Research Article

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On sub-class sizes of mutually permutable products

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Abstract: In this paper, we investigate the influence of sub-class sizes on a mutually permutable factorized group in which the sub-class sizes of some elements of its factors have certain quantitative properties. Some criteria for a group to be p -nilpotent or p -supersolvable are given.

Keywords: finite groups, sub-class sizes, mutually permutable products

MSC 2020: 20D10, 20D20, 20D40, 20E45

1 Introduction

All groups considered in this paper are finite.

We first fix some notation and terminology. Let G be a group. For an element x of G , we denote by $x^G = \{x^g \mid g \in G\}$ the conjugacy class of G containing x and denote by $|x^G|$ the class size in which x lies. If p is a prime, we say that x is a p -regular element or p' -element of G if p does not divide the order of x . We say that x is p -singular in G if its order is divisible by p , and a p -element of G is a p -singular element of prime power order. For a natural number $n = p^a \cdot m$ with $(p, m) = 1$, we call p^a the p -part of n and denote it by n_p . $|G|$ denotes the order of G , and $|G|_p$ is the order of Sylow p -subgroups of G . Let H/K be a chief factor of G . If p divides $|H/K|$, then H/K is called a p -chief factor of G . In addition, $\pi(G)$ denotes the set of all primes dividing the order of G . The other notation and terminology are standard.

The influence of certain arithmetical conditions of conjugacy class sizes on the structure of groups has been studied by many authors with lots of results available (see a nice survey of Camina and Camina [1] for the details). For example, Itô proved in [2] that for a group G and a prime p , $|x^G|$ is not divisible by p for every $x \in G$ if and only if G has a central Sylow p -subgroup. If q is a prime different from p and pq does not divide $|x^G|$ for every $x \in G$, then G is either p -nilpotent or q -nilpotent. But the converse of this result is not true. Later on, Camina proved in [3] that $|x^G|_p = 1$ for every p' -element $x \in G$ if and only if $G = O_p(G) \times O_{p'}(G)$. This result was improved by Liu et al. in [4] for every p' -element $x \in G$ of prime power order. It was also proved by Chillag and Herzog in [5] that G is supersolvable if $|x^G|$ is square-free for each $x \in G$. An improvement of this result was given in [4].

Recently, Qian and Yang in [6] generalized the usual concepts of “conjugacy class” and “conjugacy class size” and introduced the new concepts of “sub-class” and “sub-class size.” For a group G and an element $x \in G$, it is easy to observe that G has a unique minimal subnormal subgroup that contains x . We denote this

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subnormal subgroup by G_x , which is uniquely associated with x . Instead of considering the class x^G in G , one can just consider the class

$$x^{G^*} = \{x^g \mid g \in G_x\} = x^{G_x},$$

which is called the sub-class of G containing x . $|x^{G^*}|$ or $|x^{G_x}|$ is called the sub-class size. By Remarks 1.8–1.10 in [6], we know that sub-classes and sub-class sizes have many interesting properties which are different from that of conjugacy classes and conjugacy class sizes. Therefore, it is natural to ask what happens for the structure of a group G if the restrictions on the usual conjugacy sizes of G as in the results mentioned in the last paragraph are replaced by the restrictions on the sub-class sizes. It was shown in [6] that some known results are still true and some new phenomena emerge as can be seen from the following two results (see [6, Theorem 1.4] and [6, Theorem 1.5], respectively).

Theorem 1.1. *Let G be a group and p be a prime. Then, the following statements are equivalent.*

- (1) p does not divide $|x^{G^*}|$ for every x of G .
- (2) p does not divide $|x^{G^*}|$ for every p' -element $x \in G$ of prime power order.
- (3) G is p -nilpotent.

