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Research Article

Ruijuan Li*, Yanqin Cao, and Xinhong Zhang

On kernels by rainbow paths in arc-coloured digraphs

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Abstract: In 2018, Bai, Fujita and Zhang [Discrete Math. **341** (2018), no. 6, 1523–1533] introduced the concept of a kernel by rainbow paths (for short, RP-kernel) of an arc-coloured digraph D, which is a subset S of vertices of D such that (a) there exists no rainbow path for any pair of distinct vertices of S, and (b) every vertex outside S can reach S by a rainbow path in D. They showed that it is NP-hard to recognize whether an arc-coloured digraph has an RP-kernel and it is NP-complete to decide whether an arc-coloured tournament has an RP-kernel. In this paper, we give the sufficient conditions for the existence of an RP-kernel in arc-coloured unicyclic digraphs, semicomplete digraphs, quasi-transitive digraphs and bipartite tournaments, and prove that these arc-coloured digraphs have RP-kernels if certain "short" cycles and certain "small" induced subdigraphs are rainbow.

Keywords: arc-coloured digraphs, kernels, kernels by rainbow paths

MSC 2020: 05C20, 05C12, 05C07

1 Introduction

For convenience of the reader, some necessary terminologies and notations not mentioned in this section can be found in Section 2. All digraphs considered in this paper are finite. In this paper, all paths, walks and cycles are always directed. For terminology and notation, we refer the reader to Bang-Jensen and Gutin [1].

Let D be a digraph. A *kernel* of D is a subset $S \subseteq V(D)$ such that (a) for any pair of distinct vertices $x, y \in S$ are non-adjacent, and (b) for each vertex $v \in V(D) \setminus S$, there exists a vertex $s \in S$ such that $(v, s) \in A(D)$. This notion was originally introduced in the game theory by von Neumann and Morgenstern [2] in 1944. Kernels have found many applications and several sufficient conditions for the existence of a kernel have been proved, see [3]. In this paper, we will need the following result.

Theorem 1.1. [2] *Let D be a digraph. If D has no cycle, then D has a unique kernel.*

Let D be a digraph and m a positive integer. An arc-colouring of D is a mapping $C : A(D) \to \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. $C(D)$ and $C(x, y)$ denote the set of colours appearing on all the arcs of D and the colour appearing on the arc $(x, y) \in A(D)$. We call D an *m-arc-coloured digraph* if $|C(D)| = m$. An arc-coloured digraph is called *monochromatic* if all arcs are assigned the same colour. Define a *kernel by monochromatic paths* of an arc-coloured digraph D to be a subset $S \subseteq V(D)$ such that (a) there exists no

*** Corresponding author: Ruijuan Li,** School of Mathematical Science, Shanxi University, 030006, Taiyuan, P. R. China, e-mail: ruijuanli@sxu.edu.cn
Yanqin Cao: School of Mathematical Science, Shanxi University, 030006, Taiyuan, P. R. China
Xinhong Zhang: Department of Applied Mathematics, Taiyuan University of Science and Technology, 030024, Taiyuan, P. R. China

monochromatic path for any pair of vertices of S, and (b) for each vertex outside S can reach S by a monochromatic path.

The concept of a kernel by monochromatic paths of an arc-coloured digraph was introduced by Sands et al. [4] in 1982 as a generalization of the concept of a kernel. They showed that every 2-coloured digraph has a kernel by monochromatic paths. As a corollary, they showed that every 2-coloured tournament has a one-vertex kernel by monochromatic paths. They also proposed the problem asking whether a 3-coloured tournament with no rainbow triangle has a one-vertex kernel by monochromatic paths. In 1988, Shen [5] proved that for $m \geqslant 3$ every m-coloured tournament with no rainbow triangle and no rainbow transitive triangle has a one-vertex kernel by monochromatic paths, and also showed that the condition "with no rainbow triangle and no rainbow transitive triangle" cannot be improved for $m \geqslant 5$. In 2004, Galeana-Sánchez and Rojas-Monroy [6] showed that the condition of Shen cannot be improved for $m = 4$ by constructing a family of counterexamples. For more results about kernels by monochromatic paths of an arc-coloured digraph can be found in [7–11].

An arc-coloured digraph is called *properly coloured* if any two consecutive arcs have distinct colours. Define a *kernel by properly coloured paths* of an arc-coloured digraph D to be a subset $S \subseteq V(D)$ such that (a) there exists no properly coloured path for any pair of vertices of S, and (b) for each vertex outside S can reach S by a properly coloured path.

The concept of a kernel by properly coloured paths of an arc-coloured digraph was introduced by Delgado-Escalante and Galeana-Sánchez [12] in 2009 as a generalization of the concept of a kernel. Bai et al. [13] showed in 2018 that it is NP-hard to recognize whether an arc-coloured digraph has a kernel by properly coloured paths. They conjecture that every arc-coloured digraph with all cycles properly coloured has a kernel by properly coloured paths and verified the conjecture for unicyclic digraphs, semicomplete digraphs and bipartite tournaments. In 2018, Delgado-Escalante et al. [14] gave some sufficient conditions for the existence of a kernel by properly coloured paths in arc-coloured tournaments, quasi-transitive digraphs and k-partite tournaments.

An arc-coloured digraph is called *rainbow* if all arcs have distinct colours. Define a *kernel by rainbow paths* (for short, *RP-kernel*) of an arc-coloured digraph D to be a subset $S \subseteq V(D)$ such that (a) there exists no rainbow path for any pair of vertices of S, and (b) for each vertex outside S can reach S by a rainbow path.

The concept of an RP-kernel of an arc-coloured digraph was introduced by Bai et al. [13] in 2018 as a generalization of the concept of kernel. They showed that it is NP-hard to recognize whether an arc-coloured digraph has an RP-kernel. Recently, Bai et al. [15] proposed the following theorem.

Theorem 1.2. [15] *It is NP-complete to decide whether an arc-coloured tournament has an RP-kernel.*

Problem 1.3. [15] Is it true that every arc-coloured digraph with all cycles rainbow has an RP-kernel?

Just as other NP-complete problems, we give some sufficient conditions for the existence of an RP-kernel in arc-coloured unicyclic digraphs, semicomplete digraphs, quasi-transitive digraphs and bipartite tournaments and prove that these arc-coloured digraphs have RP-kernels if certain "short" cycles and certain "small" induced subdigraphs are rainbow.

2 Terminology and preliminaries

Let D be a digraph. $V(D)$ and $A(D)$ denote its vertex and arc sets. If (x, y) is an arc of D, sometimes we use the notation $x \to y$ to denote this arc and say that x *dominates* y. The *out-neighbourhood* (resp. *in-neighbourhood*) of a vertex $x \in V(D)$ is the $N_D^+(x) = \{y \mid (x, y) \in A(D)\}$ (resp. $N_D^-(x) = \{y \mid (y, x) \in A(D)\}$). For a vertex $x \in V(D)$, the *out-degree* (resp. *in-degree*) of x is denoted by $d_D^+(x) = |N_D^+(x)|$ (resp. $d_D^-(x) = |N_D^-(x)|$). A vertex in D is called *sink* (resp. *source*) if $d_D^+(x) = 0$ (resp. $d_D^-(x) = 0$). An arc $(x, y) \in A(D)$ is called *asymmetrical* (resp. *symmetrical*) if $(y, x) \notin A(D)$ (resp. $(y, x) \in A(D)$). If S is a nonempty set of $V(D)$, then the subdigraph

$D[S]$ induced by S is the digraph having vertex set S, and whose arcs are all those arcs of D joining vertices of S.

For disjoint sets X and Y, $X \Rightarrow Y$ means that every vertex of X dominates every vertex of Y and $y \nrightarrow x$ for any $x \in X$ and $y \in Y$. If $Y = \{v\}$, we always denote $X \Rightarrow v$ instead of $X \Rightarrow \{v\}$. $X \nRightarrow Y$ means that there exists a vertex $u \in Y$ such that $X \nRightarrow u$.

For two distinct vertices $x, y \in V(D)$, a path P from x to y is denoted by (x, y)-path, and $\ell(P)$ denotes the length of path P. Let $S \subseteq V(D)$. (x, S)-path in D denotes an (x, s)-path for some $s \in S$. (S, x)-path in D denotes an (s, x)-path for some $s \in S$. We always call a cycle C of length $\ell(C)$ an $\ell(C)$-cycle.

A digraph D is called *strong* if there exists a path from x to y and a path from y to x in D for every choice of distinct vertices x, y of D. A *strong component* of a digraph D is a maximal induced subdigraph of D which is strong. For any non-strong digraph D, we can label its strong components $D_1, D_2, \ldots, D_p$, $p \geqslant 2$, in such a way that there is no arc from D_j to D_i when $j > i$.

In the following proof, we use the definition below.

Definition 2.1. For an arc-coloured digraph D, the *rainbow closure* of D denoted by $C_r(D)$, is a digraph such that:

(a) $V(C_r(D)) = V(D)$;
(b) $A(C_r(D)) = \{(u, v) \mid \text{there exists a rainbow}\ (u, v)\text{-path in}\ D\}$.

It is not hard to see the following simple and useful result.

Observation 2.2. An arc-coloured digraph D has an RP-kernel if and only if $C_r(D)$ has a kernel.

A digraph D is called a *kernel-perfect digraph* or *KP-digraph* when every induced subdigraph of D has a kernel. The following theorem gives a sufficient condition for a digraph to be a *KP*-digraph.

Theorem 2.3. [3] *Let D be a digraph such that every cycle in D has at least one symmetrical arc. Then D is a KP-digraph.*

3 Unicyclic digraphs

A digraph D is a *unicyclic digraph* if it contains only one cycle. In this section, we consider the sufficient conditions for the existence of an RP-kernel in an arc-coloured unicyclic digraph.

Theorem 3.1. *Let D be an m-arc-coloured unicyclic digraph such that the unique cycle is rainbow. Then D has an RP-kernel.*

Proof. Let D be an m-arc-coloured unicyclic digraph with the unique cycle C. We will show the result by constructing an RP-kernel S of D. If D is strong, then $D = C$. Since the cycle C is rainbow, each vertex of C forms an RP-kernel of C. The desired result follows directly.

