Research Article

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Inviscid, zero Froude number limit of the viscous shallow water system

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Abstract: In this paper, we study the inviscid and zero Froude number limits of the viscous shallow water system. We prove that the limit system is represented by the incompressible Euler equations on the whole space. Furthermore, the rate of convergence is also obtained.

Keywords: viscous shallow water equations, incompressible Euler equations, low Froude number limit, inviscid limit

MSC 2020: 35B25, 35Q35, 76B15

1 Introduction

We know that scale analysis provides a valuable insight into the behaviour of complex fluid systems in the regime, where some of the characteristic dimensionless parameters become small or infinitely large. In this paper, we study the following scaled two-dimensional viscous shallow water equations that are also called the Saint-Venant equations among the French scientific community:

$$\partial_t h^\varepsilon + \operatorname{div}(h^\varepsilon u^\varepsilon) = 0, \quad (1.1)$$

$$\partial_t(h^\varepsilon u^\varepsilon) + \operatorname{div}(h^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \frac{1}{Fr^2} h^\varepsilon \nabla h^\varepsilon - \nu \nabla \cdot (h^\varepsilon \nabla u^\varepsilon) = 0, \quad (1.2)$$

$$h^\varepsilon|_{t=0} = h_{0,\varepsilon}(x), \quad u^\varepsilon|_{t=0} = u_{0,\varepsilon}(x), \quad (1.3)$$

where $h^\varepsilon = h^\varepsilon(x, t)$ is the height of the fluid surface, $u^\varepsilon = u^\varepsilon(x, t)$ is the horizontal velocity field, $x = (x_1, x_2) \in \mathbb{R}^2$, $0 < \nu < 1$ is the viscous coefficient, and $0 < Fr \ll 1$ is the Froude number that measures the inverse pressure forcing and brings in fast oscillations. The strictly positive function $h_{0,\varepsilon}$ describes the bottom topography. Moreover, the energy for the system (1.1)–(1.3) can be defined as follows:

$$E^\varepsilon(t) = \frac{1}{2} \int_{\mathbb{R}^2} \left[h^\varepsilon |u^\varepsilon|^2 + \frac{1}{Fr^2} (h^\varepsilon - 1)^2 \right] dx, \quad (1.4)$$

which satisfies

$$E^\varepsilon(t) + \nu \int_0^t \int_{\mathbb{R}^2} h^\varepsilon |\nabla u^\varepsilon|^2 dx d\tau = E^\varepsilon(0), \quad \frac{d}{dt} E^\varepsilon(t) + \nu \int_{\mathbb{R}^2} h^\varepsilon |\nabla u^\varepsilon|^2 dx = 0, \quad (1.5)$$

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with

$$E^\varepsilon(0) = \frac{1}{2} \int_{\mathbb{R}^2} \left[h_{0,\varepsilon} |u_{0,\varepsilon}|^2 + \frac{1}{Fr^2} (h_{0,\varepsilon} - 1)^2 \right] dx \leq C,$$

where C is a positive constant independent of ε .

We will study the case when $Fr = \varepsilon$ and $\nu = \varepsilon$. Our aim is to identify the viscous shallow water equations (1.1) and (1.2) in the limit $\varepsilon \rightarrow 0$, meaning the inviscid, zero Froude number limit. More precisely, if we set $\varepsilon = 0$ in the system (1.1)–(1.2), then $h^\varepsilon = 1$. Thus, the above system will become the incompressible Euler equations (if ν , the limit of u^ε , exists)

$$\operatorname{div} v = 0, \quad (1.6)$$

$$\partial_t v + v \cdot \nabla v + \nabla \pi = 0. \quad (1.7)$$

The question of the singular asymptotic limits, such as incompressible, low Froude number limit, in fluid mechanics has received considerable attention. In [1], Cheng proved the low Froude number limit and convergence rates of compressible Euler and rotating shallow water equations, ill-prepared initial data towards their incompressible counterparts. In [2], Wu performed the mathematical derivation of the rotating lake equations from the classical solution of the rotating shallow water and Euler equations via the low Froude number limit. In this present paper, we consider the viscous version. The objective of this paper is to prove the low Froude number limit of the shallow water system (1.1)–(1.2). We shall apply the so-called relative-entropy method, first introduced by Brenier [3] to prove the convergence of the Vlasov-Poisson system to the incompressible Euler equations. The method has also been applied to prove various singular limits of the other equations, e.g., [4–8] and to study weak-strong uniqueness problem, e.g., [9,10]. For the singular limit problems of the shallow water model, we also refer to [11,12].

For any vector field ϕ , we denote by $Q\phi$ and $P\phi$, respectively, the gradient part of ϕ and the divergence-free part of ϕ , namely,

$$Q\phi = \nabla \Delta^{-1}(\operatorname{div} \phi), \quad P\phi = \phi - Q\phi.$$

The rest of this paper is organized as follows. In the next section, we state some useful known results and our main results. Finally, Section 3 is devoted to the proof of our main result.