Theorem 1.2. *Suppose that G is a group and p, q are different primes. Then, pq does not divide $|x^{G^*}|$ for each x of G if and only if G is either p -nilpotent or q -nilpotent.*

Our purpose in this paper is to present some results on the structure of a mutually permutable factorized finite group in which the sub-class sizes of some elements of its factors have certain arithmetical properties. Recall that two subgroups A and B of a group G are said to be mutually permutable if A is permutable with all subgroups of B and B is permutable with all subgroups of A . Let $G = G_1 G_2 \cdots G_r$ be the product of its pairwise permutable subgroups $G_1, G_2, \dots, G_r$. Then, G is said to be the mutually permutable product of $G_1, G_2, \dots, G_r$ if G_i and G_j are mutually permutable subgroups of G for all $i, j \in \{1, 2, \dots, r\}$. During the past three decades, the structure of mutually permutable products has been extensively developed according to the monograph by Ballester-Bolinches et al. [7]. Recently, Ballester-Bolinches et al. in [8] analyzed the structure of mutually permutable product $G = G_1 G_2 \cdots G_r$ under the assumption that p or p^2 does not divide the conjugacy class sizes of some given elements in the factors G_i for some fixed prime p . Theorems 1.1 and 1.3 in [8] together with the aforementioned Theorems 1.1 and 1.2 motivate us to further investigate the influence of sub-class sizes on p -nilpotence and p -supersolvability of a mutually permutable factorized group. Recall that a group G is called p -nilpotent if each p -chief factor H/K of G is central in G , that is $H/K \leq Z(G/K)$, or equivalently, if each Sylow p -subgroup of G has a normal p -complement in G . G is said to be p -supersolvable if each p -chief factor H/K has order p .

Our main results are as follows.

Theorem 1.3. *Let $G = G_1 G_2 \cdots G_r$ be the mutually permutable product of its subgroups $G_1, G_2, \dots, G_r$ and p be a prime. Then, p does not divide $|x^{G^*}|$ for every p' -element $x \in \bigcup_{i=1}^r G_i$ of prime power order if and only if G is p -nilpotent.*

Theorem 1.4. *Let $G = AB$ be the mutually permutable product of its p -solvable subgroups A and B and p be a prime. Suppose that for each p' -element x in $A \cup B$, p^2 does not divide $|x^{G^*}|$. Then, G is p -supersolvable.*

Theorem 1.5. *Let $G = AB$ be the mutually permutable product of its subgroups A and B and let p be a prime such that $(p-1, |G|) = 1$. Suppose that p^2 does not divide $|x^{G^*}|$ for each p' -element $x \in A \cup B$ of prime power order. Then, G is a solvable p -nilpotent group.*

Theorem 1.3 can be viewed as an extension of Theorem 1.1 mentioned above by Qian and the second author. Theorem 1.4 is a sub-class size version of [8, Theorem 1.3] with the additional restriction that A and B

are p -solvable. It was proved by Qian and Wang in [9] that for a group G and a prime p with $(p-1, |G|) = 1$, if p^2 does not divide $|x^G|$ for every p' -element $x \in G$ of prime power order, then G is a solvable p -nilpotent group and a Sylow p -subgroup of $G/O_p(G)$ is elementary abelian. Therefore, Theorem 1.5 can be partially regarded as a sub-class version of this result on mutually permutable products.

As an application of Theorems 1.4 and 1.5, we have the following corollary.

Corollary 1.6. *Let $G = AB$ be the mutually permutable product of its subgroups A and B . Suppose that for every $p \in \pi(G)$ and each p' -element $x \in A \cup B$, p^2 does not divide $|x^{G^*}|$. Then, G is supersolvable.*

Proof. Let p be the smallest prime dividing the order of G . Then, the hypothesis and Theorem 1.5 imply that G is solvable. It follows from Theorem 1.4 that G is p -supersolvable for every prime $p \in \pi(G)$. Hence, G is supersolvable. $\square$

For a group G and every element $x \in G$, we know from Remark 1.10 in [6] that $|x^{G^*}|$ divides $|x^G|$ since G_x is subnormal in G , and in many cases, $|x^{G^*}|$ is much smaller than $|x^G|$. As two immediate consequences of 1.6, we have:

Corollary 1.7. [8, Corollary 1.5] *Let $G = AB$ be the mutually permutable product of its subgroups A and B . Suppose that for every prime $p \in \pi(G)$ and every p -regular element $x \in A \cup B$, $|x^G|$ is not divisible by p^2 . Then, G is supersolvable.*

Corollary 1.8. [4, Theorem 10] *Let $G = AB$ be a product of two subgroups A and B which are permutable in G . Suppose that for every prime $p \in \pi(G)$ and every element $x \in A \cup B$, $|x^G|$ is not divisible by p^2 . Then, G is supersolvable.*

2 Preliminaries

In this section, we collect some results that will be used in our proof. We start by citing some basic properties about sub-classes and sub-class sizes that are contained in [6].

