Research Article

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New criteria-based $\mathcal{H}$ -tensors for identifying the positive definiteness of multivariate homogeneous forms

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Abstract: Positive definite polynomials are important in the field of optimization. $\mathcal{H}$ -tensors play an important role in identifying the positive definiteness of an even-order homogeneous multivariate form. In this paper, we propose an iterative scheme for identifying $\mathcal{H}$ -tensor and prove that the algorithm can terminate within finite iterative steps. Some numerical examples are provided to illustrate the efficiency and validity of methods.

Keywords: positive definiteness, homogeneous multivariate form, $\mathcal{H}$ -tensors, generalized diagonal dominance, iterative scheme

MSC 2020: 15A69, 15A18, 65F15, 65H17

1 Introduction

The m th degree homogeneous polynomial of n variables $f(x)$ is denoted as

$$f(x) = \sum_{i_1, i_2, \dots, i_m \in N} a_{i_1 i_2 \dots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}, \quad (1)$$

where $x = (x_1, x_2, \dots, x_n) \in R^n$. The form in (1) can be represented as

$$f(x) \equiv \mathcal{A}x^m, \quad (2)$$

where $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ is an m th-order n -dimensional tensor [1,2] with entries

$$a_{i_1 i_2 \dots i_m} \in R, \quad i_j = 1, 2, \dots, n, \quad j = 1, 2, \dots, m.$$

The homogeneous polynomial $f(x)$ in (2) is said to be positive definiteness if $f(x) > 0$ for any $x \in R^n$ ($x \neq 0$). Positive-definite homogeneous polynomial has very important application in the field of dynamical and positive semi-definite polynomial as well as in the field of optimization. Since each homogeneous polynomial can be associated with a symmetric tensor, we can identify the positive (semi-)definiteness of homogeneous polynomial by identifying positive definiteness of the symmetric tensor associated with it [3–9].

For $n \leq 3$, the positive definiteness of the homogeneous polynomial form can be checked by Sturm theorem [10]. However, for $n > 3$ and $m \geq 4$, the problem of determining a given even-order multivariate polynomial $f(x)$ is positive semi-definiteness or not is NP-hard [7]. For this case, by $\mathcal{H}$ -tensors, Li et al. [11] proposed a method to identify the positive definiteness of homogeneous polynomial forms. It is well known

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that $\mathcal{H}$ -tensor is a special kind of tensors and an even order symmetric $\mathcal{H}$ -tensor with positive diagonal entries is positive definite [11,12]. Due to this, we may identify the positive definiteness of homogeneous polynomial forms via identifying $\mathcal{H}$ -tensors. Recently, with the help of generalized diagonally dominant tensor, various criteria for $\mathcal{H}$ -tensors are provided [13–18]. In this paper, we continue finding new non-parameter-involved iterative criterion for identifying $\mathcal{H}$ -tensors. Its validity will be theoretically guaranteed, and its performance will be illustrated in a set of numerical tests.

2 Preliminaries

In this section, some notations, definitions and lemmas are given. Let $\mathcal{C}^{[m,n]}(\mathcal{R}^{[m,n]})$ be the set of all complex (real) m th-order n -dimensional tensors and $\mathcal{I} \in \mathcal{C}^{[m,n]}$ be the unit tensor, where

$$\delta_{i_1 i_2 \dots i_m} = \begin{cases} 1, & \text{if } i_1 = \dots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

Let Q be a nonempty subset of N and $N \setminus Q$ be the complement of Q in N . For a tensor $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathcal{C}^{[m,n]}$, we denote

$$\begin{aligned} R_i(\mathcal{A}) &= \sum_{\substack{i_2, \dots, i_m \in N \\ \delta_{i i_2 \dots i_m} = 0}} |a_{i i_2 \dots i_m}| = \sum_{i_2, \dots, i_m \in N} |a_{i i_2 \dots i_m}| - |a_{i i \dots i}|, \\ N_1 &= \{i \in N : 0 < |a_{i i \dots i}| = R_i(\mathcal{A})\}, \\ N_2 &= \{i \in N : 0 < |a_{i i \dots i}| < R_i(\mathcal{A})\}, \\ N_3 &= \{i \in N : |a_{i i \dots i}| > R_i(\mathcal{A})\}, \\ Q^{m-1} &= \{i_2 i_3 \dots i_m : i_j \in S, j = 2, 3, \dots, m\}, \\ N^{m-1} \setminus Q^{m-1} &= \{i_2 i_3 \dots i_m : i_2 i_3 \dots i_m \in N^{m-1} \text{ and } i_2 i_3 \dots i_m \notin S^{m-1}\}, \\ N_0^{m-1} &= N^{m-1} \setminus (N_2^{m-1} \cup N_3^{m-1}), \\ r &= \max_{i \in N_3} \left\{ \frac{\sum_{i_2 \dots i_m \in N_0^{m-1}} |a_{i i_2 \dots i_m}| + \sum_{i_2 \dots i_m \in N_2^{m-1}} |a_{i i_2 \dots i_m}|}{|a_{i i \dots i}| - \sum_{\substack{i_2 \dots i_m \in N_3^{m-1} \\ \delta_{i i_2 \dots i_m} = 0}} |a_{i i_2 \dots i_m}|} \right\}, \\ P_{i,r}(\mathcal{A}) &= \sum_{i_2 \dots i_m \in N_0^{m-1}} |a_{i i_2 \dots i_m}| + \sum_{i_2 \dots i_m \in N_2^{m-1}} |a_{i i_2 \dots i_m}| + r \sum_{\substack{i_2 \dots i_m \in N_3^{m-1} \\ \delta_{i i_2 \dots i_m} = 0}} |a_{i i_2 \dots i_m}|, \quad \forall i \in N_3, \\ r_1 &= \max_{i \in N_3} \left\{ \frac{\sum_{i_2 \dots i_m \in N_0^{m-1}} |a_{i i_2 \dots i_m}| + \sum_{i_2 \dots i_m \in N_2^{m-1}} |a_{i i_2 \dots i_m}|}{|a_{i i \dots i}| - \sum_{\substack{i_2 \dots i_m \in N_3^{m-1} \\ \delta_{i i_2 \dots i_m} = 0}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \left\{ \frac{P_{j,r}(\mathcal{A})}{|a_{j j \dots j}|} \right\} |a_{i i_2 \dots i_m}|} \right\}, \end{aligned}$$

