Research Article

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Existence of a solution of Hilfer fractional hybrid problems via new Krasnoselskii-type fixed point theorems

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Abstract: This work intends to treat the existence of mild solutions for the Hilfer fractional hybrid differential equation (HFHDE) with linear perturbation of first and second type in partially ordered Banach spaces. First, we establish the results concerning the actuality of fixed point of sum and product of operators via the concepts of measure of noncompactness and simulation functions in partially ordered spaces. Then combining these fixed point theorems with the concepts in fractional calculus, new existence results for mild solutions of HFHDE's are established. Furthermore, the presented fixed point results and existence results improve and extend the present state-of-art in the literature. Competent examples in support of theory are illustrated for better understanding.

Keywords: partially ordered spaces (algebra), coupled fixed point, measure of noncompactness, Hilfer fractional derivative, fractional hybrid differential equation

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1 Introduction

Differential equations of fractional order found immense importance and productive applications in different branches, such as physics, chemistry, electrodynamics, aerodynamics, and other branches of engineering. This branch of mathematics is indeed emerging as a significant area of study in the recent time; see [1,2]. Although the concepts of fractional derivative have been mentioned in classical literature, two notions of fractional derivatives have attracted much attention of researchers due to Riemann-Liouville and Caputo.

In a parallel development, Hilfer [3] introduced a generalized version of Riemann-Liouville fractional derivative, famously called as Hilfer fractional derivative. It is well known that this derivative includes both the fractional derivatives that are of Riemann-Liouville and Caputo. Furati et al. [4] studied a Cauchy-type problem involving Hilfer fractional derivative. Recently, some noteworthy works on Hilfer fractional differential equations have been appeared in the literature, which deal with various problems; see [5–7] and references therein.

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Recently, perturbation techniques emerged as a key player in nonlinear analysis and found distinguished importance in the study of the dynamical systems. One of the differential equations perturbed with linear perturbation is called as hybrid differential equations, see [8]. Such equations attracted a lot of attention and many works on the theory have been published so far. One of the necessary tools while solving such problems is fixed point result.

The fixed point theorems are generally based on some specific properties such as continuity, contractiveness, compactness, monotonicity, etc. The two most cited and influential results of Banach and Schauder from fixed point theory are based on contractiveness and compactness of operators, respectively. Many times while solving integral equations, the situation arises where the operator is split up in the form of sum of two operators ($\mathcal{T} = \mathcal{A} + \mathcal{B}$), out of which one ($\mathcal{A}$) is contraction and other ($\mathcal{B}$) is compact. But $\mathcal{T}$ itself is neither compact nor contraction. So neither of the aforementioned two theorems apply directly in such cases. Krasnoselskii [9] in 1958 made first attempt to handle such situation and successfully proved a theorem, which is named by his name. It is stated as follows:

Theorem 1.1. [9,10] *Let $M \neq \emptyset$ be a convex and closed set in a Banach space S . Let $\mathcal{A}, \mathcal{B} : M \rightarrow S$ such that $\mathcal{A}$ is contraction and $\mathcal{B}$ is compact and continuous. In addition, if $\mathcal{A}y + \mathcal{B}x \in S$ for all $y, x \in S$, then $\mathcal{A} + \mathcal{B}$ admits at least one fixed point.*

Theorem 1.1 is readily used in proving the existence of solutions to integral and differential equations. However, a large number of improvements have been appeared in the literature over past time modifying the assumptions (see [11,12] and references therein). The mixed arguments and applications in various branches of mathematical sciences are key for evolution of research in fixed point theory. Considering the possible applications and generalizations in mind, Ran and Reurings [13] proposed a hybrid fixed point result in posets. This work pioneered the subsequent research in partially ordered spaces [14]. On the line of Theorem 1.1, Dhage in [15] initiated investigations of existence of fixed points for sum and product of two mappings in partially ordered Banach spaces (algebra). Other worth mentioning works in this area are of Yang et al. [16] and Nashine et al. [17].

Acted upon by the above discussion, in this manuscript we intend to present some generalized forms of Krasnoselskii- and Dhage-type results for fixed points via the concept of partial Hausdorff measure of noncompactness (PH-MNC) with the aid of simulation function in the setting of partially ordered spaces (algebra). As an application, the existence results for the Hilfer FHDEs with linear perturbation of first and second type along with coupled system are established. Some competent examples are illustrated for better understanding of the theory. We claim that upto our best knowledge such FHDEs with Hilfer derivative are not considered in current existing literature.

The work flow of this manuscript is as follows: Section 2 presents all the prerequisites and auxiliary results from existing literature. Section 3 presents Krasnoselskii-type fixed and coupled fixed point results for nonlinear operators. Section 4 presents the Dhage-type fixed point results in partially ordered normed linear algebras. This new theory is utilized in Section 5 to investigate the existence of mild solutions for the Hilfer FHDEs with linear perturbation of first and second type. Also a coupled system of Hilfer fractional hybrid differential equations (HFHDEs) is studied.

2 Auxiliary concepts and results

Table 1 represents various notations and their meaning used throughout this article.

Recall that a fixed point for a self mapping $\mathcal{T}$ on a set $X \neq \emptyset$ is an element $\omega \in X$ which satisfies the operator equation $\omega = \mathcal{T}(\omega)$, whereas the coupled fixed point for a coupled operator $\mathcal{T} : X \times X \rightarrow X$ is a pair $(v, \omega) \in X \times X$ satisfying the system $v = \mathcal{T}(v, \omega)$ and $\omega = \mathcal{T}(\omega, v)$.

Table 1: Notations and explanation

Notations	Meaning
$B(\omega, \gamma)$	Closed ball of radius γ with center ω
$\overline{\mathcal{C}}$	Closure of the chain $\mathcal{C}$
$\overline{\text{con}}(\mathcal{C})$	Convex and closed hull of chain $\mathcal{C}$
$\text{dia}(\mathcal{C})$	Diameter of the set $\mathcal{C}$
$\mathcal{P}_{bd, cn}(\mathbb{P})$	Collection of nonempty bounded chains in $\mathbb{P}$
$\mathcal{P}_{rcp, cn}(\mathbb{P})$	Collection of nonempty relatively compact chains in $\mathbb{P}$
$\mathbb{R}^+$	Set of nonnegative real numbers
$\mathbb{N}$	Set of natural numbers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$

Let $\mathbb{P}$ be a real linear space endowed with a norm $\|\cdot\|$. It is well known that any relation $\sqsubseteq$ on $\mathbb{P}$ satisfying reflexivity, antisymmetry, transitivity, and order linearity is said to be a partial order. If the linear space $\mathbb{P}$ is endowed with a norm $\|\cdot\|$ and have a partial order $\sqsubseteq$ on its elements, then it is called a *partial ordered normed linear space* and denoted by $(\mathbb{P}, \sqsubseteq, \|\cdot\|)$. We call elements $u, v \in (\mathbb{P}, \sqsubseteq, \|\cdot\|)$ *comparable* if either $u \sqsubseteq v$ or $v \sqsubseteq u$ holds. Any nonempty subset $\mathcal{C}$ in $(\mathbb{P}, \sqsubseteq, \|\cdot\|)$ is called a *totally ordered set or chain* if its every possible pair of elements is comparable. The space $(\mathbb{P}, \sqsubseteq, \|\cdot\|)$ is called *complete* if its norm $\|\cdot\|$ induces a metric m such that $(\mathbb{P}, m)$ is complete. However, $(\mathbb{P}, \sqsubseteq, \|\cdot\|)$ is called *regular* if any nonincreasing (nondecreasing) sequence $\{a_n\} \subset \mathbb{P}$ converging to a^* , then $a^* \sqsubseteq a_n$ (respectively, $a_n \sqsubseteq a^*$) for each $n \in \mathbb{N}$.

Definition 2.1. [15] $(\mathbb{P}, \sqsubseteq, \|\cdot\|)$ is said to be $(\|\cdot\| - \sqsubseteq)$ -compatible if $\{\omega_n\}$ is a monotone sequence in $\mathbb{P}$ having a subsequence say $\{\omega_{n_k}\}$ converging to ω^* implies that $\{\omega_n\}$ converges to ω^* .

Definition 2.2. [15] An operator $\mathcal{T} : \mathbb{P} \rightarrow \mathbb{P}$ is said to be partial continuous at a point $a_0 \in \mathbb{P}$ if for each $\varepsilon > 0$, we can find a $\delta > 0$ such that $\|\mathcal{T}a - \mathcal{T}a_0\| < \varepsilon$ supposing $\|a - a_0\| < \delta$, provided $a \in \mathbb{P}$ is comparable to a_0 . $\mathcal{T}$ is said to be partial continuous on $\mathbb{P}$ if it is partial continuous at each point of $\mathbb{P}$.

Remark 2.1. $\mathcal{T}$ is continuous on every chain $\mathcal{C}$ in $\mathbb{P}$ if it is partial continuous on $\mathbb{P}$.

Definition 2.3. [15] An operator $\mathcal{T} : \mathbb{P} \rightarrow \mathbb{P}$ is called

- (i) monotone nonincreasing if $\mathcal{T}\omega \sqsubseteq \mathcal{T}v$;
- (ii) monotone nondecreasing if $\mathcal{T}v \sqsubseteq \mathcal{T}\omega$;

whenever $v \sqsubseteq \omega$, for every $v, \omega \in \mathbb{P}$. In both these cases, we simply call $\mathcal{T}$ as monotone.

Definition 2.4. [15] $\mathcal{T} : \mathbb{P} \times \mathbb{P} \rightarrow \mathbb{P}$ is said to be mixed monotone if the mapping

- (i) $v \mapsto \mathcal{T}(\omega, v)$ is nonincreasing for every $\omega \in \mathbb{P}$ and
- (ii) $\omega \mapsto \mathcal{T}(\omega, v)$ is nondecreasing for every $v \in \mathbb{P}$.

Definition 2.5. [15] An operator $\mathcal{T} : \mathbb{P} \rightarrow \mathbb{P}$ is called

- (i) partial compact if the image of each chain $\mathcal{C}$ in $\mathbb{P}$ under $\mathcal{T}$ is relatively compact in $\mathbb{P}$;
- (ii) partial bounded if the image of each chain $\mathcal{C}$ in $\mathbb{P}$ under $\mathcal{T}$ is bounded.

Remark 2.2. It is observed that, on $(\mathbb{P}, \sqsubseteq, \|\cdot\|)$

- each compact operator is partial compact;
- each partial compact operator is partial bounded.

But inverse is not true.

2.1 Partial Housdorff measure of noncompactness

Definition 2.6. The non-negative number

$$\beta_p(\mathcal{C}) = \inf \left\{ \gamma > 0 : \mathcal{C} \subset \bigcup_{i=1}^N B(\omega_i, \gamma), \omega_i \in \mathbb{P}, i = 1, \dots, N \right\},$$

assigned with a bounded chain $\mathcal{C}$ in a partially ordered normed linear space $\mathbb{P}$ is called *PH-MNC*.

