Research Article

Zhibin Du*, Carlos M. da Fonseca, Yingqiu Xu, and Jiahao Ye

Disproving a conjecture of Thornton on Bohemian matrices

<https://doi.org/10.1515/math-2021-0045>

received November 27, 2020; accepted May 6, 2021

Abstract: In this paper, we disprove a remaining conjecture about Bohemian matrices, in which the numbers of distinct determinants of a normalized Bohemian upper-Hessenberg matrix were conjectured.

Keywords: Bohemian matrix, integral matrix, normalized upper Hessenberg matrix, determinant

MSC 2020: 15A15

1 Introduction

A matrix is called Bohemian if its entries are of a bounded height, typically drawn from a discrete set. The term “Bohemian” is intended as a mnemonic and derived from “BOunded HEight Matrix of Integers.” These matrices are of relevant interest in many areas where their spectral properties are relevant. The consideration to this type of properties goes back to more than one century ago, as pointed out in [1,2]. For a more global historical perspective and current trends, the reader is referred to [3–5] and references therein.

Meanwhile, the study of these matrices has become independent and, based on numerous and hard computational experiments, many conjectures have emerged in the literature. The most important collection can be found in the so-called *Characteristic Polynomial Database* [6].

Quite recently, this catalogue attracted the attention of several researchers, namely on those conjectures concerned to Hessenberg matrices. Indeed, they are extended and proved in [7–9]. This claim has two exceptions though. One of them is solved and generalized in [10], the remaining unsolved one is [6, Conjecture 9].

Conjecture 1. [6, Conjecture 9] *The number of distinct determinants of an $n \times n$ upper-Hessenberg matrix with entries from the set $\{0, 1\}$, subdiagonal entries fixed at 1, and diagonal entries fixed at 0 is given by sequence A212264.*

We recall that the first 11 terms of sequence [11, A212264] are listed in Table 1, where we understand $A212264_n$ as the n th term of the sequence A212264.

* **Corresponding author: Zhibin Du**, School of Software, South China Normal University, Foshan, Guangdong 528225, China, e-mail: zhibindu@126.com

Carlos M. da Fonseca: Kuwait College of Science and Technology, Doha District, Block 4, P.O. Box 27235, Safat 13133, Kuwait; Chair of Computational Mathematics, University of Deusto, 48007 Bilbao, Spain, e-mail: c.dafonseca@kcst.edu.kw

Yingqiu Xu: School of Software, South China Normal University, Foshan, Guangdong 528225, China, e-mail: 1275284510@qq.com

Jiahao Ye: School of Software, South China Normal University, Foshan, Guangdong 528225, China, e-mail: 17725399897@163.com

Table 1: The first few values of A212264

n	0	1	2	3	4	5	6	7	8	9	10
A212264 _{n}	1	1	2	2	3	4	6	9	12	18	26

In this paper, we will show that, for $n = 10$, this conjecture is false. Next, we analyze the Hessenberg matrices aforementioned in detail. Then we establish several results which will lead to a counterexample, concluding that the conjecture is false. In Section 5, we provide a theoretical approach for such evidence. Moreover, we claim that the conjecture is also false for $n = 11$. More generally, our methods are valid for any n as well.

2 Preliminaries

In this section, we analyze the Hessenberg matrices mentioned in Conjecture 1 in detail. We recall that these matrices have relevant properties and applications (cf., e.g., [12–16]).

Let $\mathcal{M}_n$ be the set of all $n \times n$ upper-Hessenberg matrices with entries from the set $\{0, 1\}$, subdiagonal entries all equal to 1, and diagonal entries all equal to 0. That is to say, each matrix in $\mathcal{M}_n$ is of the following form:

$$\begin{pmatrix} 0 & * & * & \cdots & * & * \\ 1 & 0 & * & \cdots & * & * \\ 0 & 1 & 0 & \cdots & * & * \\ & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & & 1 & 0 & * \\ & & & & 0 & 1 & 0 \end{pmatrix}_{n \times n}, \quad (2.1)$$

where $*$ takes either the value 0 or 1. Several open problems and results involving these matrices can be found, for example, in [17,18].