and

$$\omega_i = \frac{R_i(\mathcal{A})}{R_i(\mathcal{A}) + |a_{i i \dots i}|}, \quad \forall i \in N_2, \quad (3)$$

$$\delta_1 = \max\{\omega_i, r_1\}, \quad \forall i \in N_2, \quad (4)$$

$$P_{i,r_1}(\mathcal{A}) = \sum_{i_2 \dots i_m \in N_0^{m-1}} |a_{i i_2 \dots i_m}| + \sum_{i_2 \dots i_m \in N_2^{m-1}} |a_{i i_2 \dots i_m}| + r_1 \sum_{\substack{i_2 \dots i_m \in N_3^{m-1} \\ \delta_{i i_2 \dots i_m} = 0}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \left\{ \frac{P_{j,r_1}(\mathcal{A})}{|a_{j j \dots j}|} \right\} |a_{i i_2 \dots i_m}|, \quad \forall i \in N_3, \quad (5)$$

$$h_1 = \max_{i \in N_3} \left\{ \frac{\delta_1 \sum_{i_2 \dots i_m \in N_0^{m-1}} |a_{ii_2 \dots i_m}| + \sum_{i_2 \dots i_m \in N_2^{m-1}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \{\omega_j\} |a_{ii_2 \dots i_m}|}{P_{i, r_1}(\mathcal{A}) - \sum_{\substack{i_2 \dots i_m \in N_3^{m-1} \\ \delta_{ii_2 \dots i_m} = 0}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \left\{ \frac{P_{j, r_1}(\mathcal{A})}{|a_{jj \dots j}|} \right\} |a_{ii_2 \dots i_m}|} \right\}. \quad (6)$$

In this paper, we always assume that $N_1 \cup N_2 \neq \emptyset$ and $N_3 \neq \emptyset$. In addition, we also assume that $\mathcal{A}$ satisfies: $a_{ii \dots i} \neq 0$, $R_i(\mathcal{A}) \neq 0$, $\forall i \in N$.

Definition 1. [12] Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in C^{[m, n]}$. $\mathcal{A}$ is called a diagonally dominant tensor if

$$|a_{ii \dots i}| \geq \sum_{\substack{i_2, \dots, i_m \in N \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}|, \quad \forall i \in N. \quad (7)$$

$\mathcal{A}$ is a strictly diagonally dominant tensor if all inequalities in (7) hold.

Definition 2. [12] Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in C^{[m, n]}$. $\mathcal{A}$ is called an $\mathcal{H}$ -tensor if there exists an entrywise positive vector $x = (x_1, x_2, \dots, x_n)^T \in R^n$ such that

$$|a_{ii \dots i}| x_i^{m-1} > \sum_{\substack{i_2, \dots, i_m \in N \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}| x_{i_2} \dots x_{i_m}, \quad \forall i \in N.$$

Definition 3. [17] Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in C^{[m, n]}$, $X = \text{diag}(x_1, x_2, \dots, x_n)$. Denote

$$\mathcal{B} = (b_{i_1 \dots i_m}) = \mathcal{A} X^{m-1}, \quad b_{i_1 i_2 \dots i_m} = a_{i_1 i_2 \dots i_m} x_{i_2} x_{i_3} \dots x_{i_m}, \quad i_j \in N, \quad j \in N.$$

We call $\mathcal{B}$ the product of the tensor $\mathcal{A}$ and the matrix X .

Definition 4. [11] Tensor $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in C^{[m, n]}$ is called reducible if there exists a nonempty proper index subset $I \subset N$ such that

$$a_{i_1 i_2 \dots i_m} = 0, \quad \forall i_1 \in I, \quad \forall i_2, \dots, i_m \notin I.$$

Otherwise, we say $\mathcal{A}$ is irreducible.

Definition 5. [18] Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in C^{[m, n]}$, for $i, j \in N (i \neq j)$, if there exist indices $k_1, k_2, \dots, k_r$ with

$$\sum_{\substack{i_2, \dots, i_m \in N \\ \delta_{k_s i_2 \dots i_m} = 0, k_{s+1} \in \{i_2, \dots, i_m\}}} |a_{k_s i_2 \dots i_m}| \neq 0, \quad s = 0, 1, \dots, r,$$

where $k_0 = i$, $k_{r+1} = j$, we say there is a nonzero element chain from i to j .

Lemma 1. [18] Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in C^{[m, n]}$. If

- (i) $|a_{ii \dots i}| \geq R_i(\mathcal{A})$, $\forall i \in N$,
 - (ii) $J(\mathcal{A}) = \{i \in N : |a_{ii \dots i}| > R_i(\mathcal{A})\} \neq \emptyset$,
 - (iii) for any $i \notin J(\mathcal{A})$, there is a nonzero element chain from i to j such that $j \in J(\mathcal{A})$,
- then $\mathcal{A}$ is an $\mathcal{H}$ -tensor.

Lemma 2. [17] Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in C^{[m, n]}$. If there is a positive diagonal matrix X such that $\mathcal{A} X^{m-1}$ is strictly diagonally dominant, then $\mathcal{A}$ is an $\mathcal{H}$ -tensor.

Lemma 3. [11] Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in C^{[m, n]}$. If $\mathcal{A}$ is irreducible diagonally dominant tensor such that the inequality in (7) holds for at least one i , then $\mathcal{A}$ is an $\mathcal{H}$ -tensor.