Now assume D is not strong. Then D has strong components $D_1, D_2, \ldots, D_k$ $(k \geqslant 2)$ such that there exists no arc from D_i to D_j for any $i > j$. Since D is unicyclic, then one of the strong components $D_1, D_2, \ldots, D_k$ containing the unique cycle C and any other strong component is a single vertex. Let $k = j_1$. If D_k is a single vertex, say $D_k = D_{j_1} = \{v_{j_1}\}$, we put v_{j_1} into S. If $D_k = D_{j_1} = C$, we put an arbitrary vertex of C, say also v_{j_1}, into S. Since C is rainbow, $V(C) \setminus \{v_{j_1}\}$ can reach v_{j_1} by a rainbow path. In the ith step, say $S = \{v_{j_1}, v_{j_2}, \ldots, v_{j_{i-1}}\}$. Let $j_i \in \{1, 2, \ldots, j_{i-1} - 1\}$ be the largest integer such that there exists no rainbow path from some vertex of D_{j_i} to S. If D_{j_i} is a single vertex, say $D_{j_i} = \{v_{j_i}\}$, we put v_{j_i} into S. If $D_{j_i} = C$, we put an arbitrary vertex of C, say also v_{j_i}, into S. The procedure can be completed after finite steps. The set S constructed by the aforementioned procedure is an RP-kernel of D. □

4 Semicomplete digraphs

A digraph D is *semicomplete* if for any pair of vertices there exists at least one arc between them. A *tournament* is a semicomplete digraph with no 2-cycle. In this section, we consider the sufficient conditions for the existence of an RP-kernel in an arc-coloured semicomplete digraph. Since each pair of vertices in a semicomplete digraph are adjacent, it follows that an RP-kernel of a semicomplete digraph consists of only one vertex.

Theorem 4.1. *Let D be an m-arc-coloured semicomplete digraph with all 3-cycles are rainbow in D. Then D has a one-vertex RP-kernel.*

Proof. Let v be a vertex of D with maximum in-degree. Since D is a semicomplete digraph, it follows that $N_D^+(v) \cup N_D^-(v) \cup \{v\} = V(D)$. Moreover, for any $u \in N_D^+(v)$, there exists a vertex $w \in N_D^-(v)$ such that $u \to w$. If not, then for any $w \in N_D^-(v)$, we have $w \to u$. This implies that $N_D^-(v) \cup \{v\} \subseteq N_D^-(u)$, which contradicts the choice of v. So (u, w, v, u) is a rainbow 3-cycle. It follows (u, w, v) is a rainbow (u, v)-path. Now for any $u \in N_D^+(v)$, there exists a rainbow (u, v)-path. Clearly, for any $w \in N_D^-(v)$, there exists a rainbow (w, v)-path. Combining with $N_D^+(v) \cup N_D^-(v) \cup \{v\} = V(D)$, we have for any $x \in V(D) \setminus \{v\}$, there exists a rainbow (x, v)-path. Thus, $\{v\}$ is a one-vertex RP-kernel of D. □

Corollary 4.2. [15] *Let T be an m-arc-coloured tournament in which all 3-cycles are rainbow. Then T has a one-vertex RP-kernel.*

5 Quasi-transitive digraphs

A digraph D is a *quasi-transitive digraph* if whenever $\{(u, v), (v, w)\} \subseteq A(D)$, then either $(u, w) \in A(D)$ or $(w, u) \in A(D)$. In this section, we consider the sufficient conditions for the existence of an RP-kernel in an arc-coloured quasi-transitive digraph.

Lemma 5.1. [1] *Let D be a quasi-transitive digraph. If x and y are a pair of distinct vertices of D such that D has an (x, y)-path but x does not dominate y, then either $y \to x$, or there exist vertices $u, v \in V(D) \setminus \{x, y\}$ such that $x \to u \to v \to y$ and $y \to u \to v \to x$.*

Let QT_4 be the quasi-transitive digraph, which has $V(QT_4) = \{x, y, u, v\}$ and $A(QT_4) = \{(x, u), (u, v), (v, y), (y, u), (v, x)\}$ (Figure 1).

Lemma 5.2. *Let D be an m-arc-coloured quasi-transitive digraph with all 3-cycles and all induced subdigraphs QT_4 are rainbow in D. If x and y are a pair of distinct vertices of D such that D has a rainbow (x, y)-path but no rainbow (y, x)-path, then $(x, y) \in A(D)$.*

Proof. Suppose to the contrary that $(x, y) \notin A(D)$. Since there exists no rainbow (y, x)-path, then x, y are non-adjacent. Let $P = (x = x_0, x_1, x_2, \ldots, x_n = y) \subseteq D$ be a rainbow (x, y)-path. By Lemma 5.1, there exist vertices $u, v \in V(D) \setminus \{x, y\}$ such that $x \to u \to v \to y$ and $y \to u \to v \to x$. This implies that $D[x, u, v, y]$ is QT_4 which is rainbow. It follows (y, u, v, x) is a rainbow (y, x)-path, which is a contradiction. Thus, $(x, y) \in A(D)$. □

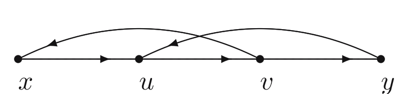

Figure 1: A quasi-transitive digraph QT_4.

Theorem 5.3. *Let D be an m-arc-coloured quasi-transitive digraph with all 3-cycles and all induced subdigraphs QT_4 are rainbow in D. Then $C_r(D)$ is a KP-digraph.*

Proof. Suppose to the contrary that $C_r(D)$ is not a *KP*-digraph. By Theorem 2.3, there exists a cycle with no symmetrical arc. Let $C = (u_1, u_2, \ldots, u_\ell, u_1)$ be a shortest cycle with no symmetrical arc in $C_r(D)$. We will get a contradiction by showing that C has a symmetrical arc.

Claim 1. $C \subseteq D$ and $\ell \geqslant 5$.

Proof. Since C has no symmetrical arc, for each $i \in \{1, 2, \ldots, \ell\}$, there exists a rainbow (u_i, u_{i+1})-path and no rainbow (u_{i+1}, u_i)-path in D. By Lemma 5.2, we have $(u_i, u_{i+1}) \in A(D)$. Then $C \subseteq D$.

Now we prove $\ell \geqslant 5$. Since C has no symmetrical arc, we have $\ell \geqslant 3$.

If $\ell = 3$, combining with $C \subseteq D$, we have $C = (u_1, u_2, u_3, u_1)$ is a rainbow 3-cycle in D. This implies that (u_2, u_3, u_1) is a rainbow (u_2, u_1)-path and hence $(u_2, u_1) \in A(C_r(D))$. Note that $(u_1, u_2) \in A(C)$, which contradicts that C has no symmetrical arc.

If $\ell = 4$, by the proof above, we have $C = (u_1, u_2, u_3, u_4, u_1)$ is a cycle in D. Since D is quasi-transitive, we have u_1, u_3 are adjacent. If $(u_1, u_3) \in A(D)$, then (u_1, u_3, u_4, u_1) is a rainbow 3-cycle. This implies that (u_4, u_1, u_3) is a rainbow (u_4, u_3)-path and hence $(u_4, u_3) \in A(C_r(D))$. Note that $(u_3, u_4) \in A(C)$, which contradicts that C has no symmetrical arc. If $(u_3, u_1) \in A(D)$, then (u_1, u_2, u_3, u_1) is a rainbow 3-cycle. This implies that (u_2, u_3, u_1) is a rainbow (u_2, u_1)-path and hence $(u_2, u_1) \in A(C_r(D))$. Note that $(u_1, u_2) \in A(C)$, which contradicts that C has no symmetrical arc.

Thus, $\ell \geqslant 5$. □

By Claim 1, we have $\ell - 1 \geqslant 4$ and $C \subseteq D$. Considering $\{(u_{\ell-1}, u_\ell), (u_\ell, u_1)\} \subseteq A(D)$, we have $u_1, u_{\ell-1}$ are adjacent.

If $(u_1, u_{\ell-1}) \in A(D)$, then $(u_{\ell-1}, u_\ell, u_1, u_{\ell-1})$ is a rainbow 3-cycle. This implies that $(u_\ell, u_1, u_{\ell-1})$ is a rainbow $(u_\ell, u_{\ell-1})$-path and hence $(u_\ell, u_{\ell-1}) \in A(C_r(D))$. Note that $(u_{\ell-1}, u_\ell) \in A(C)$, which contradicts that C has no symmetrical arc.

If $(u_{\ell-1}, u_1) \in A(D)$, since $(u_1, u_2) \in A(D)$, there exists $i \in \{3, 4, \ldots, \ell - 1\}$ such that $(u_i, u_1) \in A(D)$. Let

$$i_0 = \min\{i \in \{3, 4, \ldots, \ell - 1\} \mid (u_i, u_1) \in A(D)\}.$$

Considering $\{(u_{i_0-1}, u_{i_0}), (u_{i_0}, u_1)\} \subseteq A(D)$, we have u_1, u_{i_0-1} are adjacent. By the choice of i_0, we have $(u_1, u_{i_0-1}) \in A(D)$. It follows $(u_1, u_{i_0-1}, u_{i_0}, u_1)$ is a rainbow 3-cycle. This implies that $(u_{i_0}, u_1, u_{i_0-1})$ is a rainbow (u_{i_0}, u_{i_0-1})-path and hence $(u_{i_0}, u_{i_0-1}) \in A(C_r(D))$. Note that $(u_{i_0-1}, u_{i_0}) \in A(C)$, which contradicts that C has no symmetrical arc.

Thus, $C_r(D)$ is a *KP*-digraph. □

By Observation 2.2 and Theorem 5.3, the following corollary is direct.

Corollary 5.4. *Let D be an m-arc-coloured quasi-transitive digraph with all 3-cycles and all induced subdigraphs QT_4 are rainbow in D. Then D has an RP-kernel.*

6 Bipartite tournaments

A digraph D is a *bipartite tournament* if there exists a partition of $V(D)$ into two sets $\{X, Y\}$ such that there exists no arc between any two vertices in the same set and there exists an arc between any two vertices in different sets. In this section, we consider the sufficient conditions for the existence of an RP-kernel in an arc-coloured bipartite tournament. We begin with two simple observations.

Observation 6.1. [13] Let D be an m-arc-coloured digraph and $v \in V(D)$ a source. Then D has an RP-kernel if and only if $D - v$ has an RP-kernel.

Observation 6.2. 1-arc-coloured bipartite tournament $D = (X, Y)$ has an RP-kernel.

Proof. Obviously, a kernel of D is also an RP-kernel of D. We claim that either X or Y is a kernel of D and hence an RP-kernel of D. If X is not a kernel, then there exists $v \in Y$ such that $X \Rightarrow v$. This implies that Y is a kernel of D. □

In the following, we may assume $m \geqslant 2$.

Observation 6.3. Let $D = (X, Y)$ be an m-arc-coloured bipartite tournament with $\min\{|X|, |Y|\} = 1$. Then D has an RP-kernel.