Definition 2.7. [15] A *PH-MNC* satisfies the following axioms on a partially ordered Banach space $\mathbb{P}$

1. $\beta_p(\mathcal{C}) = 0$ iff $\mathcal{C} \in \mathcal{P}_{rcp, cn}(\mathbb{P})$;
2. $\beta_p(\mathcal{C}) = \beta_p(\overline{\mathcal{C}})$, $\mathcal{C} \in \mathcal{P}_{bd, cn}(\mathbb{P})$;
3. $\mathcal{C} \subset \mathcal{D}$ implies $\beta_p(\mathcal{C}) \leq \beta_p(\mathcal{D})$, $\mathcal{C}, \mathcal{D} \in \mathcal{P}_{bd, cn}(\mathbb{P})$;
4. If $\lim_{n \rightarrow \infty} \beta_p(\mathcal{C}_n) = 0$ for a sequence of closed chains $\{\mathcal{C}_n\}$ in $\mathcal{P}_{bd, cn}(\mathbb{P})$ such that $\mathcal{C}_{n+1} \subset \mathcal{C}_n$, then $\mathcal{C}_\infty = \bigcap_{n=1}^\infty \mathcal{C}_n$ is partial compact and nonempty;
5. $\beta_p(\mathcal{C} \cup \mathcal{D}) = \max\{\beta_p(\mathcal{C}), \beta_p(\mathcal{D})\}$, where $\mathcal{C}, \mathcal{D} \in \mathcal{P}_{bd, cn}(\mathbb{P})$;
6. $\beta_p(\lambda \mathcal{C}) = |\lambda| \beta_p(\mathcal{C})$ for any real number λ and $\mathcal{C} \in \mathcal{P}_{bd, cn}(\mathbb{P})$;
7. $\beta_p(\mathcal{C} + \mathcal{D}) \leq \beta_p(\mathcal{C}) + \beta_p(\mathcal{D})$, $\mathcal{C}, \mathcal{D} \in \mathcal{P}_{bd, cn}(\mathbb{P})$.

Definition 2.8. [15] An operator $\mathcal{T} : \mathbb{P} \rightarrow \mathbb{P}$ is said to be partial condensing if

$$\beta_p(\mathcal{T}(\mathcal{C})) < \beta_p(\mathcal{C}),$$

for each $\mathcal{C} \in \mathcal{P}_{bd, cn}(\mathbb{P})$ such that $\beta_p(\mathcal{C}) > 0$, where β_p is *PH-MNC*.

Definition 2.9. [18] A coupled operator $\mathcal{T} : \mathbb{P} \times \mathbb{P} \rightarrow \mathbb{P}$ is said to be coupled partial condensing if

$$\beta_p(\mathcal{T}(\mathcal{C} \times \mathcal{D})) + \beta_p(\mathcal{T}(\mathcal{D} \times \mathcal{C})) < \beta_p(\mathcal{C}) + \beta_p(\mathcal{D}),$$

for all $\mathcal{C}, \mathcal{D} \in \mathcal{P}_{bd, cn}(\mathbb{P})$ such that $\beta_p(\mathcal{C}) + \beta_p(\mathcal{D}) > 0$, where β_p is *PH-MNC*.

We also refer to [19–21] for some new classes of condensing operators.

Theorem 2.1. [15] Let $\mathcal{T}$ be a nondecreasing, partial bounded, partial continuous, and partial condensing operator on a $(\|\cdot\| - \sqsubseteq)$ -compatible, regular, and complete partial ordered normed linear space $(\mathbb{P}, \sqsubseteq, \|\cdot\|)$. Then $\mathcal{T}$ admits at least one fixed point if $\mathbb{P}$ contains ω_0 such that $\omega_0 \sqsubseteq \mathcal{T}(\omega_0)$ or $\mathcal{T}(\omega_0) \sqsubseteq \omega_0$.

Theorem 2.2. [18] Let $\mathcal{T} : \mathbb{P} \times \mathbb{P} \rightarrow \mathbb{P}$ be a partial bounded, partial continuous, mixed monotone, and coupled partial condensing operator, where $(\mathbb{P}, \sqsubseteq, \|\cdot\|)$ be a $(\|\cdot\| - \sqsubseteq)$ -compatible, regular, and complete partial ordered normed linear space. Then $\mathcal{T}$ admits at least one coupled fixed point if $\mathbb{P}^2$ contains (ω_0, ν_0) such that $\omega_0 \sqsubseteq \mathcal{T}(\omega_0, \nu_0)$ and $\mathcal{T}(\nu_0, \omega_0) \sqsubseteq \nu_0$.

2.2 Simulation function

In 2015, Khojasteh et al. [22] brought a new concept which they termed as “simulation function.” Using this new notion they obtained generalizations of various fixed point theorems. A slight modification is made in definition of simulation function by R.-L. de Hierro and Samet in [23]. Later, observing the redundancy in definition of simulation function of [22], Argoubi et al. [24] redefined the notion by removing first condition from [22, Definition 2.1], which we present here.

Definition 2.10. [24] A mapping $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfying

(ζ_1) $\zeta(t_1, t_2) < t_2 - t_1$ for all $t_1, t_2 > 0$,

(ζ_2) if there are two sequences $\{s_j\}$ and $\{t_j\}$ in $\mathbb{R}^+ \setminus \{0\}$ such that $\lim_{j \rightarrow \infty} s_j = \lim_{j \rightarrow \infty} t_j > 0$ and $t_j < s_j$, then $\lim_{j \rightarrow \infty} \sup \zeta(t_j, s_j) < 0$,

is called simulation function. Let $\mathcal{Z}$ denotes the family of all mappings in Definition 2.10.

Example 2.1. A monotonic nondecreasing upper semicontinuous function $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying $\eta(0) = 0$ is called a D -function [15]. Let $\zeta_D(t, s) = \eta(s) - t$, for $s, t \in \mathbb{R}^+$, where η be a D -function with $\eta(r) < r$. Then $\zeta_D \in \mathcal{Z}$.

2.3 Terminologies and results from fractional calculus

Definition 2.11. [1,2] Let $g \in \mathcal{L}^1(a, b)$. The integral

$$\mathbb{I}_{a^+}^\mu g(t) = \frac{1}{\Gamma(\mu)} \int_a^t \frac{g(s)}{(t-s)^{1-\mu}} ds, \quad t > a,$$

is the left side Riemann-Liouville fractional integral of order $\mu > 0$ of the function g .

Definition 2.12. [1,2] The left side Riemann-Liouville fractional derivative of order μ of g is given by the expression

$$\mathbb{D}_{a^+}^\mu g(t) = \mathbb{D}_{a^+}^{1-\mu} g(t), \quad 0 < \mu < 1, \quad t > a, \quad \mathbb{D} = \frac{d}{dt},$$

provided right-hand side exists.

Lemma 2.1. [4] For $s > a$, we have

$$\begin{aligned} \mathbb{I}_{a^+}^\mu (t-a)^{\nu-1}(s) &= \frac{\Gamma(\nu)}{\Gamma(\nu+\mu)} (s-a)^{\nu+\mu-1}, \quad \mu \geq 0, \quad \nu > 0, \\ [\mathbb{D}_{a^+}^\mu (t-a)^{\nu-1}](s) &= 0, \quad 0 < \mu < 1. \end{aligned}$$

Definition 2.13. [3] The right-hand side Hilfer fractional derivative operator of order $0 < \mu < 1$ and type $0 \leq \nu \leq 1$ is defined by

$$\mathbb{D}_{a^+}^{\mu,\nu} = \mathbb{I}_{a^+}^{\nu(1-\mu)} \mathbb{D}_{a^+}^{(1-\nu)(1-\mu)}.$$

Alternatively, we can write

$$\mathbb{D}_{a^+}^{\mu,\nu} = \mathbb{I}_{a^+}^{\nu(1-\mu)} \mathbb{D}_{a^+}^{(1-\gamma)} = \mathbb{I}_{a^+}^{\nu(1-\mu)} \mathbb{D}_{a^+}^\gamma, \quad \gamma = \mu + \nu - \mu\nu.$$

Remark 2.3. [3]

(i) If $\nu = 0$, then $\mathbb{D}_{a^+}^{\mu,\nu} = \mathbb{D}_{a^+}^\mu$ which is Riemann-Liouville derivative.

(ii) If $\nu = 1$, then $\mathbb{D}_{a^+}^{\mu,\nu} = \mathbb{I}_{a^+}^{1-\mu}$ which is Caputo derivative.

That is, the Hilfer fractional derivative plays a role of interpolator between the Caputo and Riemann-Liouville derivatives.

(iii) Following relation holds for the parameter γ :

$$0 < \gamma \leq 1, \quad \gamma \geq \mu, \quad \gamma > \nu, \quad 1 - \gamma < 1 - \nu(1 - \mu).$$

Throughout this article, let $a, b \in (-\infty, \infty)$. Let $C[a, b]$, $C^k[a, b]$, and $\mathcal{AC}[a, b]$ represent the spaces of continuous, k times continuously differentiable and absolutely continuous functions on $[a, b]$, respectively. The space $C[a, b]$ is the Banach space with the norm

$$\|w\|_C = \max\{|w(t)| : t \in [a, b]\}.$$

The space $\mathcal{L}^1(a, b)$ of Lebesgue integrable functions from (a, b) to $\mathbb{R}$ is a Banach space with the norm

$$\|w\|_1 = \int_a^b |w(s)| ds.$$

We will consider the following weighted spaces of continuous functions

$$C_\gamma[a, b] = \{g : (a, b) \rightarrow \mathbb{R} \mid (t-a)^\gamma g(t) \in C[a, b]\}, \quad 0 \leq \gamma < 1, \quad (2.1)$$

$$C_\gamma^k[a, b] = \{g \in C^{k-1}[a, b] \mid g^{(k)} \in C_\gamma[a, b]\}, \quad k \in \mathbb{N}, \quad (2.2)$$

which are Banach spaces with the norms

$$\|g\|_{C_\gamma} = \|(t-a)^\gamma g(t)\|_C$$

and

$$\|g\|_{C_\gamma^k} = \sum_{i=1}^{k-1} \|g^{(i)}\|_C + \|g^{(k)}\|_{C_\gamma}.$$

We will consider the partially ordered relation $\sqsubseteq$ on $C_\gamma[a, b]$ by $v \sqsubseteq w$ if and only if $(t-a)^\gamma v(t) \leq (t-a)^\gamma w(t)$. Then the space $(C_\gamma[a, b], \sqsubseteq, \|\cdot\|_{C_\gamma})$ is a partially ordered Banach space.

The following relations hold: $C_0[a, b] = C[a, b]$, $C_\gamma^k[a, b] \subset \mathcal{AC}^k[a, b]$, $C_\gamma^0[a, b] = C_\gamma[a, b]$, $C_{\gamma_1}[a, b] \subset C_{\gamma_2}[a, b]$, $0 \leq \gamma_1 < \gamma_2 < 1$.

Lemma 2.2. [25] Let $\gamma \in [0, 1]$, $a < b < c$, $g \in C_\gamma[a, b]$, $g \in C_\gamma[b, c]$, and g is continuous at b . Then $g \in C_\gamma[a, c]$.

Lemma 2.3. [25] For $\mu > 0$, $\mathbb{I}_{a^+}^\mu$ maps $C[a, b]$ into $C[a, b]$.