2 Main results

For this, we first recall the following classical result on the existence of sufficiently regular solutions of the incompressible Euler system (1.6)–(1.7) with the initial data $v(0) = v_0$.

Proposition 2.1. [13] *Let $v_0 \in \bigcap_{s>0} H^s(\mathbb{R}^2) \subset C^\infty(\mathbb{R}^2)$ satisfying $\operatorname{div} v_0 = 0$. Then, for any $T > 0$, the incompressible Euler system (1.6)–(1.7) exists a unique classical solution v satisfying*

$$v, \pi \in \bigcap_{s>0} C([0, T]; H^s(\mathbb{R}^2)). \quad (2.1)$$

For a fixed $\varepsilon > 0$, Liu and Yin [14] proved the global existence of the classical solution of the shallow water system (1.1)–(1.2) with small initial data in $H^s(\mathbb{R}^2)$ for $s > 1$.

Proposition 2.2. *Let $h_{0,\varepsilon} - 1, u_{0,\varepsilon} \in H^s(\mathbb{R}^2)$, $s > 1$. Then, there exist a positive time T_ε and a unique solution $(h^\varepsilon, u^\varepsilon)$ of Cauchy problem (1.1)–(1.3) such that*

$$(h^\varepsilon - 1, u^\varepsilon) \in L^\infty([0, T_\varepsilon]; H^s(\mathbb{R}^2)), u^\varepsilon \in L^2([0, T_\varepsilon]; H^{s+1}(\mathbb{R}^2)).$$

Furthermore, there exists a constant M such that if

$$\|h_{0,\varepsilon} - 1\|_{H^s(\mathbb{R}^2)} + \|u_{0,\varepsilon}\|_{H^s(\mathbb{R}^2)} \leq M$$

is uniform for any $\varepsilon > 0$, then we can choose $T_\varepsilon = +\infty$.

Motivated by [3,9], we introduce the following relative entropy functional:

$$\mathcal{E}(t) = \frac{1}{2} \int_{\mathbb{R}^2} h^\varepsilon |u^\varepsilon - v - \nabla \Psi|^2 dx + \frac{1}{2\varepsilon^2} \int_{\mathbb{R}^2} (h^\varepsilon - 1 - \varepsilon\sigma)^2 dx, \quad (2.2)$$

where σ, Ψ are the solution of the following acoustic system related to the inviscid Euler and shallow water system (1.1)–(1.2) by the following linear relations [7]:

$$\partial_t \sigma + \frac{1}{\varepsilon} \Delta \Psi = 0, \quad (2.3)$$

$$\partial_t \nabla \Psi + \frac{1}{\varepsilon} \nabla \sigma = 0 \quad (2.4)$$

supplemented with the initial data

$$\sigma(0) = \sigma_0 = h_0^1, \quad \nabla \Psi(0) = \nabla \Psi_0 = u_0 - v_0, \quad (2.5)$$

where $h_0^1 \in C_c^\infty(\mathbb{R}^2)$, $u_0 \in \bigcap_{s>0} H^s(\mathbb{R}^2)$ are given functions and $v_0 = P[u_0]$. Note that Ψ and σ depend on ε , that is, $\Psi = \Psi_\varepsilon$ and $\sigma = \sigma_\varepsilon$. For convenience, the notation with ε for Ψ and σ will not be used in what follows. Similar to [8], our goal is to use the relative entropy inequality to deduce the strong convergence to the limit system claimed in Theorem 2.3. The initial data $(h_0^1, \nabla \Psi_0)$ can be regularized in the following way:

$$h_0^1 = h_{0,\eta}^1 = \chi_\eta * (\psi_\eta h_0^1), \quad \nabla \Psi_0 = \nabla \Psi_{0,\eta} = \chi_\eta * (\psi_\eta \nabla \Psi_0), \quad \eta > 0, \quad (2.6)$$

where $\{\chi_\eta\}$ is a family of regularizing kernels and $\psi_\eta \in C_c^\infty(\mathbb{R}^2)$ is the standard cut-off function.

The total change in energy of the fluid caused by acoustic wave is given by

$$\frac{1}{2} \int_{\mathbb{R}^2} (|\sigma|^2 + |\nabla \Psi|^2) dx,$$

which is conserved in time, namely,

$$\frac{1}{2} \int_{\mathbb{R}^2} (|\sigma|^2 + |\nabla \Psi|^2) dx = \frac{1}{2} \int_{\mathbb{R}^2} (|h_0^1|^2 + |\nabla \Psi_0|^2) dx. \quad (2.7)$$

In addition, for any $t > 0$, the dispersive estimates hold [8]

$$\|\nabla \Psi\|_{W^{k,p}} + \|\sigma\|_{W^{k,p}} \leq C \left(1 + \frac{t}{\varepsilon}\right)^{\frac{1}{p}-\frac{1}{q}} (\|\nabla \Psi_0\|_{W^{k,q}} + \|h_0^1\|_{W^{k,q}}), \quad (2.8)$$

where

$$2 \leq p \leq +\infty, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad k = 0, 1, \dots$$

Hereafter, C stands for the generic positive constants independent of ε and may be depends on $T > 0$, $\|v\|_{L^2(\mathbb{R}^2)}$, $\|v\|_{L^\infty(\mathbb{R}^2)}$, $\|\pi\|_{L^2(\mathbb{R}^2)}$, $\|v\|_{L^2(\mathbb{R}^2)}$ and other constants.