Proof. W.l.o.g., assume $|X| = \min\{|X|, |Y|\} = 1$. Obviously, D has no cycle. This implies that $C_r(D)$ also has no cycle. By Theorem 1.1, $C_r(D)$ has a unique kernel. By Observation 2.2, D has an RP-kernel. □

Now we consider the RP-kernel of an m-arc-coloured bipartite tournament $D = (X, Y)$ with $\min\{|X|, |Y|\} = 2$. W.l.o.g., assume $\min\{|X|, |Y|\} = |X| = 2$ and $X = \{x_1, x_2\}$. If $X \Rightarrow Y$, then Y is an RP-kernel of D. If $Y \Rightarrow X$, then X is an RP-kernel of D. So we assume $X \not\Rightarrow Y$ and $Y \not\Rightarrow X$. Let

$$Y_0 = \{y \in Y \mid X \Rightarrow y\},$$

$$Y_1 = \{y \in Y \setminus Y_0 \mid \text{there exists a rainbow}\,(y, Y_0)\text{-path in}\,D\},$$

$$Y_2 = Y \setminus (Y_0 \cup Y_1).$$

By Observation 6.1, we also assume that D has no source in Y. By the definition of Y_0, the following claim holds directly.

Each vertex in $Y \setminus Y_0 = Y_1 \cup Y_2$ has exactly one out-neighbour and one in-neighbour in X. (∗)

We will show that D has an RP-kernel if every 4-cycle contained in D is coloured with at least three colours. For this purpose, we divide the proof into two lemmas.

Lemma 6.4. Let $D = (X, Y)$ be an m-arc-coloured bipartite tournament with $\min\{|X|, |Y|\} = 2$ satisfying that every 4-cycle contained in D is coloured with at least three colours. Y_0, Y_1 and Y_2 are defined as above. If $Y_0 \neq \emptyset$, then D has an RP-kernel.

Proof. Note that there exists no rainbow (y, Y_0)-path for any $y \in Y_2$. If $Y_2 = \emptyset$, then Y_0 is an RP-kernel of D. So we assume that $Y_2 \neq \emptyset$. Clearly, the following claim holds directly.

Claim 1. There exist rainbow paths from $X \cup Y_1$ to Y_0; there exists no rainbow path from Y_0 to $Y \setminus Y_0$ and there exists no rainbow path from Y_2 to Y_0.

Claim 2. For some $x_i \in X$, if x_i has an in-neighbour in Y_2, then all arcs from x_i to Y_0 are assigned the common colour.

Proof. Suppose to the contrary that $C(x_i, y_1) \neq C(x_i, y_2)$ for some $y_1, y_2 \in Y_0$. Let $y \in Y_2$ with $y \to x_i$. Since $C(x_i, y_1) \neq C(x_i, y_2)$, y can reach Y_0 by a rainbow path passing through (y, x_i) as well as either (x_i, y_1) or (x_i, y_2). This contradicts that there exists no rainbow (y, Y_0)-path for any $y \in Y_2$. Thus, all arcs from x_i to Y_0 are assigned the common colour. □

For convenience, we will denote the common colour assigned the arcs from x_i to Y_0 by $C(x_i, Y_0)$ for $x_i \in X$ with an in-neighbour in Y_2. By the definition of Y_2, the following claim holds directly.

Claim 3. For any $y \in Y_2$ with $y \to x_i$ for some $x_i \in X$, $C(y, x_i) = C(x_i, Y_0)$.

Let $S \subseteq Y_2$ be the maximal subset such that there exists no rainbow path for any pair of vertices of S in D. Let

$$R = \{r \in Y_2 \setminus S \mid \text{there exists no rainbow}\,(r, S)\text{-path in}\,D\}.$$

If $R = \emptyset$, then $Y_0 \cup S$ is an RP-kernel of D. Assume that $R \neq \emptyset$ and let $r \in R \subseteq Y_2$ be arbitrary. By $(*)$, w.l.o.g., we assume

$$x_1 \to r \to x_2.$$

By the choice of S, there exists a rainbow (s, r)-path P for some $s \in S$ in D.

If $\ell(P) = 4$, w.l.o.g., assume $P = (s, x_2, y, x_1, r)$ where $y \in Y \setminus Y_0$. It is clear that $C(x_1, r) \neq C(x_1, Y_0)$, since otherwise, in the rainbow path P, we replace the arc (x_1, r) with (x_1, y_0) for any $y_0 \in Y_0$ and get a rainbow (s, y_0)-path, which contradicts $s \in Y_2$. Now we claim that $Y_0 \cup \{r\}$ is an RP-kernel of D. By Claim 1, it is sufficient to show that there exists a rainbow (z, r)-path for any $z \in Y_2 \setminus \{s, r\}$. By $(*)$, we have either $z \to x_1$ or $z \to x_2$. If $z \to x_1$, by Claim 3, we have $C(z, x_1) = C(x_1, Y_0)$. Combining with $C(x_1, r) \neq C(x_1, Y_0)$, we have (z, x_1, r) is a rainbow (z, r)-path. If $z \to x_2$, by Claim 3, we have $C(z, x_2) = C(s, x_2) = C(x_2, Y_0)$. In the rainbow path P, we replace the arc (s, x_2) with (z, x_2) and get a rainbow (z, r)-path (z, x_2, y, x_1, r). This implies that $Y_0 \cup \{r\}$ is an RP-kernel of D.

If $\ell(P) = 2$, now (s, x_1, r) is the rainbow (s, r)-path. Note that $s \in S \subseteq Y_2$ and $s \to x_1$. Let $y \in Y_2$ with $y \to x_1$. By Claim 3, we have $C(y, x_1) = C(s, x_1) = C(x_1, Y_0)$. In the rainbow path (s, x_1, r), we replace the arc (s, x_1) with (y, x_1) and get a rainbow path (y, x_1, r). This means that all vertices dominating x_1 in Y_2 can reach r by a rainbow path. Let

$$Q_1 = \{y \in Y_2 \setminus \{r\} \mid y \to x_1\}, \quad Q_2 = \{y \in Y_2 \setminus \{r\} \mid y \to x_2\}.$$

Clearly, each vertex of Q_1 can reach r by a rainbow path. By $(*)$, we have $Q_1 \cup Q_2 = Y_2 \setminus \{r\}$ and $Q_1 \cap Q_2 = \emptyset$. If $Q_2 = \emptyset$, then $Y_0 \cup \{r\}$ is an RP-kernel of D. So assume that $Q_2 \neq \emptyset$. Also by $(*)$,

$$x_1 \Rightarrow Q_2 \Rightarrow x_2.$$

If there exists a rainbow (q_2, r)-path P' for some $q_2 \in Q_2$, we claim that $Y_0 \cup \{r\}$ is an RP-kernel of D. Since $x_1 \to r \to x_2$ and $x_1 \to q_2 \to x_2$, we have $\ell(P') \neq 2$ and hence $\ell(P') = 4$. W.l.o.g., assume $P' = (q_2, x_2, y, x_1, r)$ where $y \in Y \setminus Y_0$. It is sufficient to show that there exists a rainbow (q_2', r)-path for any $q_2' \in Q_2 \setminus \{q_2, r\}$. By Claim 3, we have $C(q_2, x_2) = C(q_2', x_2) = C(x_2, Y_0)$. In the rainbow path P', we replace the arc (q_2, x_2) with (q_2', x_2) and get a rainbow (q_2', r)-path (q_2', x_2, y, x_1, r). This implies that $Y_0 \cup \{r\}$ is an RP-kernel of D.

If there exists a rainbow (r, q_2)-path P'' for some $q_2 \in Q_2$, we claim that $Y_0 \cup \{q_2\}$ is an RP-kernel of D. Since $x_1 \to r \to x_2$ and $x_1 \to q_2 \to x_2$, we have $\ell(P'') \neq 2$ and hence $\ell(P'') = 4$. W.l.o.g., assume $P'' = (r, x_2, y, x_1, q_2)$ where $y \in Y \setminus Y_0$. It is clear that $C(x_1, q_2) \neq C(x_1, Y_0)$, since otherwise, in the rainbow path P'', we replace the arc (x_1, q_2) with (x_1, y_0) for any $y_0 \in Y_0$ and get a rainbow (r, y_0)-path, which contradicts $r \in Y_2$. Now it is sufficient to show that there exists a rainbow (w, q_2)-path for any $w \in Y_2 \setminus \{r, q_2\}$. By $(*)$, we have either $w \to x_1$ or $w \to x_2$. If $w \to x_1$, by Claim 3, we have $C(w, x_1) = C(x_1, Y_0)$. Combining with $C(x_1, q_2) \neq C(x_1, Y_0)$, we have (w, x_1, q_2) is a rainbow (w, q_2)-path. If $w \to x_2$, by Claim 3, we have $C(w, x_2) = C(r, x_2) = C(x_2, Y_0)$. In the rainbow path P'', we replace the arc (r, x_2) with (w, x_2) and get a rainbow (w, q_2)-path (w, x_2, y, x_1, q_2). This implies that $Y_0 \cup \{q_2\}$ is an RP-kernel of D.

If there exists no rainbow (q_2, r)-path and no rainbow (r, q_2)-path for any $q_2 \in Q_2$, we claim that $Y_0 \cup Q_2 \cup \{r\}$ is an RP-kernel of D.

Claim 4. If there exists no rainbow (Q_2, r)-path and there exists no rainbow (r, Q_2)-path, then there exists no rainbow path for any pair of vertices of Q_2.

Proof. Suppose to the contrary that there exists a rainbow path for some $q_2, q_2' \in Q_2$, say (q_2, x_2, y, x_1, q_2'), where $y \in Y \backslash Y_0$. Note that $y \neq r$ since there exists no rainbow (Q_2, r)-path. Since $r, q_2 \in Y_2$, by Claim 3, we have $C(r, x_2) = C(q_2, x_2) = C(x_2, Y_0)$. In the rainbow path (q_2, x_2, y, x_1, q_2'), we replace the arc (q_2, x_2) with (r, x_2) and get a rainbow (r, q_2')-path (r, x_2, y, x_1, q_2'), which contradicts that there exists no rainbow (r, Q_2)-path. □

Recall that $Q_1 \cup Q_2 = Y_2 \backslash \{r\}$ and each vertex of Q_1 can reach r by a rainbow path. By Claims 1 and 4, $Y_0 \cup Q_2 \cup \{r\}$ is an RP-kernel of D. □

Lemma 6.5. *Let $D = (X, Y)$ be an m-arc-coloured bipartite tournament with $\min\{|X|, |Y|\} = 2$ satisfying that every 4-cycle contained in D is coloured with at least three colours. Y_0, Y_1 and Y_2 are defined as above. If $Y_0 = \emptyset$, then D has an RP-kernel.*

Proof. By $(*)$, each vertex of Y has one out-neighbour and one in-neighbour in X. We give a partition of Y as follows

$$Y' = \{y \in Y \mid x_1 \to y \to x_2\}, \quad Y'' = \{y \in Y \mid x_2 \to y \to x_1\}.$$

If $Y' = \emptyset$, then $Y = Y''$ and $x_2 \Rightarrow Y \Rightarrow x_1$. If there exists a rainbow (x_2, x_1)-path, then $\{x_1\}$ is an RP-kernel of D. If there exists no rainbow (x_2, x_1)-path, then $\{x_1, x_2\}$ is an RP-kernel of D. If $Y'' = \emptyset$, we can prove that either $\{x_2\}$ or $\{x_1, x_2\}$ is an RP-kernel of D. So we assume $Y' \neq \emptyset$ and $Y'' \neq \emptyset$.

