Research Article

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Infinitesimals via Cauchy sequences: Refining the classical equivalence

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Abstract: A refinement of the classic equivalence relation among Cauchy sequences yields a useful infinitesimal-enriched number system. Such an approach can be seen as formalizing Cauchy’s sentiment that a null sequence “becomes” an infinitesimal. We signal a little-noticed construction of a system with infinitesimals in a 1910 publication by Giuseppe Peano, reversing his earlier endorsement of Cantor’s belittling of infinitesimals.

Keywords: Cauchy sequence, hyperreal, infinitesimal

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1 Historical background

Robinson developed his framework for analysis with infinitesimals in his 1966 book [1]. There have been various attempts either

- (1) To obtain clarity and uniformity in developing analysis with infinitesimals by working in a version of set theory with a concept of infinitesimal built in syntactically (see, e.g., [2,3]), or
- (2) To define nonstandard universes in a simplified way by providing mathematical structures that would not have all the features of a Hewitt-Luxemburg-style ultrapower [4,5], but nevertheless would suffice for basic application and possibly could be helpful in teaching.

The second approach has been carried out, for example, by James Henle in his non-nonstandard analysis [6] (based on Schmieden-Laugwitz [7]; see also [8]), as well as by Terry Tao in his “cheap nonstandard analysis” [9]. These frameworks are based upon the identification of real sequences whenever they are eventually equal.¹ A precursor of this approach is found in the work of Giuseppe Peano. In his 1910 paper [10] (see also [11]) he introduced the notion of the end (*fine*; pl. *fini*) of a function (i.e., sequence) based upon its eventual behavior. Equality and order between such *fini* are defined as follows:

- $\text{fine}(f) = \text{fine}(g)$ if and only if $f(n) = g(n)$ eventually;
- $\text{fine}(f) < \text{fine}(g)$ if and only if $f(n) < g(n)$ eventually.

The collection

$$\text{Pe} = \{\text{fine}(f) : \mathbb{N} \xrightarrow{f} \mathbb{R}\} \quad (1.1)$$

¹ I.e., modulo the Fréchet filter.

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extends the set of real numbers (included as constant sequences), is partially ordered, and includes infinite and infinitesimal elements. For instance, if $f(n) = n$, then $\text{fine}(f) > r$ for every $r \in \mathbb{R}$, since eventually $f(n) > r$. Similarly, if $g(n) = n^{-1}$, then $0 < \text{fine}(g) < r$ for every positive $r \in \mathbb{R}$, since eventually $0 < g(n) < r$. Moreover, Peano defines operations on the *fini*. For instance, $\text{fine}(f) + \text{fine}(g)$ can be defined as $\text{fine}(f + g)$, and $\text{fine}(f) \cdot \text{fine}(g)$, as $\text{fine}(fg)$. The construction results in Peano's partially ordered non-Archimedean ring Pe of (1.1) with zero divisors that extend $\mathbb{R}$. Commenting on Peano's 1910 construction, Fisher notes that here

Peano contradicts his contention of 1892, following Cantor, that constant infinitesimals are impossible. [12, p. 154]

Peano's 1910 article seems to have been overlooked by Freguglia who claims "to put Peano's opinion about the unacceptability of the actual infinitesimal notion into evidence" [13, p. 145].

2 Refining the equivalence relation on Cauchy sequences

The present text belongs to neither of the categories summarized in Section 1. Rather, we propose to exploit a concept that is a household word for most of the mathematical audience to a greater extent than either Fréchet filters or ends of functions, namely Cauchy sequences. More precisely, we propose to factor the classical homomorphism

$$C \rightarrow \mathbb{R},$$

from the ring C of Cauchy sequences to its quotient space $\mathbb{R}$, through an intermediate integral domain D , by refining the traditional equivalence relation on C . The composition

$$C \rightarrow D \rightarrow \mathbb{R}$$

("undoing" the refinement) is the classical homomorphism.

The classical construction of the real line $\mathbb{R}$ involves declaring Cauchy sequences $u = (u_i)$ and $v = (v_i)$ to be equivalent if the sequence $(u_i - v_i)$ tends to zero. Instead, we define an equivalence relation $\sim$ on C by setting $u \sim v$ if and only if they actually coincide on a *dominant* set of indices i (this would be true in particular if $\text{fine}(u) = \text{fine}(v)$). The notion of dominance is relative to a nonprincipal ultrafilter $\mathcal{U}$ on the set of natural numbers.² The complement of a dominant set is called negligible. Namely, we have

$$u \sim v \text{ if and only if } \{i \in \mathbb{N} : u_i = v_i\} \in \mathcal{U}. \quad (2.1)$$

We set $D = C/\sim$. Then a null sequence generates an infinitesimal of D . By further identifying all null sequences with the constant sequence (0), we obtain an epimorphism

$$\mathbf{sh} : D \rightarrow \mathbb{R}, \quad (2.2)$$

assigning to each Cauchy sequence, the value of its limit in $\mathbb{R}$. It turns out that such a framework is sufficient to develop infinitesimal analysis in the sense specified in Section 3.

