Research Article

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Path homology theory of edge-colored graphs

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Abstract: In this paper, we introduce the category and the homotopy category of edge-colored digraphs and construct the functorial homology theory on the foundation of the path homology theory provided by Grigoryan, Muranov, and Shing-Tung Yau. We give the construction of the path homology theory for edge-colored graphs that follows immediately from the consideration of natural functor from the category of graphs to the subcategory of symmetrical digraphs. We describe the natural filtration of path homology groups of any digraph equipped with edge coloring, provide the definition of the corresponding spectral sequence, and obtain commutative diagrams and braids of exact sequences.

Keywords: path homology groups, edge coloring, edge-colored path, homotopy-colored graphs, spectral sequence

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1 Introduction

In this paper, we continue to study the homological properties of colored digraphs and graphs. It should be mentioned that, on the basic concepts of the path homology theory introduced in [1–3], the collection of path homology theories for vertex colored (di)graphs have been already constructed in [4].

The path homology theory is a homology theory for digraphs that computes the simplicial homology of a finite simplicial complex S if applied to its incidence digraph $G = G_S$ defined in the following way (see [1,3]). For a finite simplicial complex S , let V be the set of its simplexes. Consider a digraph $G = (V, E)$ with the set of vertices V as above and the set of arrows E such that $\sigma \rightarrow \tau$ is an arrow if and only if $\tau \subset \sigma$. Then the path homology of the digraph G is naturally isomorphic to the simplicial homology of the simplicial complex S . The cohomology theory of digraphs that is dual to the path homology theory was introduced in previous studies [5–8]. This cohomology theory was motivated by the physical applications of discrete mathematics. This theory provides a differential calculus on digraphs and discrete sets which are considered as discretizations of topological spaces. Thus, various physical theories can be formulated on a discrete set analogous to the continuum case.

Recently, the path homology theory has been used in applications of the persistent homology to the various types of networks (see, for e.g., [9,10]). So in [10] the directed networks related to applications are considered and efficient algorithms for computing one-dimensional path homology and its persistent version are developed.

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In what follows, we provide the category of edge-colored digraphs together with the notion of the homotopy and the homotopy category of edge-colored digraphs. For any edge-colored digraph, we define the collection of edge-colored path homology groups and indicate the possibility of the functorial passage from the category of graphs to the subcategory of symmetrical digraphs. We also prove the colored homotopy invariance of the colored path homology and describe its algebraic properties. More precisely, we discuss the natural filtration of the path homology groups of the edge-colored digraph, construct commutative diagrams and braids of exact sequences for those homology groups, and describe the spectral sequence that is associated with the filtration. This paper contains many examples to illustrate the non-triviality of edge-colored homology groups. We discuss also possible applications of the constructed theory.

2 Preliminaries

In this section, for the sake of convenience, we review some basic notions of various categories of digraphs, graphs (see [11–13]), and the path homology theory (see [1–3]).

A *graph* $G = (V, E)$ is a nonempty set V of objects called *vertices* together with a set E of non ordered pairs $\{v, w\} \in E$ of distinct vertices $v, w \in V$ called *edges*.

A *directed graph* or simply a *digraph* $G = (V, E)$ is a nonempty set V of objects called *vertices* together with a set E of ordered pairs of distinct vertices of V called *directed edges* or *arrows*. A pair $(v, w) \in E$ is denoted $v \rightarrow w$, whereas the vertices v and w are called, respectively, the *origin* and the *end* of the given arrow. Accordingly, we write $v = \text{orig}(v \rightarrow w)$, $w = \text{end}(v \rightarrow w)$.

Two different edges (arrows) $e, e' \in E_G$ of a graph (digraph) G are called *incident* if they have at least one common vertex.

Let $G = (V, E)$ be a graph. The corresponding *symmetric digraph* $\tilde{G} = (\tilde{V}, \tilde{E})$ is built from the same set of vertices $\tilde{V} = V$ but each edge $\{v, w\} \in E$ gives rise to two arrows, namely $(v \rightarrow w), (w \rightarrow v) \in \tilde{E}$.

Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be any graphs. A *mapping* $f : G \rightarrow H$ between two graphs is defined as a mapping $f : V_G \rightarrow V_H$ such that for any edge $\{v, w\} \in E_G$, we have either $\{f(v), f(w)\} \in E_H$ or $f(v) = f(w) \in V_H$. The mapping f is called a *homomorphism* if $\{f(v), f(w)\} \in E_H$ for any $\{v, w\} \in E_G$.

Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be any digraphs this time. A *digraph mapping* $f : G \rightarrow H$ (or simply a *mapping*) from a digraph G to a digraph H is a mapping $f : V_G \rightarrow V_H$ such that for any arrow $(v \rightarrow w) \in E_G$, we have either $(f(v) \rightarrow f(w)) \in E_H$ or $f(v) = f(w) \in V_H$. The mapping f is called a *homomorphism* if $(f(v) \rightarrow f(w)) \in E_H$ for any $(v \rightarrow w) \in E_G$.

Having in mind all the considerations above, we point out that we are thus provided with the following categories: the category $\mathcal{G}$ of graphs and graph mappings, the subcategory $\mathcal{NG}$ of $\mathcal{G}$ with the same objects and with the morphisms given by homomorphisms, the category $\mathcal{D}$ of digraphs and digraph mappings, and the subcategory $\mathcal{ND}$ of $\mathcal{D}$ with the same objects and with the morphisms given by homomorphisms. It is easy to check that the passing from a graph G to a symmetric digraph $\tilde{G}$ naturally defines the mapping of morphisms and gives the functors $S : \mathcal{G} \rightarrow \mathcal{D}$ and $NS : \mathcal{NG} \rightarrow \mathcal{ND}$.

Let G and H be any graphs. We define their *Box product* $G \square H$ as a graph with the set of vertices $V_{G \square H} = V_G \times V_H$ and the set of edges $E_{G \square H}$ such that $\{(x, y), (x', y')\} \in E_{G \square H}$ if and only if either $x = x'$ and $\{y, y'\} \in E_H$ or $\{x, x'\} \in E_G$ and $y = y'$.

We define the *Box product* $G \square H$ of two digraphs G and H in a similar fashion, namely as a digraph with the set of vertices $V_{G \square H} = V_G \times V_H$ and the set of edges $E_{G \square H}$ such that there is an arrow $((x, y) \rightarrow (x', y')) \in E_{G \square H}$ if and only if either $x = x'$ and $y \rightarrow y'$ or $x \rightarrow x'$ and $y = y'$.

Fix $n \geq 0$. Denote by J_n a graph with the set of vertices $\{0, 1, \dots, n\}$ and the set of edges $\{\{i, i+1\} \mid 0 \leq i \leq n-1\}$. From now on J_n will be called a *segment graph*. Denote by I_n any digraph with the set of vertices $\{0, 1, \dots, n\}$ and the set of edges in which there is exactly one of the arrows: $i \rightarrow i+1$ or $i+1 \rightarrow i$ for $0 \leq i \leq n-1$. Such I_n will be called a *segment digraph*.

Two graph mappings $f_0, f_1 : G \rightarrow H$ are called *homotopic* and denoted accordingly $f_0 \simeq f_1$ if there exists a segment graph J_n and a mapping $F : G \square J_n \rightarrow H$ called a *homotopy* between f_0 and f_1 such that

$$F|_{G \square \{0\}} = f_0 : G \square \{0\} \rightarrow H, \quad F|_{G \square \{n\}} = f_1 : G \square \{n\} \rightarrow H.$$

Two graph homomorphisms are called *strongly homotopic* if the homotopy between them is also a homomorphism.