In particular, we consider the following subsets of Y' and Y'', respectively

$$Y^* = \{y \in Y' \mid C(x_1, y) \neq C(y, x_2)\}, \quad Y^{**} = \{y \in Y'' \mid C(x_2, y) \neq C(y, x_1)\}.$$

Let $y' \in Y'$ and $y'' \in Y''$ be arbitrary. If $Y^* = \emptyset$, then $C(x_1, y') = C(y', x_2)$. Note that (x_1, y', x_2, y'', x_1) is a 4-cycle. Since every 4-cycle is coloured with at least three colours, we have (y', x_2, y'', x_1) is a rainbow path. Clearly, (y'', x_1) is a rainbow path. It follows that $\{x_1\}$ is an RP-kernel of D. Similarly, if $Y^{**} = \emptyset$, we can prove that $\{x_2\}$ is an RP-kernel of D. So we assume $Y^* \neq \emptyset$ and $Y^{**} \neq \emptyset$.

Case 1: $Y' \backslash Y^* \neq \emptyset$ or $Y'' \backslash Y^{**} \neq \emptyset$.

W.l.o.g., assume $Y' \backslash Y^* \neq \emptyset$. Let $y' \in Y' \backslash Y^*$ be arbitrary. We assume that $C(x_1, y') = C(y', x_2) = \alpha$. Let

$$Y'_\alpha = \{y' \in Y' \mid C(x_1, y') = C(y', x_2) = \alpha\}.$$

Clearly, $Y'_\alpha \neq \emptyset$. Let $y'' \in Y''$ be arbitrary. Note that $x_1 \Rightarrow y' \Rightarrow x_2 \Rightarrow y'' \Rightarrow x_1$. Since every 4-cycle in D is coloured with at least three colours, we have $C(x_2, y'') \neq C(y'', x_1)$, $C(x_2, y'') \neq \alpha$ and $C(y'', x_1) \neq \alpha$. Let $C(x_2, y'') = \beta$ and $C(y'', x_1) = \gamma$. Then α, β, γ are pairwise distinct.

Define the following vertex subsets, which are shown in Figure 2, in which a box represents a set of vertices while dotted, dashed, thick dotted and solid arcs represent, respectively, the arcs coloured by α, β, γ and ω, where ω stands for an arbitrary colour not in $\{\alpha, \beta, \gamma\}$. Let

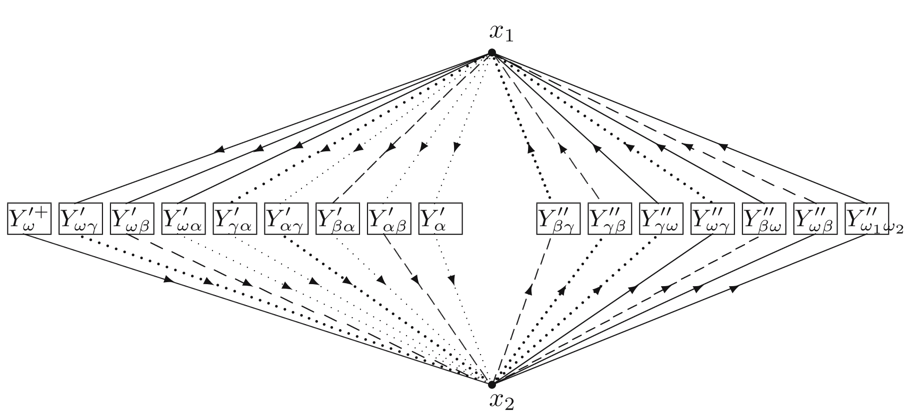

Figure 2: An arc-coloured bipartite tournament for Case 1 of the proof of Lemma 6.5.

$$Y'_{c_1c_2} = \{y' \in Y' \mid C(x_1, y') = c_1,\ C(y', x_2) = c_2\},$$

where $(c_1, c_2) = \{(\alpha, \beta), (\beta, \alpha), (\alpha, \gamma), (\gamma, \alpha), (\omega, \alpha), (\omega, \beta), (\omega, \gamma)\}$;

$$Y'^{+}_{\omega} = \{y' \in Y' \mid C(x_1, y') \text{ is arbitrary},\ C(y', x_2) \notin \{\alpha, \beta, \gamma\}\};$$
$$Y''_{c_1c_2} = \{y'' \in Y'' \mid C(x_2, y'') = c_1,\ C(y'', x_1) = c_2\},$$

where $(c_1, c_2) = \{(\beta, \gamma), (\gamma, \beta), (\gamma, \omega), (\omega, \gamma), (\beta, \omega), (\omega, \beta)\}$;

$$Y''_{\omega_1\omega_2} = \{y'' \in Y'' \mid C(x_2, y''), C(y'', x_1) \notin \{\alpha, \beta, \gamma\} \text{ and } C(x_2, y'') \neq C(y'', x_1)\}.$$

Note that $Y'_{\alpha} \neq \emptyset$ and $Y''_{\beta\gamma} \neq \emptyset$. Since every 4-cycle is coloured with at least three colours, we have

$$Y' = Y'_{\alpha} \cup Y'_{\alpha\beta} \cup Y'_{\beta\alpha} \cup Y'_{\alpha\gamma} \cup Y'_{\gamma\alpha} \cup Y'_{\omega\alpha} \cup Y'_{\omega\beta} \cup Y'_{\omega\gamma} \cup Y'^{+}_{\omega},$$

$$Y'' = Y''_{\beta\gamma} \cup Y''_{\gamma\beta} \cup Y''_{\beta\omega} \cup Y''_{\gamma\omega} \cup Y''_{\omega\beta} \cup Y''_{\omega\gamma} \cup Y''_{\omega_1\omega_2}.$$

For convenience, a vertex in $Y'_{c_1c_2}$ (resp. $Y''_{c_1c_2}, Y'_{\alpha}, Y'^{+}_{\omega}, Y''_{\omega_1\omega_2}$) we denote by $y'_{c_1c_2}$ (resp. $y''_{c_1c_2}, y'_{\alpha}, y'^{+}_{\omega}, y''_{\omega_1\omega_2}$).

If $Y''_{\omega_1\omega_2} \neq \emptyset$, then for any $y' \in Y' \setminus Y'^{+}_{\omega}$, $(y', x_2, y''_{\omega_1\omega_2}, x_1)$ is a rainbow (y', x_1)-path; for any $y'^{+}_{\omega} \in Y'^{+}_{\omega}$, $(y'^{+}_{\omega}, x_2, y''_{\beta\gamma}, x_1)$ is a rainbow (y'^{+}_{ω}, x_1)-path. This implies that $\{x_1\}$ is an RP-kernel of D. So we assume $Y''_{\omega_1\omega_2} = \emptyset$.

Subcase 1.1: $Y''_{\omega\beta} \cup Y''_{\beta\omega} \neq \emptyset$ and $Y''_{\omega\gamma} \cup Y''_{\gamma\omega} \neq \emptyset$.

Then for any $y' \in Y'_{\alpha} \cup Y'_{\beta\alpha} \cup Y'_{\gamma\alpha} \cup Y'_{\omega\alpha} \cup Y'^{+}_{\omega}$, $(y', x_2, y''_{\beta\gamma}, x_1)$ is a rainbow (y', x_1)-path; for any $y' \in Y'_{\alpha\beta} \cup Y'_{\omega\beta}$, (y', x_2, y'', x_1) is a rainbow (y', x_1)-path, where $y'' \in Y''_{\omega\gamma} \cup Y''_{\gamma\omega}$; for any $y' \in Y'_{\alpha\gamma} \cup Y'_{\omega\gamma}$, (y', x_2, y'', x_1) is a rainbow (y', x_1)-path, where $y'' \in Y''_{\omega\beta} \cup Y''_{\beta\omega}$. This implies that $\{x_1\}$ is an RP-kernel of D.

Subcase 1.2: Exactly one of the subsets $Y''_{\omega\beta} \cup Y''_{\beta\omega}$ and $Y''_{\omega\gamma} \cup Y''_{\gamma\omega}$ is not an empty set.

W.l.o.g., assume $Y''_{\omega\beta} \cup Y''_{\beta\omega} \neq \emptyset$ and $Y''_{\omega\gamma} \cup Y''_{\gamma\omega} = \emptyset$. Now $Y'' = Y''_{\beta\gamma} \cup Y''_{\gamma\beta} \cup Y''_{\beta\omega} \cup Y''_{\omega\beta}$.

Subcase 1.2.1: $Y'_{\omega\beta} = \emptyset$ and $Y'_{\alpha\beta} = \emptyset$.

Now $Y' = Y'_{\alpha} \cup Y'_{\beta\alpha} \cup Y'_{\alpha\gamma} \cup Y'_{\gamma\alpha} \cup Y'_{\omega\alpha} \cup Y'_{\omega\gamma} \cup Y'^{+}_{\omega}$. For any $y' \in Y'_{\alpha} \cup Y'_{\beta\alpha} \cup Y'_{\gamma\alpha} \cup Y'_{\omega\alpha} \cup Y'^{+}_{\omega}$, $(y', x_2, y''_{\beta\gamma}, x_1)$ is a rainbow (y', x_1)-path. For any $y' \in Y'_{\alpha\gamma} \cup Y'_{\omega\gamma}$, (y', x_2, y'', x_1) is a rainbow (y', x_1)-path, where $y'' \in Y''_{\omega\beta} \cup Y''_{\beta\omega}$. This implies that $\{x_1\}$ is an RP-kernel of D.

Subcase 1.2.2: $Y'_{\omega\beta} \neq \emptyset$ and $Y'_{\alpha\beta} \neq \emptyset$.