Let us denote by a_n the number of distinct determinants among all matrices in $\mathcal{M}_n$. Conjecture 1 claims that $\{a_n\}_{n \geq 0}$ agrees with A212264.

It is easy to see that, after expanding the determinant of the matrix defined in (2.1) along the first column and the last row, one can obtain:

$$\begin{vmatrix} 0 & * & * & \cdots & * & * \\ 1 & 0 & * & \cdots & * & * \\ 0 & 1 & 0 & \cdots & * & * \\ & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & & 1 & 0 & * \\ & & & & 0 & 1 & 0 \end{vmatrix}_{n \times n} = \begin{vmatrix} * & * & * & \cdots & * & * \\ 1 & 0 & * & \cdots & * & * \\ 0 & 1 & 0 & \cdots & * & * \\ & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & & 1 & 0 & * \\ & & & & 0 & 1 & * \end{vmatrix}_{(n-2) \times (n-2)}. \quad (2.2)$$

Based on this observation, we introduce a modified version of $\mathcal{M}_n$, say $\mathcal{N}_n$. The only difference between $\mathcal{M}_n$ and $\mathcal{N}_n$ lies on the first and last diagonal entries, which are 0 in $\mathcal{M}_n$, but 1 in $\mathcal{N}_n$. Hence, each matrix in $\mathcal{N}_n$ is of the following form:

$$\begin{pmatrix} 1 & * & * & \cdots & * & * \\ 1 & 0 & * & \cdots & * & * \\ 0 & 1 & 0 & \cdots & * & * \\ & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & & 1 & 0 & * \\ & & & & 0 & 1 & 1 \end{pmatrix}_{n \times n}. \quad (2.3)$$

A close relation about the determinants of matrices in $\mathcal{M}_n$ and $\mathcal{N}_n$ is revealed by (2.2). So, the investigation about the determinants of the matrices in $\mathcal{N}_n$ can assist us during the study of Conjecture 1.

3 Matrices in $\mathcal{N}_n$

Let $\mathcal{DN}_n$ be the set of all possible determinants of matrices in $\mathcal{N}_n$ and set $b_n = |\mathcal{DN}_n|$.

Lemma 1. For any $n \geq 1$,

$$\mathcal{DN}_n \subset \mathcal{DN}_{n+1}.$$

Proof. The cases when $n = 1$ or 2 can be verified trivially. We consider $n \geq 3$ in the following. Assume that $k \in \mathcal{DN}_n$, that is to say, there exists some matrix, say $A \in \mathcal{N}_n$, such that $\det A = k$. For example, write A as

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ 1 & 0 & a_{23} & \cdots & a_{2,n-1} & a_{2n} \\ 0 & 1 & 0 & \cdots & a_{3,n-1} & a_{3n} \\ & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & 1 & 0 & a_{n-1,n} \\ & & & 0 & 1 & 1 \end{pmatrix}_{n \times n}.$$

Based on A , we construct B as

$$B = \begin{pmatrix} 1 & 1 & 0 & a_{23} & \cdots & a_{2,n-1} & a_{2n} \\ 1 & 0 & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & a_{3,n-1} & a_{3n} \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 1 & 0 & a_{n-1,n} \\ & & & 0 & 1 & 1 & 1 \end{pmatrix}_{(n+1) \times (n+1)}.$$

It is easy to see that $B \in \mathcal{N}_{n+1}$ and $\det A = \det B = k$. So $\mathcal{DN}_n \subset \mathcal{DN}_{n+1}$ follows. $\square$

Remark 3.1. In the above proof, based on $A \in \mathcal{N}_n$ with $\det A = k$, we construct a matrix $B \in \mathcal{N}_{n+1}$ such that $\det B = k$. Alternatively, it is also possible to construct another matrix $C \in \mathcal{N}_{n+1}$ such that $\det C = -k$, for example, set

$$C = \begin{pmatrix} 1 & 1 & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ 1 & 0 & 0 & a_{23} & \cdots & a_{2,n-1} & a_{2n} \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & a_{3,n-1} & a_{3n} \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 1 & 0 & a_{n-1,n} \\ & & & 0 & 1 & 1 & 1 \end{pmatrix}_{(n+1) \times (n+1)}.$$

The next corollary is an immediate consequence of the previous lemma.