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Two digraph homomorphisms are called *strongly homotopic* if the homotopy between them is a homomorphism. In the case of homotopy F for which $n = 1$, the map F is called the *one-step homotopy of (di)graphs*.

Now we give a brief explanation of the notion of homotopy in the case of graphs that is similar to the case of digraph given in [2]. There exists a *one-step homotopy* F between graph mappings $f : G \rightarrow H$ and $g : G \rightarrow H$ if and only if either $f(x) = g(x) \in V_H$ or $\{f(x), g(x)\} \in E_H$ for every $x \in V_G$. It follows that two graph mappings f and g are homotopic if there is a finite sequence of graph mappings $f = f_0, f_1, \dots, f_n = g$ from G to H such that f_k and f_{k+1} are one-step homotopic (see also Examples 3.6, 3.7, and 3.9).

This time, it is easy to check that we are provided with the following categories: the category $\mathcal{G}'$ of graphs and classes of homotopic mappings, the category $\mathcal{NG}'$ with the same objects and with the morphisms given by the classes of strongly homotopic homomorphisms, the category $\mathcal{D}'$ of digraphs and classes of homotopic mappings, and the category $\mathcal{ND}'$ with the same objects and with the morphisms given by the classes of strongly homotopic homomorphisms.

It follows easily from [2, Proposition 6.5] that the functors S and NS preserve the relation to be homotopic and hence define functors $S' : \mathcal{G}' \rightarrow \mathcal{D}'$ and $NS' : \mathcal{NG}' \rightarrow \mathcal{ND}'$.

Now we turn our attention to the path homology groups of any (di)graph and provide necessary definitions. Let $G = (V, E)$ be a digraph and R be a commutative ring.

An *elementary p -path* denoted $e_{i_0 \dots i_p}$ is defined as any sequence $i_0, \dots, i_p$ of vertices. Let $\Lambda_p(V, R)$ be a free R -module generated by all elementary p -paths. The elements of Λ_p are called *p -paths*. We set $\Lambda_{-1} = 0$ and define the *boundary operator* $\partial : \Lambda_p \rightarrow \Lambda_{p-1}$ in the following way:

$$\partial e_{i_0 \dots i_p} = \begin{cases} \sum_{q=0}^p (-1)^q e_{i_0 \dots \widehat{i_q} \dots i_p} & \text{for } p \geq 1, \\ 0 & \text{for } p = 0. \end{cases}$$

The operator defined above has the property $\partial^2 = 0$.

For $p \geq 1$, let $I_p = I_p(V)$ be the submodule of Λ_p that is generated by all elementary paths for which two consecutive vertices are equal. Set $I_0 = I_{-1} = 0$. Then $\partial(I_p) \subset I_{p-1}$ and we obtain a chain factor complex $\mathcal{R}_* = \mathcal{R}_p(V, R) = \Lambda_p/I_p$ with the differential that is induced by ∂ . The elements of this module are called *regular paths* and *regular elementary paths* for basic elements. Now we return to the consideration of the digraph $G = (V, E)$. For $p \geq 1$, a regular elementary path $e_{i_0 \dots i_p}$ is called *allowed* if $(i_k \rightarrow i_{k+1})$ is an arrow of the digraph G for $0 \leq k \leq p-1$. For $p \geq 0$, let $\mathcal{A}_p = \mathcal{A}_p(G, R)$ be a submodule of $\mathcal{R}_p(V, R)$ that is generated by all the allowed elementary p -paths and set $\mathcal{A}_{-1} = 0$. Define a submodule $\Omega_p = \Omega_p(G, R) = \{v \in \mathcal{A}_p : \partial v \in \mathcal{A}_{p-1}\}$ of the module $\mathcal{A}_p$. The submodule Ω_p consists of all linear combinations v of allowed paths for which ∂v is a linear combination of allowed paths as well. Having all the above in mind, we obtain a chain complex $\Omega_*(G, R)$. The homologies of this chain complex are called *path homologies of the digraph G* and denoted by

$$H_p(G, R) := \ker \partial|_{\Omega_p} / \text{im } \partial|_{\Omega_{p+1}}.$$

For a graph G we define the *path homology* $H_p(G, R)$ setting $H_p(G, R) := H_p(\tilde{G}, R)$, where $\tilde{G}$ denotes the corresponding symmetric digraph.