Now $Y' = Y'_{\alpha} \cup Y'_{\alpha\beta} \cup Y'_{\beta\alpha} \cup Y'_{\alpha\gamma} \cup Y'_{\gamma\alpha} \cup Y'_{\omega\alpha} \cup Y'_{\omega\beta} \cup Y'_{\omega\gamma} \cup Y'^{+}_{\omega}$. Note that there exists no rainbow path for any pair of vertices of $Y'_{\omega\beta} \cup Y'_{\alpha\beta}$ since $C(Y'_{\omega\beta} \cup Y'_{\alpha\beta}, x_2) = \beta \in C(x_2, Y'') \cup C(Y'', x_1)$. Since every 4-cycle is coloured with at least three colours, we have $C(x_1, y'_{\omega\beta}) \notin C(x_2, y''_{\omega\beta}) \cup C(y''_{\beta\omega}, x_1)$ for any $y'_{\omega\beta} \in Y'_{\omega\beta}$. For any $y' \in Y'_{\alpha} \cup Y'_{\beta\alpha} \cup Y'_{\alpha\gamma} \cup Y'_{\gamma\alpha} \cup Y'_{\omega\alpha} \cup Y'_{\omega\gamma}$, $(y', x_2, y'', x_1, y'_{\omega\beta})$ is a rainbow $(y', y'_{\omega\beta})$-path, where $y'' \in Y''_{\omega\beta} \cup Y''_{\beta\omega}$. For any $y'^{+}_{\omega} \in Y'^{+}_{\omega}$, $(y'^{+}_{\omega}, x_2, y''_{\beta\gamma}, x_1, y'_{\alpha\beta})$ is a rainbow $(y'^{+}_{\omega}, y'_{\alpha\beta})$-path. This implies that $Y'_{\omega\beta} \cup Y'_{\alpha\beta}$ is an RP-kernel of D.

Subcase 1.2.3: $Y'_{\omega\beta} \neq \emptyset$ and $Y'_{\alpha\beta} = \emptyset$.

Now $Y' = Y'_{\alpha} \cup Y'_{\beta\alpha} \cup Y'_{\alpha\gamma} \cup Y'_{\gamma\alpha} \cup Y'_{\omega\alpha} \cup Y'_{\omega\beta} \cup Y'_{\omega\gamma} \cup Y'^{+}_{\omega}$. By the proof above, there exists no rainbow path for any pair of vertices of $Y'_{\omega\beta}$ and each vertex of $Y'_{\alpha} \cup Y'_{\beta\alpha} \cup Y'_{\alpha\gamma} \cup Y'_{\gamma\alpha} \cup Y'_{\omega\alpha} \cup Y'_{\omega\gamma}$ can reach $Y'_{\omega\beta}$ by a rainbow path passing through a vertex of $Y''_{\omega\beta} \cup Y''_{\beta\omega}$. If $|C(x_1, Y'_{\omega\beta})| \geqslant 2$, let $y'_{\omega\beta 1}, y'_{\omega\beta 2} \in Y'_{\omega\beta}$ with $C(x_1, y'_{\omega\beta 1}) \neq C(x_1, y'_{\omega\beta 2})$. Let $y'^{+}_{\omega} \in Y'^{+}_{\omega}$ be arbitrary. Note that either $(y'^{+}_{\omega}, x_2, y''_{\beta\gamma}, x_1, y'_{\omega\beta 1})$ or $(y'^{+}_{\omega}, x_2, y''_{\beta\gamma}, x_1, y'_{\omega\beta 2})$ is a rainbow $(y'^{+}_{\omega}, Y'_{\omega\beta})$-path.

This implies that $Y'_{\omega\beta}$ is an RP-kernel of D. If $|C(x_1, Y'_{\omega\beta})| = 1$, let $U = \{y'^{+}_{\omega} \in Y'^{+}_{\omega} \mid C(y'^{+}_{\omega}, x_2) = C(x_1, Y'_{\omega\beta})\}$. For any $y' \in Y'^{+}_{\omega} \setminus U$, $C(y', x_2) \neq C(x_1, Y'_{\omega\beta})$ and $(y', x_2, y''_{\beta\gamma}, x_1, y'_{\omega\beta})$ is a rainbow path from $Y'^{+}_{\omega} \setminus U$ to $Y'_{\omega\beta}$. If there exists a vertex $u \in U$ with $C(x_1, u) \neq \beta$, for any $u' \in U \setminus \{u\}$, either (u', x_2, y''_1, x_1, u) or (u', x_2, y''_2, x_1, u) is a rainbow (u', u)-path, where $y''_1 \in Y''_{\omega\beta} \cup Y''_{\beta\omega}$ and $y''_2 \in Y''_{\beta\gamma} \cup Y''_{\gamma\beta}$. Note that there exists no rainbow path for any pair of vertices of $Y'_{\omega\beta} \cup \{u\}$. This implies that $Y'_{\omega\beta} \cup \{u\}$ is an RP-kernel of D. If $C(x_1, U) = \beta$, then there exists no rainbow path for any pair of vertices of $Y'_{\omega\beta} \cup U$. This implies that $Y'_{\omega\beta} \cup U$ is an RP-kernel of D.

Subcase 1.2.4: $Y'_{\omega\beta} = \varnothing$ and $Y'_{\alpha\beta} \neq \varnothing$.

Now $Y' = Y'_{\alpha} \cup Y'_{\alpha\beta} \cup Y'_{\beta\alpha} \cup Y'_{\alpha\gamma} \cup Y'_{\gamma\alpha} \cup Y'_{\omega\alpha} \cup Y'_{\omega\gamma} \cup Y'^{+}_{\omega}$. If $Y'_{\omega\alpha} \neq \varnothing$ or $Y'_{\gamma\alpha} \neq \varnothing$, then for any $y'' \in Y''_{\gamma\beta} \cup Y''_{\omega\beta}$, (y'', x_1, y', x_2) is a rainbow (y'', x_2)-path, where $y' \in Y'_{\gamma\alpha} \cup Y'_{\omega\alpha}$; for any $y'' \in Y''_{\beta\gamma} \cup Y''_{\beta\omega}$, $(y'', x_1, y'_{\alpha\beta}, x_2)$ is a rainbow (y'', x_2)-path. This implies that $\{x_2\}$ is an RP-kernel of D. If $Y'_{\omega\alpha} = \varnothing$ and $Y'_{\gamma\alpha} = \varnothing$, now $Y' = Y'_{\alpha} \cup Y'_{\alpha\beta} \cup Y'_{\beta\alpha} \cup Y'_{\alpha\gamma} \cup Y'_{\omega\gamma} \cup Y'^{+}_{\omega}$. Note that there exists no rainbow path for any pair of vertices of $Y'_{\alpha} \cup Y'_{\alpha\beta} \cup Y'_{\beta\alpha}$. For any $y' \in Y'_{\alpha\gamma} \cup Y'_{\omega\gamma}$, $(y', x_2, y'', x_1, y'_{\alpha\beta})$ is a rainbow $(y', Y'_{\alpha\beta})$-path, where $y'' \in Y''_{\omega\beta} \cup Y''_{\beta\omega}$. For any $y'^{+}_{\omega} \in Y'^{+}_{\omega}$, $(y'^{+}_{\omega}, x_2, y''_{\beta\gamma}, x_1, y'_{\alpha\beta})$ is a rainbow $(y'^{+}_{\omega}, Y'_{\alpha\beta})$-path. This implies that $Y'_{\alpha} \cup Y'_{\alpha\beta} \cup Y'_{\beta\alpha}$ is an RP-kernel of D.

Subcase 1.3: $Y''_{\omega\beta} \cup Y''_{\beta\omega} = \varnothing$ and $Y''_{\omega\gamma} \cup Y''_{\gamma\omega} = \varnothing$.

Then $Y'' = Y''_{\beta\gamma} \cup Y''_{\gamma\beta}$. This implies that there exists no rainbow path for any pair of vertices of $Y'_{\omega\beta} \cup Y'_{\omega\gamma} \cup Y'_{\alpha\gamma} \cup Y'_{\alpha\beta}$.

Subcase 1.3.1: $Y'_{\alpha\gamma} \cup Y'_{\alpha\beta} \neq \varnothing$ and $Y'_{\omega\beta} \cup Y'_{\omega\gamma} \neq \varnothing$.

Now $Y' = Y'_{\alpha} \cup Y'_{\alpha\beta} \cup Y'_{\beta\alpha} \cup Y'_{\alpha\gamma} \cup Y'_{\gamma\alpha} \cup Y'_{\omega\alpha} \cup Y'_{\omega\beta} \cup Y'_{\omega\gamma} \cup Y'^{+}_{\omega}$. For any $y' \in Y' \setminus Y'^{+}_{\omega}$, $(y', x_2, y''_{\beta\gamma}, x_1, y'_{\omega\beta})$ and $(y', x_2, y''_{\beta\gamma}, x_1, y'_{\omega\gamma})$ are rainbow $(y', Y'_{\omega\beta} \cup Y'_{\omega\gamma})$-paths; for any $y'^{+}_{\omega} \in Y'^{+}_{\omega}$, $(y'^{+}_{\omega}, x_2, y''_{\beta\gamma}, x_1, y'_{\alpha\gamma})$ and $(y'^{+}_{\omega}, x_2, y''_{\beta\gamma}, x_1, y'_{\alpha\beta})$ are rainbow $(y', Y'_{\alpha\gamma} \cup Y'_{\alpha\beta})$-paths. This implies that $Y'_{\omega\beta} \cup Y'_{\omega\gamma} \cup Y'_{\alpha\gamma} \cup Y'_{\alpha\beta}$ is an RP-kernel of D.

Subcase 1.3.2: $Y'_{\alpha\gamma} \cup Y'_{\alpha\beta} = \varnothing$ and $Y'_{\omega\beta} \cup Y'_{\omega\gamma} = \varnothing$.

Now $Y' = Y'_{\alpha} \cup Y'_{\beta\alpha} \cup Y'_{\gamma\alpha} \cup Y'_{\omega\alpha} \cup Y'^{+}_{\omega}$. For any $y' \in Y'$, $(y', x_2, y''_{\beta\gamma}, x_1)$ is a rainbow (y', x_1)-path. This implies that $\{x_1\}$ is an RP-kernel of D.

Subcase 1.3.3: $Y'_{\alpha\gamma} \cup Y'_{\alpha\beta} \neq \varnothing$ and $Y'_{\omega\beta} \cup Y'_{\omega\gamma} = \varnothing$.

