Research Article

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The (1, 2)-step competition graph of a hypertournament

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Abstract: In 2011, Factor and Merz [Discrete Appl. Math. **159** (2011), 100–103] defined the (1, 2)-step competition graph of a digraph. Given a digraph $D = (V, A)$, the (1, 2)-step competition graph of D , denoted $C_{1,2}(D)$, is a graph on $V(D)$, where $xy \in E(C_{1,2}(D))$ if and only if there exists a vertex $z \neq x, y$ such that either $d_{D-y}(x, z) = 1$ and $d_{D-x}(y, z) \leq 2$ or $d_{D-x}(y, z) = 1$ and $d_{D-y}(x, z) \leq 2$. They also characterized the (1, 2)-step competition graphs of tournaments and extended some results to the (i, j) -step competition graphs of tournaments. In this paper, the definition of the (1, 2)-step competition graph of a digraph is generalized to a hypertournament and the (1, 2)-step competition graph of a k -hypertournament is characterized. Also, the results are extended to (i, j) -step competition graphs of k -hypertournaments.

Keywords: k -hypertournament, (1, 2)-step competition graph, (i, j) -step competition graph

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1 Terminology and introduction

Let $G = (V, E)$ be an undirected graph, or a graph for short. $V(G)$ and $E(G)$ are the vertex set and the edge set of G , respectively. The complement G^c of a graph G is the graph with vertex set $V(G)$ in which two vertices are adjacent if and only if they are not adjacent in G . Let G_1 and G_2 be two graphs. The union of G_1 and G_2 , denoted by $G_1 \cup G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.

Let $D = (V, A)$ be a directed graph, or a digraph for short. $V(D)$ and $A(D)$ are the vertex set and the arc set of D , respectively. Let $i \geq 1, j \geq 1$. The (i, j) -step competition graph of D , denoted $C_{i,j}(D)$, is a graph on $V(D)$, where $xy \in E(C_{i,j}(D))$ if and only if there exists a vertex $z \neq x, y$ such that either $d_{D-y}(x, z) \leq i$ and $d_{D-x}(y, z) \leq j$ or $d_{D-x}(y, z) \leq i$ and $d_{D-y}(x, z) \leq j$. When $(i, j) = (1, 1)$, it is also called the competition graph of D .

The notion of competition graph was introduced by Cohen [1] as a means of determining the smallest dimension of ecological phase space. In recent years, many researchers investigated m -step competition graphs of some special digraphs and the competition numbers of some graphs etc. (see [2–4]). Particularly, in 1998, Fisher, Lundgren, Merz and Reid [5] studied the domination graphs and competition graphs of a tournament. Recall that a tournament is an orientation of a complete graph. In 2011, Factor and Merz [6] gave the definition of the (i, j) -step competition graph of a digraph. They also characterized the (1, 2)-step competition graph of a tournament and extended some results to the (i, j) -step competition graph of a tournament. They proved the following theorems related to this paper.

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Theorem 1.1. [6] *A graph G on $n \geq 5$ vertices is the $(1, 2)$ -step competition graph of some strong tournament if and only if G is K_n , $K_n - E(P_2)$, or $K_n - E(P_3)$.*

Theorem 1.2. [6] *G , a graph on n vertices, is the $(1, 2)$ -step competition graph of some tournament if and only if G is one of the following graphs:*

1. K_n , where $n \neq 2, 3, 4$;
2. $K_{n-1} \cup K_1$, where $n > 1$;
3. $K_n - E(P_3)$, where $n > 2$;
4. $K_n - E(P_2)$, where $n \neq 1, 4$, or
5. $K_n - E(K_3)$, where $n \geq 3$.

Theorem 1.3. [6] *If T is a tournament with n vertices, $i \geq 1$ and $j \geq 2$, then $C_{i,j}(T) = C_{1,2}(T)$.*

Given two integers n and k , $n \geq k > 1$, a k -hypertournament T on n vertices is a pair (V, A) , where V is a set of vertices, $|V| = n$ and A is a set of k -tuples of vertices, called arcs, so that for any k -subset S of V , A contains exactly one of the $k!$ k -tuples whose entries belong to S . As usual, we use $V(T)$ and $A(T)$ to denote the vertex set and the arc set of T , respectively. Clearly, a 2-hypertournament is merely a tournament. When $k = n$, the hypertournament has only one arc and it does not have much significance to study. Thus, in what follows, we consider $3 \leq k \leq n - 1$.

Let $T = (V, A)$ be a k -hypertournament on n vertices. For an arc a of T , $T - a$ denotes a hyperdigraph obtained from T by removing the arc a and $\bar{a}$ denotes the set of vertices contained in a . If $v_i, v_j \in \bar{a}$ and v_i precedes v_j in a , we say that v_i dominates v_j in a . We also say the vertex v_j is an out-neighbour of v_i and use the following notation:

$$N_T^+(v_i) = \{v_j \in V \setminus \{v_i\} : v_i \text{ precedes } v_j \text{ in some arc } a \in A(T)\}.$$

We will omit the subscript T if the k -hypertournament T is known from the context.

