Some results on semigroups of transformations with restricted range

Research Article

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Abstract: Let \( X \) be a non-empty set and \( Y \) a non-empty subset of \( X \). Denote the full transformation semigroup on \( X \) by \( T(X) \) and write \( f(X) = \{ f(x) | x \in X \} \) for each \( f \in T(X) \). It is well known that \( T(X, Y) = \{ f \in T(X) | f(X) \subseteq Y \} \) is a subsemigroup of \( T(X) \) and \( RT(X, Y) \), the set of all regular elements of \( T(X, Y) \), also forms a subsemigroup of \( T(X, Y) \). Green’s \(*\)-relations and Green’s \(~\)-relations (with respect to a non-empty subset \( U \) of the set of idempotents) were introduced by Fountain in 1979 and Lawson in 1991, respectively. In this paper, we intend to present certain characterizations of these two sets of Green’s relations of the semigroup \( T(X, Y) \). This investigation proves that the semigroup \( T(X, Y) \) is always a left Ehresmann semigroup, and \( RT(X, Y) \) is orthodox (resp. completely regular) if and only if \( Y \) contains at most two elements.

Keywords: Green’s \(~\)-relation, left (resp. right) Ehresmann (resp. restriction) semigroup, orthodox, completely regular

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1 Introduction

Let \( S \) be a semigroup and \( E(S) \) the set of idempotents in \( S \). Then an element \( a \) in \( S \) is called regular if there exists \( x \in S \) such that \( a = axa \). Moreover, if every element in \( S \) is regular, then we call \( S \) regular. It is well known that Green’s relations \( L, R, H, D, J \) play an important role in the theory of semigroups (see [1]). Recall that \( aLb \) (resp. \( aRb \)) if and only if \( S^a = S^b \) (resp. \( S^a = S^b \)), and \( H = L \cap R \). A semigroup \( S \) is regular if and only if every \( L\)-class of \( S \) contains idempotents (if and only if every \( R\)-class of \( S \) contains idempotents). A regular semigroup \( S \) is called orthodox (resp. completely regular) if the set of idempotents forms a subsemigroup of \( S \) (resp. if every \( H\)-class of \( S \) contains an idempotent).

As generalizations of Green’s relations, Green’s \(*\)-relations \( L^*, R^*, H^*, D^*, J^* \) were introduced and investigated in the 1970s (see [2,3]). Recall that the relations \( L^* \) and \( R^* \) on a semigroup \( S \) are defined as follows:

\[
L^* = \{(a, b) \in S \times S | ax = ay \text{ if and only if } bx = by \text{ for all } x, y \in S^3\},
\]

\[
R^* = \{(a, b) \in S \times S | xa = ya \text{ if and only if } xb = yb \text{ for all } x, y \in S^3\}.
\]

Obviously, \( L^* \) and \( R^* \) are equivalences on \( S \). According to Fountain [3], an element \( a \) in \( S \) is said to be right abundant (resp. left abundant) if the \( L^*\)-class (resp. \( R^*\)-class) containing \( a \) has an idempotent. Moreover, an abundant element of \( S \) is an element which is both left and right abundant. Furthermore, if every element of a semigroup \( S \) is left abundant (resp. right abundant, abundant), then we call \( S \) left abundant (resp. right abundant, abundant).

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In 1991, Lawson [4] introduced a new set of Green’s relations. Let $S$ be a semigroup and $U$ be a non-empty subset of $E(S)$. Define the relations $\mathcal{L}^U$ and $\mathcal{R}^U$ on $S$ as follows:

$$\mathcal{L}^U = \{(a, b) \in S \times S | ab = a \text{ if and only if } bu = b \text{ for all } u \in U\},$$
$$\mathcal{R}^U = \{(a, b) \in S \times S | ba = a \text{ if and only if } ub = b \text{ for all } u \in U\}.$$

Then $\mathcal{L}^U$ and $\mathcal{R}^U$ are also equivalences on $S$. In particular, we write $\mathcal{L}^{E(S)}$ and $\mathcal{R}^{E(S)}$ as $\mathcal{L}$ and $\mathcal{R}$, respectively. Moreover, we call $\mathcal{L}$ and $\mathcal{R}$ Green’s $\sim$-relations of $S$. By Lawson [4],

$$\mathcal{L} \subseteq \mathcal{L}^* \subseteq \mathcal{L}^U, \quad \mathcal{R} \subseteq \mathcal{R}^* \subseteq \mathcal{R}^U \quad (1.1)$$

for every non-empty subset $U$ of $E(S)$. For more information on Green’s relations and their generalizations, the reader can see the literature [5]. According to Lawson [4], an element $a$ in $S$ is left semiabundant (resp. right semiabundant) if the $\mathcal{R}$-class (resp. $\mathcal{L}$-class) containing $a$ has an idempotent. Moreover, a semiabundant element of $S$ is an element which is both left and right semiabundant. Furthermore, if every element of a semigroup $S$ is left semiabundant (resp. right semiabundant, semiabundant), then we call $S$ left semiabundant (resp. right semiabundant, semiabundant). By (1.1), for a semigroup $S$, regular elements are always abundant, and left abundant (resp. right abundant, abundant) elements are always left semiabundant (resp. right semiabundant, semiabundant).