Lemma 2.4. [25] Let $\mu > 0$ and $\gamma \in [0, 1]$, then $\mathbb{I}_{a^+}^\mu$ maps $C_\gamma[a, b]$ into $C_\gamma[a, b]$ and is bounded.

Lemma 2.5. [25] Let $\mu > 0$ and $0 \leq \gamma < 1$, then $\mathbb{I}_{a^+}^\mu$ is bounded from $C_\gamma[a, b]$ into $C[a, b]$ if $\gamma \leq \mu$.

Lemma 2.6. [25] Let $\gamma \in [0, 1]$ and $g \in C_\gamma[a, b]$. Then $\mathbb{I}_{a^+}^\mu g(a) = \lim_{s \rightarrow a} \mathbb{I}_{a^+}^\mu g(s) = 0$, $\gamma < \mu$.

Lemma 2.7. Let $g \in L^1(a, b)$. Then $\lim_{t \rightarrow b^+} \int_a^b (t-s)^{\mu-1} g(s) ds = \int_a^b (b-s)^{\mu-1} g(s) ds = \Gamma(\mu) \mathbb{I}_{a^+}^\mu g(b)$, $\mu > 0$.

Lemma 2.8. [4, 25] Let $\mu, \nu \geq 0$ and $g \in L^1(a, b)$. Then $\mathbb{I}_{a^+}^\mu \mathbb{I}_{a^+}^\nu g(t) = \mathbb{I}_{a^+}^{\mu+\nu} g(t)$ a.e., $t \in [a, b]$. In particular, if g lies in $C_\gamma[a, b]$ or $C[a, b]$, then equality holds at every t in (a, b) or $[a, b]$.

Lemma 2.9. [4] Let $\mu > 0$, $\gamma \in [0, 1]$, and $g \in C_\gamma[a, b]$. Then $\mathbb{D}_{a^+}^\mu \mathbb{I}_{a^+}^\mu g(t) = g(t)$, for all $t \in (a, b)$.

Lemma 2.10. [4] Let $\mu \in (0, 1)$, $\gamma \in [0, 1]$, $g \in C_\gamma[a, b]$, and $\mathbb{I}_{a^+}^{1-\mu} g \in C_\gamma^1[a, b]$. Then

$$\mathbb{I}_{a^+}^\mu \mathbb{D}_{a^+}^\mu g(t) = g(t) - \frac{\mathbb{I}_{a^+}^{1-\mu} g(a)}{\Gamma(\mu)} (t-a)^{\mu-1},$$

for all $t \in (a, b)$.

We will also consider the following spaces

$$C_{1-\gamma}^{\mu,\nu}[a, b] = \{g \in C_{1-\gamma}[a, b] : \mathbb{D}_{a^+}^{\mu,\nu} g \in C_{1-\gamma}[a, b]\}$$

and

$$C_{1-\gamma}^{\gamma}[a, b] = \{g \in C_{1-\gamma}[a, b] : \mathbb{D}_{a^+}^{\gamma} g \in C_{1-\gamma}[a, b]\}.$$

Here $C_{1-\gamma}^{\gamma}[a, b] \subset C_{1-\gamma}^{\mu,\nu}[a, b]$.

Lemma 2.11. [4] Let $\mu \in (0, 1)$, $\nu \in [0, 1]$, and $\gamma = \mu + \nu - \mu\nu$. If $g \in C_{1-\gamma}^{\gamma}[a, b]$, then

$$\mathbb{I}_{a^+}^{\gamma} \mathbb{D}_{a^+}^{\gamma} g = \mathbb{I}_{a^+}^{\mu} \mathbb{D}_{a^+}^{\mu,\nu} g$$

and

$$\mathbb{D}_{a^+}^{\gamma} \mathbb{I}_{a^+}^{\mu} g = \mathbb{D}_{a^+}^{\nu(1-\mu)} g.$$

Lemma 2.12. [4] Let $g \in \mathcal{L}^1(a, b)$. If $\mathbb{D}_{a^+}^{\nu(1-\mu)} g$ exists and lies in $\mathcal{L}^1(a, b)$, then

$$\mathbb{D}_{a^+}^{\mu,\nu} \mathbb{I}_{a^+}^{\mu} g = \mathbb{I}_{a^+}^{\nu(1-\mu)} \mathbb{D}_{a^+}^{\nu(1-\mu)} g.$$

Lemma 2.13. [4] Let $\mu \in (0, 1)$, $\nu \in [0, 1]$, and $\gamma = \mu + \nu - \mu\nu$. If $g \in C_{1-\gamma}^{\gamma}[a, b]$ and $\mathbb{I}_{a^+}^{1-\nu(1-\mu)} g \in C_{1-\gamma}^1[a, b]$, then

$$\mathbb{D}_{a^+}^{\mu,\nu} \mathbb{I}_{a^+}^{\mu} g(t) = g(t), \quad t \in (a, b].$$

3 Fixed point results

In this section, let $(\mathbb{P}, \sqsubseteq, \|\cdot\|)$ denote partially ordered normed linear space and we consider the following families of operators:

$$\begin{aligned} \mathcal{M}_{nd,bd}^{par}(\mathbb{P}) &= \{\mathcal{T} : \mathbb{P} \rightarrow \mathbb{P} \mid \mathcal{T} \text{ is nondecreasing and partial bounded}\}; \\ \mathcal{M}_{nd,ct}^{par}(\mathbb{P}) &= \{\mathcal{T} : \mathbb{P} \rightarrow \mathbb{P} \mid \mathcal{T} \text{ is nondecreasing and partial continuous}\}; \\ \mathcal{M}_{mm,bd}^{c,par}(\mathbb{P}) &= \{\mathcal{T} : \mathbb{P} \times \mathbb{P} \rightarrow \mathbb{P} \mid \mathcal{T} \text{ is mixed monotone and partial bounded}\}; \\ \mathcal{M}_{mm,ct}^{c,par}(\mathbb{P}) &= \{\mathcal{T} : \mathbb{P} \times \mathbb{P} \rightarrow \mathbb{P} \mid \mathcal{T} \text{ is mixed monotone and partial continuous}\}. \end{aligned}$$

Let us enunciate with the following definition which is defined in the settings of metric space in [22].

Definition 3.1. A self-operator $\mathcal{T}$ on $(\mathbb{P}, \sqsubseteq, \|\cdot\|)$ is called a partial nonlinear $\mathcal{Z}$ -contraction if for all comparable elements $\omega, \nu \in \mathbb{P}$ it satisfies

$$\zeta(\|\mathcal{T}\omega - \mathcal{T}\nu\|, \|\omega - \nu\|) \geq 0,$$

where $\zeta \in \mathcal{Z}$.

Definition 3.2. A coupled operator $\mathcal{T} : \mathbb{P} \times \mathbb{P} \rightarrow \mathbb{P}$ is said to be a partial nonlinear coupled $\mathcal{Z}$ -contraction if for all comparable elements $(u, \nu), (\omega, \gamma) \in \mathbb{P} \times \mathbb{P}$ it satisfies

$$\zeta(\|\mathcal{T}(u, \nu) - \mathcal{T}(\omega, \gamma)\|, \|u - \omega\| + \|\nu - \gamma\|) \geq 0,$$

where $\zeta \in \mathcal{Z}$.

3.1 Fixed point theorems of Krasnoselskii type

Let us begin with the following lemma which is useful in proving Krasnoselskii-type result.

Lemma 3.1. *Let $\mathcal{T}$ be a nondecreasing and partial nonlinear $\mathcal{Z}$ -contraction on $(\mathbb{P}, \sqsubseteq, \|\cdot\|)$, then $\mathcal{T}$ is partial condensing with respect to PH-MNC β_p defined on $\mathbb{P}$, that is,*

$$\beta_p(\mathcal{T}(C)) < \beta_p(C),$$

for every $C \in \mathcal{P}_{bd, cn}(\mathbb{P})$ with $\beta_p(C) > 0$.

Proof. The details of proof are similar to that in [26, Theorem 2.1] for metric space setting. □

We now present our first fixed point result for nonlinear operators.

Theorem 3.1. *Assume that $(\mathbb{P}, \sqsubseteq, \|\cdot\|)$ is $(\|\cdot\| - \sqsubseteq)$ -compatible, regular, and complete. Let $\mathcal{A} \in \mathcal{M}_{nd, bd}^{par}(\mathbb{P})$ be a partial nonlinear $\mathcal{Z}$ -contraction and $\mathcal{B} \in \mathcal{M}_{nd, ct}^{par}(\mathbb{P})$ be partial compact. Then $(\mathcal{A} + \mathcal{B})$ admits at least one fixed point if $\mathbb{P}$ contains ω_0 such that $\omega_0 \sqsubseteq \mathcal{A}\omega_0 + \mathcal{B}\omega_0$ or $\mathcal{A}\omega_0 + \mathcal{B}\omega_0 \sqsubseteq \omega_0$.*

Proof. From the hypotheses assumed on operators $\mathcal{A}$ and $\mathcal{B}$, we can conclude that $(\mathcal{A} + \mathcal{B})$ is non-decreasing partial bounded and continuous. We also have $\omega_0 \sqsubseteq (\mathcal{A} + \mathcal{B})\omega_0$. Let $\mathcal{C}$ be any chain in $\mathbb{P}$ such that $\beta_p(\mathcal{C}) > 0$. Then we have

$$\beta_p((\mathcal{A} + \mathcal{B})\mathcal{C}) \leq \beta_p(\mathcal{A}(\mathcal{C}) + \mathcal{B}(\mathcal{C})) \leq \beta_p(\mathcal{A}(\mathcal{C})) + \beta_p(\mathcal{B}(\mathcal{C})). \quad (3.1)$$

As $\mathcal{B}$ is partial compact and $\mathcal{A}$ is nonlinear partial $\mathcal{Z}$ -contraction, so by application of Lemma 3.1, we get from (3.1),

$$\beta_p((\mathcal{A} + \mathcal{B})\mathcal{C}) \leq \beta_p(\mathcal{A}(\mathcal{C})) < \beta_p(\mathcal{C}).$$

Thus, necessary requirements of Theorem 2.1 are fulfilled. Hence, $(\mathcal{A} + \mathcal{B})$ admits at least one fixed point in $\mathbb{P}$. □

Following corollary is an improvement of [16, Theorem 4].