A path P in a k -hypertournament T is a sequence $v_1 a_1 v_2 a_2 v_3 \cdots v_{t-1} a_{t-1} v_t$ of distinct vertices $v_1, v_2, \dots, v_t$, $t \geq 1$ and distinct arcs $a_1, a_2, \dots, a_{t-1}$ such that v_i precedes v_{i+1} in a_i , $1 \leq i \leq t - 1$. Meanwhile, let the vertex set $V(P) = \{v_1, v_2, \dots, v_t\}$ and the arc set $A(P) = \{a_1, a_2, \dots, a_{t-1}\}$. The length of a path P is the number of its arcs, denoted $\ell(P)$. A path from x to y is an (x, y) -path. The k -hypertournament T is called strong if T has an (x, y) -path for every pair x, y of distinct vertices in T .

A k -hypertournament T is said to be transitive if its vertices are labeled $v_1, v_2, \dots, v_n$ in such an order so that $i < j$ if and only if v_i precedes v_j in each arc containing v_i and v_j .

Now we generalize the $(1, 2)$ -step competition graph of a digraph to a k -hypertournament.

Definition 1.4. The (i, j) -step competition graph of a k -hypertournament T with $i \geq 1$ and $j \geq 1$, denoted $C_{i,j}(T)$, is a graph on $V(T)$, where $xy \in E(C_{i,j}(T))$ if and only if there exist a vertex $z \neq x, y$ and an (x, z) -path P and a (y, z) -path Q satisfying the following:

- (a) $y \notin V(P)$, $x \notin V(Q)$;
- (b) $\ell(P) \leq i$ and $\ell(Q) \leq j$, or $\ell(Q) \leq i$ and $\ell(P) \leq j$;
- (c) P and Q are arc-disjoint.

If $xy \in E(C_{i,j}(T))$, we say that x and y (i, j) -step compete. In particular, we say that x and y compete if $\ell(P) = 1$ and $\ell(Q) = 1$. $C_{1,1}(T)$ is also called the competition graph of the k -hypertournament T . Clearly, when $k = 2$, T is a tournament and $C_{i,j}(T)$ is the (i, j) -step competition graph of T .

The k -hypertournaments form one of the most interesting class of digraphs. For the class of k -hypertournaments, the popular topics are the Hamiltonicity and vertex-pancyclicity (see [7–10]). Besides, some researchers investigated the degree sequences and score sequences of k -hypertournaments (see [11,12]). Recently, the H -force set of a hypertournament was also studied (see [13]). In this paper, we study the $(1, 2)$ -step competition graph of a k -hypertournament and extend Theorems 1.1–1.3 to k -hypertournaments.

In Sections 2 and 3, useful lemmas are provided to make the proof of the main results easier. In Sections 4 and 5, the $(1, 2)$ -step competition graph of a (strong) k -hypertournament is characterized. In Section 6, the main results are extended to the (i, j) -step competition graph of a k -hypertournament.

2 The missing edges of $C_{1,2}(T)$

Let $T = (V, A)$ be a k -hypertournament. For a pair of distinct vertices x and y in T , $A_T(x, y)$ denotes the set of all arcs of T in which x precedes y , $A_T\{x, y\}$ denotes the set of all arcs containing x, y in T and $A_T^*\{x, y\}$ denotes the set of all arcs containing x, y in T and in which neither x nor y is the last entry.

Lemma 2.1. *Let T be a k -hypertournament with n vertices, where $3 \leq k \leq n - 1$. Then $xy \notin E(C_{1,2}(T))$ if and only if one of the following holds:*

- (a) $N^+(x) = \emptyset$;
- (b) $N^+(y) = \emptyset$;
- (c) $N^+(x) = \{y\}$;
- (d) $N^+(y) = \{x\}$;
- (e) $A_T^*\{x, y\}$ contains exactly an arc a , and $N_{T-a}^+(x) \subseteq \{y\}$, $N_{T-a}^+(y) \subseteq \{x\}$.

Proof. First, we show the “if” part. Clearly, if one of (a)–(d) holds, we have $xy \notin E(C_{1,2}(T))$. Now we assume that the argument (e) holds. Since $A_T^*\{x, y\}$ contains exactly an arc a , and $N_{T-a}^+(x) \subseteq \{y\}$, $N_{T-a}^+(y) \subseteq \{x\}$, we have to use the unique arc a to obtain the out-neighbour except y of x and the out-neighbour except x of y . So x and y are impossible to (1, 2)-step compete and hence $xy \notin E(C_{1,2}(T))$.