3 Criteria for identifying $\mathcal{H}$ -tensors

In this section, we give some new criteria for $\mathcal{H}$ -tensors.

Theorem 1. Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in C^{[m, n]}$ and let $\omega_i, \delta_1, P_{j, r_1}(\mathcal{A}), h_1$ be defined in (3)–(6), respectively. If

$$\begin{aligned} |a_{ii \dots i}| \omega_i &> \delta_1 \sum_{i_2 \dots i_m \in N_0^{m-1}} |a_{ii_2 \dots i_m}| + \sum_{\substack{i_2 \dots i_m \in N_2^{m-1} \\ \delta_{ii_2 \dots i_m} = 0}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \{\omega_j\} |a_{ii_2 \dots i_m}| \\ &+ h_1 \sum_{i_2 \dots i_m \in N_3^{m-1}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \left\{ \frac{P_{j, r_1}(\mathcal{A})}{|a_{jj \dots j}|} \right\} |a_{ii_2 \dots i_m}|, \quad \forall i \in N_2, \end{aligned} \quad (8)$$

and

$$|a_{ii \dots i}| \neq \sum_{\substack{i_2 \dots i_m \in N_0^{m-1} \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}|, \quad \forall i \in N_1 \neq \emptyset \text{ (or } N_1 = \emptyset), \quad (9)$$

then $\mathcal{A}$ is an $\mathcal{H}$ -tensor.

Proof. By the definitions of $r, P_{i, r}(\mathcal{A}), r_1$ and $P_{i, r_1}(\mathcal{A})$, we obtain

$$0 < r_1 \leq r < 1,$$

and

$$r_1 |a_{ii \dots i}| \geq \sum_{i_2 \dots i_m \in N_0^{m-1}} |a_{ii_2 \dots i_m}| + \sum_{i_2 \dots i_m \in N_2^{m-1}} |a_{ii_2 \dots i_m}| + r_1 \sum_{\substack{i_2 \dots i_m \in N_3^{m-1} \\ \delta_{ii_2 \dots i_m} = 0}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \left\{ \frac{P_{j, r_1}(\mathcal{A})}{|a_{jj \dots j}|} \right\} |a_{ii_2 \dots i_m}|, \quad \forall i \in N_3,$$

that is, $P_{i, r_1}(\mathcal{A}) \leq r_1 |a_{ii \dots i}|$. Then,

$$0 < \frac{P_{i, r_1}(\mathcal{A})}{|a_{ii \dots i}|} \leq r_1 \leq r < 1, \quad \forall i \in N_3.$$

By the definitions of ω_i and δ_1 , we get that for each $i \in N_3$,

$$\begin{aligned} &\frac{\delta_1 \sum_{i_2 \dots i_m \in N_0^{m-1}} |a_{ii_2 \dots i_m}| + \sum_{\substack{i_2 \dots i_m \in N_2^{m-1} \\ \delta_{ii_2 \dots i_m} = 0}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \{\omega_j\} |a_{ii_2 \dots i_m}|}{P_{i, r_1}(\mathcal{A}) - \sum_{\substack{i_2 \dots i_m \in N_3^{m-1} \\ \delta_{ii_2 \dots i_m} = 0}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \left\{ \frac{P_{j, r_1}(\mathcal{A})}{|a_{jj \dots j}|} \right\} |a_{ii_2 \dots i_m}|} \\ &\leq \frac{\delta_1 \left(P_{i, r_1}(\mathcal{A}) - r_1 \sum_{\substack{i_2 \dots i_m \in N_3^{m-1} \\ \delta_{ii_2 \dots i_m} = 0}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \left\{ \frac{P_{j, r_1}(\mathcal{A})}{|a_{jj \dots j}|} \right\} |a_{ii_2 \dots i_m}| \right)}{P_{i, r_1}(\mathcal{A}) - \sum_{\substack{i_2 \dots i_m \in N_3^{m-1} \\ \delta_{ii_2 \dots i_m} = 0}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \left\{ \frac{P_{j, r_1}(\mathcal{A})}{|a_{jj \dots j}|} \right\} |a_{ii_2 \dots i_m}|} < 1. \end{aligned}$$

By the definition of h_1 , for $\forall i \in N_3$, we obtain $0 < h_1 < 1$ and

$$\begin{aligned} h_1 P_{i, r_1}(\mathcal{A}) &> \delta_1 \sum_{i_2 \dots i_m \in N_0^{m-1}} |a_{ii_2 \dots i_m}| + \sum_{i_2 \dots i_m \in N_2^{m-1}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \{\omega_j\} |a_{ii_2 \dots i_m}| \\ &+ h_1 \sum_{\substack{i_2 \dots i_m \in N_3^{m-1} \\ \delta_{ii_2 \dots i_m} = 0}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \left\{ \frac{P_{j, r_1}(\mathcal{A})}{|a_{jj \dots j}|} \right\} |a_{ii_2 \dots i_m}|. \end{aligned} \quad (10)$$

Let

$$M_i = \frac{1}{\sum_{i_2 \dots i_m \in N_3^{m-1}} |a_{ii_2 \dots i_m}|} \left(|a_{ii \dots i}| \omega_i - \delta_1 \sum_{i_2 \dots i_m \in N_0^{m-1}} |a_{ii_2 \dots i_m}| - \sum_{\substack{i_2 \dots i_m \in N_2^{m-1} \\ \delta_{ii_2 \dots i_m} = 0}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \{\omega_j\} |a_{ii_2 \dots i_m}| \right. \\ \left. - h_1 \sum_{i_2 \dots i_m \in N_3^{m-1}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \frac{P_{j, r_1}(\mathcal{A})}{|a_{jj \dots j}|} |a_{ii_2 \dots i_m}| \right), \quad \forall i \in N_2. \quad (11)$$

If $\sum_{i_2 i_3 \dots i_m \in N_3^{m-1}} |a_{ii_2 \dots i_m}| = 0$, we denote $M_i = +\infty$. From inequalities (8), (9) and (11), we obtain $M_i > 0$ ($\forall i \in N_2$).