Now $Y' = Y'_{\alpha} \cup Y'_{\alpha\beta} \cup Y'_{\beta\alpha} \cup Y'_{\alpha\gamma} \cup Y'_{\gamma\alpha} \cup Y'_{\omega\alpha} \cup Y'^{+}_{\omega}$. If $Y'_{\omega\alpha} = \varnothing$, then $Y' = Y'_{\alpha} \cup Y'_{\alpha\beta} \cup Y'_{\beta\alpha} \cup Y'_{\alpha\gamma} \cup Y'_{\gamma\alpha} \cup Y'^{+}_{\omega}$. Note that there exists no rainbow path for any pair of vertices of $Y'_{\alpha} \cup Y'_{\alpha\beta} \cup Y'_{\beta\alpha} \cup Y'_{\alpha\gamma} \cup Y'_{\gamma\alpha}$. For any $y'^{+}_{\omega} \in Y'^{+}_{\omega}$, $(y'^{+}_{\omega}, x_2, y''_{\beta\gamma}, x_1, y'_{\alpha})$ is a rainbow $(y'^{+}_{\omega}, y'_{\alpha})$-path. This implies that $Y'_{\alpha} \cup Y'_{\alpha\beta} \cup Y'_{\beta\alpha} \cup Y'_{\alpha\gamma} \cup Y'_{\gamma\alpha}$ is an RP-kernel of D. If $Y'_{\omega\alpha} \neq \varnothing$, for any $y'' \in Y''$, $(y'', x_1, y'_{\omega\alpha}, x_2)$ is a rainbow (y'', x_2)-path. This implies that $\{x_2\}$ is an RP-kernel of D.

Subcase 1.3.4: $Y'_{\alpha\gamma} \cup Y'_{\alpha\beta} = \varnothing$ and $Y'_{\omega\beta} \cup Y'_{\omega\gamma} \neq \varnothing$.

Now $Y' = Y'_{\alpha} \cup Y'_{\beta\alpha} \cup Y'_{\gamma\alpha} \cup Y'_{\omega\alpha} \cup Y'_{\omega\beta} \cup Y'_{\omega\gamma} \cup Y'^{+}_{\omega}$. For any $y' \in Y'_{\alpha} \cup Y'_{\beta\alpha} \cup Y'_{\gamma\alpha} \cup Y'_{\omega\alpha}$, $(y', x_2, y''_{\beta\gamma}, x_1, y'_{\omega\beta})$ and $(y', x_2, y''_{\beta\gamma}, x_1, y'_{\omega\gamma})$ are rainbow $(y', Y'_{\omega\beta} \cup Y'_{\omega\gamma})$-paths. If $|C(x_1, Y'_{\omega\beta} \cup Y'_{\omega\gamma})| \geqslant 2$, let $y'_1, y'_2 \in Y'_{\omega\beta} \cup Y'_{\omega\gamma}$ with $C(x_1, y'_1) \neq C(x_1, y'_2)$. Let $y'^{+}_{\omega} \in Y'^{+}_{\omega}$ be arbitrary. Note that either $(y'^{+}_{\omega}, x_2, y''_{\beta\gamma}, x_1, y'_1)$ or $(y'^{+}_{\omega}, x_2, y''_{\beta\gamma}, x_1, y'_2)$ is a rainbow $(y'^{+}_{\omega}, Y'_{\omega\beta} \cup Y'_{\omega\gamma})$-path. This implies that $Y'_{\omega\beta} \cup Y'_{\omega\gamma}$ is an RP-kernel of D. If $|C(x_1, Y'_{\omega\beta} \cup Y'_{\omega\gamma})| = 1$, let $U = \{y'^{+}_{\omega} \in$

$Y_{\omega}^{'+} \mid C(y_{\omega}^{'+}, x_2) = C(x_1, Y'_{\omega\beta} \cup Y'_{\omega\gamma})\}$. Then for any $y' \in Y_{\omega}^{'+}\backslash U$, we have $C(y', x_2) \neq C(x_1, Y'_{\omega\beta} \cup Y'_{\omega\gamma})$. Note that $(y', x_2, y''_{\beta\gamma}, x_1, y'_{\omega\beta})$ and $(y', x_2, y''_{\beta\gamma}, x_1, y'_{\omega\gamma})$ are two rainbow $(y', Y'_{\omega\beta} \cup Y'_{\omega\gamma})$-paths. If there exists a vertex $u \in U$ with $C(x_1, u) \notin \{\beta, \gamma, C(x_1, Y'_{\omega\beta} \cup Y'_{\omega\gamma})\}$, then for any $u' \in U\backslash\{u\}$, $(u', x_2, y''_{\beta\gamma}, x_1, u)$ is a rainbow (u', u)-path. This implies that $Y'_{\omega\beta} \cup Y'_{\omega\gamma} \cup \{u\}$ is an RP-kernel of D. If $C(x_1, U) \subseteq \{\beta, \gamma, C(x_1, Y'_{\omega\beta} \cup Y'_{\omega\gamma})\}$, there exists no rainbow path for any pair of vertices of $Y'_{\omega\beta} \cup Y'_{\omega\gamma} \cup U$. This implies that $Y'_{\omega\beta} \cup Y'_{\omega\gamma} \cup U$ is an RP-kernel of D.

Case 2: $Y'\backslash Y^* = Y''\backslash Y^{**} = \varnothing$.

Now $Y' = Y^*$, $Y'' = Y^{**}$. This means $C(x_1, y') \neq C(y', x_2)$ and $C(x_2, y'') \neq C(y'', x_1)$ for any $y' \in Y'$ and $y'' \in Y''$. Let $y' \in Y'$ be arbitrary. Assume $C(x_1, y') = \alpha$ and $C(y', x_2) = \beta$.

Define the following vertex subsets, which are shown in Figure 3 in which a box represents a set of vertices, while dotted, dashed, solid arcs represent, respectively, the arcs coloured by α, β and ω, where ω stands for an arbitrary colour not in $\{\alpha, \beta\}$.

$$Y'_{c_1c_2} = \{y' \in Y' \mid C(x_1, y') = c_1,\ \ C(y', x_2) = c_2\},$$

where $(c_1, c_2) \in \{(\alpha, \beta), (\beta, \alpha), (\alpha, \omega), (\beta, \omega), (\omega, \alpha), (\omega, \beta)\}$;

$$Y'_{\omega_1\omega_2} = \{y' \in Y' \mid C(x_1, y'), C(y', x_2) \notin \{\alpha, \beta\} \text{ and } C(x_1, y') \neq C(y', x_2)\}.$$

Since every 4-cycle is coloured with at least three colours and $Y'_{\alpha\beta} \neq \varnothing$, Y'' can be divided into the following vertex subsets.

$$Y''_{c_1c_2} = \{y'' \in Y'' \mid C(x_2, y'') = c_1,\ \ C(y'', x_1) = c_2\},$$

where $(c_1, c_2) \in \{(\alpha, \omega), (\beta, \omega), (\omega, \alpha), (\omega, \beta)\}$ and ω is an arbitrary colour not in $\{\alpha, \beta\}$;

$$Y''_{\omega_1\omega_2} = \{y'' \in Y'' \mid C(x_2, y''), C(y'', x_1) \notin \{\alpha, \beta\} \text{ and } C(x_2, y'') \neq C(y'', x_1)\}.$$

Now,

$$Y' = Y'_{\alpha\beta} \cup Y'_{\beta\alpha} \cup Y'_{\alpha\omega} \cup Y'_{\beta\omega} \cup Y'_{\omega\alpha} \cup Y'_{\omega\beta} \cup Y'_{\omega_1\omega_2},$$
$$Y'' = Y''_{\alpha\omega} \cup Y''_{\beta\omega} \cup Y''_{\omega\alpha} \cup Y''_{\omega\beta} \cup Y''_{\omega_1\omega_2}.$$

For convenience a vertex in $Y'_{c_1c_2}$ ($Y''_{c_1c_2}$, $Y'_{\omega_1\omega_2}$, $Y''_{c_1c_2}$) we denote by $y'_{c_1c_2}$ ($y''_{c_1c_2}$, $y'_{\omega_1\omega_2}$, $y''_{c_1c_2}$).

If $Y'_{\omega_1\omega_2} \neq \varnothing$ or $Y'_{\omega_1\omega_2} \cup Y''_{\omega\alpha} \cup Y''_{\omega\beta} = \varnothing$, then for any $y'' \in Y''_{\omega\alpha} \cup Y''_{\omega\beta}$, $(y'', x_1, y'_{\omega_1\omega_2}, x_2)$ is a rainbow (y'', x_2)-path; for any $y'' \in Y''_{\alpha\omega} \cup Y''_{\beta\omega} \cup Y''_{\omega_1\omega_2}$, $(y'', x_1, y'_{\alpha\beta}, x_2)$ is a rainbow (y'', x_2)-path. This implies that $\{x_2\}$ is an RP-kernel of D. So we assume $Y'_{\omega_1\omega_2} = \varnothing$ and $Y''_{\omega\alpha} \cup Y''_{\omega\beta} \neq \varnothing$.

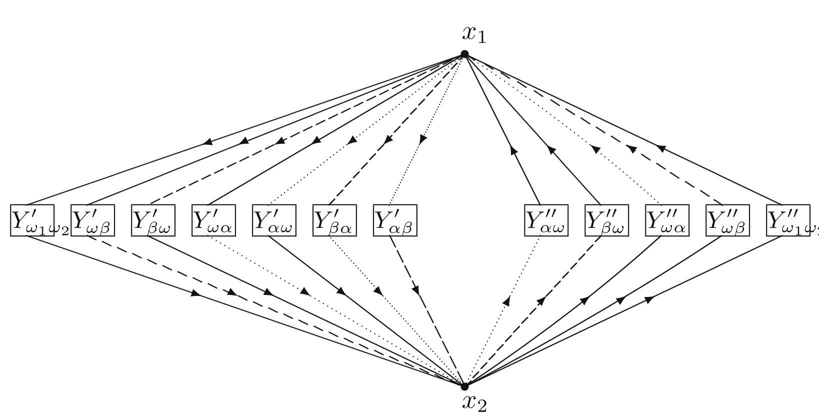

Figure 3: An arc-coloured bipartite tournament for Case 2 of the proof of Lemma 6.5.

Subcase 2.1: $Y''_{\omega\alpha} \neq \emptyset$ and $Y''_{\omega\beta} \neq \emptyset$.