Now we show the “only if” part. Assume that $xy \notin E(C_{1,2}(T))$. Also, assume that x and y do not satisfy (a)–(d). That means $N^+(x) \setminus \{y\} \neq \emptyset$, $N^+(y) \setminus \{x\} \neq \emptyset$. Now we show that x and y satisfy (e). Suppose $A_T^*\{x, y\}$ consists of at least two arcs, say $a_1, a_2 \in A_T^*\{x, y\}$. Let w_i be the last entry of a_i for $i = 1, 2$. If $w_1 = w_2$, then x and y compete, a contradiction. So assume $w_1 \neq w_2$. Note that $\binom{n-2}{k-2} - 2 \leq |A_T\{w_1, w_2\} \setminus \{a_1, a_2\}| \leq \binom{n-2}{k-2}$. For $(n, k) \neq (4, 3)$, we have $|A_T\{w_1, w_2\} \setminus \{a_1, a_2\}| \geq 1$. For $(n, k) = (4, 3)$, since both a_1 and a_2 contain x, y and $a_1 \neq a_2$, we have $a_1, a_2 \notin A_T\{w_1, w_2\}$ and hence $|A_T\{w_1, w_2\} \setminus \{a_1, a_2\}| = |A_T\{w_1, w_2\}| = 2$. Let $a_3 \in A_T\{w_1, w_2\} \setminus \{a_1, a_2\}$. W.l.o.g., $a_3 \in A_T(w_1, w_2)$. Then $P = xa_1w_1a_3w_2$ and $Q = ya_2w_2$ are the paths such that x and y (1, 2)-step compete, a contradiction. So $A_T^*\{x, y\}$ consists of exactly an arc a . Suppose $N_{T-a}^+(x) \setminus \{y\} \neq \emptyset$. Then there exists an arc b distinct from a such that x has an out-neighbour distinct from y in b . Similar to the aforementioned proof, whether or not the last entries of a and b are same, we always have $xy \in E(C_{1,2}(T))$, a contradiction. So $N_{T-a}^+(x) \subseteq \{y\}$. Similarly, $N_{T-a}^+(y) \subseteq \{x\}$. Thus, $A_T^*\{x, y\}$ contains exactly an arc a , and $N_{T-a}^+(x) \subseteq \{y\}$, $N_{T-a}^+(y) \subseteq \{x\}$.

The lemma holds. □

By the proof of Lemma 2.1, we obtain the following result.

Corollary 2.2. *Let T be a strong k -hypertournament with n vertices, where $3 \leq k \leq n - 1$. Then $xy \notin E(C_{1,2}(T))$ if and only if one of the following holds:*

- (a) $N^+(x) = \{y\}$;
- (b) $N^+(y) = \{x\}$;
- (c) $A_T^*\{x, y\}$ contains exactly an arc a , and $N_{T-a}^+(x) \subseteq \{y\}$, $N_{T-a}^+(y) \subseteq \{x\}$.

3 The forbidden subgraphs of $(C_{1,2}(T))^c$

Lemma 3.1. *Let G on n vertices be the (1, 2)-step competition graph of some k -hypertournament T , where $3 \leq k \leq n - 1$. Then the complement G^c of G does not contain a pair of disjoint edges.*

Proof. Suppose the complement G^c of G contains a pair of disjoint edges, say xy and zw . So $xy, zw \notin E(G)$ and x, y, z, w are distinct. By Lemma 2.1, we have xy satisfies one of the cases (a)–(e) and zw satisfies one of the cases (a)–(e).

Suppose that at least one of xy and zw satisfies one of the cases (a)–(d). W.l.o.g., we assume that $N^+(x) \subseteq \{y\}$. Then it must be true that the vertex z dominates x in each arc containing x, z but not containing w . Meanwhile, it must be true that the vertex w dominates x in each arc containing x, w but not containing z . So z and w compete and hence $zw \in E(C_{1,2}(T)) = E(G)$, a contradiction. Thus, both xy and zw satisfy (e).

However, since $A_T^*\{x, y\}$ contains exactly an arc a , and $N_{T-a}^+(x) \subseteq \{y\}$, $N_{T-a}^+(y) \subseteq \{x\}$, we have the vertex x must be the last entry in each arc containing x, z, w but not containing y and the vertex y must be the last entry in each arc containing y, z, w but not containing x . Thus, $A_T^*\{z, w\}$ contains at least two arcs, which contradicts the fact that zw satisfies (e).

The lemma holds. $\square$

Lemma 3.2. *Let G on n vertices be the $(1, 2)$ -step competition graph of some k -hypertournament T , where $3 \leq k \leq n - 1$. Then the complement G^c of G does not contain 3-cycle.*

Proof. Suppose to the contrary that the complement G^c of G contains 3-cycle, say $xyzx$. So $xy, xz, yz \notin E(G)$ and x, y, z are distinct. By Lemma 2.1, we have xy, xz and yz satisfy one of the cases (a)–(e), respectively.

Claim 1. None of xy, xz and yz satisfies the case (a) or (b).