Corollary 3.1. *Assume that $(\mathbb{P}, \sqsubseteq, \|\cdot\|)$ is $(\|\cdot\| - \sqsubseteq)$ -compatible, regular, and complete. Let $\mathcal{A} \in \mathcal{M}_{nd, bd}^{par}(\mathbb{P})$ be a partial nonlinear $\mathcal{D}$ -contraction and $\mathcal{B} \in \mathcal{M}_{nd, ct}^{par}(\mathbb{P})$ be partial compact. Then $(\mathcal{A} + \mathcal{B})$ admits at least one fixed point if $\mathbb{P}$ contains ω_0 such that $\omega_0 \sqsubseteq \mathcal{A}\omega_0 + \mathcal{B}\omega_0$ or $\mathcal{A}\omega_0 + \mathcal{B}\omega_0 \sqsubseteq \omega_0$.*

Corollary 3.2. *Assume that $(\mathbb{P}, \sqsubseteq, \|\cdot\|)$ is $(\|\cdot\| - \sqsubseteq)$ -compatible, regular, and complete. Let $\mathcal{A} \in \mathcal{M}_{nd, bd}^{par}(\mathbb{P})$ be a partial nonlinear contraction mapping and $\mathcal{B} \in \mathcal{M}_{nd, ct}^{par}(\mathbb{P})$ be partial compact. Then $(\mathcal{A} + \mathcal{B})$ admits at least one fixed point if $\mathbb{P}$ contains ω_0 such that $\omega_0 \sqsubseteq \mathcal{A}\omega_0 + \mathcal{B}\omega_0$ or $\mathcal{A}\omega_0 + \mathcal{B}\omega_0 \sqsubseteq \omega_0$.*

3.2 Coupled fixed point theorems of Krasnoselskii type

In this section, we present a new coupled fixed point theorem for sum of a partial nonlinear coupled $\mathcal{Z}$ -contraction with partial compact coupled operator.

Theorem 3.2. *Assume that $(\mathbb{P}, \sqsubseteq, \|\cdot\|)$ is $(\|\cdot\| - \sqsubseteq)$ -compatible, regular, and complete. Let $\mathcal{A} \in \mathcal{M}_{mm, bd}^{c, par}(\mathbb{P})$ be a partial nonlinear coupled $\mathcal{Z}$ -contraction and $\mathcal{B} \in \mathcal{M}_{mm, ct}^{c, par}(\mathbb{P})$ be partial compact. Then $(\mathcal{A} + \mathcal{B})$ admits at least one coupled fixed point if $\mathbb{P}^2$ contains (ω_0, ν_0) such that $\omega_0 \sqsubseteq \mathcal{A}(\omega_0, \nu_0) + \mathcal{B}(\omega_0, \nu_0)$ and $\mathcal{A}(\nu_0, \omega_0) + \mathcal{B}(\nu_0, \omega_0) \sqsubseteq \nu_0$.*

Proof. Let us define $\mathcal{T} : \mathbb{P} \times \mathbb{P} \rightarrow \mathbb{P}$ by

$$\mathcal{T}(w, v) = \mathcal{A}(w, v) + \mathcal{B}(w, v).$$

Also

$$\mathcal{T}(v, w) = \mathcal{A}(v, w) + \mathcal{B}(v, w).$$

From the definition of $\mathcal{T}$, it can be observed that $\mathcal{T}$ is well defined and $\mathcal{T}$ is mixed monotone, partial bounded, and partial continuous coupled operator. We need to show first that $\mathcal{T}$ is coupled partial condensing operator.

Let $\mathcal{C}_1, \mathcal{C}_2 \in \mathcal{P}_{bd, cn}$ such that $\beta_p(\mathcal{C}_1) > 0$ and $\beta_p(\mathcal{C}_2) > 0$, where β_p is PH-MNC. Then clearly $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$ is nonempty bounded chain in $\mathbb{P}^2$. Let us define $\tilde{\beta}_p(\mathcal{C}) = \max\{\beta_p(\mathcal{C}_1), \beta_p(\mathcal{C}_2)\}$. Then $\tilde{\beta}_p$ is a PH-MNC on $\mathbb{P}^2$ and $\tilde{\beta}_p(\mathcal{C}) > 0$.

Let us define two sequences $\{t_n\}$ and $\{s_n\}$ defined by $t_n = \tilde{\beta}_p(\mathcal{C}) - \varepsilon_n > 0$ and $s_n = \tilde{\beta}_p(\mathcal{C}) + \varepsilon_n > 0$, where $\{\varepsilon_n\}$ is such that $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Then $\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} s_n = \tilde{\beta}_p(\mathcal{C}) > 0$ and $t_n < s_n$. Then by (ζ_2) in Definition 2.10 for a $\zeta \in \mathcal{Z}$, we have

$$\limsup_{n \rightarrow \infty} \zeta(t_n, s_n) < 0. \quad (3.2)$$

Choosing $\varepsilon = \sup \varepsilon_n$ sufficiently small, we can deduce from inequality (3.2) that there exists $\Delta < 0$ so that

$$\zeta(t, s) < \Delta, \quad (3.3)$$

where, $t \in [\tilde{\beta}_p(\mathcal{C}) - \varepsilon, \tilde{\beta}_p(\mathcal{C})]$ and $s \in [\tilde{\beta}_p(\mathcal{C}), \tilde{\beta}_p(\mathcal{C}) + \varepsilon]$. Let $\mathcal{R} = \tilde{\beta}_p(\mathcal{C}) + \varepsilon$. Assume that $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$ has a $\mathcal{R}$ -net, that is,

$$\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2 \subset \bigcup_{j=1}^K B(u_j, \mathcal{R}), \quad (3.4)$$

where $u_j \in \mathbb{P} \times \mathbb{P}$, $j = 1, 2, \dots, K$.

Let $\mathcal{R}^+ = \tilde{\beta}_p(\mathcal{C}) - \varepsilon$. We show that $\mathcal{A}(\mathcal{C})$ has a $\mathcal{R}^+$ -net. Let $y = (y_1, y_2) \in \mathcal{A}(\mathcal{C})$. Then for every $y \in \mathcal{A}(\mathcal{C})$, there exists $x = (x_1, x_2) \in \mathcal{C}$ such that $\mathcal{A}x = \mathcal{A}(x_1, x_2) = (y_1, y_2) = y$. Then from (3.4), there exists $u_j = (u_{j_1}, u_{j_2})$ such that

$$\|u_j - x\| = \|u_{j_1} - x_1\| + \|u_{j_2} - x_2\| < \mathcal{R}.$$

Now if $\mathcal{A}(x) = \mathcal{A}(x_1, x_2) = \mathcal{A}(u_{j_1}, u_{j_2}) = \mathcal{A}(u_j)$, then $\|\mathcal{A}(u_j) - \mathcal{A}(x)\| < \mathcal{R}^+$. Suppose $\mathcal{A}(x) \neq \mathcal{A}(u_j)$, then we have two cases.

Case I: If $0 < \|u_j - x\| < \mathcal{R}^+$, then $\mathcal{Z}$ -contractivity of $\mathcal{A}$ gives us

$$0 \leq \zeta(\|\mathcal{A}(u_j) - \mathcal{A}(x)\|, \|u_j - x\|),$$

which implies

$$\|\mathcal{A}(u_j) - \mathcal{A}(x)\| \leq \|u_j - x\| < \mathcal{R}^+.$$

Case II: If $\mathcal{R}^+ < \|u_j - x\| < \mathcal{R}$, then either $\|\mathcal{A}u_j - \mathcal{A}x\| < \mathcal{R}^+$ or $\|\mathcal{A}u_j - \mathcal{A}x\| \geq \mathcal{R}^+$. Suppose $\|\mathcal{A}u_j - \mathcal{A}x\| \geq \mathcal{R}^+$, then (3.3) gives us

$$\zeta(\|\mathcal{A}u_j - \mathcal{A}x\|, \|u_j - x\|) < \Delta,$$

which is contradictory to the fact that $\mathcal{A}$ is $\mathcal{Z}$ -contraction. Thus, in both the cases we get $\|\mathcal{A}u_j - \mathcal{A}x\| < \mathcal{R}^+$, that is, $\mathcal{A}(\mathcal{C})$ has a $\mathcal{R}^+$ -net. This means

$$\tilde{\beta}_p(\mathcal{A}(\mathcal{C})) \leq \mathcal{R}^+ < \tilde{\beta}_p(\mathcal{C}),$$

or

$$\tilde{\beta}_p(\mathcal{A}(\mathcal{C}_1 \times \mathcal{C}_2)) < \tilde{\beta}_p(\mathcal{C}_1 \times \mathcal{C}_2). \quad (3.5)$$

We are now in a condition to show that $\mathcal{T}$ is a condensing coupled operator. Let $\mathcal{C}, \mathcal{D} \in \mathcal{P}_{bd, cn}(\mathbb{P})$. Then $\mathcal{C} \times \mathcal{D} \in \mathcal{P}_{bd, cn}(\mathbb{P}^2)$. From definition of $\mathcal{T}$, partial compactness of $\mathcal{B}$ and (3.5) we have

$$\begin{aligned}\tilde{\beta}_p(\mathcal{T}(\mathcal{C} \times \mathcal{D})) &= \tilde{\beta}_p(\mathcal{A}(\mathcal{C} \times \mathcal{D}) + \mathcal{B}(\mathcal{C} \times \mathcal{D})) \\ &\leq \tilde{\beta}_p(\mathcal{A}(\mathcal{C} \times \mathcal{D})) + \tilde{\beta}_p(\mathcal{B}(\mathcal{C} \times \mathcal{D})) \\ &\leq \tilde{\beta}_p(\mathcal{A}(\mathcal{C} \times \mathcal{D})) \\ &< \tilde{\beta}_p(\mathcal{C} \times \mathcal{D}).\end{aligned}\tag{3.6}$$

Similarly,

$$\tilde{\beta}_p(\mathcal{T}(\mathcal{D} \times \mathcal{C})) < \tilde{\beta}_p(\mathcal{D} \times \mathcal{C}).\tag{3.7}$$

Adding (3.6) and (3.7) we get

$$\tilde{\beta}_p(\mathcal{T}(\mathcal{C} \times \mathcal{D})) + \tilde{\beta}_p(\mathcal{T}(\mathcal{D} \times \mathcal{C})) < \tilde{\beta}_p(\mathcal{C} \times \mathcal{D}) + \tilde{\beta}_p(\mathcal{D} \times \mathcal{C}).$$

Thus, the operator $\mathcal{T}$ is a coupled partial condensing on $\mathbb{P} \times \mathbb{P}$ with respect to PH-MNC $\tilde{\beta}_p$. Now by hypotheses, $\mathbb{P} \times \mathbb{P}$ contains (ω_0, ν_0) such that

$$\omega_0 \sqsubseteq \mathcal{A}(\omega_0, \nu_0) + \mathcal{B}(\omega_0, \nu_0) = \mathcal{T}(\omega_0, \nu_0)$$

and

$$\mathcal{T}(\nu_0, \omega_0) = \mathcal{A}(\nu_0, \omega_0) + \mathcal{B}(\nu_0, \omega_0) \sqsubseteq \nu_0.$$

Thus, the coupled operator $\mathcal{T}$ fulfills all the requirements of Theorem 2.2 and therefore the coupled operator $\mathcal{T} = \mathcal{A} + \mathcal{B}$ has at least one coupled fixed point. $\square$

Following corollary is the main result in [18].

Corollary 3.3. Assume that $(\mathbb{P}, \sqsubseteq, \|\cdot\|)$ is $(\|\cdot\| - \sqsubseteq)$ -compatible, regular, and complete. Let $\mathcal{A} \in \mathcal{M}_{mm, bd}^{c, par}(\mathbb{P})$ be a partial nonlinear coupled D -contraction and $\mathcal{B} \in \mathcal{M}_{mm, ct}^{c, par}(\mathbb{P})$ be partial compact. Then $(\mathcal{A} + \mathcal{B})$ admits at least one fixed point if $\mathbb{P}^2$ contains (ω_0, ν_0) such that $\omega_0 \sqsubseteq \mathcal{A}(\omega_0, \nu_0) + \mathcal{B}(\omega_0, \nu_0)$ and $\mathcal{A}(\nu_0, \omega_0) + \mathcal{B}(\nu_0, \omega_0) \sqsubseteq \nu_0$.