For a digraph mapping $f : G \rightarrow H$, define for every $p \geq 0$ the induced map $f_* : \Lambda_p(V_G) \rightarrow \Lambda_p(V_H)$ given on the basic elements by the rule

$$f_*(e_{i_0 \dots i_p}) = e_{f(i_0) \dots f(i_p)}. \quad (2.1)$$

The map f_* is a morphism of chain complexes and it is clear that $f_*(I_p(V_G)) \subset I_p(V_H)$ [2]. Hence, we obtain an induced chain map of quotient chain complexes $f_* : \mathcal{R}_*(V_G) \rightarrow \mathcal{R}_*(V_H)$ that is defined on basic elements by the rule

$$f_*(e_{i_0 \dots i_p}) = \begin{cases} e_{f(i_0) \dots f(i_p)}, & \text{if } e_{f(i_0) \dots f(i_p)} \text{ is regular,} \\ 0, & \text{if } e_{f(i_0) \dots f(i_p)} \text{ otherwise.} \end{cases} \quad (2.2)$$

It follows from (2.2) that if a path $e_{i_0 \dots i_p} \in \mathcal{A}_p(G)$, then $f_*(e_{i_0 \dots i_p})$ is $0 \in \mathcal{A}_p(H)$ or is allowed. Hence, $f_*(\mathcal{A}_p(G)) \subset \mathcal{A}_p(H)$. Now the standard line of arguments provides a morphism of chain complexes $\Omega_*(G) \rightarrow \Omega_*(H)$ and a homomorphism of homology groups $H_*(G, R) \rightarrow H_*(H, R)$.

Thus, the path homology groups of digraphs and graphs are functorial and these groups are homotopy invariant [2].

3 Categories of edge-colored digraphs and graphs

In this section, we introduce several categories of edge-colored graphs and digraphs, describe their basic properties, and provide examples. In the next section, we shall use these categories for constructing the collection of edge-colored path homology theories.

An *edge coloring* of a graph (digraph) $G = (V_G, E_G)$ is given by an assignment of a color to each edge (arrow) $e \in E_G$. An edge coloring is called *proper* if incident edges (arrows) have distinct colors. An edge coloring is called *k-improper* if for any edge (arrow) $e \in E_G$ there exist at most k incident edges (arrows) having the same color as e . An edge coloring that uses k colors is called a *k-edge coloring*.

An edge coloring of a (graph) digraph $G = (V_G, E_G)$ can be considered as a pair (G, φ) , where $\varphi : E_G \rightarrow \mathbb{N}$ is a function. For the k -edge coloring, we assume that $\varphi : E_G \rightarrow \{1, \dots, k\}$. In what follows, we shall consider proper coloring as the k -improper coloring with $k = 0$. Since from now on only edge colorings will be considered, the word edge will be accordingly omitted.

Definition 3.1. Let (G, φ) and (H, ψ) be any colored graphs (digraphs). A graph (digraph) mapping $f : G \rightarrow H$ is called a *colored morphism* (or simply a *morphism*) if $\psi(f(e)) = \varphi(e)$ for all $e \in E_G$ with $f(e) \in E_H$.

Thus, we obtain the following categories: the category $\mathcal{CG}$ of colored graphs and the colored morphisms, the subcategory $\mathcal{CNG}$ with the same objects and with the colored morphisms that are homomorphisms, the category $\mathcal{CD}$ of colored digraphs and the colored morphisms, and the subcategory $\mathcal{CND}$ with the same objects and with the colored morphisms given by homomorphisms.