Proof. Suppose at least one of xy, xz and yz satisfies the case (a) or (b). W.l.o.g., we assume that xy satisfies (a), i.e. $N^+(x) = \emptyset$. Then it must be true that the vertex y dominates x in each arc containing x, y but not containing z . Also, the vertex z dominates x in each arc containing x, z but not containing y . So y and z compete and $yz \in E(C_{1,2}(T)) = E(G)$, a contradiction. Thus, none of xy, xz and yz satisfies the case (a) or (b). $\square$

Claim 2. At most one of xy, xz and yz satisfies the case (c) or (d).

Proof. Suppose at least two edges among xy, xz and yz satisfy the case (c) or (d). W.l.o.g., we assume that both xy and yz satisfy (c) or (d). We consider the following four cases.

Case 1: Both xy and yz satisfy (c). It means that $N^+(x) = \{y\}$, $N^+(y) = \{z\}$. If xz satisfies (c), i.e. $N^+(x) = \{z\}$, it contradicts $N^+(x) = \{y\}$. If xz satisfies (d), i.e. $N^+(z) = \{x\}$, the arcs containing simultaneously x, y, z do not satisfy $N^+(x) = \{y\}$, $N^+(y) = \{z\}$ and $N^+(z) = \{x\}$, a contradiction. If xz satisfies (e), i.e. $A_T^*\{x, z\}$ contains exactly an arc a , then there exists a vertex w such that $w \in N^+(x)$. Since $N^+(x) = \{y\}$, we have $w = y$. So x is the second last entry, y is the last entry and z is any other entry in a . Then the vertex z dominates y in a . Also, the vertex x dominates y in each arc containing x, y but not containing z . So $xz \in E(C_{1,2}(T)) = E(G)$, a contradiction.

Case 2: Both xy and yz satisfy (d). It means that $N^+(y) = \{x\}$, $N^+(z) = \{y\}$. Similarly to Case 1, we can also get a contradiction.

Case 3: xy satisfies (c) and yz satisfies (d). It means that $N^+(x) = \{y\}$, $N^+(z) = \{y\}$. Now the arcs containing simultaneously x, y, z do not satisfy $N^+(x) = \{y\}$, $N^+(z) = \{y\}$, a contradiction.

Case 4: xy satisfies (d) and yz satisfies (c). It means that $N^+(y) = \{x\}$ and $N^+(y) = \{z\}$. Then $x = z$, a contradiction.

Thus, at most one of xy, xz and yz satisfies the case (c) or (d). $\square$

Claim 3. At most one of xy, xz and yz satisfies the case (e).

Proof. Suppose at least two edges among xy, xz and yz satisfy the case (e). W.l.o.g., we assume that both xz and yz satisfy (e). From the assumption that xz satisfies (e), we get the vertex y dominates x in each arc containing x, y but not containing z . From the assumption that yz satisfies (e), we get the vertex x dominates y in each arc containing x, y but not containing z . This is a contradiction. Thus, at most one of xy, xz and yz satisfies the case (e). $\square$

By Claims 1–3, it is impossible that $xy, xz, yz \notin E(G)$ hold simultaneously. Thus, the complement G^c of G does not contain 3-cycle. The lemma holds. $\square$

Lemma 3.3. *Let G on n vertices be the (1, 2)-step competition graph of some k -hypertournament T , where $3 \leq k \leq n - 1$. Then the complement G^c of G does not contain $K_{1,3}$, unless $G = K_{n-1} \cup K_1$.*

Proof. Let T be a k -hypertournament on n vertices, where $3 \leq k \leq n - 1$, and G the (1, 2)-step competition graph of T . Assume $G \neq K_{n-1} \cup K_1$. Now we show that the complement G^c of G does not contain $K_{1,3}$. Suppose not. Let $\{x, y, z, w\}$ and $\{xy, xz, xw\}$ be the vertex set and edge set of the subgraph $K_{1,3}$, respectively. So $xy, xz, xw \notin E(G)$. By Lemma 2.1, we have xy, xz and xw satisfy one of the cases (a)–(e).

Claim 1. None of xy, xz and xw satisfies the case (a) or (b).

Proof. Suppose at least one of xy, xz and xw satisfies the case (a) or (b). W.l.o.g., we assume that xy satisfies (a), i.e. $N^+(x) = \emptyset$. Let $V(T) = \{v_1, v_2, \dots, v_n\}$ and $x = v_n$. By Lemma 2.1, for all $1 \leq i \leq n - 1$, we have $v_i v_n \notin E(C_{1,2}(T))$. We claim that for all $1 \leq i < j \leq n - 1$, $v_i v_j \in E(C_{1,2}(T))$. Indeed,