Lemma 2.1. *Let G be a group and N be a subgroup of G . Let x be any element of G .*

- (1) *If G is a simple group, then $x^{G^*} = x^{G_x} = x^G$ and therefore $|x^{G^*}| = |x^{G_x}| = |x^G|$.*
- (2) *If N is a subnormal subgroup of G such that $x \in N$, then $N_x = G_x$, $x^{N^*} = x^{G^*}$ and $|x^{N^*}| = |x^{G^*}|$.*
- (3) *Assume that N is normal in G and set $\bar{G} = G/N$. Then, $\bar{G}_x \leq \bar{G}_x$, $|\bar{x}^{\bar{G}^*}| \mid |x^{G^*}|$, and $|\bar{x}^{\bar{G}}| \mid |x^G|$.*

Proof. (1) and (2) are obvious by the definition and (3) follows from [6, Lemma 2.1]. $\square$

Lemma 2.2. *Let $G = G_1 G_2 \cdots G_r$ be the product of the pairwise mutually permutable subgroups of $G_1, G_2, \dots, G_r$. Then,*

- (1) *If N is a normal subgroup of G , then G/N is the mutually permutable product of $G_1 N/N, G_2 N/N, \dots, G_r N/N$.*
- (2) *If each G_i is p -solvable for a prime p , then G is p -solvable.*
- (3) *If N is a non-abelian minimal normal subgroup of G , then there exists some $j \in \{1, 2, \dots, r\}$ such that $N \leq G_j$.*

Proof. Assertion (1) follows from [7, Theorem 4.1.11]. Assertion (2) follows from [7, Theorem 4.1.15]. Assertion (3) is [7, Theorem 4.3.8]. $\square$

For a group G , we let H_G denote the largest normal subgroup of G contained in H , where H is a subgroup of G .

Lemma 2.3. *If $G = AB$ is a nontrivial group such that G is the product of the mutually permutable subgroups A and B , then $A_G B_G$ is nontrivial.*

Proof. See [7, Theorem 4.3.11]. □

Lemma 2.4. *Let G be a group and p be a prime. Then, p does not divide $|x^G|$ for every p' -element $x \in G$ of prime power order if and only if $G = O_p(G) \times O_{p'}(G)$.*

Proof. See [4, Theorem 5]. □

As usual, let $F(G)$ denote the Fitting subgroup of a group G , which is generated by all nilpotent normal subgroups of G . Here, we denote by $F_p(G)$ the largest p -nilpotent normal subgroup of G , where p is a prime.

Lemma 2.5. *Let G be a p -solvable group. Then, for given $x \in G$, $|x^{G^*}|$ is not divisible by p if and only if $x \in F_p(G)$.*

Proof. This is [6, Lemma 3.1]. □

Lemma 2.6. [10, Lemma 2.9] *If P is a nilpotent normal subgroup of a group G and $P \cap \Phi(G) = 1$, then P is the direct product of some minimal normal subgroups of G .*

Lemma 2.7. *Let p be a prime and A be a p' -group acting faithfully on an elementary abelian p -group N such that $||[x, N]|| = p$ for all nontrivial $x \in A$. Then, A is cyclic.*

Proof. This is [8, Lemma 2.4]. □

Lemma 2.8. *Let S be a non-abelian simple group. Then, there exists an element $x \in S$ of odd prime-power order such that $4 ||x^S|$.*