Corollary 3.4. Assume that $(\mathbb{P}, \sqsubseteq, \|\cdot\|)$ is $(\|\cdot\| - \sqsubseteq)$ -compatible, regular, and complete. Let $\mathcal{A} \in \mathcal{M}_{mm, bd}^{c, par}(\mathbb{P})$ be a partial nonlinear coupled contraction and $\mathcal{B} \in \mathcal{M}_{mm, ct}^{c, par}(\mathbb{P})$ be partial compact. Then $(\mathcal{A} + \mathcal{B})$ admits at least one fixed point if $\mathbb{P}^2$ contains (ω_0, ν_0) such that $\omega_0 \sqsubseteq \mathcal{A}(\omega_0, \nu_0) + \mathcal{B}(\omega_0, \nu_0)$ and $\mathcal{A}(\nu_0, \omega_0) + \mathcal{B}(\nu_0, \omega_0) \sqsubseteq \nu_0$.

4 Product of two operators and fixed points

In this section, let $(\mathbb{A}, \sqsubseteq, \|\cdot\|)$ denote a partially ordered normed linear algebra. Let

$$\mathbb{A}^+ = \{\omega \in \mathbb{A} : \omega \sqsupseteq 0\}$$

and

$$\kappa = \{\mathbb{A}^+ \subset \mathbb{A} : \omega\nu \in \mathbb{A}^+ \text{ for all } \omega, \nu \in \mathbb{A}^+\},$$

where 0 is the zero element in $\mathbb{A}$. Every member of κ is called positive vector of $\mathbb{A}$. An operator $\mathcal{T} : \mathbb{A} \rightarrow \mathbb{A}$ is positive if its range is contained in κ .

Remark 4.1. It is well known that if $\omega_1, \omega_2, \nu_1, \nu_2 \in \kappa$ such that $\omega_1 \sqsubseteq \nu_1$ and $\omega_2 \sqsubseteq \nu_2$, then $\omega_1\omega_2 \sqsubseteq \nu_1\nu_2$.

Let $\mathcal{C}$ and $\mathcal{D}$ be two chains in $\mathbb{A}$, then

$$\mathcal{C}\mathcal{D} = \{\omega\nu \in \mathbb{P} : \omega \in \mathcal{C} \text{ and } \nu \in \mathcal{D}\}$$

and

$$\|\mathcal{C}\| = \sup\{\|\nu\| : \nu \in \mathcal{C}\}.$$

Lemma 4.1. [15] Let $\mathcal{C}$ and $\mathcal{D}$ be two chains in $(\mathbb{A}, \sqsubseteq, \|\cdot\|)$ and let β_p be a PH-MNC on $\mathbb{A}$ then

$$\beta_p(\mathcal{C}\mathcal{D}) \leq \|\mathcal{D}\|\beta_p(\mathcal{C}) + \|\mathcal{C}\|\beta_p(\mathcal{D}).$$

Theorem 4.1. Assume that $(\mathbb{A}, \sqsubseteq, \|\cdot\|)$ is $(\|\cdot\| - \sqsubseteq)$ -compatible, regular, and complete. Let $\mathcal{G} \in \mathcal{M}_{nd,bd}^{par}(\mathbb{A})$ be partial nonlinear $\mathcal{Z}$ -contraction and $\mathcal{H} \in \mathcal{M}_{nd,ct}^{par}(\mathbb{A})$ be partial compact. Suppose $\mathbb{A}$ contains ω_0 such that $\omega_0 \sqsubseteq \mathcal{G}\omega_0\mathcal{H}\omega_0$ or $\mathcal{G}\omega_0\mathcal{H}\omega_0 \sqsubseteq \omega_0$ and there is an $M = \|\mathcal{H}(\mathcal{C})\| < 1$, for any chain $\mathcal{C}$ in $\mathbb{A}$. Then, $\mathcal{G}\mathcal{H}$ admits at least one fixed point in $\mathbb{A}$.

Proof. Observing assumption on operators $\mathcal{G}$ and $\mathcal{H}$, it is clear that $\mathcal{G}\mathcal{H}$ is a nondecreasing, partial bounded, and partial continuous operator on $\mathbb{P}$. Also there exists $\omega_0 \in \mathbb{A}$ such that $\omega_0 \sqsubseteq \mathcal{G}\omega_0\mathcal{H}\omega_0 = (\mathcal{G}\mathcal{H})\omega_0$.

Now, let β_p be a PH-MNC on $\mathbb{A}$ and $\mathcal{C}$ be any chain in $\mathbb{A}$. Then we have

$$\beta_p((\mathcal{G}\mathcal{H})\mathcal{C}) = \beta_p(\mathcal{G}(\mathcal{C})\mathcal{H}(\mathcal{C})) \leq \|\mathcal{G}(\mathcal{C})\|\beta_p(\mathcal{H}(\mathcal{C})) + \|\mathcal{H}(\mathcal{C})\|\beta_p(\mathcal{G}(\mathcal{C})). \quad (4.1)$$

Considering the fact that $\mathcal{G}$ is partial $\mathcal{Z}$ -contraction, then it can be easily shown that $\mathcal{G}$ is partial condensing with respect to PH-MNC β . As $\mathcal{H}$ is partial compact so $\beta_p(\mathcal{H}(\mathcal{C})) = 0$ and $\|\mathcal{H}(\mathcal{C})\| < 1$. Applying these facts in (4.1) we get

$$\beta_p((\mathcal{G}\mathcal{H})\mathcal{C}) < \beta_p(\mathcal{C}).$$

Thus, existence of fixed point is ensured by application of Theorem 2.1. $\square$

Corollary 4.1. Assume that $(\mathbb{A}, \sqsubseteq, \|\cdot\|)$ is $(\|\cdot\| - \sqsubseteq)$ -compatible, regular, and complete. Let $\mathcal{G} \in \mathcal{M}_{nd,bd}^{par}(\mathbb{A})$ be partial nonlinear D -contraction and $\mathcal{H} \in \mathcal{M}_{nd,ct}^{par}(\mathbb{A})$ be partial compact. Suppose $\mathbb{A}$ contains ω_0 such that $\omega_0 \sqsubseteq \mathcal{G}\omega_0\mathcal{H}\omega_0$ or $\mathcal{G}\omega_0\mathcal{H}\omega_0 \sqsubseteq \omega_0$ and there is an $M = \|\mathcal{H}(\mathcal{C})\| < 1$, for any chain $\mathcal{C}$ in $\mathbb{A}$. Then $\mathcal{G}\mathcal{H}$ admits at least one fixed point in $\mathbb{A}$.

Corollary 4.2. Assume that $(\mathbb{A}, \sqsubseteq, \|\cdot\|)$ is $(\|\cdot\| - \sqsubseteq)$ -compatible, regular, and complete. Let $\mathcal{G} \in \mathcal{M}_{nd,bd}^{par}(\mathbb{A})$ be partial nonlinear contraction and $\mathcal{H} \in \mathcal{M}_{nd,ct}^{par}(\mathbb{A})$ be partial compact. Suppose $\mathbb{A}$ contains ω_0 such that $\omega_0 \sqsubseteq \mathcal{G}\omega_0\mathcal{H}\omega_0$ or $\mathcal{G}\omega_0\mathcal{H}\omega_0 \sqsubseteq \omega_0$ and there is an $M = \|\mathcal{H}(\mathcal{C})\| < 1$, for any chain $\mathcal{C}$ in $\mathbb{A}$. Then $\mathcal{G}\mathcal{H}$ admits at least one fixed point in $\mathbb{A}$.

5 Applications

In this section, we study the HFHDEs with linear perturbation of first and second type along with coupled system of HFHDEs with linear perturbation of second type.

5.1 HFHDE of second type

Consider the following HFHDE with linear perturbation of second type

$$\mathbb{D}_{a^+}^{\mu,\nu}[w(t) - f(t, w(t))] = g(t, w(t)), \quad \text{a.e. } t \in [a, b], \quad (5.1)$$

$$\mathbb{I}_{a^+}^{1-\gamma}[w(t) - f(t, w(t))](a) = W_a, \quad \gamma = \mu + \nu - \mu\nu, \quad (5.2)$$

where $\mathbb{D}_{a^+}^{\mu,\nu}$ is the Hilfer fractional derivative operator.

First we derive a result which brings the equivalence of HFHDE (5.1)–(5.2) with the following integral equation:

$$w(t) = f(t, w(t)) + \frac{W_a}{\Gamma(\gamma)}(t-a)^{\gamma-1} + \frac{1}{\Gamma(\mu)} \int_a^t \frac{g(s, w(s))}{(t-s)^{1-\mu}} ds, \quad t > a. \quad (5.3)$$

Lemma 5.1. Let $\gamma = \mu + \nu - \mu\nu$ where $0 < \mu < 1$ and $0 \leq \nu \leq 1$. Let function $g : (a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ be such that $g(\cdot, w(\cdot)) \in C_{1-\gamma}[a, b]$ for any $w \in C_{1-\gamma}[a, b]$. If $w \in C_{1-\gamma}^\gamma[a, b]$ is such that $w - f(\cdot, w(\cdot)) \in C_{1-\gamma}^\gamma[a, b]$ where $f : (a, b] \times \mathbb{R} \rightarrow \mathbb{R}$, then w satisfies (5.1)–(5.2) if and only if it satisfies (5.3).

Proof. First, let us prove necessary part. Let $w \in C_{1-\gamma}^\gamma[a, b]$ be a solution to (5.1)–(5.2). We wish to show that w is also a solution of (5.3). By definition of space $C_{1-\gamma}^\gamma[a, b]$, we have

$$\mathbb{D}_{a^+}^\gamma[w - f(t, w(t))] \in C_{1-\gamma}[a, b], \quad t \in [a, b].$$

Then Lemma 2.5 and Definition 2.12 yield us

$$\mathbb{D} \mathbb{I}_{a^+}^{1-\gamma}[w(t) - f(t, w(t))] \in C_{1-\gamma}[a, b].$$

Thus,

$$\mathbb{I}_{a^+}^{1-\gamma}[w(t) - f(t, w(t))] \in C_{1-\gamma}^1[a, b].$$

Applying Lemma 2.10, we get

$$\mathbb{I}_{a^+}^\gamma \mathbb{D}_{a^+}^\gamma[w(t) - f(t, w(t))] = [w(t) - f(t, w(t))] - \frac{\mathbb{I}_{a^+}^\gamma[w(s) - f(s, w(s))](a)}{\Gamma(\gamma)}(t-a)^{\gamma-1}. \quad (5.4)$$

By our hypothesis and Lemma 2.11, we get

$$\mathbb{I}_{a^+}^\gamma \mathbb{D}_{a^+}^\gamma[w(t) - f(t, w(t))] = \mathbb{I}_{a^+}^\mu \mathbb{D}_{a^+}^{\mu, \nu}[w(t) - f(t, w(t))] = \left[\mathbb{I}_{a^+}^\mu g(s, w(s)) \right](t). \quad (5.5)$$

From (5.4) and (5.5), we obtain

$$w(t) = f(t, w(t)) + \frac{W_a}{\Gamma(\gamma)}(t-a)^{\gamma-1} + \left[\mathbb{I}_{a^+}^\mu g(s, w(s)) \right](t), \quad (5.6)$$

which is (5.3).