- For $1 \leq i < j \leq n - (k - 1)$, the vertex v_i dominates v_n by the arc consisting of $v_i, \dots, v_{i+(k-3)}, v_{n-1}, v_n$ and the vertex v_j dominates v_n by the arc consisting of $v_j, \dots, v_{j+(k-3)}, v_{n-1}, v_n$. Then v_i and v_j compete and $v_i v_j \in E(C_{1,2}(T))$.
- For $n - (k - 2) \leq i < j \leq n - 1$, the vertex v_i dominates v_n by the arc consisting of $v_{n-k}, \dots, v_{n-3}, v_{n-1}, v_n$ for $n - (k - 2) \leq i \leq n - 3$ and by the arc consisting of $v_1, \dots, v_{1+(k-3)}, v_{n-2}, v_n$ for $i = n - 2$ and the vertex v_j dominates v_n by the arc consisting of $v_{n-(k-1)}, \dots, v_{n-2}, v_{n-1}, v_n$. Then v_i and v_j compete and $v_i v_j \in E(C_{1,2}(T))$.
- For $1 \leq i \leq n - k$ and $n - (k - 2) \leq j \leq n - 1$, the vertex v_i dominates v_n by the arc consisting of $v_i, \dots, v_{i+(k-3)}, v_{n-1}, v_n$ and the vertex v_j dominates v_n by the arc consisting of $v_{n-(k-1)}, \dots, v_{n-2}, v_{n-1}, v_n$. Then v_i and v_j compete and $v_i v_j \in E(C_{1,2}(T))$.
- For $i = n - (k - 1)$ and $n - (k - 2) \leq j \leq n - 1$, the vertex $v_{n-(k-1)}$ dominates v_n by the arc consisting of $v_1, \dots, v_{1+(k-3)}, v_{n-2}, v_n$ for $k = 3$ and by the arc consisting of $v_{n-k}, v_{n-(k-1)}, \dots, v_{n-3}, v_{n-1}, v_n$ for $4 \leq k \leq n - 1$ and the vertex v_j dominates v_n by the arc consisting of $v_{n-(k-1)}, \dots, v_{n-2}, v_{n-1}, v_n$. Then v_j and $v_{n-(k-1)}$ compete and $v_j v_{n-(k-1)} \in E(C_{1,2}(T))$.

Then $C_{1,2}(T) = K_{n-1} \cup K_1$, a contradiction. Thus, none of xy, xz and xw satisfies the case (a) or (b). $\square$

Claim 2. At most one of xy, xz and xw satisfies the case (c) or (d).

Proof. Suppose at least two edges among xy, xz and xw satisfy the case (c) or (d). W.l.o.g., we assume that both xy and xz satisfy the case (c) or (d). We consider the following four cases.

Case 1: Both xy and xz satisfy (c). It means that $N^+(x) = \{y\}$ and $N^+(x) = \{z\}$. Then $y = z$, a contradiction.

Case 2: Both xy and xz satisfy (d). It means that $N^+(y) = \{x\}$ and $N^+(z) = \{x\}$. Now the arcs containing simultaneously x, y, z do not satisfy $N^+(y) = \{x\}$, $N^+(z) = \{x\}$, a contradiction.

Case 3: xy satisfies (c) and xz satisfies (d). It means that $N^+(x) = \{y\}$ and $N^+(z) = \{x\}$. If xw satisfies (c), then $N^+(x) = \{w\}$, contradicting $N^+(x) = \{y\}$. If xw satisfies (d), then $N^+(w) = \{x\}$. Now the arcs containing simultaneously x, z, w do not satisfy $N^+(z) = \{x\}$, $N^+(w) = \{x\}$, a contradiction. If xw satisfies (e). Let b be an arc containing x, z, w . $N^+(x) = \{y\}$ yields z dominates x in b . $N^+(z) = \{x\}$ yields z is the second last entry, x is the last entry and w is any other entry of b . Clearly, $b \notin A_T^*(x, w)$, i.e. $z \in N_{T-a}^+(w)$, which contradicts the fact that $N_{T-a}^+(w) \subseteq \{x\}$.

Case 4: xy satisfies (d) and xz satisfies (c). Similarly to Case 3, we can also get a contradiction.

Thus, at most one of xy, xz and xw satisfies the case (c) or (d). $\square$

Claim 3. At most one of xy, xz and xw satisfies the case (e).

Proof. Suppose at least two edges among xy, xz and xw satisfy the case (e). Assume that xy and xz satisfy (e). From the assumption that xy satisfies (e), we get the vertex z dominates y in each arc containing y, z but

not containing x . From the assumption that xz satisfies (e), we get the vertex y dominates z in each arc containing y, z but not containing x . This is a contradiction. Thus, at most one of xy, xz and xw satisfies the case (e). $\square$

By Claims 1–3, it is impossible that $xy, xz, xw \notin E(G)$ hold simultaneously. Thus, the complement G^c of G does not contain $K_{1,3}$ unless $G = K_{n-1} \cup K_1$. The lemma holds. $\square$

By Corollary 2.2 and the proof of Lemma 3.3, we obtain the following result.