Proof. This is [9, Lemma 2.1]. □

3 Proofs of main results

Proof of Theorem 1.3. The sufficiency part of Theorem 1.3 is clear in view of Theorem 1.1, and so we just need to consider the necessity part. Assume that it is false and let G be a counterexample of minimal order. By the hypotheses, we have that $G = G_1 G_2 \cdots G_r$ is the mutually permutable product of its subgroups $G_1, G_2, \dots, G_r$ and p does not divide $|x^{G^*}|$ for every p' -element $x \in \bigcup_{i=1}^r G_i$ of prime power order, where p is a prime. Let N be a minimal normal subgroup of G . Then, we can assume that N is non-trivial in G by Lemmas 2.2(3) and 2.4. By Lemma 2.2(1), $G/N = (G_1 N/N)(G_2 N/N) \cdots (G_r N/N)$ is the mutually permutable product of $G_1 N/N, G_2 N/N, \dots, G_r N/N$. In view of Lemma 2.1(3), we see that G/N satisfies the hypotheses and so G/N is p -nilpotent by the choice of G . If G has a minimal normal subgroup M different from N , then G/M is also p -nilpotent. It follows that G is p -nilpotent, since $N \cap M = 1$ and so $G = G/(N \cap M) \leq G/N \times G/M$, a contradiction. Hence, we have that N is the unique minimal subgroup of G .

We claim that N must be abelian. Otherwise, $N \leq G_j$ for some $j \in \{1, 2, \dots, r\}$. Therefore, for every p' -element $x \in N$ of prime power order, $|x^{N^*}| = |x^{G^*}|$ is not divisible by p by the hypotheses. It follows

from Theorem 1.1 that N is p -nilpotent and so N has a normal p -complement $O_{p'}(N)$. This yields that $N = O_{p'}(N)$ by the minimality of N , which implies that G is p -nilpotent since G/N is p -nilpotent.

Hence, N is an abelian elementary p -group. Since G/N is p -nilpotent, we see that G is p -solvable. Since G is not p -nilpotent, we get that $L = F_p(G) < G$. Let $L_{p'}$ be the normal p -complement of L . Then, by Lemma 2.5, we conclude that $L_{p'}$ contains every p' -element of prime power order in $\bigcup_{i=1}^l G_i$ and so $L_{p'}$ is nontrivial, a contradiction since $L_{p'}$ is normal in G , and N is the unique normal subgroup of G . Thus, the proof is complete. $\square$

Proof of Theorem 1.4. Assume the result is false and let G be a counterexample of minimal order. By Lemma 2.2(2), we see that G is p -solvable.

Let N be an arbitrary nontrivial minimal normal subgroup of G . By Lemmas 2.1(3) and 2.2(1), we see that the quotient group G/N fulfills our conditions on G and so G/N is p -supersolvable by the minimality of G . Suppose that G has another minimal normal subgroup M with $M \neq N$, then G/M is p -supersolvable by the foregoing argument. It follows that G is p -supersolvable since $G = G/(N \cap M) \leq G/N \times G/M$. This contradiction shows that G has a unique minimal normal subgroup, say N , such that $|N| \geq p^2$. We assert that $\Phi(G) = 1$. If not, then $N \leq \Phi(G)$ and so $G/\Phi(G)$ is p -supersolvable since G/N is p -supersolvable. It follows from [11, Satz 1] that G is p -supersolvable, a contradiction. Hence, we have $\Phi(G) = 1$. By Lemma 2.6, we see that $F(G)$ is the direct product of some minimal normal subgroup of G . It follows that $N = F(G)$ and so $C_G(N) = N$ by [12, Theorem 3.21].

Let L/N be a minimal normal subgroup of $G/N = G/C_G(N)$. Since G/N is p -supersolvable, we have that L/N is either a p -group of order p or a p' -group. By [13, Ch. A, Lemma 13.6], we have that $O_p(G/C_G(N)) = 1$, and therefore, G/N is a p' -chief factor of G . By Lemma 2.3, we may further assume that $L/N \leq AN/N$. We first suppose that L/N is not solvable. Then, $L/N = L_1/N \times L_2/N \times \cdots \times L_t/N$ is a direct product of isomorphic non-abelian simple groups L_i/N for $i \in \{1, 2, \dots, t\}$. It is clear that L_i/N is perfect. Let x be a nontrivial p' -element in L_1 . Then, G_x is contained in L_1 and $N \leq N_G(G_x)$ by [12, Theorem 2.6]. Since G_x is subnormal in G , we have that $G_x N/N = L_1/N$. Since L_1/N is perfect and G_x is normal in $G_x N$, we see that N must be contained in G_x and consequently $L_1/N = G_x/N$. Let $L_1 = N \rtimes T$, where T is a Hall p' -subgroup of L_1 . Then $T \leq A$ and T acts faithfully on N . By the hypotheses, we have that p^2 does not divide $|x^{G^*}| = |x^{G_x}|$ for every nontrivial $x \in T$ and so $|N/C_N(x)| = p$ since $N \leq G_x$. Now, invoking Lemma 2.7, we get that T is cyclic, a contradiction since L_1/N is not solvable.