Now we prove sufficient part. Let $w \in C_{1-\gamma}^\gamma[a, b]$ be a solution to (5.3), which can also be written as (5.6). Applying the operator $\mathbb{D}_{a^+}^\gamma$ on both sides of (5.6) and then using Lemmas 2.1, 2.3, and Definition 2.12, we get

$$\mathbb{D}_{a^+}^\gamma[w(t) - f(t, w(t))] = \mathbb{D}_{a^+}^{v(1-\mu)} g(t, w(t)). \quad (5.7)$$

Since $\mathbb{D}_{a^+}^\gamma[w(t) - f(t, w(t))] \in C_{1-\gamma}[a, b]$, we have

$$\mathbb{D} \mathbb{I}_{a^+}^{1-v(1-\mu)} g = \mathbb{I}_{a^+}^{v(1-\mu)} g \in C_{1-\gamma}[a, b]. \quad (5.8)$$

As $g \in C_{1-\gamma}[a, b]$, Lemma 2.10 implies that

$$\mathbb{I}_{a^+}^{1-v(1-\mu)} g \in C_{1-\gamma}[a, b]. \quad (5.9)$$

From (5.8) and (5.9), it follows that

$$\mathbb{I}_{a^+}^{1-v(1-\mu)} g \in C_{1-\gamma}^1[a, b].$$

Thus, g and $\mathbb{I}_{a^+}^{v(1-\mu)} g$ satisfy the conditions of Lemma 2.10. Now apply $\mathbb{I}_{a^+}^{v(1-\mu)}$ on both sides of (5.7), we get

$$\mathbb{I}_{a^+}^{v(1-\mu)} \mathbb{D}_{a^+}^\gamma[w(t) - f(t, w(t))] = \mathbb{I}_{a^+}^{v(1-\mu)} \mathbb{D}_{a^+}^{v(1-\mu)} g(t, w(t)).$$

Use of Definition 2.13 and Lemma 2.10 gives us

$$\mathbb{D}_{a^+}^{\mu, \nu} [w(t) - f(t, w(t))] = g(t, w(t)) - \frac{[\mathbb{I}_{a^+}^{1-\nu(1-\mu)} g(s, w(s))](a)}{\Gamma(1-\nu(1-\mu))} (t-a)^{\nu(1-\mu)-1}. \quad (5.10)$$

Since $1-\gamma < 1-\nu(1-\mu)$, Lemma 2.6 yields $[\mathbb{I}_{a^+}^{1-\nu(1-\mu)} g(s, w(s))](a) = 0$. Hence, (5.10) reduces to

$$\mathbb{D}_{a^+}^{\mu, \nu} [w(t) - f(t, w(t))] = g(t, w(t)), \quad t \in [a, b].$$

It remains to show that the initial condition (5.2) also holds. Applying $\mathbb{I}_{a^+}^{1-\gamma}$ to both sides of (5.6), we get

$$\mathbb{I}_{a^+}^{1-\gamma} [w(t) - f(t, w(t))] = \mathbb{I}_{a^+}^{1-\gamma} \frac{W_a}{\Gamma(\gamma)} (t-a)^{\gamma-1} + \mathbb{I}_{a^+}^{1-\gamma} \mathbb{I}_{a^+}^{\mu} g(t, w(t)).$$

By Lemma 2.1, we have

$$\mathbb{I}_{a^+}^{1-\gamma} [w(t) - f(t, w(t))] = W_a + \mathbb{I}_{a^+}^{1-\gamma+\mu} g(t, w(t)) = W_a + \mathbb{I}_{a^+}^{1-\nu(1-\mu)} g(t, w(t)).$$

Varying $t \rightarrow a$ and using Lemma 2.6 we obtain

$$\mathbb{I}_{a^+}^{1-\gamma} [w(t) - f(t, w(t))] = W_a. \quad \square$$

Definition 5.1. An $w \in C_{1-\gamma}^{\gamma}$ is a mild solution of HFHDE with linear perturbation of second type (5.1)–(5.2) if it satisfies Volterra integral equation (5.3).

Theorem 5.1. Let $\gamma = \mu + \nu - \mu\nu$ where $0 < \mu < 1$ and $0 \leq \nu \leq 1$. Let function $f : (a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ be a non-decreasing in second variable and bounded such that $f(\cdot, w(\cdot)) \in C_{1-\gamma}^{\nu(1-\mu)}[a, b]$ for any $w \in C_{1-\gamma}[a, b]$. Let function $g : (a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ be nondecreasing in second variable such that $g(\cdot, w(\cdot)) \in C_{1-\gamma}^{\nu(1-\mu)}[a, b]$ for any $w \in C_{1-\gamma}[a, b]$. Then (5.1)–(5.2) has at least one mild solution in the space $C_{1-\gamma}^{\gamma}[a, b] \subseteq C_{1-\gamma}^{\mu, \nu}[a, b]$, if the following assumptions are satisfied:

- (A1) mapping $t \mapsto w - f(t, w)$ is nondecreasing;
- (A2) there exists a function $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ which is upper semicontinuous with $\phi(t) < t$ and $\phi(0) = 0$ such that

$$|f(t, w_1) - f(t, w_2)| \leq \phi(|w_1 - w_2|), \quad t \in [a, b], \quad w_1, w_2 \in C_{1-\gamma}^{\gamma}[a, b];$$

- (A3) there exists a function $\kappa \in C[a, b]$ such that $|g(t, w)| \leq \kappa(t)$, $t \in [a, b]$;
- (A4) there exists $w_0 \in C_{1-\gamma}^{\gamma}[a, b]$ satisfying

$$\mathbb{D}^{\mu, \nu} [w_0(t) - f(t, w_0(t))] \leq g(t, w_0(t)), \quad \text{for all } t \in [a, b],$$

and

$$\mathbb{I}_{a^+}^{1-\gamma} [w_0(s) - f(s, w_0(s))](a) \leq W_a^0.$$

Proof. As per Lemma 5.1, it suffices to prove the result for equivalent integral equation (5.3). Let us define two operators $\mathcal{A}, \mathcal{B} : C_{1-\gamma}^{\gamma}[a, b] \rightarrow C_{1-\gamma}^{\gamma}[a, b]$ by

$$\mathcal{A}w(t) = f(t, w(t))$$

and

$$\mathcal{B}w(t) = \frac{W_a}{\Gamma(\gamma)} (t-a)^{\gamma-1} + \frac{1}{\Gamma(\mu)} \int_a^t \frac{g(s, w(s))}{(t-s)^{1-\mu}} ds, \quad t \in [a, b].$$

Then equation (5.3) can be written as

$$\mathcal{A}w(t) + \mathcal{B}w(t) = w(t), \quad t \in [a, b].$$

Step 1: Operator $\mathcal{A}$ is nondecreasing, partial bounded, and partial $\mathcal{Z}$ -contraction.

Since f is nondecreasing in second variable and bounded so $\mathcal{A}$ is nondecreasing and partial bounded. Let $w_1, w_2 \in C_{1-\gamma}^\gamma[a, b]$, then by assumption (A_2) ,

$$|\mathcal{A}w_1(t) - \mathcal{A}w_2(t)| = |f(t, w_1(t)) - f(t, w_2(t))| \leq \phi(|w_1 - w_2|).$$

Taking supremum over $t \in [a, b]$, and $\zeta(t, s) = \phi(s) - t$, we obtain

$$\zeta(\|\mathcal{A}w_1 - \mathcal{A}w_2\|, \|w_1 - w_2\|) > 0.$$

Thus, $\mathcal{A}$ is $\mathcal{Z}$ -contraction.

Step 2: Operator $\mathcal{B}$ is nondecreasing, partial continuous, and partial compact.

Considering (A_1) and assumption on mapping g , it is clear that $\mathcal{B}$ is nondecreasing and partial continuous. We now need to show that $\mathcal{B}$ is partial compact, for that it is sufficient to show that $\mathcal{B}(\mathcal{C})$ is uniformly bounded and equicontinuous for any bounded chain $\mathcal{C}$ in $C_{1-\gamma}^\gamma[a, b]$. Let $v \in \mathcal{B}(\mathcal{C})$. Then

$$\begin{aligned} |(t-a)^{1-\gamma}v(t)| &= |\mathcal{B}w(t)(t-a)^{1-\gamma}| \\ &= \left| (t-a)^{1-\gamma} \frac{W_a}{\Gamma(\gamma)} (t-a)^{\gamma-1} + \frac{(t-a)^{1-\gamma}}{\Gamma(\mu)} \int_a^t (t-s)^{\mu-1} g(s, w(s)) ds \right| \\ &\leq \left| \frac{W_a}{\Gamma(\gamma)} \right| + \frac{(t-a)^{1-\gamma}}{\Gamma(\mu)} \int_a^t (t-s)^{\mu-1} |g(s, w(s))| ds \\ &\leq \left| \frac{W_a}{\Gamma(\gamma)} \right| + \frac{1}{\Gamma(\mu)} \int_a^t (t-a)^{1-\gamma} (t-s)^{\mu-1} \kappa(s) ds \\ &\leq \left| \frac{W_a}{\Gamma(\gamma)} \right| + \frac{(t-a)^{\mu-1}}{\Gamma(\mu)} \|\kappa\|_{C_{1-\gamma}}. \end{aligned}$$

Taking supremum on both sides

$$\|v(t)\| \leq \frac{1}{\Gamma(\gamma)} |W_a| + \frac{(b-a)^{\mu-1}}{\Gamma(\mu)} \|\kappa\|_{C_{1-\gamma}}.$$

Therefore, $\mathcal{B}(\mathcal{C})$ is uniformly bounded. Now, let $t_1, t_2 \in [a, b]$, $t_2 \leq t_1$, and let $w \in \mathcal{C}$. Then

$$\begin{aligned} |(t_1-a)^{1-\gamma}\mathcal{B}w(t_1) - (t_2-a)^{1-\gamma}\mathcal{B}w(t_2)| &= \left| (t_1-a)^{1-\gamma} \left(\frac{W_a}{\Gamma(\gamma)} (t_1-a)^{\gamma-1} + \frac{1}{\Gamma(\mu)} \int_a^{t_1} (t_1-s)^{\mu-1} g(s, w(s)) ds \right) \right. \\ &\quad \left. - (t_2-a)^{1-\gamma} \left(\frac{W_a}{\Gamma(\gamma)} (t_2-a)^{\gamma-1} + \frac{1}{\Gamma(\mu)} \int_a^{t_2} (t_2-s)^{\mu-1} g(s, w(s)) ds \right) \right| \\ &\leq \left| \frac{(t_1-a)^{1-\gamma}}{\Gamma(\mu)} \int_a^{t_1} (t_1-s)^{\mu-1} g(s, w(s)) ds - \frac{(t_2-a)^{1-\gamma}}{\Gamma(\mu)} \int_a^{t_2} (t_2-s)^{\mu-1} g(s, w(s)) ds \right| \\ &\leq \frac{1}{\Gamma(\mu)} \left| \int_a^{t_1} [(t_1-a)^{1-\gamma} (t_1-s)^{\mu-1} - (t_2-a)^{1-\gamma} (t_2-s)^{\mu-1}] g(s, w(s)) ds \right| \\ &\quad - \frac{1}{\Gamma(\mu)} \left| \int_{t_1}^{t_2} (t_2-a)^{1-\gamma} (t_2-s)^{\mu-1} g(s, w(s)) ds \right|. \end{aligned}$$

As $t_1 \rightarrow t_2$, the right-hand side of the above inequality tends to zero. So $\mathcal{B}(\mathcal{C})$ is equicontinuous.