On the other hand, Lawson [4] also introduced left and right Ehresmann semigroups by using $\mathcal{L}^U$ and $\mathcal{R}^U$, respectively. In 2010, Gould [6] characterized left and right Ehresmann semigroups from the view of variety. A unary semigroup $(S, \cdot, ^*)$ is called left Ehresmann if for all $x, y \in S$,

$$x'x = x, \quad (xy)^* = (xy')^*, \quad (x'y)^* = x'y^* = y^*x, \quad x'(xy)^* = (xy')^*.$$ (1.2)

A left Ehresmann semigroup $(S, \cdot, ^*)$ is called a left restriction semigroup if $(xy)^*x = xy^*$ for all $x, y \in S$. Dually, a unary semigroup $(S, \cdot, ^*)$ is called right Ehresmann if for all $x, y \in S$,

$$xx^* = x, \quad (xy)^* = (x'y)^*, \quad (x'y)^* = x'y^* = y^*x, \quad (xy)^*y^* = (xy)^*.$$ (1.3)

A right Ehresmann semigroup $(S, \cdot, ^*)$ is called a right restriction semigroup if $x'y = y(xy)^*$ for all $x, y \in S$.

Let $X$ be a non-empty set and $Y$ a non-empty subset of $X$. Denote the full transformation semigroup on $X$ by $T(X)$ and use $|X|$ to represent the cardinality of $X$. For all $U \subseteq X$ and $f \in T(X)$, write $\ker f = \{(x, y) \in X \times X | f(x) = f(y)\}$, $f(U) = \{f(x) | x \in U\}$ and $X \setminus U = \{x \in X | x \notin U\}$. Obviously, the set

$$T(X, Y) = \{f \in T(X) | f(X) \subseteq Y\}$$

is a subsemigroup of $T(X)$. In [7,8], the authors investigated Green’s relations and the regularity, and Green’s $^*$-relations and the abundance of $T(X, Y)$, respectively. In particular, Sanwong and Sommanee proved in [7] that the set $RT(X, Y)$ of all regular elements of $T(X, Y)$ in fact a regular subsemigroup of $T(X, Y)$.

The aim of this paper is to study $T(X, Y)$ along this direction. We first give some characterizations of Green’s $\sim$-relations $\mathcal{L}$ and $\mathcal{R}$ on $T(X, Y)$ and obtain a necessary and sufficient condition under which $\mathcal{L} = \mathcal{L}^*$ (resp. $\mathcal{R} = \mathcal{R}^*$, $\mathcal{L}^* = \mathcal{L}$ and $\mathcal{R} = \mathcal{R}^*$). Next, we prove that $T(X, Y)$ is always a left Ehresmann semigroup but it is neither a left restriction semigroup nor a right Ehresmann semigroup unless $|Y| = 1$ or $Y = X$. Finally, we also consider the orthodoxy and complete regularity of $RT(X, Y)$. In the continuation of this section, we recall some known results from [5] and [7,8].

**Lemma 1.1.** (Theorem 4.2.2 in [5] and its dual) Let $S$ be a semigroup and $a, b \in S$. If $a$ and $b$ are regular (resp. right abundant; left abundant), then

$$a\mathcal{L}b \Leftrightarrow a\mathcal{L}^*b \Leftrightarrow a\mathcal{L}^*b, \quad a\mathcal{R}b \Leftrightarrow a\mathcal{R}^*b \Leftrightarrow a\mathcal{R}^*b \quad (\text{resp. } a\mathcal{L}^*b \Leftrightarrow a\mathcal{L}^*b; a\mathcal{R}^*b \Leftrightarrow a\mathcal{R}^*b).$$

As consequences, if $S$ is regular (resp. right abundant; left abundant), then $\mathcal{L} = \mathcal{L}^* = \mathcal{L}, \mathcal{R} = \mathcal{R}^* = \mathcal{R}^*$ (resp. $\mathcal{L}^* = \mathcal{L}; \mathcal{R}^* = \mathcal{R}$).
Lemma 1.2. (Theorems 2.4, 3.2 and 3.3 in [7]) Let $f, g \in T(X, Y)$. Then the following statements hold:

1. $f$ is regular in $T(X, Y)$ if and only if $f(X) = f(Y)$.
2. $f \not\equiv g$ if and only if $f = g$ or $(f, g)$ are regular and $f(X) = g(X)$.
3. $f \not\equiv g$ if and only if $\ker f = \ker g$.