The main result of this paper can be stated as follows:

Theorem 2.3. *Let $M > 0$ be a constant in Proposition 2.2. Let the initial data of the shallow water system (1.1)–(1.2) be of the following form:*

$$h^\varepsilon(0) = h_{0,\varepsilon} = 1 + \varepsilon h_{0,\varepsilon}^1, \quad u^\varepsilon(0) = u_{0,\varepsilon}, \quad (2.9)$$

satisfying (1.5) with

$$\|h_{0,\varepsilon}^1\|_{H^s(\mathbb{R}^2)} + \|u_{0,\varepsilon}\|_{H^s(\mathbb{R}^2)} \leq M \quad \text{for } s > 1 \quad \text{and} \quad \forall \varepsilon > 0. \quad (2.10)$$

Let all requirements of Proposition 2.1 be satisfied with the initial data $v_0 = Pu_0$ for the incompressible Euler system (1.6)–(1.7). Let (σ, Ψ) be the solution of the acoustic system (2.3)–(2.4) with the initial data (2.6). Furthermore, let both system (1.1)–(1.2) with initial data (2.9)–(2.10) and system (1.6)–(1.7) with initial data $v_0 = P[u_0]$ have classical solution on the time interval $[0, T]$. Then, there exists $\varepsilon_0 > 0$ such that

$$\begin{aligned} & \|\sqrt{h^\varepsilon}(u^\varepsilon - v - \nabla\Psi)(\cdot, t)\|_{L^2(\mathbb{R}^2)}^2 + \left\| \left(\frac{h^\varepsilon - 1}{\varepsilon} \right)(\cdot, t) - \sigma(\cdot, t) \right\|_{L^2(\mathbb{R}^2)}^2 \\ & \leq C(\|u_{0,\varepsilon} - u_0\|_{L^2(\mathbb{R}^2)}^2 + \|h_{0,\varepsilon}^1 - h_0^1\|_{L^2(\mathbb{R}^2)}^2 + \varepsilon^{1-\delta}), \quad t \in [0, T] \end{aligned} \quad (2.11)$$

for any $\delta \in (0, 1)$ and $0 < \varepsilon \leq \varepsilon_0$.

Corollary 2.4. In addition to the hypotheses of Theorem 2.3, if $u_0 = v_0$ and $h_0^1 = 0$, then Ψ and σ are both equal to zero and we have

$$\|\sqrt{h^\varepsilon}(u^\varepsilon - v)\|_{L^2(\mathbb{R}^2)}^2 \leq C\varepsilon^{1-\delta} \quad (2.12)$$

and

$$\|h^\varepsilon - 1\|_{L^2(\mathbb{R}^2)}^2 \leq C\varepsilon^{3-\delta} \quad (2.13)$$

for any $t \in [0, T]$.

3 Proof of main results

By (1.5) or the initial conditions in Theorem 2.3, we can obtain

$$\|\sqrt{h^\varepsilon}u^\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}^2)} \leq C \quad (3.1)$$

and

$$\|(h^\varepsilon - 1)(\cdot, t)\|_{L^2(\mathbb{R}^2)} \leq C\varepsilon \quad (3.2)$$

for $t \in [0, T]$.

Using the energy equality (1.5) and $\operatorname{div} v = 0$, the relative entropy functional can be rewritten as

$$\mathcal{E}(t) = E^\varepsilon(t) - \int_{\mathbb{R}^2} h^\varepsilon u^\varepsilon \cdot (v + \nabla\Psi) dx + \frac{1}{2} \int_{\mathbb{R}^2} h^\varepsilon |v + \nabla\Psi|^2 dx - \frac{1}{\varepsilon} \int_{\mathbb{R}^2} (h^\varepsilon - 1) \sigma dx + \frac{1}{2} \int_{\mathbb{R}^2} |\sigma|^2 dx. \quad (3.3)$$

Differentiating the relative entropy functional (3.3) with respect to t , we obtain