Corollary 2. For any $1 \leq s < t$,

$$\mathcal{DN}_s \subset \mathcal{DN}_t.$$

Lemma 3. For $n \geq 5$,

$$a_n = b_{n-2}.$$

Proof. The application of (2.2) is a key point here. Analogous to the notation $\mathcal{DN}_n$, we denote by $\mathcal{DM}_n$ the set of all possible determinants of matrices in $\mathcal{M}_n$, i.e., $a_n = |\mathcal{DM}_n|$. Assume that $t \in \mathcal{DM}_n$, i.e., $\det A = t$, for some $A \in \mathcal{M}_n$. From (2.2) (expanding along the first column and the last row), we get

$$\det A = \begin{vmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ 1 & 0 & a_{23} & \cdots & a_{2,n-1} & a_{2n} \\ 0 & 1 & 0 & \cdots & a_{3,n-1} & a_{3n} \\ & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & & 1 & 0 \\ & & & & 0 & 1 \end{vmatrix}_{n \times n} = \begin{vmatrix} a_{12} & a_{13} & a_{14} & \cdots & a_{1,n-2} & a_{1n} \\ 1 & 0 & a_{34} & \cdots & a_{3,n-2} & a_{3n} \\ 0 & 1 & 0 & \cdots & a_{4,n-2} & a_{4n} \\ & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & & 1 & 0 \\ & & & & 0 & 1 \end{vmatrix}_{(n-2) \times (n-2)}. \quad (3.1)$$

Set

$$B = \begin{pmatrix} a_{12} & a_{13} & a_{14} & \cdots & a_{1,n-2} & a_{1n} \\ 1 & 0 & a_{34} & \cdots & a_{3,n-2} & a_{3n} \\ 0 & 1 & 0 & \cdots & a_{4,n-2} & a_{4n} \\ & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & & 1 & 0 \\ & & & & 0 & 1 \end{pmatrix}_{(n-2) \times (n-2)},$$

whose determinant, from (3.1), is the same as that of A .

If $a_{12} = a_{n-1,n} = 1$, then $B \in \mathcal{N}_{n-2}$ and $\det B = \det A = t$, i.e., $t \in \mathcal{DN}_{n-2}$. Otherwise, a_{12} and/or $a_{n-1,n}$ are equal to 0, at this time we continue to do expansion(s), for the first column and/or the last row with only one 1, until the first and the last diagonal entries are both equal to 1 or a matrix of order 1 is left. This implies that there exists some matrix, say $C \in \mathcal{N}_s$, with $1 \leq s \leq n-3$, satisfying $\det C = \pm \det A = \pm t$. If $\det C = t$, then by Corollary 2, $t \in \mathcal{DN}_s \subset \mathcal{DN}_{n-2}$. If $\det C = -t$, then by Remark 3.1, $t \in \mathcal{DN}_{s+1} \subset \mathcal{DN}_{n-2}$.

In conclusion, the above arguments tell us $\mathcal{DM}_n \subset \mathcal{DN}_{n-2}$ or, equivalently, $a_n \leq b_{n-2}$. As to its reversed version: $a_n \geq b_{n-2}$, it can be proven by using (3.1) and setting $a_{12} = a_{n-1,n} = 1$, i.e., given a matrix $B \in \mathcal{N}_{n-2}$, we may construct a matrix $A \in \mathcal{M}_n$ with the same determinant.

The desired equality $a_n = b_{n-2}$ follows now. $\square$

Remark 3.2. The reason why we require $n \geq 5$ is because, $t = -1$ (following the same notation as in the above proof) is possible, for $\mathcal{DM}_n$ with $n \geq 2$, but $-1 \notin \mathcal{DN}_2$, so the requirement $n-2 > 2$, i.e., $n \geq 5$, is needed.