Lemma 1.3. (Lemma 2, Corollary 1 and Lemma 3 in [8]) Let $f, g \in T(X, Y)$. Then the following statements hold:

1. $(f, g) \in R^*$ if and only if $f(X) = g(X)$.
2. $T(X, Y)$ is left abundant.
3. $(f, g) \in L^*$ if and only if one of the following conditions holds:
   
   - $f(X) = f(Y), g(X) = g(Y)$ and $\ker f = \ker g$;
   
   - $f(X) \neq f(Y), g(X) \neq g(Y)$ and $\ker f \cap (Y \times Y) = \ker g \cap (Y \times Y)$.

2 Main results

Throughout the rest of the paper, let $X$ be a fixed non-empty set and $Y$ a fixed non-empty subset of $X$. Denote the set of all idempotents in $T(X, Y)$ by $E(T)$. We first characterize Green’s ~-relations $\mathcal{L}$ and $\mathcal{R}$ on $T(X, Y)$. By Lemmas 1.1 and 1.3 (2), we have $\mathcal{R}^* = \mathcal{R}$, and so $(f, g) \in \mathcal{R}$ if and only if $f(X) = g(X)$ for all $f, g \in T(X, Y)$. To give a characterization of $\mathcal{L}$, we need the following lemma.

Lemma 2.1. Let $f \in T(X, Y)$. Then $f(X) = f(Y)$ if and only if there exists $e \in E(T)$ such that $fe = f$.

Proof. If $f(X) = f(Y)$, then by Lemma 1.2, $f$ is regular in $T(X, Y)$, and so there is $h \in T(X, Y)$ such that $f = fhf$. Take $e = hf$. Then $f = fe$ and $e \in E(T)$. Conversely, if there exists $e \in E(T)$ such that $fe = f$, then $e$ is regular, and so $e(X) = e(Y)$ by Lemma 1.2. Therefore, $f(X) = fe(X) = fe(Y) = f(Y)$. □

Theorem 2.2. Let $f, g \in T(X, Y)$. Then $(f, g) \in \mathcal{L}$ if and only if one of the following conditions holds:

1. $f(X) = f(Y), g(X) = g(Y)$, $\ker f = \ker g$;
2. $f(X) \neq f(Y), g(X) \neq g(Y)$.

Proof. Let $(f, g) \in \mathcal{L}$. If $f(X) = f(Y)$, then by Lemma 2.1 there exists $e \in E(T)$ such that $fe = f$. Since $(f, g) \in \mathcal{L}$, it follows that $g = ge$. Using Lemma 2.1 again, we have $g(X) = g(Y)$. Similarly, it follows that $g(X) = g(Y)$ implies $f(X) = f(Y)$. Thus, $f(X) = f(Y)$ if and only if $g(X) = g(Y)$. If this is the case, there exists $h \in T(X, Y)$ such that $f = fhf$ by Lemma 1.2. Since $(f, g) \in \mathcal{L}$ and $hf \in E(T)$, it follows that $g = ghf$. If $(x, y) \in \ker f$, then $f(x) = f(y)$, and so $g(x) = ghf(x) = ghf(y) = g(y)$.

This implies that $\ker f \subseteq \ker g$. Dually, $\ker g \subseteq \ker f$, and hence $\ker f = \ker g$. In other words, we have shown that one of the above two conditions must be true if $(f, g) \in \mathcal{L}$.

Conversely, if (1) is true, by Lemma 1.3 we have $(f, g) \in \mathcal{L}^*$, and so $(f, g) \in \mathcal{L}$ by (1.1). If (2) holds, then by Lemma 2.1 we have $fe \neq f$ and $ge \neq g$ for all $e \in E(T)$. This also gives that $(f, g) \in \mathcal{L}$.

The following corollary gives the semiabundance of $T(X, Y)$.

Corollary 2.3. If $|Y| \neq 1$ and $Y \neq X$, then $T(X, Y)$ is left semiabundant but not right semiabundant.