3 Null sequences, infinitesimals, and P-points

For the purposes of the definition below, it is convenient to distinguish notationally between $\mathbb{N}$ used as the index set of sequences and ω used as the collection on which our filters are defined.

² An ultrafilter on $\mathbb{N}$ is a maximal collection of subsets of $\mathbb{N}$ which is closed under finite intersection and passage to a superset. An ultrafilter is nonprincipal if and only if it includes no finite subsets of $\mathbb{N}$.

Definition 3.1. A non-principal ultrafilter $\mathcal{U}$ on ω is called a *P-point* if for every partition $\{C_n : n \in \mathbb{N}\}$ of ω such that

$$(\forall n \in \mathbb{N}) C_n \notin \mathcal{U},$$

there exists some $A \in \mathcal{U}$ such that $A \cap C_n$ is a finite set for each n .

Informally, $\mathcal{U}$ being a P-point means that every partition of ω into negligible sets is actually a partition into *finite* sets, up to a $\mathcal{U}$ -negligible subset of ω . The existence of P-point ultrafilters is consistent with the Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC) as is their nonexistence, as shown by Shelah; see [14]. Their existence is guaranteed by the Continuum Hypothesis by Rudin [15].

Theorem 3.2 originates with Choquet [16]; see Benci and Di Nasso [17, Proposition 6.6], and the end of this section for additional historical remarks.

Denote by $C \subseteq \mathbb{R}^{\mathbb{N}}$ the ring of Cauchy sequences (i.e., convergent sequences) of real numbers, and by D its quotient by the relation (2.1).

Theorem 3.2. $\mathcal{U}$ is a P-point ultrafilter if and only if D is isomorphic to the ring of finite hyperreals in $\mathbb{R}^{\mathbb{N}}/\mathcal{U}$ where the epimorphism $\mathbf{sh}$ of (2.2) corresponds to the shadow/standard part.

Proof. Suppose every bounded sequence becomes Cauchy when restricted to a suitable $\mathcal{U}$ -dominant set $Y \subseteq \mathbb{N}$. Let $X_1 = \mathbb{N}$, $X_2, X_3, \dots$ be an inclusion-decreasing sequence of sets in $\mathcal{U}$. Define a sequence f by setting $f(i) = \frac{1}{n}$ for $i \in X_n \setminus X_{n+1}$. By hypothesis, there is a $Y \in \mathcal{U}$ such that f is Cauchy on Y . Suppose the intersection $Y \cap (X_n \setminus X_{n+1})$ is infinite for some n . Then there are infinitely many i where $f(i) = \frac{1}{n}$ and also infinitely many i where $f(i) < \frac{1}{n+1}$ (this happens for all $i \in Y \cap X_{n+2} \in \mathcal{U}$). Such a sequence is not Cauchy on Y . The contradiction shows that every intersection with Y must be finite, proving that $\mathcal{U}$ is a P-point.

Conversely, suppose $\mathcal{U}$ is a P-point. Let f be a bounded sequence. By binary divide-and-conquer, we construct inductively a sequence of nested compact segments $S_i \subseteq \mathbb{R}$ of length tending to zero such that for each i , the sequence f takes values in S_i for a dominant set $X_i \subseteq \mathbb{N}$ of indices. By hypothesis, there exists $Y \in \mathcal{U}$ such that each complement $S_i \setminus Y$ is finite. Then the restriction of f to the dominant set Y necessarily tends to the intersection point $\bigcap_{i \in \mathbb{N}} S_i \in \mathbb{R}$.

Benci and Di Nasso gave a similar argument relating *monotone* sequences and selective ultrafilters in [17, p. 376]. $\square$

If $\mathcal{U}$ is not a P-point, the natural monomorphism $D \rightarrow \mathbb{R}^{\mathbb{N}}/\mathcal{U}$ will not be onto the finite part of the ultrapower $\mathbb{R}^{\mathbb{N}}/\mathcal{U}$.

Let $I \subseteq D$ be the ideal of infinitesimals, and denote by I^{-1} the set of inverses of nonzero infinitesimals.

Corollary 3.3. If $\mathcal{U}$ is a P-point ultrafilter, then the hyperreal field $\mathbb{R}^{\mathbb{N}}/\mathcal{U}$ decomposes as a disjoint union $D \sqcup I^{-1}$.