Hence, by $0 < \frac{h_1 P_{i, r_1}(\mathcal{A})}{|a_{ii \dots i}|} < 1$ ($\forall i \in N_3$), there exists a positive number ε such that

$$0 < \varepsilon < \min_{i \in N_2} \{M_i\} \leq +\infty, \quad 0 < \max_{i \in N_3} \left\{ \frac{h_1 P_{i, r_1}(\mathcal{A})}{|a_{ii \dots i}|} + \varepsilon \right\} < \delta_1 < 1. \quad (12)$$

Set $X = \text{diag}(x_1, x_2, \dots, x_n)$, where

$$x_i = \begin{cases} (\delta_1)^{\frac{1}{m-1}}, & i \in N_1, \\ (\omega_i)^{\frac{1}{m-1}}, & i \in N_2, \\ \left(\frac{h_1 P_{i, r_1}(\mathcal{A})}{|a_{ii \dots i}|} + \varepsilon \right)^{\frac{1}{m-1}}, & i \in N_3. \end{cases}$$

Let $\mathcal{B} = (b_{i_1 i_2 \dots i_m}) = \mathcal{A} X^{m-1}$. Now, we prove that $\mathcal{B}$ is a diagonally dominant matrix.

For $i \in N_1$, by inequalities (9) and (12), we obtain

$$R_i(\mathcal{B}) = \sum_{\substack{i_2 i_3 \dots i_m \in N_0^{m-1} \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}| x_{i_2} \dots x_{i_m} + \sum_{i_2 i_3 \dots i_m \in N_2^{m-1}} |a_{ii_2 \dots i_m}| (\omega_{i_2})^{\frac{1}{m-1}} \dots (\omega_{i_m})^{\frac{1}{m-1}} \\ + \sum_{i_2 i_3 \dots i_m \in N_3^{m-1}} |a_{ii_2 \dots i_m}| \left(\varepsilon + \frac{h_1 P_{i_2, r_1}(\mathcal{A})}{|a_{i_2 i_2 \dots i_2}|} \right)^{\frac{1}{m-1}} \dots \left(\varepsilon + \frac{h_1 P_{i_m, r_1}(\mathcal{A})}{|a_{i_m i_m \dots i_m}|} \right)^{\frac{1}{m-1}} \\ \leq \delta_1 \sum_{\substack{i_2 i_3 \dots i_m \in N_0^{m-1} \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}| + \sum_{i_2 i_3 \dots i_m \in N_2^{m-1}} \max_{j \in \{i_2, \dots, i_m\}} \{\omega_j\} |a_{ii_2 \dots i_m}| \\ + \sum_{i_2 i_3 \dots i_m \in N_3^{m-1}} \left(\varepsilon + \max_{j \in \{i_2, \dots, i_m\}} \frac{h_1 P_{j, r_1}(\mathcal{A})}{|a_{jj \dots j}|} \right) |a_{ii_2 \dots i_m}| \leq |a_{ii \dots i}| \delta_1 = |b_{ii \dots i}|. \quad (13)$$

For $i \in N_2$, by inequalities (11) and (12), we have

$$R_i(\mathcal{B}) = \sum_{i_2 i_3 \dots i_m \in N_0^{m-1}} |a_{ii_2 \dots i_m}| x_{i_2} \dots x_{i_m} + \sum_{\substack{i_2 i_3 \dots i_m \in N_2^{m-1} \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}| (\omega_{i_2})^{\frac{1}{m-1}} \dots (\omega_{i_m})^{\frac{1}{m-1}} \\ + \sum_{i_2 i_3 \dots i_m \in N_3^{m-1}} |a_{ii_2 \dots i_m}| \left(\varepsilon + \frac{h_1 P_{i_2, r_1}(\mathcal{A})}{|a_{i_2 i_2 \dots i_2}|} \right)^{\frac{1}{m-1}} \dots \left(\varepsilon + \frac{h_1 P_{i_m, r_1}(\mathcal{A})}{|a_{i_m i_m \dots i_m}|} \right)^{\frac{1}{m-1}} \\ \leq \delta_1 \sum_{i_2 i_3 \dots i_m \in N_0^{m-1}} |a_{ii_2 \dots i_m}| + \sum_{\substack{i_2 i_3 \dots i_m \in N_2^{m-1} \\ \delta_{ii_2 \dots i_m} = 0}} \max_{j \in \{i_2, \dots, i_m\}} \{\omega_j\} |a_{ii_2 \dots i_m}| \\ + \sum_{i_2 i_3 \dots i_m \in N_3^{m-1}} \left(\varepsilon + \max_{j \in \{i_2, \dots, i_m\}} \frac{h_1 P_{j, r_1}(\mathcal{A})}{|a_{jj \dots j}|} \right) |a_{ii_2 \dots i_m}| < |a_{ii \dots i}| \omega_i = |b_{ii \dots i}|. \quad (14)$$