Proof. By Lemma 1.3 (2), $T(X, Y)$ is left abundant, and so it is left semiabundant. We next show that $T(X, Y)$ is not right semiabundant. Since $|Y| \geq 2$, we can take distinct elements $a, b$ in $Y$. Consider the mapping $f : X \to X$ defined by
Then \( f \in T(X, Y) \) and \([a, b] = f(X) \neq f(Y) = \{a\} \) (note that \( Y \neq X \)). Suppose that the \( \overline{L} \)-class containing \( f \) has an idempotent \( e \). By Theorem 2.2, we have \( e(X) \neq e(Y) \). However, by Lemma 2.1, \( e(X) = e(Y) \), a contradiction. This implies that the \( \overline{L} \)-class containing \( f \) has no idempotent. Thus, \( T(X, Y) \) is not right semiabundant.

**Remark 2.4.** Corollary 2.3 shows that the semigroup \( T(X, Y) \) with \(|Y| \neq 1 \) and \( Y \neq X \) provides an explicit example of a left semiabundant semigroup, which is not right semiabundant. Certainly, its dual semigroup gives an explicit example of a right semiabundant semigroup, which is not left semiabundant.

By (1.1), Lemmas 1.2 (1) and 2.1, the following corollary is clear.

**Corollary 2.5.** Let \( f \in T(X, Y) \). Then

\[
\text{f is regular} \iff \text{f is abundant} \iff \text{f is right abundant} \iff \text{f is right semiabundant}.
\]

We have already observed that \( \mathcal{R}^* = \overline{R} \) in \( T(X, Y) \). We now consider the conditions under which the equality \( \mathcal{L} = \mathcal{L}^* \) (resp. \( \mathcal{R} = \mathcal{R}^* \), \( \mathcal{L}^* = \overline{L} \)) holds in \( T(X, Y) \). We start with the following simple result: if \(|Y| = 1 \) or \( Y = X \), then \( \mathcal{L} = \mathcal{L}^* = \overline{L} \) and \( \mathcal{R} = \mathcal{R}^* = \overline{R} \). In fact, in this case, for all \( f \in T(X, Y) \), we have \( f(X) \subseteq Y \) with \(|Y| = 1 \) or \( Y = X \), which implies that \( f(X) = f(Y) \). So \( T(X, Y) \) is regular by Lemma 1.2 (1). In view of Lemma 1.1, the desired result holds. Moreover, we can prove the following results.

**Theorem 2.6.** For the semigroup \( T(X, Y) \) in which \(|Y| \neq 1 \) and \( Y \neq X \), the following statements hold:

1. \( \mathcal{R} \neq \mathcal{R}^* \).
2. \( \mathcal{L} = \mathcal{L}^* \) if and only if \(|X \setminus Y| = 1 \).
3. \( \mathcal{L}^* = \overline{L} \) if and only if \(|Y| = 2 \).

**Proof.** (1) By hypothesis, \(|Y| \geq 2 \) and \(|X \setminus Y| \geq 1 \). Take different elements \( a, b \) in \( Y \) and define \( f, g \in T(X, Y) \) by

\[
f = \begin{bmatrix} X \setminus Y \times Y \\ a & b \end{bmatrix}, \quad g = \begin{bmatrix} X \setminus Y \times Y \\ b & a \end{bmatrix}.
\]

Then \( f(X) = g(X) \). By Lemma 1.3 (1), we have \((f, g) \in \mathcal{R}^* \). However, by (1) and (2) of Lemma 1.2, neither \( f \) nor \( g \) is regular, and so \((f, g) \notin \mathcal{R} \). Thus, \( \mathcal{R} \neq \mathcal{R}^* \).

2. **Necessity.** By hypothesis, \(|Y| \geq 2 \) and \(|X \setminus Y| \geq 1 \). If \(|X \setminus Y| \geq 2 \), then we can take different elements \( a, b \) in \( Y \) and different elements \( c, d \) in \( X \setminus Y \). Define \( f, g \in T(X, Y) \) by

\[
f = \begin{bmatrix} X & \{a, b, c, d\} \\ a & a & b & c & d \\ c & b & b & a & a \\ b & b & b & a & a \end{bmatrix}, \quad g = \begin{bmatrix} X & \{a, b, c, d\} \\ a & a & b & c & d \\ c & b & b & a & a \\ b & b & b & a & a \end{bmatrix}.
\]

Obviously, \( \text{ker} f \neq \text{ker} g \), \( f(Y) = \{a\} \neq \{a, b\} = f(X) \) and \( g(X) = \{a, b\} \neq \{b\} = g(Y) \), and

\[
\text{ker} f \cap (Y \times Y) = Y \times Y = \text{ker} g \cap (Y \times Y).
\]

This yields that \((f, g) \in \mathcal{L}^* \) and \((f, g) \notin \mathcal{L} \) by Lemmas 1.3 (3) and 1.2 (3).