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(t) &= -\varepsilon \int_{\mathbb{R}^2} h^\varepsilon |\nabla u^\varepsilon|^2 dx - \frac{d}{dt} \int_{\mathbb{R}^2} h^\varepsilon u^\varepsilon \cdot (v + \nabla\Psi) dx \\ &\quad + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} h^\varepsilon |v + \nabla\Psi|^2 dx - \frac{1}{\varepsilon} \frac{d}{dt} \int_{\mathbb{R}^2} (h^\varepsilon - 1) \sigma dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |\sigma|^2 dx \\ &= -\varepsilon \int_{\mathbb{R}^2} h^\varepsilon |\nabla u^\varepsilon|^2 dx + \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |\sigma|^2 dx. \end{aligned} \quad (3.4)$$

Using the momentum equation (1.2), by integration by parts, we have

$$\begin{aligned} \mathbf{I}_1 &= - \int_{\mathbb{R}^2} h^\varepsilon u^\varepsilon \otimes u^\varepsilon : \nabla(v + \nabla\Psi) dx - \frac{1}{2\varepsilon^2} \int_{\mathbb{R}^2} (h^\varepsilon - 1)^2 \Delta\Psi dx - \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} (h^\varepsilon - 1) \Delta\Psi dx \\ &\quad - \int_{\mathbb{R}^2} h^\varepsilon u^\varepsilon \cdot \partial_t(v + \nabla\Psi) dx + \varepsilon \int_{\mathbb{R}^2} h^\varepsilon \nabla u^\varepsilon : \nabla(v + \nabla\Psi) dx \end{aligned}$$

$$\begin{aligned}
&= - \int_{\mathbb{R}^2} h^\varepsilon (u^\varepsilon - v - \nabla \Psi) \otimes (u^\varepsilon - v - \nabla \Psi) : \nabla (v + \nabla \Psi) dx \\
&\quad - \int_{\mathbb{R}^2} h^\varepsilon u^\varepsilon \otimes (v + \nabla \Psi) : \nabla (v + \nabla \Psi) dx - \int_{\mathbb{R}^2} h^\varepsilon v \otimes (u^\varepsilon - v - \nabla \Psi) : \nabla v dx \\
&\quad - \int_{\mathbb{R}^2} h^\varepsilon v \otimes (u^\varepsilon - v - \nabla \Psi) : \nabla^2 \Psi dx - \int_{\mathbb{R}^2} h^\varepsilon \nabla \Psi \otimes (u^\varepsilon - v - \nabla \Psi) : \nabla v dx \\
&\quad - \int_{\mathbb{R}^2} h^\varepsilon \nabla \Psi \otimes (u^\varepsilon - v - \nabla \Psi) : \nabla^2 \Psi dx - \frac{1}{2\varepsilon^2} \int_{\mathbb{R}^2} (h^\varepsilon - 1)^2 \Delta \Psi dx - \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} (h^\varepsilon - 1) \Delta \Psi dx \\
&\quad - \int_{\mathbb{R}^2} h^\varepsilon u^\varepsilon \cdot \partial_t (v + \nabla \Psi) dx + \varepsilon \int_{\mathbb{R}^2} h^\varepsilon \nabla u^\varepsilon : \nabla (v + \nabla \Psi) dx \\
&= - \int_{\mathbb{R}^2} h^\varepsilon (u^\varepsilon - v - \nabla \Psi) \otimes (u^\varepsilon - v - \nabla \Psi) : \nabla (v + \nabla \Psi) dx \\
&\quad - \frac{1}{2} \int_{\mathbb{R}^2} h^\varepsilon u^\varepsilon \cdot \nabla |v + \nabla \Psi|^2 dx - \int_{\mathbb{R}^2} h^\varepsilon (u^\varepsilon - v - \nabla \Psi) \cdot ((v \cdot \nabla) v) dx \\
&\quad - \int_{\mathbb{R}^2} h^\varepsilon v \otimes (u^\varepsilon - v - \nabla \Psi) : \nabla^2 \Psi dx - \int_{\mathbb{R}^2} h^\varepsilon \nabla \Psi \otimes (u^\varepsilon - v - \nabla \Psi) : \nabla v dx \\
&\quad - \int_{\mathbb{R}^2} h^\varepsilon \nabla \Psi \otimes (u^\varepsilon - v - \nabla \Psi) : \nabla^2 \Psi dx - \frac{1}{2\varepsilon^2} \int_{\mathbb{R}^2} (h^\varepsilon - 1)^2 \Delta \Psi dx \\
&\quad - \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} (h^\varepsilon - 1) \Delta \Psi dx - \int_{\mathbb{R}^2} h^\varepsilon u^\varepsilon \cdot \partial_t (v + \nabla \Psi) dx + \varepsilon \int_{\mathbb{R}^2} h^\varepsilon \nabla u^\varepsilon : \nabla (v + \nabla \Psi) dx.
\end{aligned} \tag{3.5}$$