Finally, for $i \in N_3$, combining (10) and (12), we have

$$\begin{aligned}
 R_i(\mathcal{B}) &= \sum_{i_2 i_3 \dots i_m \in N_0^{m-1}} |a_{ii_2 \dots i_m}| x_{i_2} \dots x_{i_m} + \sum_{i_2 i_3 \dots i_m \in N_2^{m-1}} |a_{ii_2 \dots i_m}| (\omega_{i_2})^{\frac{1}{m-1}} \dots (\omega_{i_m})^{\frac{1}{m-1}} \\
 &\quad + \sum_{\substack{i_2 i_3 \dots i_m \in N_3^{m-1} \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}| \left(\varepsilon + \frac{h_1 P_{i_2, r_1}(\mathcal{A})}{|a_{i_2 i_2 \dots i_2}|} \right)^{\frac{1}{m-1}} \dots \left(\varepsilon + \frac{h_1 P_{i_m, r_1}(\mathcal{A})}{|a_{i_m i_m \dots i_m}|} \right)^{\frac{1}{m-1}} \\
 &\leq \delta_1 \sum_{i_2 i_3 \dots i_m \in N_0^{m-1}} |a_{ii_2 \dots i_m}| + \sum_{i_2 i_3 \dots i_m \in N_2^{m-1}} \max_{j \in \{i_2, \dots, i_m\}} \{\omega_j\} |a_{ii_2 \dots i_m}| \\
 &\quad + \sum_{\substack{i_2 i_3 \dots i_m \in N_3^{m-1} \\ \delta_{ii_2 \dots i_m} = 0}} \left(\varepsilon + \max_{j \in \{i_2, \dots, i_m\}} \frac{h_1 P_{j, r_1}(\mathcal{A})}{|a_{jj \dots j}|} \right) |a_{ii_2 \dots i_m}| \\
 &< \varepsilon \sum_{\substack{i_2 i_3 \dots i_m \in N_3^{m-1} \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}| + h_1 P_{i, r_1}(\mathcal{A}) \\
 &< \varepsilon |a_{ii \dots i}| + h_1 P_{i, r_1}(\mathcal{A}) = |b_{ii \dots i}|.
 \end{aligned} \tag{15}$$

Therefore, from inequalities (13)–(15), we obtain

$$|b_{ii \dots i}| \geq R_i(\mathcal{B}) (\forall i \in N_1), \quad |b_{ii \dots i}| > R_i(\mathcal{B}) (\forall i \in N_2 \cup N_3).$$

Combining (9), for all $i \in N_1$, there exists $i_2 \dots i_m \in N_2^{m-1} \cup N_3^{m-1}$ such that $a_{ii_2 \dots i_m} \neq 0$. Since $\mathcal{B} = (b_{ii_2 \dots i_m}) = \mathcal{A} \mathcal{X}^{m-1}$, then for $\forall i \in N_1$, there exists $i_2 \dots i_m \in N_2^{m-1} \cup N_3^{m-1}$ such that $b_{ii_2 \dots i_m} \neq 0$. Hence, $\mathcal{B}$ is a diagonally dominant tensor with nonzero elements chain, and by Lemma 1, $\mathcal{B}$ is an $\mathcal{H}$ -tensor. Furthermore, by Lemma 2, $\mathcal{A}$ is an $\mathcal{H}$ -tensor. $\square$

By a similar proof to that of Theorem 1 and Lemmas 1, 2 and 3, we can obtain the following theorem.

Theorem 2. Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in C^{[m, n]}$ with

$$\begin{aligned}
 |a_{ii \dots i}| \omega_i &\geq \delta_1 \sum_{i_2 \dots i_m \in N_0^{m-1}} |a_{ii_2 \dots i_m}| + \sum_{\substack{i_2 \dots i_m \in N_2^{m-1} \\ \delta_{ii_2 \dots i_m} = 0}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \{\omega_j\} |a_{ii_2 \dots i_m}| \\
 &\quad + h_1 \sum_{i_2 \dots i_m \in N_3^{m-1}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \left\{ \frac{P_{j, r_1}(\mathcal{A})}{|a_{jj \dots j}|} \right\} |a_{ii_2 \dots i_m}|, \quad \forall i \in N_2.
 \end{aligned} \tag{16}$$

If $\mathcal{A}$ satisfies one of the following conditions, then $\mathcal{A}$ is an $\mathcal{H}$ -tensor,

- (i) $\mathcal{A}$ is irreducible and $N_1 \neq \emptyset$ or a strict inequality holds for at least one $i \in N_2$ in inequality (12);
- (ii) $(N_1 \cup N_2) \setminus (\overline{N}_1 \cup \overline{N}_2) \neq \emptyset$, and for all $i \in (\overline{N}_1 \cup \overline{N}_2 \cup N_3)$, there exists a nonzero element chain from i to j such that $j \in (N_1 \cup N_2) \setminus (\overline{N}_1 \cup \overline{N}_2)$, where

$$\begin{aligned}
 \overline{N}_1 &= \left\{ i \in N_1 : |a_{ii \dots i}| = \sum_{\substack{i_2 \dots i_m \in N_0^{m-1} \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}| \right\}, \\
 \overline{N}_2 &= \left\{ i \in N_2 : |a_{ii \dots i}| \omega_i = \delta_1 \sum_{i_2 \dots i_m \in N_0^{m-1}} |a_{ii_2 \dots i_m}| + \sum_{\substack{i_2 \dots i_m \in N_2^{m-1} \\ \delta_{ii_2 \dots i_m} = 0}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \{\omega_j\} |a_{ii_2 \dots i_m}| \right. \\
 &\quad \left. + h_1 \sum_{i_2 \dots i_m \in N_3^{m-1}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \left\{ \frac{P_{j, r_1}(\mathcal{A})}{|a_{jj \dots j}|} \right\} |a_{ii_2 \dots i_m}| \right\}.
 \end{aligned}$$

Example 1. Consider a tensor $\mathcal{A} = (a_{ijk}) \in C^{[3,3]}$ defined as follows:

$$\mathcal{A} = [A(1, :, :), A(2, :, :), A(3, :, :)],$$

$$A(1, :, :) = \begin{pmatrix} 12 & 1 & 0 \\ 1 & 6 & 0 \\ 1 & 0 & 12 \end{pmatrix}, \quad A(2, :, :) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 10 & 2 \\ 0 & 2 & 2 \end{pmatrix}, \quad A(3, :, :) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