**Sufficiency.** Let \(|X \setminus Y| = 1 \), \( f, g \in T(X, Y) \) and \((f, g) \in \mathcal{L}^* \). In view of Lemma 1.3 (3), we have one of the following two cases:

1. **Case 1.** \( f(X) = f(Y) \), \( g(X) = g(Y) \) and \( \text{ker} f = \text{ker} g \). In this case, we have \((f, g) \in \mathcal{L} \) by Lemma 1.2 (3).
2. **Case 2.** \( f(X) \neq f(Y) \), \( g(X) \neq g(Y) \) and \( \text{ker} f \cap (Y \times Y) = \text{ker} g \cap (Y \times Y) \). Let \( x_2, x_2 \in X \) and \((x_1, x_2) \in \text{ker} f \). Then \( f(x_1) = f(x_2) \). If both \( x_1 \) and \( x_2 \) are in \( Y \), then \((x_1, x_2) \in \text{ker} f \cap (Y \times Y) = \text{ker} g \cap (Y \times Y) \), and so \((x_1, x_2) \in \text{ker} g \). If neither \( x_1 \) nor \( x_2 \) is in \( Y \), then \( x_1 = x_2 \) by the fact \(|X \setminus Y| = 1 \), and certainly \((x_1, x_2) \in \text{ker} g \). If \( x_1 \in Y \) and \( x_2 \notin Y \), then we have \( X = Y \cup \{x_2\} \) by the fact \(|X \setminus Y| = 1 \). This implies that \( f(X) = f(Y) \) since \( f(x_1) = f(x_2) \).
which leads to a contradiction. Dually, it is also impossible that $x_1 \notin Y$ and $x_2 \in Y$. Thus, $\ker f \subseteq \ker g$ in this case. This together with its dual give that $\ker f = \ker g$. By Lemma 1.2 (3), $(f, g) \in \mathcal{L}$.

We have shown that $\mathcal{L}^* \subseteq \mathcal{L}$ in both cases. By (1.1), we have $\mathcal{L} = \mathcal{L}^*$.

(3) Let $|Y| = 2$, $f, g \in T(X, Y)$ and $(f, g) \in \mathcal{L}^*$. Then $f(Y) = f(Y)$, $g(Y) = g(Y)$, $\ker f = \ker g$ or $f(X) \neq f(Y)$, $g(X) \neq g(Y)$ by Theorem 2.2. If the former case holds, then $(f, g) \in \mathcal{L}^*$ by Lemma 1.3 (3). Now assume that $f(X) \neq f(Y)$ and $g(X) \neq g(Y)$. Since $f(Y) \subseteq f(X) \subseteq Y$ and $|Y| = 2$, we have $|f(X)| = 2$ and $|g(Y)| = 1$. Similarly, $|g(X)| = 2$ and $|g(Y)| = 1$. Thus,

$$\ker f \cap (Y \times Y) = Y \times Y = \ker g \cap (Y \times Y).$$

By Lemma 1.3 (3), $(f, g) \in \mathcal{L}^*$. Hence, we have $\mathcal{L} \subseteq \mathcal{L}^*$ in both cases. By (1.1), $\mathcal{L} = \mathcal{L}^*$.

Conversely, we need to prove that $\mathcal{L}^* \not= \mathcal{L}$ when $|Y| \geq 3$ as $|Y| \neq 1$ and $Y \neq X$ by hypothesis. Take different elements $a, b, c$ in $Y$. Define $f, g \in T(X, Y)$ by

$$f = \begin{pmatrix} Y & X \setminus Y \\ a & b \end{pmatrix}, \quad g = \begin{pmatrix} \{a, b\} & Y \setminus \{a, b\} \\ a & b \end{pmatrix}.$$ 

Then $f(X) \neq f(Y)$, $g(X) \neq g(Y)$. By Theorem 2.2, we have $(f, g) \in \mathcal{L}$. Since $(a, c) \in Y \times Y$, $(a, c) \in \ker f$ and $(a, c) \notin \ker g$, it follows that

$$\ker f \cap (Y \times Y) \neq \ker g \cap (Y \times Y).$$

This implies that $(f, g) \notin \mathcal{L}^*$ by Lemma 1.3. Therefore, $\mathcal{L}^* \neq \mathcal{L}^*$.

The next result shows that $T(X, Y)$ is always a left Ehresmann semigroup but it is neither a left restriction semigroup nor a right Ehresmann semigroup unless $Y| = 1$ or $Y = X$.

**Theorem 2.7.** $T(X, Y)$ is a left Ehresmann semigroup. However, if $Y| \neq 1$ and $Y \neq X$, then $T(X, Y)$ is neither right Ehresmann nor left restriction.