Employing the continuity equation (1.1), integration by parts, one gets

$$\mathbf{I}_2 = \int_{\mathbb{R}^2} h^\varepsilon (v + \nabla \Psi) \cdot \partial_t (v + \nabla \Psi) dx + \frac{1}{2} \int_{\mathbb{R}^2} h^\varepsilon u^\varepsilon \cdot \nabla |v + \nabla \Psi|^2 dx. \tag{3.6}$$

For $\mathbf{I}_3$, by the continuity equation (1.1) and the acoustic equations (2.3) and (2.4), we have

$$\begin{aligned}
\mathbf{I}_3 &= -\frac{1}{\varepsilon} \int_{\mathbb{R}^2} \partial_t h^\varepsilon \sigma dx - \frac{1}{\varepsilon} \int_{\mathbb{R}^2} (h^\varepsilon - 1) \partial_t \sigma dx \\
&= -\frac{1}{\varepsilon} \int_{\mathbb{R}^2} h^\varepsilon u^\varepsilon \cdot \nabla \sigma dx - \frac{1}{\varepsilon} \int_{\mathbb{R}^2} (h^\varepsilon - 1) \partial_t \sigma dx \\
&= \int_{\mathbb{R}^2} h^\varepsilon u^\varepsilon \cdot \partial_t \nabla \Psi dx + \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} (h^\varepsilon - 1) \Delta \Psi dx.
\end{aligned} \tag{3.7}$$

Consequently, by (3.4)–(3.7), we have

$$\begin{aligned}
\frac{d}{dt} \mathcal{E}(t) &= -\varepsilon \int_{\mathbb{R}^2} h^\varepsilon \nabla u^\varepsilon : \nabla (u^\varepsilon - v - \nabla \Psi) dx - \int_{\mathbb{R}^2} h^\varepsilon (u^\varepsilon - v - \nabla \Psi) \otimes (u^\varepsilon - v - \nabla \Psi) : \nabla (v + \nabla \Psi) dx \\
&\quad - \int_{\mathbb{R}^2} h^\varepsilon (u^\varepsilon - v - \nabla \Psi) \cdot [\partial_t v + (v \cdot \nabla) v] dx - \int_{\mathbb{R}^2} h^\varepsilon v \otimes (u^\varepsilon - v - \nabla \Psi) : \nabla^2 \Psi dx \\
&\quad - \int_{\mathbb{R}^2} h^\varepsilon \nabla \Psi \otimes (u^\varepsilon - v - \nabla \Psi) : \nabla v dx - \int_{\mathbb{R}^2} h^\varepsilon \nabla \Psi \otimes (u^\varepsilon - v - \nabla \Psi) : \nabla^2 \Psi dx \\
&\quad - \frac{1}{2\varepsilon^2} \int_{\mathbb{R}^2} (h^\varepsilon - 1)^2 \Delta \Psi dx + \int_{\mathbb{R}^2} h^\varepsilon (v + \nabla \Psi) \cdot \partial_t \nabla \Psi dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |\sigma|^2 dx \\
&= \mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_3 + \mathbf{K}_4 + \mathbf{K}_5 + \mathbf{K}_6 + \mathbf{K}_7 + \mathbf{K}_8 + \mathbf{K}_9.
\end{aligned} \tag{3.8}$$

For $\mathbf{K}_1$, in view of (2.1), (2.8), and (3.2), applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
 \mathbf{K}_1 &= -\varepsilon \int_{\mathbb{R}^2} h^\varepsilon \nabla u^\varepsilon : \nabla(u^\varepsilon - v - \nabla \Psi) dx \\
 &= -\varepsilon \int_{\mathbb{R}^2} h^\varepsilon |\nabla u^\varepsilon|^2 dx + \varepsilon \int_{\mathbb{R}^2} h^\varepsilon \nabla u^\varepsilon : \nabla(v + \nabla \Psi) dx \\
 &= -\varepsilon \int_{\mathbb{R}^2} h^\varepsilon |\nabla u^\varepsilon|^2 dx + \varepsilon \int_{\mathbb{R}^2} (\sqrt{h^\varepsilon} - 1) \sqrt{h^\varepsilon} \nabla u^\varepsilon : (v + \nabla \Psi) dx + \varepsilon \int_{\mathbb{R}^2} \sqrt{h^\varepsilon} \nabla u^\varepsilon : \nabla(v + \nabla \Psi) dx \\
 &\leq -\varepsilon \int_{\mathbb{R}^2} h^\varepsilon |\nabla u^\varepsilon|^2 dx + \frac{\varepsilon}{2} \int_{\mathbb{R}^2} h^\varepsilon |\nabla u^\varepsilon|^2 dx + \frac{\varepsilon}{2} \|v + \nabla \Psi\|_\infty^2 \int_{\mathbb{R}^2} |\sqrt{h^\varepsilon} - 1|^2 dx \\
 &\quad + \frac{\varepsilon}{2} \int_{\mathbb{R}^2} h^\varepsilon |\nabla u^\varepsilon|^2 dx + \frac{\varepsilon}{2} \int_{\mathbb{R}^2} |v + \nabla \Psi|^2 dx \leq C \int_{\mathbb{R}^2} |h^\varepsilon - 1|^2 dx + C\varepsilon \leq C\varepsilon.
 \end{aligned} \tag{3.9}$$