Obviously,

$$|a_{111}| = 12, \quad R_1(\mathcal{A}) = 21, \quad |a_{222}| = 10, \quad R_2(\mathcal{A}) = 8, \quad |a_{333}| = 8, \quad R_3(\mathcal{A}) = 2.$$

So, $N_1 = \emptyset$, $N_2 = \{1\}$, $N_3 = \{2, 3\}$ and

$$\omega_1 = \frac{7}{11}, \quad r = \frac{1}{4}, \quad P_{2,r}(\mathcal{A}) = \frac{5}{2}, \quad P_{3,r}(\mathcal{A}) = \frac{5}{4}, \quad r_1 = \frac{4}{31}, \quad \delta_1 = \frac{7}{11},$$

$$P_{2,r_1}(\mathcal{A}) = \frac{145}{124}, \quad P_{3,r_1}(\mathcal{A}) = \frac{32}{31}, \quad h_1(\mathcal{A}) = \frac{1736}{2497}.$$

Since

$$\begin{aligned} & \delta_1 \sum_{jk \in N_0^2} |a_{ijk}| + \sum_{\substack{jk \in N_2^2 \\ \delta_{ijk}=0}} \max_{l \in \{j,k\}} \{\omega_l\} |a_{ijk}| + h_1 \sum_{jk \in N_3^2} \max_{l \in \{j,k\}} \left\{ \frac{P_{l,r_1}(\mathcal{A})}{|a_{lll}|} \right\} |a_{ijk}| \\ &= \frac{7}{11}(1 + 1 + 1) + \frac{1736}{2497} \times \left(6 \times \frac{145}{124} \times \frac{1}{10} + 12 \times \frac{32}{31} \times \frac{1}{8} \right) = \frac{8673}{2497} \\ &< \frac{84}{11} = |a_{111}| \omega_1, \end{aligned}$$

we know that $\mathcal{A}$ satisfies the conditions of Theorem 1, then $\mathcal{A}$ is an $\mathcal{H}$ -tensor. But

$$\sum_{\substack{jk \in N_0^2 \setminus N_3^2 \\ \delta_{ijk}=0}} |a_{ijk}| + \sum_{jk \in N_3^2} \max_{l \in \{j,k\}} \left\{ \frac{R_l(\mathcal{A})}{|a_{lll}|} \right\} |a_{ijk}| = (1 + 1 + 1) + \frac{7}{10} \times (6 + 12) = \frac{78}{5} > 12 = |a_{111}|.$$

So, $\mathcal{A}$ does not satisfy the conditions of Theorem 3 in [13].

4 An iterative scheme for identifying $\mathcal{H}$ -tensors

In this section, we provide an iterative algorithm for identifying $\mathcal{H}$ -tensors on the basis of the results in Section 3. The numerical experiment will be performed via Matlab R2015 on a 2.2 GHz Intel computer with a 6-core i7-8750H processor.

Algorithm 4.1.

INPUT: A tensor $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in C^{[m,n]}$ with $a_{ii \dots i} \neq 0$ for all $i \in N$.

OUTPUT: A positive diagonal matrix $X = X^{(0)} X^{(1)} \dots X^{(p-1)}$ if $\mathcal{A}$ is an $\mathcal{H}$ -tensor.

Step 1. If $N_3(\mathcal{A}) = \emptyset$, then $\mathcal{A}$ is not an $\mathcal{H}$ -tensor, stop; otherwise,

Step 2. Set $\mathcal{A}^{(0)} = \mathcal{A}$, $X^{(0)} = I$, $p = 1$.

Step 3. Compute $\mathcal{A}^{(p)} = \mathcal{A}^{(p-1)} (X^{(p-1)})^{m-1} = (a_{i_1 i_2 \dots i_m}^{(p)})$.

Step 4. If $N_2(\mathcal{A}^{(p)}) = \emptyset$, then $\mathcal{A}$ is an $\mathcal{H}$ -tensor, stop; otherwise,

Step 5. If $N_3(\mathcal{A}^{(p)}) = \emptyset$, then $\mathcal{A}$ is not an $\mathcal{H}$ -tensor, stop; otherwise,

Step 6. Compute $\delta_1^{(p)}$, $\omega_i^{(p)}$, $h_1^{(p)}$, $\delta_1^{(p)}$, $P_{j,r}(\mathcal{A}^{(p)})$.

Step 7. For $\forall i \in N_2(\mathcal{A}^{(p)})$, if $\mathcal{A}^{(p)}$ satisfies

$$\begin{aligned}
|a_{ii \dots i}^{(p)}| \omega_i^{(p)} &> \delta_1^{(p)} \sum_{i_2 \dots i_m \in N_0^{m-1}(\mathcal{A}^{(p)})} |a_{ii_2 \dots i_m}^{(p)}| \\
&+ \sum_{\substack{i_2 \dots i_m \in N_2^{m-1}(\mathcal{A}^{(p)}) \\ \delta_{ii_2 \dots i_m} = 0}} \max_{j \in \{i_2, i_3, \dots, i_m\}} \{\omega_j^{(p)}\} |a_{ii_2 \dots i_m}^{(p)}| \\
&+ h_1^{(p)} \sum_{i_2 \dots i_m \in N_3^{m-1}(\mathcal{A}^{(p)})} \max_{j \in \{i_2, i_3, \dots, i_m\}} \left\{ \frac{P_{j, i_1}(\mathcal{A}^{(p)})}{|a_{jj \dots j}^{(p)}|} \right\} |a_{ii_2 \dots i_m}^{(p)}|,
\end{aligned}$$