4 A counterexample

For any integers a and b with $a \leq b$, we use $[a, b]$ to represent the set of all integers between a and b , that is, $\{a, a+1, \dots, b-1, b\}$. The first few values of $\mathcal{DN}_n$ and b_n up to $n = 8$ can be found in Table 2 (we will introduce how to obtain them in Section 5).

Taking into account the relation $a_n = b_{n-2}$, for $n \geq 5$, presented in Lemma 3, we obtain the first few values of a_n up to $n = 10$ immediately (Table 3). Note that the values of a_n with n up to 4 are obtained directly.

If we compare now the values of a_n with A212264 shown in Tables 1 and 3, respectively, one can see that they agree when $0 \leq n \leq 9$, but not when $n = 10$. This evidence indicates that Conjecture 1 is true for small n ($0 \leq n \leq 9$), but false when $n = 10$. Consequently, we may state

Table 2: The first few values of $\mathcal{DN}_n$ and b_n up to $n = 8$

n	0	1	2	3	4	5	6	7	8
$\mathcal{DN}_n$	[1, 1]	[1, 1]	[0, 1]	[-2, 1]	[-2, 3]	[-4, 4]	[-6, 5]	[-8, 9]	[-12, 12]
b_n	1	1	2	4	6	9	12	18	25

Table 3: The first few values of a_n up to $n = 10$

n	0	1	2	3	4	5	6	7	8	9	10
a_n	1	1	2	2	3	4	6	9	12	18	25

Proposition 4. Conjecture 1 is true for $0 \leq n \leq 9$, but false when $n = 10$.

5 A theoretical analysis

In this section, we present how to obtain the data in Table 2 theoretically.

Write $A \in \mathcal{N}_n$ as

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} & \cdots & a_{1,n-1} & a_{1n} \\ 1 & 0 & a_{23} & a_{24} & \cdots & a_{2,n-1} & a_{2n} \\ 0 & 1 & 0 & a_{34} & \cdots & a_{3,n-1} & a_{3n} \\ & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & \ddots & 0 & a_{n-2,n-1} & a_{n-2,n} \\ & & & \ddots & 1 & 0 & a_{n-1,n} \\ & & & & 0 & 1 & 1 \end{pmatrix}_{n \times n}.$$

The expansion of the determinant of A , for $n \geq 4$, along the second column, leads to

$$\begin{aligned} \det A &= \begin{vmatrix} 1 & a_{12} & a_{13} & a_{14} & \cdots & a_{1,n-1} & a_{1n} \\ 1 & 0 & a_{23} & a_{24} & \cdots & a_{2,n-1} & a_{2n} \\ 0 & 1 & 0 & a_{34} & \cdots & a_{3,n-1} & a_{3n} \\ & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & \ddots & 0 & a_{n-2,n-1} & a_{n-2,n} \\ & & & \ddots & 1 & 0 & a_{n-1,n} \\ & & & & 0 & 1 & 1 \end{vmatrix}_{n \times n} \\ &= a_{12} \begin{vmatrix} a_{34} & a_{35} & \cdots & a_{3,n-1} & a_{3n} \\ 1 & 0 & \cdots & a_{5,n-1} & a_{5n} \\ & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & 1 & 0 & a_{n-1,n} \\ & & & 0 & 1 & 1 \end{vmatrix}_{(n-3) \times (n-3)} - \begin{vmatrix} 1 & a_{13} & a_{14} & \cdots & a_{1,n-1} & a_{1n} \\ 1 & a_{23} & a_{24} & \cdots & a_{2,n-1} & a_{2n} \\ 0 & 1 & 0 & \cdots & a_{4,n-1} & a_{4n} \\ & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & \ddots & 1 & 0 & a_{n-1,n} \\ & & & & 0 & 1 & 1 \end{vmatrix}_{(n-1) \times (n-1)}. \end{aligned} \quad (5.1)$$

Let us consider the determinants of the two matrices in (5.1) separately.