For $\mathbf{K}_2$, thanks to the Sobolev imbedding theorem, the Minkowski inequality, (2.1), and (2.8), we get

$$\begin{aligned}
 \mathbf{K}_2 &= - \int_{\mathbb{R}^2} h^\varepsilon (u^\varepsilon - v - \nabla \Psi) \otimes (u^\varepsilon - v - \nabla \Psi) : \nabla(v + \nabla \Psi) dx \\
 &\leq \int_{\mathbb{R}^2} h^\varepsilon |u^\varepsilon - v - \nabla \Psi|^2 |\nabla(v + \nabla \Psi)| dx \\
 &\leq C\mathcal{E}(t) \|\nabla(v + \nabla \Psi)\|_{L^\infty(\mathbb{R}^2)} \\
 &\leq C\mathcal{E}(t) \|\nabla v\|_{L^\infty(\mathbb{R}^2)} + C\mathcal{E}(t) \|\nabla^2 \Psi\|_{L^\infty(\mathbb{R}^2)} \leq C\mathcal{E}(t).
 \end{aligned} \tag{3.10}$$

For $\mathbf{K}_3$, using (1.7), the acoustic system (2.3)–(2.4), and (2.8), (3.2), the Hölder inequality, and $\operatorname{div} v = 0$, integration by parts, we have

$$\begin{aligned}
 \mathbf{K}_3 &= - \int_{\mathbb{R}^2} h^\varepsilon (u^\varepsilon - v - \nabla \Psi) \cdot \underbrace{[\partial_t v + (v \cdot \nabla) v]}_{=\nabla \pi} dx \\
 &= \int_{\mathbb{R}^2} h^\varepsilon u^\varepsilon \cdot \nabla \pi dx - \int_{\mathbb{R}^2} h^\varepsilon (v + \nabla \Psi) \cdot \nabla \pi dx \\
 &= \int_{\mathbb{R}^2} \partial_t (h^\varepsilon - 1) \pi dx - \int_{\mathbb{R}^2} (h^\varepsilon - 1) (v + \nabla \Psi) \cdot \nabla \pi dx - \varepsilon \int_{\mathbb{R}^2} \partial_t \sigma \pi dx \\
 &= \frac{d}{dt} \left[\int_{\mathbb{R}^2} (h^\varepsilon - 1) \pi dx - \varepsilon \int_{\mathbb{R}^2} \sigma \pi dx \right] - \int_{\mathbb{R}^2} (h^\varepsilon - 1) \partial_t \pi dx + \varepsilon \int_{\mathbb{R}^2} \sigma \partial_t \pi dx \\
 &\quad - \int_{\mathbb{R}^2} (h^\varepsilon - 1) (v + \nabla \Psi) \cdot \nabla \pi dx \\
 &\leq \frac{d}{dt} \left[\int_{\mathbb{R}^2} (h^\varepsilon - 1) \pi dx - \varepsilon \int_{\mathbb{R}^2} \sigma \pi dx \right] + C\varepsilon.
 \end{aligned} \tag{3.11}$$

In view of (3.2), we get

$$\|\sqrt{h^\varepsilon} - 1\|_{L^2(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} |\sqrt{h^\varepsilon} - 1|^2 dx \right)^{\frac{1}{2}} \leq \left(\int_{\mathbb{R}^2} |h^\varepsilon - 1|^2 dx \right)^{\frac{1}{2}} \leq C\varepsilon.$$