Here every element of I^{-1} can be represented by a sequence tending to infinity. In this sense, Cauchy sequences of reals (together with a refined equivalence relation) suffice to construct the hyperreal field $\mathbb{R}^* = \mathbb{R}^{\mathbb{N}}/\mathcal{U}$ and enable analysis with infinitesimals. Thus, if f is a continuous real function on $[0, 1]$, its extension f^* to the hyperreal interval $[0, 1]^* \subseteq \mathbb{R}^*$ maps the equivalence class of a Cauchy sequence (u_n) to the equivalence class of the Cauchy sequence $(f(u_n))$. The derivative of f at a real point $x \in [0, 1]$ is the shadow

$$\mathbf{sh} \left(\frac{f^*(x + \varepsilon) - f^*(x)}{\varepsilon} \right)$$

for infinitesimal $\varepsilon \neq 0$, while the integral $\int_0^1 f(x) dx$ is the shadow of the sum $\sum_{i=1}^{\mu} f^*(x_i) \varepsilon$ where the x_i are the partition points of $[0, 1]^*$ into (an infinite integer) μ subintervals and $\varepsilon = \frac{1}{\mu}$. For the definition of infinite Riemann sums see Keisler [18].

We elaborate on the history of the problem. According to Cleave's hypothesis [19], every infinitesimal $\varepsilon \in \mathbb{R}^*$ is determined by a sequence of real numbers tending to zero. Cutland et al. [20] showed that the Cleave hypothesis requires $\mathcal{U}$ to be a P-point ultrafilter. Benci and Di Nasso [17, p. 376] developed the problem further by relating the Cleave hypothesis to the so-called selective ultrafilters and monotone sequences.

4 Concluding remarks

1. Our approach can be seen as a formalization of Cauchy's sentiment that a null sequence "becomes" an infinitesimal, while a sequence tending to infinity becomes an infinite number; see, e.g., Bair et al. [21].
2. A perspective on hyperreal numbers via Cauchy sequences has value as an educational tool according to the post [22]. Formally, one could start with the full ring of sequences (Cauchy or not), and form the traditional quotient modulo a nonprincipal ultrafilter, to obtain the standard construction of the hyperreal field where every null sequence generates an infinitesimal (see, e.g., [18, p. 913], [23]; but there may be additional infinitesimals). However, pedagogical experience shows that the full ring of sequences sometimes appears as a nebulous object to students who are not yet familiar with the ultrafilter construction (see, e.g., the discussion at [22]); starting with more familiar objects such as Cauchy sequences builds upon the students' previous experience and may be more successful in facilitating intuitions about infinitesimals.
3. The refinement of the equivalence relation on Cauchy sequences (modulo the existence of P-points) enables one to obtain the (finite) hyperreals as the set of equivalence classes of Cauchy sequences, and both equivalence classes and Cauchy sequences are household concepts for any mathematics sophomore. We still use ultrafilters; however, in this setting, they are conceptually similar to maximal ideals, routinely used in undergraduate algebra.
4. The distinction between procedures and ontology is a key related issue. Historical mathematical pioneers from Fermat to Cauchy applied certain procedures in their mathematics, and while modern set-theoretic *ontology* appears to be beyond their conceptual world, many of the *procedures* of modern mathematics are not (see [24] for a more detailed discussion). Arguably, the procedures in modern infinitesimal analysis are closer to theirs than procedures in modern Weierstrassian analysis. Note that the mathematical pioneers would have been just as puzzled by the Cantorian set theory as by ultrafilters. For example, Leibniz would have likely rejected the Cantorian set theory as incoherent because contrary to the part-whole axiom; see [25].
5. Bertrand Russell accepted, as a matter of ontological certainty, Cantor's position concerning the non-existence of infinitesimals. For an analysis of a dissenting opinion by contemporary neo-Kantians see [26].
6. Robinson's framework for analysis with infinitesimals is the first (and currently the only) framework meeting the Klein-Fraenkel criteria for a successful theory of infinitesimals in terms of an infinitesimal treatment of the mean-value theorem and an approach to the definite integral via partitions into infinitesimal segments; see [27].
7. Arguably, Robinson's framework is the most successful theory of infinitesimals in applications to natural science, probability, and related fields; see [28] as well as [29,30].
8. Robinson explained his choice of the name for his theory as follows: "The resulting subject was called by me Non-standard Analysis since it involves and was, in part, inspired by the so-called Non-standard models of Arithmetic whose existence was first pointed out by T. Skolem" [1, p. vii].
9. Currently there are two popular approaches to Robinson's mathematics: model-theoretic and axiomatic/syntactic. Since Skolem's construction [31] did not use either the Axiom of Choice or ultrafilters, it is natural to ask whether one can develop an approach to analysis with infinitesimals, meeting the Klein-Fraenkel criteria, which does not refer to the notion of an ultrafilter at all. The answer is affirmative and

was provided in [3] via an internal axiomatic approach. Hrbacek and Katz [32] presented a construction of Loeb measures and nonstandard hulls in internal set theories. The effectiveness of infinitesimal methods in analysis has recently been explored in reverse mathematics; see e.g., Sanders [33].

Conflict of interest: Authors state no conflict of interest.

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