We start looking at the first matrix. If $a_{34} = 1$, then this matrix is in $\mathcal{N}_{n-3}$. Otherwise, we go ahead with further expansion(s) for the first column with only one 1, until the first diagonal entry is equal to 1 or a matrix of order 1 is left. This implies that the resulting matrix should be in $\mathcal{N}_s$, with $1 \leq s \leq n-3$, or it is the matrix of order 1 whose unique entry is equal to 0.

As to the second matrix, it belongs to $\mathcal{N}_{n-1}$ when $a_{23} = 0$. Otherwise, we have $a_{23} = 1$ and, under such condition, we may assume that $a_{13} = 0$ (if $a_{13} = 1$, denote by C_i the i th column of A , after applying the column elementary transformation $C_3 - C_1$ to A , the $(2, 3)$ -entry becomes 0, which is equivalent to the case when $a_{23} = 0$), interchanging the first two rows of the second matrix leads to a matrix in $\mathcal{N}_{n-1}$ again, but the determinant of this new matrix and that of the initial one are symmetric about the origin.

Based on the above arguments, we may decompose $\det A$ into two parts as

$$\det A = a_{12}(-1)^{n-s+1}x \pm y, \quad (5.2)$$

where $x \in \mathcal{DN}_s$, with $1 \leq s \leq n-3$, or $x = 0$, and $y \in \mathcal{DN}_{n-1}$.

Table 4: The values of $b_{n,k}$ for $2 \leq n \leq 8$

k	0	1							
$b_{2,k}$	1	1							
k	-2	-1	0	1					
$b_{3,k}$	1	3	3	1					
k	-2	-1	0	1	2	3			
$b_{4,k}$	3	13	22	18	7	1			
k	-4	-3	-2	-1	0	1	2	3	4
$b_{5,k}$	1	16	92	240	326	240	92	16	1
k	-6	-5	-4	-3	-2	-1	0	1	2
$b_{6,k}$	2	30	214	1066	3636	7628	9548	7028	2922
k	3	4	5						
$b_{6,k}$	630	62	2						
k	-8	-7	-6	-5	-4	-3	-2	-1	0
$b_{7,k}$	1	17	232	2120	13452	63724	207224	428248	555638
k	1	2	3	4	5	6	7	8	9
$b_{7,k}$	457590	244888	90552	25964	6156	1160	168	17	1
k	-12	-11	-10	-9	-8	-7	-6		
$b_{8,k}$	2	38	381	2780	16171	80550	354039		
k	-5	-4	-3	-2	-1	0	1		
$b_{8,k}$	1377344	4649491	13445260	31419442	54090152	64559982	52316604		
k	2	3	4	5	6	7	8		
$b_{8,k}$	29101438	11739824	3815224	1092494	285729	69724	15383		
k	9	10	11	12					
$b_{8,k}$	2910	443	48	3					

Next, taking some entries in the third row as variables, we get the determinants

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & a & b & c & d & e \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{vmatrix} = -9 + 2a - b - c - d + e \quad (5.6)$$

and

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & a & b & c & d & e \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{vmatrix} = 9 + 2a - b - c - d + e. \quad (5.7)$$

After choosing suitable values for a, b, c, d, e , we can obtain matrices whose determinants are in $[-12, -10]$ or $[10, 12]$.

In order to make the conclusion that $\mathcal{DN}_8 = [-12, 12]$, we still need to show that $\det A \neq \pm 13$, for any $A \in \mathcal{N}_8$. From (5.2), if $\det A = 13$ for some matrix $A \in \mathcal{N}_8$, then $a_{12} = 1, s = 5, x = 4, \pm y = 9$, i.e., A is of the form shown in (5.7) (because after expanding along the second column, one of the obtained matrices has order 7 and determinant -9), but its maximal determinant is just 12. Similarly, if $\det A = -13$, then A is of the form shown in (5.6), and its minimal determinant is just -12 .

In conclusion, $\mathcal{DN}_8 = [-12, 12]$, and thus $b_8 = a_{10} = 25$. So Conjecture 1 is false when $n = 10$.