For a colored graph (G, φ) , we define a colored symmetric digraph $(\tilde{G}, \tilde{\varphi})$ by setting $\tilde{\varphi}(v \rightarrow w) = \tilde{\varphi}(w \rightarrow v) = \varphi(v, w)$. Thus, as in Section 2, we obtain the functors $\mathcal{CS} : \mathcal{CG} \rightarrow \mathcal{CD}$ and $\mathcal{CNS} : \mathcal{CNG} \rightarrow \mathcal{CND}$.

For a colored (di)graph (G, φ) , we can consider this (di)graph G without any coloring. Any morphism of colored (di)graphs $f : (G, \varphi) \rightarrow (H, \psi)$ is, in particular, a (di)graph mapping and we obtain a collection of forgetful functors from the categories of colored (di)graphs to the corresponding categories of (di)graphs.

Example 3.2. Consider the following colored digraph G :

$$a \xrightarrow{\text{blue}} b \xleftarrow{\text{red}} x \xrightarrow{\text{green}} y \xrightarrow{\text{red}} c$$

Let the mapping $f : G \rightarrow G$ be given on the set of vertices by

$$f(a) = a, \quad f(b) = f(c) = b, \quad f(x) = f(y) = x.$$

Then f is a morphism of colored digraphs.

For a colored digraph (G, φ) , define a function κ which returns the number of different colors that are being used for the coloring of arrows in a non empty set A of arrows in E_G . For every allowed path $v = e_{i_0 \dots i_p}$ in the colored digraph (G, φ) , we define $\kappa(v) = 0$ for $p = 0$ and, for $p \geq 1$, we put $\kappa(v) = \kappa(A)$, where A consists of arrows $i_0 \rightarrow i_1, \dots, i_{p-1} \rightarrow i_p$ of the path v . An allowed regular elementary path $v = e_{i_0 \dots i_p}$ ($p \geq 0$) is called *s-colored* if $\kappa(v) = s$.

Let $f : (G, \varphi) \rightarrow (H, \psi)$ be a colored morphism. By (2.2) and Definition 3.1, for a regular path $v = e_{i_0 \dots i_p}$ in the digraph G we have $\kappa(f_*(v)) = \kappa(v)$ if the path $f_*(v)$ is regular. In the opposite case of the non-regular path $f_*(v)$, we set $\kappa(f_*(v)) = 0$. Then we have $\kappa(f_*(v)) \leq \kappa(v)$ for any allowed path $v = e_{i_0 \dots i_p}$ in the colored digraph (G, φ) .

Now we introduce the notion of a *s-colored homotopy* between two colored morphisms of digraphs. Denote by I the segment digraph $I_1 = (0 \rightarrow 1)$. For any allowed elementary p -path $v = e_{i_0 \dots i_p}$ in a colored digraph (G, φ) , define an allowed elementary $(p+1)$ -path $\hat{v}_k$ ($0 \leq k \leq p$) in $G \square I$ by $\hat{v}_k = e_{i_0 \dots i_k i'_k \dots i'_p}$, where i_j denotes the vertex $(i_j, 0) \in G \square \{0\}$ and i'_j denotes the vertex $(i_j, 1) \in G \square \{1\}$. These notions are well defined since we have the natural identifications $G = G \square \{0\}$ and $G = G \square \{1\}$. For the product $G \square I$, we define a coloring φ_i of the subgraph $G \square \{i\} \subset G \square I$ ($i = 0, 1$) by setting $\varphi_i[(v, i) \rightarrow (w, i)] = \varphi(v \rightarrow w)$, where $((v, i) \rightarrow (w, i)) \in E_{G \square \{i\}}$. Denote by $E_{[0,1]}$ the set of edges of the digraph $G \square I$ that have the form $((v, 0) \rightarrow (v, 1))$ with $v \in V_G$. Any coloring $\varphi_{[0,1]} : E_{[0,1]} \rightarrow \mathbb{N}$ of the set of edges $E_{[0,1]}$ together with the colorings φ_i for $i = 0, 1$ induce the coloring of the digraph $G \square I$ which we denote $\Phi_{[0,1]} : E_{G \square I} \rightarrow \mathbb{N}$.