Now we assume that L/N is an elementary abelian p' -chief factor. Then, $L = N \rtimes K$ for some elementary abelian q -group K with q a prime different from p . Without loss of generality, we assume that $K \leq A$ by Lemma 2.3. Observe that K acts faithfully on N . Let x be any nontrivial element in K . Let $J = N\langle x \rangle$. Then, J is subnormal in G and so $G_x \leq J$. It follows that $G_x = G_x \cap N\langle x \rangle = (G_x \cap N)\langle x \rangle$. Since $N \leq N_G(G_x)$ by [12, Theorem 2.6], we have that $G_x \cap N$ is normal in J . If $G_x \cap N = 1$, then $G_x = \langle x \rangle$ is normal in J , and therefore, $x \in C_G(N) = N$, a contradiction. Hence, $G_x \cap N \neq 1$. By Maschke's theorem (see [12, Corollary 10.17]), there exists a $\langle x \rangle$ -invariant subgroup $U \leq N$ such that $N = (G_x \cap N) \times U$. Since $\langle x \rangle$ acts completely reducibly on N , we can assume that $U = U_1 \times \cdots \times U_r$ is a direct product of minimal normal subgroups U_i of J for $i = 1, \dots, r$. It is easy to see that $U \cap G_x = 1$. By [12, Theorem 2.6] again, we have that $U_i \leq N_j(G_x)$, and thereby, we get that $U_i \leq C_j(G_x)$. It follows that $U \leq C_j(x)$. Since p^2 does not divide $|x^{G_x}|$ by the hypotheses, we have that $||x, N|| = ||x, G_x \cap N|| = p$. This holds for every nontrivial $x \in K$ and so K is cyclic of order q by Lemma 2.7. Let $K = \langle y \rangle$ with y an element of order q . Then, $L' = N$ and $G_y = L_y$ is normalized by N by [12, Theorem 2.6]. It follows that if N is not contained in L_y , then L' is contained in L_y , a contradiction. Hence, $L_y = L$, which implies that $|y^{G_y}| = |y^{L_y}| = |y^L| = |N|$. By the hypotheses and foregoing arguments, N is of order p , a final contradiction. Thus, the proof is complete. $\square$

Proof of Theorem 1.5. We first prove that G is solvable. Since any group of odd order is solvable, we may assume that G is of even order and thus $p = 2$. By Lemmas 2.3 and 2.8, we assume that G is not a non-abelian simple group. Let N be a minimal normal subgroup of G . Then, it is easy to see that G/N satisfies our assumption on G by Lemmas 2.1(3) and 2.2(1) provided that $2 \in \pi(G/N)$ and so is solvable by induction. Note that G/N is clearly solvable if $2 \notin \pi(G/N)$. If N is a non-abelian, then Lemma 2.2(3) implies that either $N \leq A$ or $N \leq B$ and so N is solvable by induction. It follows that G is solvable.

Now we prove that G is p -nilpotent. Suppose that this is false and let G be a counterexample of minimal order. Let N be any minimal normal subgroup of G . Then, by the choice of G , G/N is p -nilpotent since G/N satisfies the conditions. Moreover, we may assume that N is the unique minimal normal subgroup of G as in the proof of Theorem 1.4. If $\Phi(G)$ is nontrivial, then N is contained in $\Phi(G)$ and so $G/\Phi(G)$ is p -nilpotent, which implies that G is also p -nilpotent by [13, Ch. A, Lemma 13.2]. Therefore, $\Phi(G) = 1$ and $N = C_G(N)$. Discussing as in the last paragraph of the proof of Theorem 1.4, we obtain that N is cyclic of order p , which implies that $G/N = G/C_G(N)$ is isomorphic to a subgroup of $\text{Aut}(N) \simeq C_{p-1}$, where C_{p-1} is a cyclic group of order $p - 1$. This contradicts $p - 1, |G| = 1$, which finishes the proof. $\square$

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