Then $Y'' = Y''_{\alpha\omega} \cup Y''_{\beta\omega} \cup Y''_{\omega\beta} \cup Y''_{\omega\alpha} \cup Y''_{\omega_1\omega_2}$. For any $y' \in Y'_{\alpha\beta} \cup Y'_{\omega\beta}$, $(y', x_2, y''_{\omega\alpha}, x_1)$ is a rainbow (y', x_1)-path. For any $y' \in Y'_{\beta\alpha} \cup Y'_{\omega\alpha}$, $(y', x_2, y''_{\omega\beta}, x_1)$ is a rainbow (y', x_1)-path. Since every 4-cycle is coloured with at least three colours, we have $C(y'_{\alpha\omega}, x_2) \notin C(x_2, Y''_{\omega\alpha})$ for any $y'_{\alpha\omega} \in Y'_{\alpha\omega}$ and $C(y'_{\beta\omega}, x_2) \notin C(x_2, Y''_{\omega\beta})$ for any $y'_{\beta\omega} \in Y'_{\beta\omega}$. It follows that $(y'_{\alpha\omega}, x_2, y''_{\omega\alpha}, x_1)$ is a rainbow $(y'_{\alpha\omega}, x_1)$-path and $(y'_{\beta\omega}, x_2, y''_{\omega\beta}, x_1)$ is a rainbow $(y'_{\beta\omega}, x_1)$-path. This implies that $\{x_1\}$ is an RP-kernel of D.

Subcase 2.2: Exactly one of the subsets $Y''_{\omega\alpha}$ and $Y''_{\omega\beta}$ is not empty set.

W.l.o.g., assume $Y''_{\omega\alpha} \neq \emptyset$ and $Y''_{\omega\beta} = \emptyset$. Now, $Y'' = Y''_{\alpha\omega} \cup Y''_{\beta\omega} \cup Y''_{\omega\alpha} \cup Y''_{\omega_1\omega_2}$.

Subcase 2.2.1: $Y'_{\beta\omega} \cup Y'_{\omega\beta} \neq \emptyset$.

We see that for any $y'' \in Y''_{\alpha\omega} \cup Y''_{\beta\omega} \cup Y''_{\omega_1\omega_2}$, $(y'', x_1, y'_{\alpha\beta}, x_2)$ is a rainbow (y'', x_2)-path; for any $y''_{\omega\alpha} \in Y''_{\omega\alpha}$, $(y''_{\omega\alpha}, x_1, y', x_2)$ is a rainbow $(y''_{\omega\alpha}, x_2)$-path, where $y' \in Y'_{\beta\omega} \cup Y'_{\omega\beta}$. This implies that $\{x_2\}$ is an RP-kernel of D.

Subcase 2.2.2: $Y''_{\beta\omega} \neq \emptyset$.

We see that for any $y' \in Y'_{\beta\alpha} \cup Y'_{\omega\alpha} \cup Y'_{\beta\omega}$, $(y', x_2, y''_{\beta\omega}, x_1)$ is a rainbow (y', x_1)-path; for any $y' \in Y'_{\alpha\beta} \cup Y'_{\alpha\omega} \cup Y'_{\omega\beta}$, $(y', x_2, y''_{\omega\alpha}, x_1)$ is a rainbow (y', x_1)-path. This implies that $\{x_1\}$ is an RP-kernel of D.

Subcase 2.2.3: $Y'_{\beta\omega} \cup Y'_{\omega\beta} = \emptyset$ and $Y''_{\beta\omega} = \emptyset$.

We see that $Y' = Y'_{\alpha\beta} \cup Y'_{\beta\alpha} \cup Y'_{\alpha\omega} \cup Y'_{\omega\alpha}$ and $Y'' = Y''_{\alpha\omega} \cup Y''_{\omega\alpha} \cup Y''_{\omega_1\omega_2}$. If $Y''_{\omega_1\omega_2} \neq \emptyset$, then for any $y' \in Y' \setminus Y'_{\alpha\omega}$, $(y', x_2, y''_{\omega_1\omega_2}, x_1)$ is a rainbow (y', x_1)-path; for any $y'_{\alpha\omega} \in Y'_{\alpha\omega}$, $(y'_{\alpha\omega}, x_2, y''_{\alpha\omega}, x_1)$ is a rainbow $(y'_{\alpha\omega}, x_1)$-path. This implies that $\{x_1\}$ is an RP-kernel of D. If $Y''_{\omega_1\omega_2} = \emptyset$, then $Y'' = Y''_{\alpha\omega} \cup Y''_{\omega\alpha}$. For any $y' \in Y'_{\alpha\beta} \cup Y'_{\beta\alpha} \cup Y'_{\omega\alpha}$, $(y', x_2, y''_{\omega\alpha})$ is a rainbow $(y', y''_{\omega\alpha})$-path. By the proof above, for any $y'_{\alpha\omega} \in Y'_{\alpha\omega}$, $(y'_{\alpha\omega}, x_2, y''_{\omega\alpha})$ is a rainbow $(y'_{\alpha\omega}, y''_{\omega\alpha})$-path. If $|C(x_2, Y''_{\omega\alpha})| \geqslant 2$, let $y''_{\omega\alpha 1}, y''_{\omega\alpha 2} \in Y''_{\omega\alpha}$ with $C(x_2, y''_{\omega\alpha 1}) \neq C(x_2, y''_{\omega\alpha 2})$. For any $y''_{\alpha\omega} \in Y''_{\alpha\omega}$, either $(y''_{\alpha\omega}, x_1, y'_{\alpha\beta}, x_2, y''_{\omega\alpha 1})$ or $(y''_{\alpha\omega}, x_1, y'_{\alpha\beta}, x_2, y''_{\omega\alpha 2})$ is a rainbow $(y''_{\alpha\omega}, y''_{\omega\alpha})$-path. This implies that $Y''_{\omega\alpha}$ is an RP-kernel of D. If $|C(x_2, Y''_{\omega\alpha})| = 1$, let $U = \{y''_{\alpha\omega} \in Y''_{\alpha\omega} \mid C(y''_{\alpha\omega}, x_1) = C(x_2, Y''_{\omega\alpha})\}$. Note that there exists no rainbow path for any pair of vertices of $Y''_{\omega\alpha} \cup U$. For any $y'' \in Y''_{\alpha\omega} \setminus U$, we have $C(y'', x_1) \neq C(x_2, Y''_{\omega\alpha})$ and $(y'', x_1, y'_{\alpha\beta}, x_2, y''_{\omega\alpha})$ is a rainbow $(y'', y''_{\omega\alpha})$-path. This implies that $Y''_{\omega\alpha} \cup U$ is an RP-kernel of D.

In any case, we can find an RP-kernel of D. This proof of Lemma 6.5 is complete. □

Theorem 6.6. *Let $D = (X, Y)$ be an m-arc-coloured bipartite tournament with* $\min\{|X|, |Y|\} = 2$. *If every 4-cycle contained in D is coloured with at least three colours, then D has an RP-kernel.*

By Theorem 6.6, the following corollary is immediate.

Corollary 6.7. *Let $D = (X, Y)$ be an m-arc-coloured bipartite tournament with* $\min\{|X|, |Y|\} = 2$. *If every 4-cycle contained in D is rainbow, then D has an RP-kernel.*

Remark 6.8. The condition “every 4-cycle is coloured with at least 3 colours” in Theorem 6.6 cannot be reduced. An arc-coloured bipartite tournament with $|X| = 2$ shown in Figure 4 satisfying “every 4-cycle is 2-arc-coloured” has no RP-kernel, in which solid and dotted arcs represent, respectively, the arcs coloured by two distinct colours. Large m-arc-coloured bipartite tournaments with no RP-kernel can be obtained by adding new vertices to Y and new colours such that these new vertices completely dominate X.

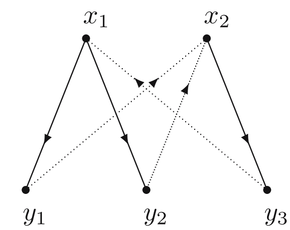

Figure 4: An arc-coloured bipartite tournament with $|X| = 2$ satisfying "every 4-cycle is 2-arc-coloured" has no RP-kernel.

In the following proof, we consider $\min\{|X|, |Y|\} \geqslant 3$.

Lemma 6.9. [7] *Let $D = (X, Y)$ be a bipartite tournament. Then the following statements hold*:

(*a*) *let $C = (u_0, u_1, u_2, \dots, u_n)$ be a walk in D. For $\{i, j\} \subseteq \{1, 2, \dots, n\}$, u_i, u_j are adjacent if and only if $j - i \equiv 1 \pmod 2$.*

(*b*) *every closed walk of length at most 6 is a cycle of D.*

Let CB_5 be a bipartite tournament, which has $V(CB_5) = \{u_1, u_2, u_3, u_4, u_5\}$ and $A(CB_5) = \{(u_1, u_2), (u_2, u_3), (u_3, u_4), (u_4, u_5), (u_4, u_1), (u_5, u_2)\}$. Let TB_4 be a bipartite tournament, which has $V(TB_4) = \{u_1, u_2, u_3, u_4\}$ and $A(TB_4) = \{(u_1, u_2), (u_2, u_3), (u_3, u_4), (u_1, u_4)\}$ (Figure 5).

Lemma 6.10. *Let $D = (X, Y)$ be an m-arc-coloured bipartite tournament with* $\min\{|X|, |Y|\} \geqslant 3$. *If all 4-cycles, 6-cycles and induced subdigraphs CB_5 in D are rainbow, and all induced subdigraphs TB_4 in D are properly coloured, then for any pair of distinct vertices $u, v \in V(D)$ satisfying there exists a rainbow (u, v)-path and no rainbow (v, u)-path in D, at least one of the following conditions holds*:

(*a*) $u \to v$;

(*b*) *there exists a (u, v)-path of length 2.*

Proof. Let $P = (u = u_0, u_1, u_2, \dots, u_n = v)$ be the shortest rainbow (u, v)-path in D. The result holds clearly for $n \leqslant 2$. Now assume $n \geqslant 3$.

If n is odd, by Lemma 6.9 (*a*), we have u_0, u_n are adjacent. Since there is no rainbow (v, u)-path in D, we have $u_0 \to u_n$. The result holds. So we assume that n is even.

Also by Lemma 6.9 (*a*), we have u_1, u_n are adjacent. If $u_1 \to u_n$, then $(u = u_0, u_1, u_n = v)$ is a (u, v)-path of length 2 and the result holds. So we assume $u_n \to u_1$.

If $n = 4$, then $P = (u = u_0, u_1, u_2, u_3, u_4 = v)$. For $u_3 \to u_0$, we see that $D[u_0, u_1, u_2, u_3, u_4]$ is an induced rainbow CB_5, which implies $(u_4, u_1, u_2, u_3, u_0)$ is a rainbow (v, u)-path, which is a contradiction. For $u_0 \to u_3$, we have (u_0, u_3, u_4) is a (u, v)-path of length 2. So we assume $n \geqslant 6$.