Step 3: By assumption (A_4) , there exists $w_0 \in C_{1-\gamma}^\gamma[a, b]$ such that

$$\mathbb{D}^{\mu, \nu}[w_0(t) - f(t, w_0(t))] \leq g(t, w_0(t)), \quad \text{for all } t \in [a, b],$$

and

$$\mathbb{I}_{a^+}^{1-\gamma}[w_0(s) - f(s, w_0(s))](a) \leq W_a^0.$$

Thus, in view of Lemma 5.1 we can reduce the above Hilfer fractional hybrid differential inequality into the following integral inequality:

$$w_0(t) \leq f(t, w_0(t)) + \frac{W_a^0}{\Gamma(\gamma)}(t-a)^{\gamma-1} + \frac{1}{\Gamma(\mu)} \int_a^t \frac{g(s, w_0(s))}{(t-s)^{1-\mu}} ds, \quad t \in [a, b].$$

The operator form of the above inequality is as follows:

$$w_0(t) \leq \mathcal{A}w_0(t) + \mathcal{B}w_0(t), \quad t \in [a, b].$$

Therefore, the operators $\mathcal{A}$ and $\mathcal{B}$ fulfill all the requirements of Theorem 3.1. This implies that operator equation $w(t) \leq \mathcal{A}w(t) + \mathcal{B}w(t)$ has a solution. Thus, (5.3) has a solution which is also a mild solution of (5.1)–(5.2). $\square$

Example 5.1. Consider the following HFHDE of second type

$$\mathbb{D}_{1^+}^{\mu, \nu} \left[w(t) - \frac{|w(t)|}{35(1 + |w(t)|)} \right] = \frac{t|w(t)|}{2 + |w(t)|}, \quad t \in (1, 2], \quad (5.11)$$

$$\mathbb{I}_{1^+}^{1-\gamma} \left[w(t) - \frac{|w(t)|}{35(1 + |w(t)|)} \right](1) = 2w(2/3), \quad (5.12)$$

with $\mu = \frac{1}{2}$, $\nu = \frac{1}{2}$. Then $\gamma = \frac{3}{4}$. We have

$$f(t, w(t)) = \frac{|w(t)|}{35(1 + |w(t)|)}$$

and

$$g(t, w(t)) = \frac{t|w(t)|}{2 + |w(t)|}.$$

Then the following can be observed easily.

- f and g are nondecreasing and f is bounded by $\frac{2}{35}$.
- $|f(t, w_1) - f(t, w_2)| \leq \phi(|w_1(t) - w_2(t)|)$, with $\phi(p) = \frac{p}{35(1 + |w_1(t)|)(1 + |w_2(t)|)}$.
- g is continuous and $|g(t, w(t))| \leq \kappa(t)$ for $\kappa(t) = t$.
- One can easily prove that $w - f(t, w)$ is nondecreasing.

If there exists a function $w_0 \in C_{\frac{1}{4}}^{\frac{3}{4}}$ such that

$$\begin{aligned} \mathbb{D}_{1^+}^{\frac{1}{2}, \frac{1}{2}} \left[w_0(t) - \frac{|w_0(t)|}{35(1 + |w_0(t)|)} \right] &\leq \frac{t|w_0(t)|}{2 + |w_0(t)|}, \\ \mathbb{I}_{1^+}^{\frac{1}{4}} \left[w_0(t) - \frac{|w_0(t)|}{35(1 + |w_0(t)|)} \right](1) &\leq 2w_0(2/3), \end{aligned}$$

then w_0 satisfies Assumption (A_4) . Therefore, all the hypotheses of Theorem 5.1 are satisfied. Thus, we can conclude that the problem (5.11)–(5.12) has at least one mild solution.

5.2 Coupled system of HFHDEs of second type

Consider the following coupled system of HFHDEs with linear perturbation of second type

$$\mathbb{D}_{a^+}^{\mu,\nu}[w(t) - f(t, w(t), v(t))] = g(t, w(t), v(t)), \quad \text{a.e. } t \in [a, b], \quad (5.13)$$

$$\mathbb{I}_{a^+}^{1-\gamma}[w(t) - f(t, w(t), v(t))](a) = U_{a^c}, \quad \gamma = \mu + \nu - \mu\nu, \quad (5.14)$$

$$\mathbb{D}_{a^+}^{\mu,\nu}[v(t) - f(t, v(t), w(t))] = g(t, v(t), w(t)), \quad \text{a.e. } t \in [a, b], \quad (5.15)$$

$$\mathbb{I}_{a^+}^{1-\gamma}[v(t) - f(t, v(t), w(t))](a) = V_{a^c}, \quad \gamma = \mu + \nu - \mu\nu. \quad (5.16)$$

Then by virtue of Lemma 5.1, the pair (w, v) is a coupled solution of the system (5.13)–(5.16) if and only if it is coupled solution of system

$$w(t) = f(t, w(t), v(t)) + \frac{U_{a^c}}{\Gamma(\gamma)}(t-a)^{\gamma-1} + \frac{1}{\Gamma(\mu)} \int_a^t \frac{g(s, w(s), v(s))}{(t-s)^{1-\mu}} ds, \quad t > a, \quad (5.17)$$

$$v(t) = f(t, v(t), w(t)) + \frac{V_{a^c}}{\Gamma(\gamma)}(t-a)^{\gamma-1} + \frac{1}{\Gamma(\mu)} \int_a^t \frac{g(s, v(s), w(s))}{(t-s)^{1-\mu}} ds, \quad t > a. \quad (5.18)$$

Theorem 5.2. Let $\gamma = \mu + \nu - \mu\nu$ where $0 < \mu < 1$ and $0 \leq \nu \leq 1$. Let $f : (a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a mixed monotone and bounded function such that $f(\cdot, w(\cdot), v(\cdot)) \in C_{1-\gamma}^{\nu(1-\mu)}[a, b]$ for any $u, v \in C_{1-\gamma}[a, b]$. Let $g : (a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a mixed monotone function such that $g(\cdot, w(\cdot), v(\cdot)) \in C_{1-\gamma}^{\nu(1-\mu)}[a, b]$ for any $w, v \in C_{1-\gamma}[a, b]$. Then system (5.13)–(5.16) has at least one mild solution in the space $C_{1-\gamma}^\gamma[a, b] \subseteq C_{1-\gamma}^{\mu,\nu}[a, b]$, if the following assumptions are satisfied:

- (A₁^c) mapping $t \mapsto w - f(t, w, v)$ is nondecreasing and mapping $t \mapsto v - f(t, w, v)$ is nonincreasing;
- (A₂^c) there exists a function $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ which is upper semicontinuous such that $\phi(t) < t$ and $\phi(0) = 0$ satisfying

$$|f(t, w, v) - f(t, x, y)| \leq \phi(|w - x| + |v - y|), \quad t \in [a, b], \quad w, v, x, y \in C_{1-\gamma}^\gamma[a, b];$$

- (A₃^c) there exists a function $\kappa \in C[a, b]$ such that $|g(t, w, v)| \leq \kappa(t)$, $t \in [a, b]$;
- (A₄^c) there exists $w_0, v_0 \in C_{1-\gamma}^\gamma[a, b]$ such that

$$\begin{aligned} \mathbb{D}_{a^+}^{\mu,\nu}[w_0(t) - f(t, w_0(t), v_0(t))] &\leq g(t, w_0(t), v_0(t)), \quad \text{for all } t \in [a, b], \\ \mathbb{I}_{a^+}^{1-\gamma}[w_0(s) - f(s, w_0(s), v_0(s))](a) &\leq U_{a^c}, \end{aligned}$$

and

$$\begin{aligned} \mathbb{D}_{a^+}^{\mu,\nu}[v_0(t) - f(t, v_0(t), w_0(t))] &\geq g(t, v_0(t), w_0(t)), \quad \text{for all } t \in [a, b], \\ \mathbb{I}_{a^+}^{1-\gamma}[v_0(s) - f(s, v_0(s), w_0(s))](a) &\geq V_{a^c}. \end{aligned}$$

Proof. It is sufficient to prove the result for the equivalent system of Volterra integral equation (5.17)–(5.18).

Let us define two coupled operators $\mathcal{A}, \mathcal{B} : C_{1-\gamma}^\gamma[a, b] \times C_{1-\gamma}^\gamma[a, b] \rightarrow C_{1-\gamma}^\gamma[a, b]$ by

$$\mathcal{A}(w, v)(t) = f(t, w(t), v(t))$$

and

$$\mathcal{B}(w, v)(t) = \frac{U_{a^c}}{\Gamma(\gamma)}(t-a)^{\gamma-1} + \frac{1}{\Gamma(\mu)} \int_a^t \frac{g(s, w(s), v(s))}{(t-s)^{1-\mu}} ds, \quad t \in [a, b].$$

Then the system (5.17)–(5.18) can be written in operator equation form as

$$\mathcal{A}(w, v)(t) + \mathcal{B}(w, v)(t) = w(t), \quad t \in [a, b], \quad (5.19)$$

$$\mathcal{A}(v, w)(t) + \mathcal{B}(v, w)(t) = v(t), \quad t \in [a, b]. \quad (5.20)$$

Step 1: The operators $\mathcal{A}$ and $\mathcal{B}$ are mixed monotone, because of the assumptions that f and g are mixed monotone.

Step 2: Also as f is bounded, $\mathcal{A}$ is partial bounded. Following step 1 in proof of Theorem 5.1, we can prove that $\mathcal{A}$ is coupled $\mathcal{Z}$ -contraction.

Step 3: Since g is continuous it is easy to show that $\mathcal{B}$ is partial continuous. Also it is easy to show $\mathcal{B}$ is partial compact by following step 2 in proof of Theorem 5.1.