**Proof.** Fix an element $a \in Y$. Define $e : X \rightarrow X$ by

$$e(x) = \begin{cases} x, & x \in Y, \\ a, & \text{otherwise}. \end{cases}$$

Then $e$ is a left identity of $T(X, Y)$. For all $f \in T(X, Y)$, let $f^+ = e$. Then the identities in (1.2) are all satisfied, and so $(T(X, Y), \cdot^+, \cdot)$ is left Ehresmann. Assume that $Y| \neq 1$, $Y \neq X$ and $(T(X, Y), \cdot^+, \cdot)$ is right Ehresmann. Then for all $f \in T(X, Y)$, we have $f^+ = f$, and so $f^+ = (f^+)^* = (f^*)^+ = f^* f^+$ by (1.3). In view of Lemma 2.1, we obtain that $f(X) = f(Y)$. This implies that $f(X) = f(Y)$ for all $f \in T(X, Y)$. However, this is impossible. In fact, fix an element $a \in X \setminus Y$ and two distinct elements $b, c \in Y$ and define $g : X \rightarrow X$ by

$$g(x) = \begin{cases} b, & x = a, \\ c, & \text{otherwise}. \end{cases}$$

Then $g \in T(X, Y)$. Obviously, $g(X) \neq g(Y)$, thus, $T(X, Y)$ is not right Ehresmann.

Let $|Y| \neq 1, Y \neq X$ and assume that $(T(X, Y), \cdot^+, \cdot)$ is a left restriction semigroup. Then by (1.2) and the “left restriction” condition, for all $u, v \in T(X, Y)$,

$$u^* u = u, \quad u^* v^* = v^* u^*, \quad (u v^+) = u^* v^*, \quad uv^* = (uv)^*.$$ 

(2.1)

Fix two distinct elements $a, b \in Y$ and define $f : X \rightarrow X, g : X \rightarrow X$ by

$$f(x) = \begin{cases} x, & x \in Y, \\ a, & \text{otherwise}, \end{cases} \quad g(x) = \begin{cases} x, & x \in Y, \\ b, & \text{otherwise}. \end{cases}$$

Then $f, g \in T(X, Y)$ and $fg = g$, $gf = f$. By (2.1), for all $x \in Y$,

$$f^*(x) = f^*(f(x)) = (f^* f)(x) = x = g(x) = (g^* g)(x) = g^*(x).$$

(2.2)
Since \( g(x) \in Y \) for all \( x \in X \), \((f^*g)(x) = f^*(g(x)) = f(g(x)) = g(x)\). This gives that \( f^*g = g \). Dually, \( g^*f = f \). By (2.1),
\[ g^* = (f^*g)^* = f^*g^* = g^*f^* = (g^*f)^* = f^*. \]
On the other hand, let \( x_0 \in X \setminus Y \). Then by (2.1), the fact \( fg = g \), the definition of \( f \) and (2.2), we have
\[ f(g^*(x_0)) = fg^*(x_0) = ((fg)^*f)(x_0) = (g^*f)(x_0) = g^*(f(x_0)) = g^*(a) = a. \]
Observe that the definition of \( f \) and the fact \( g^* \in T(X, Y) \), it follows that \( g^*(x_0) = a \). Dually, \( f^*(x_0) = a \). This implies that \( f^* \neq g^* \), a contradiction. Thus, \( T(X, Y) \) is not left restriction. \( \square \)

**Remark 2.8.** Theorem 2.7 shows that the semigroup \( T(X, Y) \), with \( |Y| = 1 \) and \( Y \neq X \) provides an explicit example of a left Ehresmann semigroup which is neither right Ehresmann nor left restriction. Certainly, its dual semigroup gives an explicit example of a right Ehresmann semigroup which is neither left Ehresmann nor right restriction.

Finally, we consider the orthodoxy and complete regularity of \( RT(X, Y) \) (the set of all regular elements of \( T(X, Y) \)). For this purpose, we need the lemma below.

**Lemma 2.9.** Let \( e \in T(X, Y) \). Then \( e \in E(T) \) if and only if \( e(X) = e(Y) \) and \( e(a) = a \) for all \( a \in e(Y) \).

**Proof.** If \( e^2 = e \), then by Lemma 2.1, we have \( e(X) = e(Y) \). Let \( a \in e(Y) = e(X) \). Then there exists \( x_0 \in X \) such that \( a = e(x_0) \). Thus, \( e(a) = ee(x_0) = e(x_0) = a \). Conversely, since \( e(X) = e(Y) \), it follows that for every \( x \in X \), there is \( y \in Y \) such that \( e(x) = e(y) \in e(Y) \), and so \( ee(x) = ee(y) = e(y) = e(x) \). This implies that \( e^2 = e \). \( \square \)

**Theorem 2.10.** For the semigroup \( T(X, Y) \), the following statements are equivalent:

1. \( RT(X, Y) \) is orthodox.
2. \( |Y| \leq 2 \).
3. \( RT(X, Y) \) is completely regular.