Definition 3.3. Let $f_0, f_1 : (G, \varphi) \rightarrow (H, \psi)$ be colored morphisms of colored digraphs and let $F : G \square I \rightarrow H$ be a homotopy.

- (i) For $s \geq 0$, we say that F defines the *s-colored one-step homotopy* from f_0 to f_1 if there exists a coloring $\varphi_{[0,1]} : E_{[0,1]} \rightarrow \mathbb{N}$ such that the mapping F is a colored morphism

$$F : (G \square I, \Phi_{[0,1]}) \rightarrow (H, \psi)$$

and, for every allowed path $v = e_{i_0 \dots i_p}$ in G with $\kappa(v) \leq s$, the condition $\kappa(F_*(\hat{v}_k)) \leq s$ is satisfied for every $0 \leq k \leq p$. We denote such a homotopy by $(F, \varphi_{[0,1]})$ or simply F^s if the mapping $\varphi_{[0,1]}$ is clear from the context.

- (ii) The colored morphisms f_0, f_1 are called *s-colored one-step homotopic* if there exists an *s-colored one-step homotopy* from f_0 to f_1 or from f_1 to f_0 .

Definition 3.4. Let $f, g : (G, \varphi) \rightarrow (H, \psi)$ be colored morphisms of colored digraphs. We say that f is *s-colored homotopic* to g if there exists a finite sequence of colored morphisms $f = f_0, f_1, \dots, f_{p-1}, f_p = g$ such that any two consequent morphisms are *s-colored one-step homotopic*.

Let f, g be *s-colored homotopic* morphisms as in Definition 3.4 with the sequence $f, f_1, \dots, f_p = g$ of *s-colored one-step homotopies*. By Definition 3.3, for every pair (f_k, f_{k+1}) of consequent morphisms, we have two possibilities. First, there is a colored morphism $F_k : (G \square I, \Phi_{[0,1]}^k) \rightarrow (H, \psi)$ such that $F_k|_{G \square \{0\}} = f_k$ and $F_k|_{G \square \{1\}} = f_{k+1}$. Second, there is a colored morphism $F_k : (G \square I, \Phi_{[0,1]}^k) \rightarrow (H, \psi)$ such that $F_k|_{G \square \{0\}} = f_{k+1}$ and $F_k|_{G \square \{1\}} = f_k$. Now, define a segment digraph I_p in the following way. For every pair of vertices $(k, k+1)$ with $0 \leq k \leq p-1$, there is an arrow $k \rightarrow k+1$ if the first case of the colored morphism F_k occurs and there is an arrow $k+1 \rightarrow k$ if the second case of the colored morphism F_k occurs. Now, the coloring $\Phi_{[0,1]}^k$ is well defined on the subgraph $k \rightarrow k+1$ or the subgraph $k+1 \rightarrow k$ of the digraph I_p whichever case occurs. The colorings $\Phi_{[0,1]}^k$ with $0 \leq k \leq p-1$ define a coloring Φ of the digraph $G \square I_p$. Moreover, the union of morphisms F_k defines a colored morphism $F : (G \square I_p, \Phi) \rightarrow (H, \psi)$ since $F_k|_{G \square \{k+1\}} = F_{k+1}|_{G \square \{k\}}$. Please note also that the condition (i) of Definition 3.3 is satisfied for the morphism F_k on the sub-digraph $k \rightarrow k+1$ (or the sub-digraph $k+1 \rightarrow k$ in the second case) of I_p under natural identification of $k \rightarrow k+1$ with I ($k+1 \rightarrow k$ with I , respectively).