Similarly, we may get $\det A \in [-18, 17]$ for $A \in \mathcal{N}_9$ ($\mathcal{DN}_9$ is not necessarily $[-18, 17]$, may be it is just a subset of $[-18, 17]$), so $b_9 = a_{11} \leq 36$, i.e., Conjecture 1 is also false when $n = 11$ (since $A_{212264_{11}} = 37$).

6 The distribution of determinants for matrices in $\mathcal{N}_n$

Let $b_{n,k}$ be the number of matrices in $\mathcal{N}_n$ whose determinants are k . Clearly,

$$\sum_k b_{n,k} = 2^{\frac{n(n-1)}{2}}.$$

We list all the $b_{n,k}$ for $2 \leq n \leq 8$ and any possible k in Table 4. These values are obtained by Algorithms 1 and 2 outlined below.

From the structure of matrices in $\mathcal{N}_n$ (see (2.3)), if the matrix is of order n , then there are $\frac{n(n-1)}{2}$ unknown entries (all of which lie above the main diagonal), taking 0 or 1. It is natural that we may associate these $\frac{n(n-1)}{2}$ values with a binary bit array of cardinality $\frac{n(n-1)}{2}$ according to the lexicographic ordering. For example, in a matrix $A = (a_{ij})$ of order 3, the unknown entries are a_{12}, a_{13}, a_{23} listed in the lexicographic ordering, we would assign $a_{12} := A[0], a_{13} := A[1], a_{23} := A[2]$ according to Algorithm 1. It provides us a way to construct a one-to-one correspondence between matrices in $\mathcal{N}_n$ and the integers ranging from 0 to $2^{\frac{n(n-1)}{2}} - 1$.

Table 5: The determinantal expressions $C[n]$, for $2 \leq n \leq 8$, used in Algorithm 2

n	$C[n]$
2	$1 - A[0]$
3	$A[1] - A[0] - A[2]$
4	$A[1] - A[2] - A[3] + A[4] + A[0] \cdot A[5]$
5	$-A[2] + A[3] + A[5] - A[6] + A[0] \cdot A[7] - A[0] \cdot A[8] - A[1] \cdot A[9] + A[4] \cdot A[9]$
6	$A[3] - A[4] - A[7] + A[8] - A[0] \cdot A[10] + A[0] \cdot A[11] - A[1] \cdot A[12] + A[1] \cdot A[13]$ $+ A[2] \cdot A[14] + A[5] \cdot A[12] - A[5] \cdot A[13] - A[6] \cdot A[14] - A[0] \cdot A[9] \cdot A[14]$
7	$A[5] - A[4] + A[9] - A[10] + A[0] \cdot A[13] - A[0] \cdot A[14] + A[1] \cdot A[16] - A[1] \cdot A[17]$ $+ A[2] \cdot A[18] - A[2] \cdot A[19] - A[6] \cdot A[16] - A[3] \cdot A[20] + A[6] \cdot A[17] - A[7] \cdot A[18]$ $+ A[7] \cdot A[19] + A[8] \cdot A[20] - A[0] \cdot A[11] \cdot A[18] + A[0] \cdot A[11] \cdot A[19]$ $+ A[0] \cdot A[12] \cdot A[20] + A[1] \cdot A[15] \cdot A[20] - A[6] \cdot A[15] \cdot A[20]$
8	$A[5] - A[6] - A[11] + A[12] - A[0] \cdot A[16] + A[0] \cdot A[17] - A[1] \cdot A[20] + A[1] \cdot A[21]$ $- A[2] \cdot A[23] + A[2] \cdot A[24] + A[7] \cdot A[20] - A[3] \cdot A[25] - A[7] \cdot A[21] + A[3] \cdot A[26]$ $+ A[4] \cdot A[27] + A[8] \cdot A[23] - A[8] \cdot A[24] + A[9] \cdot A[25] - A[9] \cdot A[26] - A[10] \cdot A[27]$ $+ A[0] \cdot A[13] \cdot A[23] - A[0] \cdot A[13] \cdot A[24] + A[0] \cdot A[14] \cdot A[25] - A[0] \cdot A[14] \cdot A[26]$ $- A[0] \cdot A[15] \cdot A[27] + A[1] \cdot A[18] \cdot A[25] - A[1] \cdot A[18] \cdot A[26] - A[1] \cdot A[19] \cdot A[27]$ $- A[7] \cdot A[18] \cdot A[25] - A[2] \cdot A[22] \cdot A[27] + A[7] \cdot A[18] \cdot A[26] + A[7] \cdot A[19] \cdot A[27]$ $+ A[8] \cdot A[22] \cdot A[27] + A[0] \cdot A[13] \cdot A[22] \cdot A[27]$