**Proof.**

(1) \( \Rightarrow \) (2). Suppose that \( |Y| \geq 3 \) and fix distinct elements \( a, b, c \in Y \). Define \( e, f \in T(X, Y) \) by
\[
e = \begin{pmatrix} c & X \setminus \{c\} \\ c & a \end{pmatrix}, \quad f = \begin{pmatrix} b & X \setminus \{b\} \\ b & c \end{pmatrix}.
\]
Then \( e \) and \( f \) are idempotents, and so \( e, f \in RT(X, Y) \). However,
\[
ef = \begin{pmatrix} b & X \setminus \{b\} \\ a & c \end{pmatrix}
\]
is not an idempotent, and this implies that \( RT(X, Y) \) is not orthodox.

(2) \( \Rightarrow \) (1), (3). If \( |Y| = 1 \), then \( T(X, Y) \) contains only one element, and \( RT(X, Y) \) is certainly orthodox and completely regular. Now let \( |Y| = 2 \) and write \( Y = \{a, b\} \). By Lemma 2.9, the set of idempotents of \( RT(X, Y) \) is
\[
E(RT(X, Y)) = \{f \in T(X, Y) \mid f(a) = a, \ f(b) = b\} \cup \{e_1, e_2\},
\]
where
\[
e_1 = \begin{pmatrix} X \\ a \end{pmatrix}, \quad e_2 = \begin{pmatrix} X \\ b \end{pmatrix}.
\]
Obviously, \( E(RT(X, Y)) \) is a subsemigroup of \( RT(X, Y) \). So \( RT(X, Y) \) is orthodox.

On the other hand, in view of Lemma 1.2 (1), we have
\[
RT(X, Y) = E(RT(X, Y)) \cup \{f \in T(X, Y) \mid f(a) = b, \ f(b) = a\}.
\]
For a given \( f \in RT(X, Y) \) satisfying \( f(a) = b \) and \( f(b) = a \), define \( e : X \to X \) by
\[
   e(x) = \begin{cases} 
     x, & x \in Y, \\
     b, & x \in X \setminus Y \text{ and } f(x) = a, \\
     a, & x \in X \setminus Y \text{ and } f(x) = b.
   \end{cases}
\]

Then \( e \in E(RT(X, Y)) \) and \( e(X) = f(X) = \{a, b\} \), \( \ker e = \ker f \). This implies that \( (e, f) \in \mathcal{H} \) in \( RT(X, Y) \) by (2) and (3) of Lemma 1.2 and Proposition 4.1.1 in [1], and so \( f \) is completely regular. Thus, \( RT(X, Y) \) is completely regular.

(3) \( \Rightarrow \) (2). We need to prove that \( RT(X, Y) \) is not completely regular when \( |Y| \geq 3 \). Let \( |Y| \geq 3 \) and take different elements \( a, b, c \) in \( Y \). Define \( f : X \to X \),
\[
   f = \begin{pmatrix}
     c & \setminus \{c\} \\
     b & a
   \end{pmatrix},
\]
then \( f \in RT(X, Y) \) by Lemma 1.2 (1). Moreover, in view of (2) and (3) of Lemma 1.2, the \( \mathcal{H} \)-class containing \( f \) just has the following two elements:
\[
   \begin{pmatrix}
     c & \setminus \{c\} \\
     b & a
   \end{pmatrix}, \begin{pmatrix}
     c & \setminus \{c\} \\
     a & b
   \end{pmatrix}.
\]
Obviously, the above two elements are not idempotents. Thus, \( RT(X, Y) \) is not completely regular. \( \square \)

To end the paper, we give a concrete example to illustrate the above results.