Then, using (3.1), (2.8), and the Hölder inequality, we have

$$\begin{aligned}
 \mathbf{K}_4 &= - \int_{\mathbb{R}^2} h^\varepsilon v \otimes (u^\varepsilon - v - \nabla \Psi) : \nabla^2 \Psi dx \\
 &= - \int_{\mathbb{R}^2} h^\varepsilon v \otimes u^\varepsilon : \nabla^2 \Psi dx + \int_{\mathbb{R}^2} h^\varepsilon v \otimes v : \nabla^2 \Psi dx + \int_{\mathbb{R}^2} h^\varepsilon v \otimes \nabla \Psi : \nabla^2 \Psi dx \\
 &= - \int_{\mathbb{R}^2} (\sqrt{h^\varepsilon} - 1) v \otimes \sqrt{h^\varepsilon} u^\varepsilon : \nabla^2 \Psi dx - \int_{\mathbb{R}^2} v \otimes \sqrt{h^\varepsilon} u^\varepsilon : \nabla^2 \Psi dx + \int_{\mathbb{R}^2} (h^\varepsilon - 1) v \otimes v : \nabla^2 \Psi dx \\
 &\quad + \int_{\mathbb{R}^2} v \otimes v : \nabla^2 \Psi dx + \int_{\mathbb{R}^2} (h^\varepsilon - 1) v \otimes \nabla \Psi : \nabla^2 \Psi dx + \int_{\mathbb{R}^2} v \otimes \nabla \Psi : \nabla^2 \Psi dx \\
 &\leq \|\sqrt{h^\varepsilon} - 1\|_{L^2(\mathbb{R}^2)} \|\sqrt{h^\varepsilon} u^\varepsilon\|_{L^2(\mathbb{R}^2)} \|v\|_{L^\infty(\mathbb{R}^2)} \|\nabla^2 \Psi\|_{L^\infty(\mathbb{R}^2)} + \|v\|_{L^2(\mathbb{R}^2)} \|\sqrt{h^\varepsilon} u^\varepsilon\|_{L^2(\mathbb{R}^2)} \|\nabla^2 \Psi\|_{L^\infty(\mathbb{R}^2)} \\
 &\quad + \|h^\varepsilon - 1\|_{L^2(\mathbb{R}^2)} \|v\|_{L^2(\mathbb{R}^2)} \|v\|_{L^\infty(\mathbb{R}^2)} \|\nabla^2 \Psi\|_{L^\infty(\mathbb{R}^2)} + \|v\|_{L^2(\mathbb{R}^2)}^2 \|\nabla^2 \Psi\|_{L^\infty(\mathbb{R}^2)} \\
 &\quad + \|h^\varepsilon - 1\|_{L^2(\mathbb{R}^2)} \|v\|_{L^\infty(\mathbb{R}^2)} \|\nabla \Psi\|_{L^2(\mathbb{R}^2)} \|\nabla^2 \Psi\|_{L^\infty(\mathbb{R}^2)} + \|v\|_{L^2(\mathbb{R}^2)} \|\nabla \Psi\|_{L^2(\mathbb{R}^2)} \|\nabla^2 \Psi\|_{L^\infty(\mathbb{R}^2)} \\
 &\leq C \left(1 + \frac{t}{\varepsilon}\right)^{-1} + C\varepsilon.
 \end{aligned} \tag{3.12}$$

Similarly to (3.12), we can obtain

$$\mathbf{K}_5 = - \int_{\mathbb{R}^2} h^\varepsilon \nabla \Psi \otimes (u^\varepsilon - v - \nabla \Psi) : \nabla v dx \leq C \left(1 + \frac{t}{\varepsilon}\right)^{-1} + C\varepsilon \tag{3.13}$$

and

$$\mathbf{K}_6 = - \int_{\mathbb{R}^2} h^\varepsilon \nabla \Psi \otimes (u^\varepsilon - v - \nabla \Psi) : \nabla^2 \Psi dx \leq C \left(1 + \frac{t}{\varepsilon}\right)^{-1} + C \left(1 + \frac{t}{\varepsilon}\right)^{-2} + C\varepsilon. \tag{3.14}$$

In view of (3.2) and (2.8), we deduce that

$$\mathbf{K}_7 = - \frac{1}{2\varepsilon^2} \int_{\mathbb{R}^2} (h^\varepsilon - 1)^2 \Delta \Psi dx \leq \frac{C}{\varepsilon^2} \|h^\varepsilon - 1\|_{L^2(\mathbb{R}^2)}^2 \|\Delta \Psi\|_{L^\infty(\mathbb{R}^2)} \leq C \left(1 + \frac{t}{\varepsilon}\right)^{-1}. \tag{3.15}$$

Using the acoustic system (2.3)–(2.4), (2.7), (3.2), and $\operatorname{div} v = 0$, we have

$$\begin{aligned}
 \mathbf{K}_8 + \mathbf{K}_9 &= \int_{\mathbb{R}^2} h^\varepsilon (v + \nabla \Psi) \cdot \partial_t \nabla \Psi dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |\sigma|^2 dx \\
 &= \int_{\mathbb{R}^2} (h^\varepsilon - 1) v \cdot \partial_t \nabla \Psi dx + \int_{\mathbb{R}^2} (h^\varepsilon - 1) \nabla \Psi \cdot \partial_t \nabla \Psi dx + \int_{\mathbb{R}^2} \nabla \Psi \cdot \partial_t \nabla \Psi dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |\sigma|^2 dx \\
 &= - \frac{1}{\varepsilon} \int_{\mathbb{R}^2} (h^\varepsilon - 1) v \cdot \nabla \sigma dx - \frac{1}{\varepsilon} \int_{\mathbb{R}^2} (h^\varepsilon - 1) \nabla \Psi \cdot \nabla \sigma dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} (|\nabla \Psi|^2 + |\sigma|^2) dx \\
 &= - \frac{1}{\varepsilon} \int_{\mathbb{R}^2} (h^\varepsilon - 1) v \cdot \nabla \sigma dx - \frac{1}{\varepsilon} \int_{\mathbb{R}^2} (h^\varepsilon - 1) \nabla \Psi \cdot \nabla \sigma dx \\
 &\leq \frac{C}{\varepsilon} \|h^\varepsilon - 1\|_{L^2(\mathbb{R}^2)} \|v\|_{L^2(\mathbb{R}^2)} \|\nabla \sigma\|_{L^\infty(\mathbb{R}^2)} + \frac{C}{\varepsilon} \|h^\varepsilon - 1\|_{L^2(\mathbb{R}^2)} \|\nabla \Psi\|_{L^2(\mathbb{R}^2)} \|\nabla \sigma\|_{L^\infty(\mathbb{R}^2)} \\
 &\leq C \left(1 + \frac{t}{\varepsilon}\right)^{-1}.
 \end{aligned} \tag{3.16}$$