Corollary 3.4. *Let G on n vertices be the $(1, 2)$ -step competition graph of some strong k -hypertournament T , where $3 \leq k \leq n - 1$. Then the complement G^c of G does not contain $K_{1,3}$.*

4 Strong k -hypertournaments

Theorem 4.1. *A graph G on n vertices is the $(1, 2)$ -step competition graph of some strong k -hypertournament T with $3 \leq k \leq n - 1$ if and only if G is $K_n, K_n - E(P_2)$, or $K_n - E(P_3)$.*

Proof. We first show the “if” part. Let T be a transitive k -hypertournament with the vertices $v_1, v_2, \dots, v_n$. Let T_1 be a k -hypertournament obtained from T by replacing the arc $(v_1, v_2, v_{n-(k-3)}, \dots, v_n)$ with $(v_n, \dots, v_{n-(k-3)}, v_2, v_1)$. It is easy to check that T_1 is strong. Now we show that $C_{1,2}(T_1) = K_n - E(P_2)$. For convenience, let $a = (v_n, \dots, v_{n-(k-3)}, v_2, v_1)$. We claim that $v_{n-1}v_n \notin E(C_{1,2}(T_1))$. Indeed, for $k = 3$, the vertex v_{n-1} has a unique out-neighbour v_n and Corollary 2.2(a) implies $v_{n-1}v_n \notin E(C_{1,2}(T_1))$. For $4 \leq k \leq n - 1$, $A_{T_1}^+(v_{n-1}, v_n)$ contains exactly an arc a , and $N_{T_1-a}^+(v_{n-1}) = \{v_n\}$, $N_{T_1-a}^+(v_n) = \emptyset$. Corollary 2.2(c) implies $v_{n-1}v_n \notin E(C_{1,2}(T_1))$. We also claim $v_i v_j \in E(C_{1,2}(T_1))$ for all $\{i, j\} \neq \{n-1, n\}$. W.l.o.g., we assume $i < j$.

- For $1 \leq i < j \leq n - (k - 1)$, v_i dominates v_n by the arc $(v_i, v_{n-(k-2)}, \dots, v_{n-1}, v_n)$ and v_j dominates v_n by the arc $(v_j, v_{n-(k-2)}, \dots, v_{n-1}, v_n)$. Then v_i and v_j compete and $v_i v_j \in E(C_{1,2}(T_1))$.
- For $n - (k - 2) \leq i < j \leq n - 1$, v_i dominates v_n by the arc $(v_1, v_{n-(k-2)}, \dots, v_i, \dots, v_j, \dots, v_{n-1}, v_n)$ and v_j dominates v_n by the arc $(v_2, v_{n-(k-2)}, \dots, v_i, \dots, v_j, \dots, v_{n-1}, v_n)$. Then v_i and v_j compete and $v_i v_j \in E(C_{1,2}(T_1))$.
- For $i = 1$ and $n - (k - 2) \leq j \leq n - 1$, v_1 dominates v_n by the arc $(v_1, v_{n-(k-2)}, \dots, v_{n-1}, v_n)$ and v_j dominates v_n by the arc $(v_2, v_{n-(k-2)}, \dots, v_j, \dots, v_{n-1}, v_n)$. Then v_1 and v_j compete and $v_1 v_j \in E(C_{1,2}(T_1))$.
- For $2 \leq i \leq n - (k - 1)$ and $n - (k - 2) \leq j \leq n - 1$, v_i dominates v_n by the arc $(v_i, v_{n-(k-2)}, \dots, v_{n-1}, v_n)$ and v_j dominates v_n by the arc $(v_1, v_{n-(k-2)}, \dots, v_j, \dots, v_{n-1}, v_n)$. Then v_i and v_j compete and $v_i v_j \in E(C_{1,2}(T_1))$.
- For $i = 1$ and $j = n$, v_n dominates v_2 by the arc a , v_2 dominates v_{n-1} by the arc $(v_2, v_{n-(k-2)}, \dots, v_{n-1}, v_n)$ and v_1 dominates v_{n-1} by the arc $(v_1, v_{n-(k-2)}, \dots, v_{n-1}, v_n)$. Then v_1 and v_n (1, 2)-step compete and $v_1 v_n \in E(C_{1,2}(T_1))$.
- For $2 \leq i \leq n - 2$ and $j = n$, v_n dominates v_1 by the arc a , v_1 dominates v_{n-1} by the arc $(v_1, v_{n-(k-2)}, \dots, v_{n-1}, v_n)$ and v_i dominates v_{n-1} by $(v_i, v_{n-(k-2)}, \dots, v_{n-1}, v_n)$ for $2 \leq i \leq n - (k - 1)$ and by $(v_2, v_{n-(k-2)}, \dots, v_i, \dots, v_{n-1}, v_n)$ for $n - (k - 2) \leq i \leq n - 2$. Then v_i and v_n (1, 2)-step compete and $v_i v_n \in E(C_{1,2}(T_1))$.

Thus, $C_{1,2}(T_1) = K_n - E(P_2)$.

Let T_2 be a k -hypertournament obtained from T_1 above by replacing the arc $(v_1, v_2, v_{n-(k-2)}, \dots, v_{n-1})$ with $(v_{n-1}, \dots, v_{n-(k-2)}, v_2, v_1)$. It is easy to check that T_2 is strong. Now we show that $C_{1,2}(T_2) = K_n$.