Based on the aforementioned one-to-one correspondence, we are ready to enumerate the number of matrices in $\mathcal{N}_n$ sharing the same determinants (i.e., generate $b_{n,k}$). This can be realized by using Algorithm 2, where we use $C[n]$ to represent the determinants of matrices in $\mathcal{N}_n$, in terms of the binary bit array $A = [A[0], A[1], \dots, A[2^{\frac{n(n-1)}{2}} - 1]]$ produced by Algorithm 1. With the help of MATLAB, we obtain all the $C[n]$, for $2 \leq n \leq 8$ (Table 5).

Algorithm 1 Num2Binary(num, r)

Input: num and r , where num is a given integer, and r is the number of binary bits of num .

Output: A binary bit array $A = [A[0], A[1], \dots, A[r - 1]]$ corresponds to integer num .

```

1:  $i = r - 1$ 
2: for  $j = 0$  to  $r - 1$  do
3:    $A[j] = 0$  //Initialize array  $A$ 
4: while  $num$  not equal to 0 do
5:    $t = num \bmod 2$ 
6:    $A[i] = t$ 
7:    $num = num \gg 1$  //Equals to  $num = num/2$ 
8:    $i = i - 1$ 
9: return  $A$ 

```

Algorithm 2 Bohemian-counter(n, k)

Input: n and k , when our object is the matrices in $\mathcal{N}_n$, and k is the determinant of some matrix in $\mathcal{N}_n$.

Output: $b_{n,k}$, the number of matrices in $\mathcal{N}_n$ whose determinants are k .

```

1:  $l = n(n - 1)/2$ 
2:  $m = 2^l$ 
3:  $count = 0$ 
4: for  $i = m - 1$  downto 0 do
5:    $A[] = \text{Num2Binary}(i, l)$ 
6:   if  $n$  equal to 4 then
7:      $temp = C[4]$  //  $C[n]$  represents the determinants of matrices in  $\mathcal{N}_n$ 
8:   else if  $n$  equal to other order  $t$  then
9:      $temp = C[t]$ 
10:    ...
11:   if  $temp$  equal to  $k$  then
12:      $count = count + 1$ 
13: return  $count$ 

```

7 Concluding remarks

In this paper, a conjecture (Conjecture 1) about Bohemian matrices is considered in detail, following the steps of several relevant publications on this topic, e.g., [3,4,6,7,10]. We disprove the conjecture. The theoretical analysis and algorithmic extension are included. More precisely, we show that Conjecture 1 is false when $n = 10$. Our technique can also be extended to disprove the case $n = 11$ and so on. In order to reveal the distribution of determinants of matrices in $\mathcal{N}_n$, we build an algorithm to generate such statistics (i.e., $b_{n,k}$), for $2 \leq n \leq 8$.

Based on these results, we propose the following problems:

- (1) What is the explicit formula for a_n (or, equivalently, for b_n)?
- (2) Is there a recurrence relation for a_n (or, equivalently, for b_n)?
- (3) Which matrix would attain the maximum or minimum determinant among matrices in $\mathcal{M}_n$ or $\mathcal{N}_n$?
Or, determine the matrix attaining the maximum absolute value of determinants?
- (4) What asymptotic distribution would the determinants of matrices in $\mathcal{M}_n$ or $\mathcal{N}_n$ (i.e., $b_{n,k}$) approach to?
Or, determine the k such that $b_{n,k}$ is maximum, for a fixed n .