Step 4: By assumption (A_4^c) , there exists a pair $(w_0, v_0) \in C_{1-\gamma}^\gamma[a, b] \times C_{1-\gamma}^\gamma[a, b]$ such that

$$\mathbb{D}^{\mu, \nu}[w_0(t) - f(t, w_0(t), v_0(t))] \leq g(t, w_0(t), v_0(t)), \quad \text{for all } t \in [a, b],$$

$$\mathbb{I}_{a^+}^{1-\gamma}[w_0(s) - f(s, w_0(s), v_0(s))](a) \leq U_{a^+}^0,$$

and

$$\mathbb{D}^{\mu, \nu}[v_0(t) - f(t, v_0(t), w_0(t))] \geq g(t, v_0(t), w_0(t)), \quad \text{for all } t \in [a, b],$$

$$\mathbb{I}_{a^+}^{1-\gamma}[v_0(s) - f(s, v_0(s), w_0(s))](a) \geq V_{a^+}^0.$$

Thus in view of Lemma 5.1, we can reduce the above system of inequalities into the system of integral inequalities

$$\begin{aligned} w_0(t) &\leq f(t, w_0(t), v_0(t)) + \frac{U_{a^+}^0}{\Gamma(\gamma)}(t-a)^{\gamma-1} + \frac{1}{\Gamma(\mu)} \int_a^t \frac{g(s, w_0(s), v_0(s))}{(t-s)^{1-\mu}} ds, \quad t > a. \\ v_0(t) &\geq f(t, v_0(t), w_0(t)) + \frac{V_{a^+}^0}{\Gamma(\gamma)}(t-a)^{\gamma-1} + \frac{1}{\Gamma(\mu)} \int_a^t \frac{g(s, v_0(s), w_0(s))}{(t-s)^{1-\mu}} ds, \quad t > a, \end{aligned}$$

which can be reduced in the following system of operator inequalities

$$\mathcal{A}(w_0, v_0)(t) + \mathcal{B}(w_0, v_0)(t) \geq w_0(t), \quad t \in [a, b],$$

$$\mathcal{A}(v_0, w_0)(t) + \mathcal{B}(v_0, w_0)(t) \leq v_0(t), \quad t \in [a, b].$$

Therefore, the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 3.2. Thus, the operator equation (5.19)–(5.20) and hence the system (5.17)–(5.18) has a coupled solution which is a mild coupled solution of the system (5.13)–(5.16). $\square$

5.3 HFHDE with linear perturbation of first type

Consider the following HFHDE with linear perturbation of first type

$$\mathbb{D}_{a^+}^{\mu, \nu} \left[\frac{w(t)}{f(t, w(t))} \right] = g(t, w(t)), \quad \text{a.e. } t \in [a, b], \quad (5.21)$$

$$\mathbb{I}_{a^+}^{1-\gamma} \left[\frac{w(t)}{f(t, w(t))} \right](a) = W^a, \quad \gamma = \mu + \nu - \mu\nu. \quad (5.22)$$

Let us derive a result which brings the equivalence of HFHDE (5.21)–(5.22) with the following integral equation:

$$w(t) = f(t, w(t)) \left(\frac{W^a}{\Gamma(\gamma)}(t-a)^{\gamma-1} + \frac{1}{\Gamma(\mu)} \int_a^t \frac{g(s, w(s))}{(t-s)^{1-\mu}} ds \right), \quad t > a. \quad (5.23)$$

Lemma 5.2. Let $\gamma = \mu + \nu - \mu\nu$ where $0 < \mu < 1$ and $0 \leq \nu \leq 1$. Let $g : (a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $g(\cdot, w(\cdot)) \in C_{1-\gamma}[a, b]$ for any $w \in C_{1-\gamma}[a, b]$. If $w \in C_{1-\gamma}^\gamma[a, b]$ is such that $w/f(\cdot, w(\cdot)) \in C_{1-\gamma}^\gamma[a, b]$ where $f : (a, b] \times \mathbb{R} \rightarrow \mathbb{R}$, then w satisfies (5.21)–(5.22) if and only if it satisfies (5.23).

Proof. The proof follows in the similar way to the proof of Lemma 5.1. $\square$

Definition 5.2. A function $w \in C_{1-\gamma}^\gamma$ is a mild solution of HFHDE with linear perturbation of first type (5.21)–(5.22) if it satisfies Volterra integral equation (5.23).

Theorem 5.3. Let $\gamma = \mu + \nu - \mu\nu$ where $0 < \mu < 1$ and $0 \leq \nu \leq 1$. Let $f : (a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ be a nondecreasing in second variable and bounded function such that $f(\cdot, w(\cdot)) \in C_{1-\gamma}^{\nu(1-\mu)}[a, b]$ for any $w \in C_{1-\gamma}[a, b]$. Let function $g : (a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ be a nondecreasing in second variable such that $g(\cdot, w(\cdot)) \in C_{1-\gamma}^{\nu(1-\mu)}[a, b]$ for any $w \in C_{1-\gamma}[a, b]$. Then (5.21)–(5.22) has at least one mild solution in the space $C_{1-\gamma}^\gamma[a, b] \subseteq C_{1-\gamma}^{\mu, \nu}[a, b]$, if the following assumptions are satisfied:

(A₁') mapping $t \mapsto w/f(t, w)$ is nondecreasing;

(A₂') there exists a function $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ which is upper semicontinuous with $\phi(t) < t$ and $\phi(0) = 0$ such that

$$|f(t, w_1) - f(t, w_2)| \leq \phi(|w_1 - w_2|), \quad t \in [a, b], \quad w_1, w_2 \in C_{1-\gamma}^\gamma[a, b];$$

(A₃') there exists a function $\kappa \in C[a, b]$ such that $|g(t, w)| \leq \kappa(t)$, $t \in [a, b]$;

(A₄') there exists $w_0 \in C_{1-\gamma}^\gamma[a, b]$ such that

$$\mathbb{D}^{\mu, \nu} \left[\frac{w_0(t)}{f(t, w_0(t))} \right] \leq g(t, w_0(t)), \quad \text{for all } t \in [a, b],$$

and

$$\mathbb{I}_{a^+}^{1-\gamma} \left[\frac{w_0(s)}{f(s, w_0(s))} \right] (a) \leq W_0^a.$$

$$(A_5') \quad \frac{W^a}{\Gamma(\gamma)} + \frac{(b-a)^{\mu-1}}{\Gamma(\mu+1)} \|\kappa\|_{C_{1-\gamma}} < 1.$$

Proof. According to Lemma 5.2, it is sufficient to prove the result for the equivalent integral equation (5.23).

Let us define two operators $\mathcal{A}, \mathcal{B} : C_{1-\gamma}^\gamma[a, b] \rightarrow C_{1-\gamma}^\gamma[a, b]$ by

$$\mathcal{A}w(t) = f(t, w(t))$$

and

$$\mathcal{B}w(t) = \frac{W^a}{\Gamma(\gamma)}(t-a)^{\gamma-1} + \frac{1}{\Gamma(\mu)} \int_a^t \frac{g(s, w(s))}{(t-s)^{1-\mu}} ds, \quad t \in [a, b].$$

Then the equation (5.23) can be written as

$$\mathcal{A}w(t)\mathcal{B}w(t) = w(t), \quad t \in [a, b].$$

We will show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 4.1.

Following steps 1 and 2 in proof of Theorem 5.1, we see that $\mathcal{A}$ is nondecreasing, partial bounded, and partial $\mathcal{Z}$ -contraction, and $\mathcal{B}$ is nondecreasing, partial continuous, and partial compact.

By assumption (A₄'), there exists $w_0 \in C_{1-\gamma}^\gamma[a, b]$ such that

$$\mathbb{D}^{\mu, \nu} \left[\frac{w_0(t)}{f(t, w_0(t))} \right] \leq g(t, w_0(t)), \quad \text{for all } t \in [a, b],$$

and

$$\mathbb{I}_{a^+}^{1-\gamma} \left[\frac{w_0(s)}{f(s, w_0(s))} \right] (a) \leq W_0^a.$$

Thus, in view of Lemma 5.2 we can reduce above Hilfer fractional hybrid differential inequality into the following integral inequality:

$$w_0(t) \leq f(t, w_0(t)) \left(\frac{W_0^a}{\Gamma(\gamma)} (t-a)^{\gamma-1} + \frac{1}{\Gamma(\mu)} \int_a^t \frac{g(s, w_0(s))}{(t-s)^{1-\mu}} ds, \quad t \in [a, b] \right).$$

The operator form of above inequality is as follows:

$$w_0(t) \leq \mathcal{A}w_0(t)\mathcal{B}w_0(t), \quad t \in [a, b].$$

Now, for any bounded chain $\mathcal{C}$ in $C_{1-\gamma}^\gamma$, we have

$$M = |\mathcal{B}(\mathcal{C})| = \sup\{\|\mathcal{B}w(t)\| : w \in \mathcal{C}\} \leq \left| \frac{W^a}{\Gamma(\gamma)} \right| + \frac{(t-a)^{\mu-1}}{\Gamma(\mu+1)} \|k\|_{C_{1-\gamma}} \leq \left| \frac{W^a}{\Gamma(\gamma)} \right| + \frac{(b-a)^{\mu-1}}{\Gamma(\mu+1)} \|k\|_{C_{1-\gamma}} < 1.$$

Thus, all the requirements of Theorem 4.1 are satisfied by the operators $\mathcal{A}$ and $\mathcal{B}$. Thus, the operator equation $\mathcal{A}w(t)\mathcal{B}w(t) = w(t)$ and hence the integral equation (5.23) has a solution which is a mild solution of (5.21)–(5.22). $\square$

Example 5.2. Let us consider the following HFHDE of first type

$$\mathbb{D}_{0^+}^{\mu, \nu} \left[\frac{3w(t)}{t \sin w(t)} \right] = \frac{t|w(t)|}{4(1 + |w(t)|)}, \quad t \in (0, 1], \quad (5.24)$$

$$\mathbb{I}_{0^+}^{1-\gamma} \left[\frac{3w(t)}{t \sin w(t)} \right] (0) = 1, \quad (5.25)$$

with $\mu = \frac{1}{2}$, $\nu = \frac{1}{2}$. Then $\gamma = \frac{3}{4}$. We have

$$f(t, w(t)) = \frac{1}{3} t \sin w(t)$$

and

$$g(t, w(t)) = \frac{t|w(t)|}{4(1 + |w(t)|)}.$$

Then the following can be observed easily.

- f and g are nondecreasing and f is bounded by $\frac{1}{3}$.
- $|f(t, w) - f(t, v)| \leq \phi(|w(t) - v(t)|)$, with $\phi(p) = \frac{1}{3}tp$.
- g is continuous and $|g(t, u(t))| \leq \kappa(t)$ for $\kappa(t) = \frac{1}{4}$.
- One can easily prove that $w/f(t, w)$ is nondecreasing.

Also one can calculate the value

$$\left| \frac{W^a}{\Gamma(\gamma)} \right| + \frac{(b-a)^{\mu-1}}{\Gamma(\mu+1)} \|h\|_{C_{1-\gamma}} \approx 0.971127 < 1.$$

If there exists a function $w_0 \in C_{\frac{1}{4}}^{\frac{3}{4}}$ such that

$$\begin{aligned} \mathbb{D}_{0^+}^{\mu, \nu} \left[\frac{3w_0(t)}{t \sin w_0(t)} \right] &\leq \frac{t|w_0(t)|}{4(1 + |w_0(t)|)}, \\ \mathbb{I}_{0^+}^{\frac{1}{4}} \left[\frac{3w_0(t)}{t \sin w_0(t)} \right] (0) &\leq 2, \end{aligned}$$

then w_0 satisfies the assumption (A_4') . Therefore, all the hypotheses of Theorem 5.3 are satisfied. Thus, we can conclude that the problem (5.24)–(5.25) has at least one mild solution.

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