908826617>.
- [8] G. Kaplan, *On T -groups, supersolvable groups, and maximal subgroups*, Arch. Math. (Basel) **96** (2011), no. 1, 19–25, DOI: <https://doi.org/10.1007/s00013-010-0207-0>.
- [9] K. Doerk, *Minimal nicht überauflösbare, endliche Gruppen*, Math. Z. **91** (1966), 198–205, DOI: <https://doi.org/10.1007/BF01312426>.
- [10] D. J. S. Robinson, *Groups which are minimal with respect to normality being intransitive*, Pacific J. Math. **31** (1969), no. 3, 777–785, DOI: <https://doi.org/10.2140/pjm.1969.31.777>.
- [11] R. A. Bryce and J. Cossey, *The Wielandt subgroup of a finite soluble group*, J. Lond. Math. Soc. **40** (1989), no. 2, 244–256, DOI: <https://doi.org/10.1112/jlms/s2-40.2.244>.
- [12] J. Buckley, *Finite groups whose minimal subgroups are normal*, Math. Z. **116** (1970), 15–17, DOI: <https://doi.org/10.1007/BF01110184>.
- [13] N. S. Narasimha Sastry, *On minimal non-PN-groups*, J. Algebra **65** (1980), no. 1, 104–109, DOI: [https://doi.org/10.1016/0021-8693\(80\)90241-0](https://doi.org/10.1016/0021-8693(80)90241-0).
- [14] P. Guo, R. Ge, and X. Zhang, *On minimal non- PN^* -groups*, Adv. Math. (China) **42** (2013), no. 1, 41–46.
- [15] A. Ballester-Bolinches and R. Esteban-Romero, *On minimal non-supersoluble groups*, Rev. Mat. Iberoam. **23** (2007), no. 1, 127–142, DOI: <https://doi.org/10.4171/RMI/488>.