- For $\{i, j\} \neq \{n-1, n\}$, similarly to the proof of T_1 , we have $v_i v_j \in E(C_{1,2}(T_2))$.
- For $i = n - 1$ and $j = n$, v_{n-1} dominates v_1 by the arc $(v_{n-1}, \dots, v_{n-(k-2)}, v_2, v_1)$ and v_n dominates v_1 by the arc $(v_n, \dots, v_{n-(k-3)}, v_2, v_1)$. Then v_{n-1} and v_n compete and $v_{n-1}v_n \in E(C_{1,2}(T_2))$.

Thus, $C_{1,2}(T_2) = K_n$.

Let T_3 be a k -hypertournament with the vertices $v_1, v_2, \dots, v_n$ satisfying the following:

- (1) Each arc excluding v_1, v_2 satisfies $i < j$ if and only if v_i precedes v_j ;

- (2) Each arc including v_1, v_2 satisfies that v_1 is the second last entry, v_2 is the last entry and the remaining $k - 2$ entries satisfy $i < j$ if and only if v_i precedes v_j ;
- (3) Each arc including v_1 but excluding v_2 satisfies that v_1 is the last entry and the remaining $k - 1$ entries satisfy $i < j$ if and only if v_i precedes v_j ;
- (4) Each arc including v_2 but excluding v_1, v_3 satisfies that v_2 is the last entry and the remaining $k - 1$ entries satisfy $i < j$ if and only if v_i precedes v_j ;
- (5) Each arc including v_2, v_3 but excluding v_1 satisfies that v_2 is the second last entry, v_3 is the last entry and the remaining $k - 2$ entries satisfy $i < j$ if and only if v_i precedes v_j .

It is easy to check that T_3 is strong. Now we show that $C_{1,2}(T_3) = K_n - E(P_3)$. Note that $N^+(v_1) = \{v_2\}$ and $N^+(v_2) = \{v_3\}$. By Corollary 2.2(a), we have $v_1v_2, v_2v_3 \notin E(C_{1,2}(T_3))$. Now we consider the arc v_iv_j for $\{i, j\} \neq \{1, 2\}$ and $\{i, j\} \neq \{2, 3\}$. W.l.o.g., we assume $i < j$.

- For $3 \leq i < j \leq n - (k - 3)$, v_i dominates v_2 by the arc $(v_i, \dots, v_{i+(k-3)}, v_1, v_2)$ and v_j dominates v_2 by the arc $(v_j, \dots, v_{j+(k-3)}, v_1, v_2)$. Then v_i and v_j compete and $v_iv_j \in E(C_{1,2}(T_3))$.
- For $n - (k - 4) \leq i < j \leq n$, v_i dominates v_2 by the arc $(v_{n-(k-2)}, \dots, v_i, \dots, v_j, \dots, v_{n-1}, v_1, v_2)$ and v_j dominates v_2 by the arc $(v_{n-(k-3)}, \dots, v_i, \dots, v_j, \dots, v_n, v_1, v_2)$. Then v_i and v_j compete and $v_iv_j \in E(C_{1,2}(T_3))$.
- For $3 \leq i \leq n - (k - 3)$ and $n - (k - 4) \leq j \leq n$, v_i dominates v_2 by the arc $(v_i, \dots, v_{i+(k-3)}, v_1, v_2)$ for $3 \leq i \leq n - (k - 2)$ and by the arc $(v_{n-(k-2)}, v_{n-(k-3)}, \dots, v_{n-1}, v_1, v_2)$ for $i = n - (k - 3)$ and v_j dominates v_2 by the arc $(v_{n-(k-3)}, \dots, v_j, \dots, v_n, v_1, v_2)$. Then v_i and v_j compete and $v_iv_j \in E(C_{1,2}(T_3))$.
- For $i = 1$ and $3 \leq j \leq n - (k - 2)$, v_1 dominates v_2 by the arc $(v_{n-(k-3)}, \dots, v_n, v_1, v_2)$ and v_j dominates v_2 by the arc $(v_j, \dots, v_{j+(k-3)}, v_1, v_2)$. Then v_1 and v_j compete and $v_1v_j \in E(C_{1,2}(T_3))$.
- For $i = 1$ and $n - (k - 3) \leq j \leq n$, v_1 dominates v_2 by the arc $(v_3, \dots, v_k, v_1, v_2)$ and v_j dominates v_2 by the arc $(v_{n-(k-3)}, \dots, v_j, \dots, v_n, v_1, v_2)$. Then v_1 and v_j compete and $v_1v_j \in E(C_{1,2}(T_3))$.
- For $i = 2$ and $4 \leq j \leq n - (k - 3)$, v_2 dominates v_3 by the arc $(v_{n-(k-3)}, \dots, v_n, v_2, v_3)$, v_3 dominates v_1 by the arc $(v_3, \dots, v_k, v_1, v_2)$ and v_j dominates v_1 by the arc $(v_j, \dots, v_{j+(k-3)}, v_1, v_2)$. Then v_2 and v_j (1, 2)-step compete and $v_2v_j \in E(C_{1,2}(T_3))$.
- For $i = 2$ and $n - (k - 4) \leq j \leq n$, v_2 dominates v_3 by the arc $(v_{n-(k-3)}, \dots, v_n, v_2, v_3)$, v_3 dominates v_1 by the arc $(v_3, \dots, v_k, v_1, v_2)$ and v_j dominates v_1 by the arc $(v_{n-(k-3)}, \dots, v_j, \dots, v_n, v_1, v_2)$. Then v_2 and v_j (1, 2)-step compete and $v_2v_j \in E(C_{1,2}(T_3))$.