Denote $Q(t) = - \left[\int_{\mathbb{R}^2} (h^\varepsilon - 1) \pi dx - \varepsilon \int_{\mathbb{R}^2} \sigma \pi dx \right]$. Using (3.8) and adding the estimates for $\mathbf{K}_1, \mathbf{K}_2, \dots, \mathbf{K}_9$, we end up with

$$\frac{d}{dt} (\mathcal{E}(t) + Q(t)) \leq C\mathcal{E}(t) + C\varepsilon + C \left(1 + \frac{t}{\varepsilon}\right)^{-1}, \tag{3.17}$$

where we have used the following fact:

$$\left(1 + \frac{t}{\varepsilon}\right)^a < \left(1 + \frac{t}{\varepsilon}\right)^b, \quad 0 < a < b$$

for any $\varepsilon \in (0, 1)$. Integrating the inequality (3.17) yields

$$\mathcal{E}(t) \leq C \int_0^t \mathcal{E}(\tau) d\tau + \mathcal{E}(0) - Q(t) + Q(0) + CT\varepsilon + C\varepsilon(\ln(\varepsilon + T) - \ln \varepsilon). \quad (3.18)$$

From the initial data (2.9), (2.10), and the inequality (3.2), one can show that $Q(t) = O(\varepsilon)$ and $Q(0) = O(\varepsilon)$. Furthermore, for any fixed $0 < \delta < 1$, we have

$$\lim_{\varepsilon \rightarrow 0} \frac{\ln(\varepsilon + T) - \ln \varepsilon}{\varepsilon^{-\delta}} = 0. \quad (3.19)$$

Then, one gets

$$\mathcal{E}(t) \leq C \int_0^t \mathcal{E}(\tau) d\tau + \mathcal{E}(0) + C\varepsilon^{1-\delta}. \quad (3.20)$$

In order to obtain the desired results, we need to estimate the initial relative entropy functional $\mathcal{E}(0)$. By (2.9), (2.10), and the Taylor formula it is easy to obtain that

$$\begin{aligned} \mathcal{E}(0) &= \frac{1}{2} \int_{\mathbb{R}^2} h_{0,\varepsilon} |u_{0,\varepsilon} - u_0|^2 dx + \frac{1}{2\varepsilon^2} \int_{\mathbb{R}^2} (h_{0,\varepsilon} - 1 - \varepsilon\sigma_0)^2 dx \\ &\leq \frac{1}{2} \int_{\mathbb{R}^2} |1 + \varepsilon h_{0,\varepsilon}^1| |u_{0,\varepsilon} - u_0|^2 dx + \frac{C}{\varepsilon^2} \int_{\mathbb{R}^2} |\varepsilon(h_{0,\varepsilon}^1 - h_0^1)|^2 dx \\ &\leq \frac{1}{2} \int_{\mathbb{R}^2} |u_{0,\varepsilon} - u_0|^2 dx + \frac{\varepsilon}{2} \int_{\mathbb{R}^2} |h_{0,\varepsilon}^1| |u_{0,\varepsilon} - u_0|^2 dx + C \|h_{0,\varepsilon}^1 - h_0^1\|_{L^2(\mathbb{R}^2)}^2 \\ &\leq \frac{1}{2} \int_{\mathbb{R}^2} |u_{0,\varepsilon} - u_0|^2 dx + \frac{\varepsilon}{2} \|h_{0,\varepsilon}^1\|_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^2} |u_{0,\varepsilon} - u_0|^2 dx + C \|h_{0,\varepsilon}^1 - h_0^1\|_{L^2(\mathbb{R}^2)}^2 \\ &\leq C \|u_{0,\varepsilon} - u_0\|_{L^2(\mathbb{R}^2)}^2 + C \|h_{0,\varepsilon}^1 - h_0^1\|_{L^2(\mathbb{R}^2)}^2. \end{aligned} \quad (3.21)$$

By (3.20) and (3.21), applying the Gronwall inequality, we can complete the proof of Theorem 2.3, and consequently, Corollary 2.4.

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