and

$$|a_{ii \dots i}^{(p)}| \neq \sum_{\substack{i_2 \dots i_m \in N_0^{m-1}(\mathcal{A}^{(p)}) \\ \delta_{ii_2 \dots i_m} = 0}} |a_{ii_2 \dots i_m}^{(p)}|, \quad \forall i \in N_1(\mathcal{A}^{(p)}) \neq \emptyset \text{ (or } N_1(\mathcal{A}^{(p)}) = \emptyset),$$

then $\mathcal{A}$ is an $\mathcal{H}$ -tensor, stop; otherwise,

Step 8. Set $X^{(p)} = \text{diag}(x_1^{(p)}, x_2^{(p)}, \dots, x_n^{(p)})$, compute

$$x_i^{(p)} = \begin{cases} (\delta_1^{(p)})^{\frac{1}{m-1}}, & \text{if } i \in N_1(\mathcal{A}^{(p)}), \\ (\omega_i^{(p)})^{\frac{1}{m-1}}, & \text{if } i \in N_2(\mathcal{A}^{(p)}), \\ \left(\frac{h_1^{(p)} P_{i, i_1}(\mathcal{A}^{(p)})}{|a_{ii \dots i}^{(p)}|} \right)^{\frac{1}{m-1}}, & \text{if } i \in N_3(\mathcal{A}^{(p)}). \end{cases}$$

Step 9. Set $X^{(p)} = \text{diag}(x_1^{(p)}, x_2^{(p)}, \dots, x_n^{(p)})$, $p = p + 1$; go to step 3.

The theoretical basis for the functionality of Algorithm 4.1 as criteria for $\mathcal{H}$ -tensor is provided by the following theorem.

Theorem 3. Let $\mathcal{A} = (a_{i_1 \dots i_m}) \in \mathbb{C}^{[m, n]}$. If $\mathcal{A}$ is an $\mathcal{H}$ -tensor, then Algorithm 4.1 terminates after a finite number of iterations by producing a strictly diagonally dominant tensor.

Proof. Without loss of generality, we assume that $\mathcal{A}$ is a non-negative tensor. Suppose, on the contrary, that Algorithm 4.1 produces the infinite sequence $\{\mathcal{A}^{(p)}\}$ with

$$\mathcal{A}^{(p)} = \mathcal{A}^{(0)} (X^{(1)})^{m-1} (X^{(2)})^{m-1} \dots (X^{(p-1)})^{m-1}.$$

From the fact that each diagonal entry of positive diagonal matrix $X^{(p)}$ is less than 1, we have

$$\mathcal{A} = \mathcal{A}^{(0)} = \mathcal{A}^{(1)} \geq \dots \geq \mathcal{A}^{(p)} \geq \dots \geq 0.$$

Hence, the sequence $\{\mathcal{A}^{(p)}\}$ has a limitation,

$$\mathcal{B} = \lim_{p \rightarrow +\infty} \mathcal{A}^{(p)} \geq 0,$$

where $\mathcal{B} = \mathcal{A} X^{m-1}$, $X = X^{(1)} X^{(2)} \dots X^{(p)} \dots$ is a positive diagonal matrix.

Next, we prove that

$$\lim_{p \rightarrow +\infty} N_3(\mathcal{A}^{(p)}) = N_3(\mathcal{B}) = \emptyset.$$

In fact, suppose that $\lim_{p \rightarrow +\infty} N_3(\mathcal{A}^{(p)}) \neq \emptyset$, then

$$1 - \delta_1^{(p)} > 0, \quad 1 - \omega_i^{(p)} > 0, \quad 1 - \frac{h_1^{(p)} P_{i, i_1}(\mathcal{A}^{(p)})}{|a_{ii \dots i}^{(p)}|} > 0.$$

Therefore, there exists $i \in N_3(\mathcal{A}^{(p)})$ and positive numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3$ such that

$$a_{ii \dots i}^{(p)} (1 - \delta_1^{(p)}) > \varepsilon_1, \quad a_{ii \dots i}^{(p)} (1 - \omega_i^{(p)}) > \varepsilon_2, \quad a_{ii \dots i}^{(p)} \left(1 - \frac{h_1^{(p)} P_{i, i_1}(\mathcal{A}^{(p)})}{|a_{ii \dots i}^{(p)}|} \right) > \varepsilon_3.$$

Let $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. By Algorithm 4.1, for any $i \in N$, we obtain

$$\begin{aligned} 0 &< a_{ii \dots i}^{(p+1)} = a_{ii \dots i}^{(p)} \delta_1^{(p)} < a_{ii \dots i}^{(p)} - \varepsilon_1 < a_{ii \dots i}^{(p)} - \varepsilon_0, \\ 0 &< a_{ii \dots i}^{(p+1)} = a_{ii \dots i}^{(p)} \omega_i^{(p)} < a_{ii \dots i}^{(p)} - \varepsilon_2 < a_{ii \dots i}^{(p)} - \varepsilon_0, \\ 0 &< a_{ii \dots i}^{(p+1)} = a_{ii \dots i}^{(p)} \frac{h_1^{(p)} P_{i, \eta_1}(\mathcal{A}^{(p)})}{|a_{ii \dots i}^{(p)}|} < a_{ii \dots i}^{(p)} - \varepsilon_3 < a_{ii \dots i}^{(p)} - \varepsilon_0. \end{aligned}$$

Hence,

$$a_{ii \dots i}^{(0)} = a_{ii \dots i}^{(1)} > a_{ii \dots i}^{(2)} + \varepsilon_0 > \dots > a_{ii \dots i}^{(p)} + (p-1)\varepsilon_0.$$

When $p \rightarrow +\infty$, one has $a_{ii \dots i}^{(0)} \rightarrow +\infty$. A contradiction arrives which means that

$$\lim_{p \rightarrow +\infty} N_3(\mathcal{A}^{(p)}) = N_3(\mathcal{B}) = \emptyset.$$