Funding information: This work was supported by the National Natural Science Foundation of China (Grant No. 11701505).

Conflict of interest: The authors state no conflict of interest. C. M. da Fonseca is an Editor of the Open Mathematics and was not involved in the review process of this article.

References

- [1] O. Taussky, *Some computational problems involving integral matrices*, J. Res. Natl. Bur. Stand. B, Math. Sci. **65** (1961), 15–17.
- [2] O. Taussky, *Matrices of rational integers*, Bull. Am. Math. Soc. **66** (1960), 327–345.
- [3] E. Y. S. Chan, R. M. Corless, L. Gonzalez-Vega, J. R. Sendra, J. Sendra, and S. E. Thornton, *Upper Hessenberg and Toeplitz Bohemians*, Linear Algebra Appl. **601** (2020), 72–100, DOI: <https://doi.org/10.1016/j.laa.2020.03.037>.
- [4] R. M. Corless and S. E. Thornton, *The Bohemian eigenvalue project*, ACM Commun. Comput. Algebra **50** (2016), no. 4, 158–160.
- [5] R. M. Corless and S. E. Thornton, *Visualizing eigenvalues of random matrices*, ACM Commun. Comput. Algebra **50** (2016), 35–39.
- [6] S. E. Thornton, *The Characteristic Polynomial Database*, 2018, available at <http://bohemianmatrices.com/cpdb>.
- [7] M. Fasi and G. M. N. Porzio, *Determinants of normalized Bohemian upper Hessenberg matrices*, Electron. J. Linear Algebra **36** (2020), 352–366, DOI: <https://doi.org/10.13001/ela.2020.5053>.
- [8] M. Fasi, J. Feng, and G. M. N. Porzio, *Corrigendum to “Determinants of normalized Bohemian upper Hessenberg matrices”*, Electron. J. Linear Algebra **37** (2021), 160–162, DOI: <https://doi.org/10.13001/ela.2021.5849>.
- [9] J. Feng and H. Fan, *A study on the determinants of some Hessenberg Toeplitz Bohemians*, Preprints 2020, 2020080272, DOI: <https://doi.org/10.20944/preprints202008.0272.v1>.
- [10] J. P. Keating and A. A. Keleş, *Maximum absolute determinants of upper Hessenberg Bohemian matrices*, <https://arxiv.org/abs/2003.00454v2>, (2020).
- [11] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences (OEIS)*, <http://oeis.org/>.
- [12] C. M. da Fonseca, *An identity between the determinant and the permanent of Hessenberg type-matrices*, Czechoslovak Math. J. **61** (2011), 917–921, DOI: <https://doi.org/10.1007/s10587-011-0059-1>.
- [13] M. Janjić, *Hessenberg matrices and integer sequences*, J. Integer Seq. **13** (2010), 10.7.8.
- [14] J. T. Jia, *Numerical algorithms for the determinant evaluation of general Hessenberg matrices*, J. Math. Chem. **56** (2018), 247–256, DOI: <https://doi.org/10.1007/s10910-017-0794-0>.
- [15] E. Kılıç and T. Arıkan, *Evaluation of Hessenberg determinants via generating function approach*, Filomat **31** (2017), no. 15, 4945–4962, DOI: <https://doi.org/10.2298/FIL1715945K>.
- [16] M. Merca, *A note on the determinant of a Toeplitz-Hessenberg matrix*, Spec. Matrices **1** (2013), 10–16, DOI: <https://doi.org/10.2478/spma-2013-0003>.
- [17] M. Andelić and C. M. da Fonseca, *Some determinantal considerations for pentadiagonal matrices*, Linear Multilinear Algebra (2021), DOI: <https://doi.org/10.1080/03081087.2019.1708845>.
- [18] Z. Du and C. M. da Fonseca, *A periodic determinantal property for $(0, 1)$ double banded matrices*, Linear Multilinear Algebra (2021), DOI: <https://doi.org/10.1080/03081087.2021.1913980>.