**Example 2.11.** Let \( X = \{1, 2, 3, 4\} \) and \( Y = \{1, 2, 3\} \). Then the semigroup \( T(X, Y) \) contains 81 elements. For the sake of simplicity, we denote a typical element 
\[
   \begin{bmatrix}
     1 & 2 & 3 & 4 \\
     i & j & k & l
   \end{bmatrix}
\]
in \( T(X, Y) \) by \( (i, j, k, l) \), where \( i, j, k, l \in Y \). With the above notation, by Lemma 1.2 (3), the 14 \( L \)-classes of \( T(X, Y) \) can be described as follows:
\[
   \begin{align*}
   L_1 &= \{(i, i, i, 0) \mid i \in Y\}, & L_2 &= \{(j, i, i, 0) \mid i \in Y, i \neq j\}, \\
   L_3 &= \{(i, j, i, 0) \mid i \in Y, i \neq j\}, & L_4 &= \{(i, i, j, 0) \mid i \in Y, i \neq j\}, \\
   L_5 &= \{(i, i, j, 0) \mid i \in Y, i \neq j\}, & L_6 &= \{(i, j, i, 0) \mid i \in Y, i \neq j\}, \\
   L_7 &= \{(i, j, i, j) \mid i \in Y, i \neq j\}, & L_8 &= \{(i, i, i, j) \mid i \in Y, i \neq j\}, \\
   L_9 &= \{(i, i, j, k) \mid i, j, k \in Y\}, & L_{10} &= \{(i, j, i, k) \mid i, j, k \in Y\}, \\
   L_{11} &= \{(i, i, i, k) \mid i, j, k \in Y\}, & L_{12} &= \{(i, j, k, i) \mid i, j, k \in Y\}, \\
   L_{13} &= \{(i, j, k, i) \mid i, j, k \in Y\}, & L_{14} &= \{(i, k, i, i) \mid i, j, k \in Y\}.
   \end{align*}
\]
Observe that \( |L_1| = 3 \) and \( |L_2| = 6 \), \( i = 2, 3, \ldots, 14 \). By Theorem 2.6 (2) and the fact that \( X \setminus Y = 1 \), we have \( L = L^* \) in \( T(X, Y) \), and so the above \( L \)-classes of \( T(X, Y) \) are also the whole \( L^* \)-classes of \( T(X, Y) \). Moreover, by Lemma 1.2 (1), the non-regular \( L \)-classes (a non-regular \( L \)-class is an \( L \)-class in which every element is not regular) of \( T(X, Y) \) are \( L_6, L_9, L_{10} \) and \( L_{11} \). In view of Theorem 2.2 and Lemma 1.2 (1), \( T(X, Y) \) has 11 \( L \)-classes which are
\[
   \bigcup_{i=8}^{11} L_i, L_{12}, L_{13}, L_{14}.
\]
On the other hand, by Lemma 1.3 (1), the \( R^* \)-classes of \( T(X, Y) \) are as follows:
\[
   \begin{align*}
   R_1^* &= \{(1, 1, 1, 1)\}, & R_2^* &= \{(2, 2, 2, 2)\}, \\
   R_3^* &= \{(3, 3, 3, 3)\}, & R_4^* &= \{(f \in T(X, Y) \mid f(X) = \{1, 2\})\}, \\
   R_5^* &= \{(f \in T(X, Y) \mid f(X) = \{1, 3\})\}, & R_6^* &= \{(f \in T(X, Y) \mid f(X) = \{2, 3\})\}, \\
   R_7^* &= \{(f \in T(X, Y) \mid f(X) = Y)\}.
   \end{align*}
\]
Note that $R_7^*$ is the union of $L_9, L_{10}, \ldots, L_{13}$ and $L_{14}$, and

$$|R_1^*| = |R_2^*| = |R_3^*| = 1, \quad |R_4^*| = |R_5^*| = 14, \quad |R_6^*| = 36.$$ 

In view of the comments before Lemma 2.1, we have $\tilde{R} = R^*$, and so the above $R^*$-classes are also the whole $\tilde{R}$-classes of $T(X, Y)$. Furthermore, observe by Lemma 1.2 (1) that the set of non-regular elements in $R_4^*$ (resp. $R_5^*$, $R_6^*$) is

$$\{(1, 1, 1, 2), (2, 2, 2, 1)\} \quad \text{resp.} \quad \{(1, 1, 1, 3), (3, 3, 3, 1)\}, \quad \{(2, 2, 2, 3), (3, 3, 3, 2)\},$$

and the set of non-regular elements in $R_7^*$ is the union of $L_9, L_{10}$ and $L_{11}$. Thus, by Lemma 1.2 (2), there are 31 $\tilde{R}$-classes in $T(X, Y)$ in which the whole non-single element $\tilde{R}$-classes are as follows:

$$R_4^* \setminus \{(1, 1, 1, 2), (2, 2, 2, 1)\}, \quad R_5^* \setminus \{(1, 1, 1, 3), (3, 3, 3, 1)\}, \quad R_6^* \setminus \{(2, 2, 2, 3), (3, 3, 3, 2)\},$$

and

$$R_7^* \setminus \{(2, 2, 2, 3), (3, 3, 3, 2)\}, \quad R_7^* = \left(\bigcup_{n=9}^{16} L_n\right) \setminus \left(\bigcup_{n=9}^{11} L_n\right) = \bigcup_{n=12}^{16} L_n.$$ 

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References