If $u_0 \to u_{i_0} \to u_n$ for some $i_0 \in \{3, 5, \dots, n-1\}$, then $(u = u_0, u_{i_0}, u_n = v)$ is a (u, v)-path of length 2 and the result holds. So we assume that either $u_i \to u_0$ or $u_n \to u_i$ for each $i \in \{3, 5, \dots, n-1\}$.

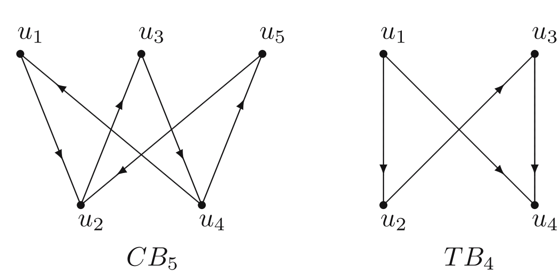

Figure 5: Two bipartite tournaments CB_5 and TB_4.

Claim 1. For each $i \in \{3, 5, \dots, n-3\}$, $u_0 \to u_i$ and $u_n \to u_i$.

Proof. It is sufficient to show that $u_0 \to u_i$. We process by induction on i. For $i = 3$, suppose to the contrary that $u_3 \to u_0$. If $u_3 \to u_n$, then $D[u_0, u_1, u_2, u_3, u_n]$ is an induced rainbow CB_5. It follows that $(u_n, u_1, u_2, u_3, u_0)$ is a rainbow (v, u)-path in D, which is a contradiction. If $u_n \to u_3$, then $D[u_n, u_3, u_0, u_1]$ is TB_4 which is properly coloured. It follows that (u_n, u_3, u_0) is a rainbow (v, u)-path in D, which is a contradiction. Thus, $(u_0, u_3) \in A(D)$.

Assume that the claim holds for $i < n-3$. We consider the case $i = n-3$.

Suppose to the contrary that $u_{n-3} \to u_0$. By the induction hypothesis, we have $u_0 \to u_{n-5}$ and $u_n \to u_{n-5}$. If $u_{n-3} \to u_n$, then $D[u_0, u_{n-5}, u_{n-4}, u_{n-3}, u_n]$ is an induced rainbow CB_5. It follows that $(u_n, u_{n-5}, u_{n-4}, u_{n-3}, u_0)$ is a rainbow (v, u)-path in D, which is a contradiction. If $u_n \to u_{n-3}$, then $D[u_n, u_{n-3}, u_0, u_{n-5}]$ is TB_4 which is properly coloured. It follows that (u_n, u_{n-3}, u_0) is a rainbow (v, u)-path in D, which is a contradiction. So $(u_0, u_{n-3}) \in A(D)$. □

Now we show $u_0 \to u_{n-1}$. Suppose to the contrary that $u_{n-1} \to u_0$. By Claim 1, we have $u_0 \to u_{n-3}$ and $u_n \to u_{n-3}$. Then $D[u_0, u_{n-3}, u_{n-2}, u_{n-1}, u_n]$ is an induced rainbow CB_5. It follows that $(u_n, u_{n-3}, u_{n-2}, u_{n-1}, u_0)$ is a rainbow (v, u)-path in D, which is a contradiction. So $u_0 \to u_{n-1}$.

Now (u_0, u_{n-1}, u_n) is a (u, v)-path of length 2. □

Theorem 6.11. *Let $D = (X, Y)$ be an m-arc-coloured bipartite tournament with $\min\{|X|, |Y|\} \geqslant 3$. If all 4-cycles, 6-cycles and induced subdigraphs CB_5 in D are rainbow, and all induced subdigraphs TB_4 in D are properly coloured, then $C_r(D)$ is a KP-digraph.*

Proof. According to Theorem 2.3, it is sufficient to prove that each cycle of $C_r(D)$ has a symmetrical arc. Suppose to the contrary that there exists a cycle C in $C_r(D)$ containing no symmetrical arc. We will get a contradiction by showing that C has a symmetrical arc. Let $C = (x_0, x_1, \dots, x_n, x_0)$. Since C has no symmetrical arc, for each $i \in \{0, 1, \dots, n\}$, there exists a rainbow (x_i, x_{i+1})-path and no rainbow (x_{i+1}, x_i)-path in D. The following claim follows directly from Lemma 6.10.

Claim 1. For each $i \in \{0, 1, \dots, n\}$, either $(x_i, x_{i+1}) \in A(D)$ or there exists a (x_i, x_{i+1})-path of length 2 in D.

Let

$$P_i = \begin{cases} (x_i, x_{i+1}), & (x_i, x_{i+1}) \in A(D); \\ (x_i, u_i, x_{i+1}), & (x_i, x_{i+1}) \notin A(D), \end{cases}$$

and $C' = P_0 P_1 \dots P_n$. Then C' is a closed walk in D.

We consider the following two cases.

Case 1. $n = 2$.

Now C is a 3-cycle. Then not all arcs of C are in D since D is a bipartite tournament. W.l.o.g., assume that $(x_0, x_1) \notin A(D)$. Then $\ell(P_0) = 2$, $\ell(P_1) \leqslant 2$ and $\ell(P_2) \leqslant 2$. Now C' is a closed walk with length at most 6. By Lemma 6.9 (b), C' is a cycle. Since all 4-cycles and 6-cycles are rainbow, we have C' is rainbow. Now $P_1 P_2$ is a rainbow (x_1, x_0)-path and hence $(x_1, x_0) \in A(C_r(D))$. Note that $(x_0, x_1) \in A(C)$, which contradicts C has no symmetrical arc.

Case 2. $n \geqslant 3$.

In this case, we set $C' = (v_0, v_1, \dots, v_k, v_0)$ where $v_0 = x_0$ and $k \geqslant n$. By Lemma 6.9 (a), k is odd since $(v_k, v_0) \in A(D)$. Also v_0, v_3 are adjacent and v_0, v_{k-2} are adjacent in D.

If $(v_3, v_0) \in A(D)$, then $(v_0, v_1, v_2, v_3, v_0)$ is a rainbow 4-cycle. This implies that (v_1, v_2, v_3, v_0) is a rainbow (v_1, v_0)-path and (v_2, v_3, v_0) is a rainbow (v_2, v_0)-path. Then $\{(v_1, v_0), (v_2, v_0)\} \subseteq A(C_r(D))$. Note that either $v_1 = x_1$ or $v_2 = x_1$. We have $(x_1, x_0) \in A(C_r(D))$. Note that $(x_0, x_1) \in A(C)$, which contradicts C has no symmetrical arc.

If $(v_0, v_{k-2}) \in A(D)$, then $(v_0, v_{k-2}, v_{k-1}, v_k, v_0)$ is a rainbow 4-cycle. This implies that $(v_0, v_{k-2}, v_{k-1}, v_k)$ is a rainbow (v_0, v_k)-path and (v_0, v_{k-2}, v_{k-1}) is a rainbow (v_0, v_{k-1})-path. Then $\{(v_0, v_k), (v_0, v_{k-1})\} \subseteq A(C_r(D))$. Note that either $v_k = x_n$ or $v_{k-1} = x_n$. We have $(x_0, x_n) \in A(C_r(D))$. Note that $(x_n, x_0) \in A(C)$, which contradicts C has no symmetrical arc.

If $(v_0, v_3) \in A(D)$ and $(v_{k-2}, v_0) \in A(D)$, we have $v_3 \neq v_{k-2}$ and hence $k-2 \geqslant 5$. Also there exists $i \in \left\{1, 2, \ldots, \frac{k-5}{2}\right\}$ such that $(v_0, v_{2i+1}) \in A(D)$ and $(v_{2i+3}, v_0) \in A(D)$. Let

$$j_0 = \max\left\{i \in \left\{1, 2, \ldots, \frac{k-5}{2}\right\} \mid (v_0, v_{2i+1}) \in A(D), \quad (v_{2i+3}, v_0) \in A(D)\right\}.$$

Then $(v_0, v_{2j_0+1}, v_{2j_0+2}, v_{2j_0+3}, v_0)$ is a rainbow 4-cycle.

If $v_{2j_0+1} \in V(C)$, let $v_{2j_0+1} = x_j$. Now $(v_{2j_0+2}, v_{2j_0+3}, v_0, v_{2j_0+1})$ is a rainbow (v_{2j_0+2}, v_{2j_0+1})-path and $(v_{2j_0+3}, v_0, v_{2j_0+1})$ is a rainbow (v_{2j_0+3}, v_{2j_0+1})-path. Then $\{(v_{2j_0+2}, v_{2j_0+1}), (v_{2j_0+3}, v_{2j_0+1})\} \subseteq A(C_r(D))$. Note that either $v_{2j_0+2} = x_{j+1}$ or $v_{2j_0+3} = x_{j+1}$. We have $(x_{j+1}, x_j) \in A(C_r(D))$. Note that $(x_j, x_{j+1}) \in A(C)$, which contradicts C has no symmetrical arc.

If $v_{2j_0+1} \notin V(C)$, by the definition of C', we have $v_{2j_0}, v_{2j_0+2} \in V(C)$. Let $v_{2j_0+2} = x_{j+1}$. By the choice of j_0, we have $(v_{2j_0+5}, v_0) \in A(D)$. This implies that $(v_0, v_{2j_0+1}, v_{2j_0+2}, v_{2j_0+3}, v_{2j_0+4}, v_{2j_0+5}, v_0)$ is a rainbow 6-cycle. So $(v_{2j_0+3}, v_{2j_0+4}, v_{2j_0+5}, v_0, v_{2j_0+1}, v_{2j_0+2})$ is a rainbow (v_{2j_0+3}, v_{2j_0+2})-path and $(v_{2j_0+4}, v_{2j_0+5}, v_0, v_{2j_0+1}, v_{2j_0+2})$ is a rainbow (v_{2j_0+4}, v_{2j_0+2})-path. Then $\{(v_{2j_0+3}, v_{2j_0+2}), (v_{2j_0+4}, v_{2j_0+2})\} \subseteq A(C_r(D))$. Note that either $v_{2j_0+3} = x_{j+2}$ or $v_{2j_0+4} = x_{j+2}$, we have $(x_{j+2}, x_{j+1}) \in A(C_r(D))$. Note that $(x_{j+1}, x_{j+2}) \in A(C)$, which contradicts C has no symmetrical arc.

In any case, we get a contradiction. Thus, $C_r(D)$ is a KP-digraph. □

By Observation 2.2 and Theorem 6.11, the following corollary is direct.

Corollary 6.12. *Let $D = (X, Y)$ be an m-arc-coloured bipartite tournament with* $\min\{|X|, |Y|\} \geqslant 3$. *If all 4-cycles, 6-cycles and induced subdigraphs CB_5 in D are rainbow, and all induced subdigraphs TB_4 in D are properly coloured, then D has an RP-kernel.*

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