That is,

$$|b_{ii \dots i}| \leq R_i(\mathcal{B}), \quad \forall i \in N.$$

So, $\mathcal{B}$ is not an $\mathcal{H}$ -tensor [11]. On the other hand, since $\mathcal{A}$ is an $\mathcal{H}$ -tensor, then there exists a positive diagonal matrix D such that $\mathcal{A}D^{m-1} = \mathcal{B}(X^{-1}D)^{m-1}$ is strictly diagonally dominant. Then, $\mathcal{B}$ is an $\mathcal{H}$ -tensor. We also obtain a contradiction, then our hypothesis does not hold. The proof is completed. $\square$

Example 2. [13] Randomly generate 100 m th-order n -dimensional tensors such that the elements of each tensor satisfying

$$a_{i_1 i_2 \dots i_m} \in \begin{cases} (-n^m \times 0.6, n^m \times 0.6), & \text{if } i_1 = \dots = i_m, \\ (-1, 1), & \text{otherwise.} \end{cases}$$

The numerical results of Algorithm 4.1 are shown in Table 1.

The numerical results of Algorithm 4.1 are shown in Table 1, where “ p_1 ” denotes the number of tensors that are $\mathcal{H}$ -tensor, “ p_2 ” denotes the number of tensors that are not $\mathcal{H}$ -tensor, “ p_3 ” denotes the number of tensors that whether they are $\mathcal{H}$ -tensor that are not checkable by using Algorithm 4.1, “ RT ” denotes the running time when algorithm terminates.

Table 1: Numerical results of Example 2

$m(\text{order})$	$n(\text{dimension})$	p_1	p_2	p_3	RT
4	6	44	56	0	25.5520
4	7	52	48	0	32.2855
4	8	57	43	0	40.4317
4	9	61	39	0	42.3266
4	10	73	27	0	39.4375
5	6	66	34	0	55.07456
5	7	81	19	0	25.2355
5	8	86	14	0	98.4053
5	9	71	29	0	30.1723
5	10	71	29	0	58.3400
6	6	81	19	0	35.7658
6	7	83	17	0	105.6540
6	8	92	8	0	34.6045
6	9	91	9	0	30.4036
6	10	93	7	0	42.3576

5 An application

Based on the criteria for $\mathcal{H}$ -tensors, we present some new conditions for the positive definiteness of even-order real symmetric tensors. First, we give the following lemma.

Lemma 4. [11] *Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in R^{[m, n]}$ be an even-order real symmetric tensor with $a_{ii \dots i} > 0$ ($\forall i \in N$). If $\mathcal{A}$ is an $\mathcal{H}$ -tensor, then $\mathcal{A}$ is positive definite.*

Combining Theorems 1, 2 and Lemma 4, we have the following result.

Theorem 4. *Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in R^{[m, n]}$ be an even-order real symmetric tensor, and $a_{ii \dots i} > 0$ ($\forall i \in N$). If $\mathcal{A}$ satisfies one of the following conditions:*

- (i) *all the conditions of Theorem 1,*
 - (ii) *all the conditions of Theorem 2,*
- then $\mathcal{A}$ is a positive definite tensor.*

Example 3. Consider polynomial $f(x) = \mathcal{A}x^6$ with $\mathcal{A}$ being a 6-order 6-dimensional real symmetric tensor with entries

$$\begin{aligned} a_{111111} &= 4, & a_{222222} &= 16, & a_{333333} &= 33, & a_{444444} &= 16, & a_{555555} &= 2, & a_{666666} &= 3, \\ a_{122222} &= a_{212222} = a_{221222} = a_{222122} = a_{222212} = a_{222221} &= -1, \\ a_{133333} &= a_{313333} = a_{331333} = a_{333133} = a_{333313} = a_{333331} &= -2, \\ a_{144444} &= a_{414444} = a_{441444} = a_{444144} = a_{444414} = a_{444441} &= -1, \\ a_{233333} &= a_{323333} = a_{332333} = a_{333233} = a_{333323} = a_{333332} &= -2, \\ a_{244444} &= a_{424444} = a_{442444} = a_{444244} = a_{444424} = a_{444442} &= -1, \\ a_{344444} &= a_{434444} = a_{443444} = a_{444344} = a_{444434} = a_{444443} &= -1, \\ a_{222333} &= a_{223233} = a_{223323} = a_{223332} = a_{232233} = a_{232323} = a_{232332} &= -1, \\ a_{233223} &= a_{233232} = a_{233322} = a_{333222} = a_{333223} = a_{333232} = a_{333223} &= -1, \\ a_{323322} &= a_{323232} = a_{323223} = a_{322332} = a_{322323} = a_{322233} &= -1, \end{aligned}$$

and other $a_{i_1 i_2 i_3 i_4 i_5 i_6} = 0$. By calculations, one has

$$a_{222222} = 16 < 18 = R_2(\mathcal{A}),$$

and

$$a_{444444} (a_{222222} - R_2(\mathcal{A}) + |a_{244444}|) = -16 < 15 = R_4(\mathcal{A}) |a_{244444}|.$$

Then, $\mathcal{A}$ is not a strictly diagonally dominated tensor or a quasi-doubly strictly diagonally dominant tensor. Hence, we cannot identify the positive definiteness of $\mathcal{A}$ by Theorem 3 in [19] and Theorem 4 in [20]. However, by using Algorithm 4.1, we obtain that $\mathcal{A}$ is an $\mathcal{H}$ -tensor. It follows from Theorem 4 that $\mathcal{A}$ is positive definite, that is, the $f(x)$ is positive definite.

6 Conclusion

In this paper, we proposed some implementable criteria for identifying $\mathcal{H}$ -tensors, which are used to identify the positive definiteness of an even degree homogeneous polynomial $f(x) \equiv \mathcal{A}x^m$. We also provided a non-parameter-involved iterative scheme for identifying $\mathcal{H}$ -tensors, which can stop within finite steps.

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