Thus, $C_{1,2}(T_3) = K_n - E(P_3)$.

Now we show the “only if” part. Let T be a strong k -hypertournament and G the (1, 2)-step competition graph of T . We show that G is K_n , $K_n - E(P_2)$, or $K_n - E(P_3)$. We claim that G^c contains at most two edges. Suppose to the contrary that G^c contains at least three edges, say $e_1, e_2, e_3 \in E(G^c)$. Let $e_i = x_iy_i$ for $i = 1, 2, 3$. By Lemma 3.1, e_1 and e_2 have a common end-point. W.l.o.g., we assume that $y_1 = x_2$. By Lemma 3.1, e_3 and e_1 have a common end-point, and e_3 and e_2 have also a common end-point. So either $e_3 = x_1y_2$ or x_2 is an end-point of e_3 . However, this implies G^c contains 3-cycle or $K_{1,3}$, which contradicts Lemma 3.2 and Corollary 3.4. So G^c contains at most two edges. Thus, if G^c contains two edges, Lemma 3.1 implies $G = K_n - E(P_3)$; if G^c contains one edge, then $G = K_n - E(P_2)$; if G^c contains no edge, then $G = K_n$.

Therefore, the theorem holds. □

5 Remaining k -hypertournaments

Theorem 5.1. *A graph G on n vertices is the (1, 2)-step competition graph of some k -hypertournament T with $3 \leq k \leq n - 1$ if and only if G is K_n , $K_n - E(P_2)$, $K_n - E(P_3)$, or $K_{n-1} \cup K_1$.*

Proof. The “if” part follows from the proof of Lemma 3.3 and Theorem 4.1. Now we show the “only if” part. Let T be a k -hypertournament and G the (1, 2)-step competition graph of T . We show that G is K_n , $K_n - E(P_2)$,

$K_n - E(P_3)$, or $K_{n-1} \cup K_1$. Similarly to the proof of “only if” of Theorem 4.1, we get G^c contains at most two edges unless $G = K_{n-1} \cup K_1$. Thus, if G^c contains two edges, Lemma 3.1 implies $G = K_n - E(P_3)$; if G^c contains one edge, then $G = K_n - E(P_2)$; if G^c contains no edge, then $G = K_n$.

Therefore, the theorem holds. $\square$

6 The (i, j) -step competition graph of a k -hypertournament

We generalize the $(1, 2)$ -step competition graph to the (i, j) -step competition graph as follows. By the definition of the (i, j) -step competition graph for a k -hypertournament T , we obtain that if $i \geq 1$, $j \geq 2$, then $E(C_{1,2}(T)) \subseteq E(C_{i,j}(T))$. It is easy to see that the proof of Lemma 2.1 implies the following corollary.

Corollary 6.1. *Let T be a k -hypertournament with n vertices satisfying $3 \leq k \leq n - 1$ and $i \geq 1$, $j \geq 2$ integers. Then $xy \notin E(C_{i,j}(T))$ if and only if one of the following holds:*

- (a) $N^+(x) = \emptyset$;
- (b) $N^+(y) = \emptyset$;
- (c) $N^+(x) = \{y\}$;
- (d) $N^+(y) = \{x\}$;
- (e) $A_T^*\{x, y\}$ contains exactly an arc a , and $N_{T-a}^+(x) \subseteq \{y\}$, $N_{T-a}^+(y) \subseteq \{x\}$.

Theorem 6.2. *Let T be a k -hypertournament with n vertices satisfying $3 \leq k \leq n - 1$ and $i \geq 1$, $j \geq 2$ integers. Then $C_{i,j}(T) = C_{1,2}(T)$.*

Proof. Clearly, $V(C_{i,j}(T)) = V(C_{1,2}(T)) = V(T)$. Since $E(C_{1,2}(T)) \subseteq E(C_{i,j}(T))$, it suffices to show that $E(C_{i,j}(T)) \subseteq E(C_{1,2}(T))$. Let $xy \in E(C_{i,j}(T))$. Suppose $xy \notin E(C_{1,2}(T))$. By Lemma 2.1, x and y must satisfy one of the cases (a)–(e). This contradicts Corollary 6.1. Thus, $xy \in E(C_{1,2}(T))$ and $E(C_{i,j}(T)) \subseteq E(C_{1,2}(T))$. $\square$

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