We would like to indicate here that the notion of s -colored homotopy essentially depends on the number $s = 0, 1, 2, \dots$. Two colored digraph morphisms f and g are 0-colored homotopic only in the case of $f = g$. Indeed, let $(F, \varphi_{[0,1]})$ be a 0-colored homotopy from f to g and $f(i) \neq g(i)$ for a vertex $i \in V_G$. Then, for $v = e_i$, we have $\kappa(v) = 0$ and $\kappa(F_*(\hat{v}_0)) = 1$. Thus, we obtain a contradiction and hence $f(i) = g(i)$ for every $i \in V_G$. Now let $f, g : G \rightarrow H$ be k -colored digraphs and $s \geq k$. Then f and g are s -colored homotopic if and only if these morphisms are homotopic as digraph mappings. It follows from Definitions 3.3, 3.4, and the definition of k -coloring that for any regular paths v in G , we have $\kappa(v) \leq s$ and $\kappa(F_*(\hat{v}_k)) \leq s$.

Now we prove that relation to be s -colored homotopic is an equivalence relation and provide several examples.

Proposition 3.5. *The relation “to be s -colored homotopic” is an equivalence relation for any $s \geq 0$ on the set of colored morphisms $f : (G, \varphi) \rightarrow (H, \psi)$ of colored digraphs.*

Proof. Let $f : G \rightarrow H$ be a colored morphism. Define a homotopy $F : G \square I \rightarrow H$ by $F(v, i) = f(v)$ where $(v, i) \in V_G \square \{i\}$ for $i = 0, 1$ and define the coloring $\varphi_{[0,1]} : E_{[0,1]} \rightarrow \mathbb{N}$ by $\varphi_{[0,1]}((v, 0) \rightarrow (v, 1)) = 1$. Since $f(e)$ is a vertex in H for any arrow $e \in E_{[0,1]}$, we conclude that for any regular path $v = e_{i_0 \dots i_p}$ the path $F_*(\hat{v}_k)$ is not regular and hence $\kappa(F_*(\hat{v}_k)) = 0 \leq s$. Thus, the pair $(F, \varphi_{[0,1]})$ is s -colored homotopy for any $s \geq 0$ and relation “to be s -homotopic” is reflexive. Note that in the place of the color “1” in the definition of the homotopy F^s we can take any another color.

Let f be s -colored homotopic to g due to the sequence of colored morphisms $f = f_0, f_1, \dots, f_{p-1}, f_p = g$ as in Definition 3.4. Then the sequence of those morphisms in reverse order $g = f_p, f_{p-1}, \dots, f_1, f_0 = f$ gives the s -colored homotopy from g to f by Definition 3.3. Thus, the relation is also symmetric.

Let $f, g, h : G \rightarrow H$ be s -colored morphisms for which f is s -colored homotopic to g by a sequence of colored morphisms $f = f_0, f_1, \dots, f_{p-1}, f_p = g$ and g is s -colored homotopic to h by a sequence of colored morphisms $g = g_0, g_1, \dots, g_{r-1}, g_r = h$, as in Definition 3.4. Then the sequence of colored morphisms $f = f_0, f_1, \dots, f_{p-1}, f_p, g_1, \dots, g_{r-1}, g_r = h$, where $f_p = g = g_0$ provides s -colored homotopy from f to h and so the transitivity is proved. $\square$

The definition of s -colored homotopy in the category of colored graphs is similar and it is an equivalence relation on the set of colored morphisms of colored graphs.

From the considerations above it follows that, for $s \geq 0$, we are provided with the collection of s -colored homotopy categories of graphs $C^s\mathcal{G}'$ in which the objects are colored graphs and morphisms are classes of s -colored homotopic morphisms. Similarly, we also obtain the collection of s -colored homotopy categories of digraphs $C^s\mathcal{D}'$ and the s -colored homotopy categories $C^s\mathcal{NG}', C^s\mathcal{ND}'$.

Below, we give an example of a non trivial one-step two-colored homotopy between colored morphisms of three-colored digraphs.

Example 3.6. Consider the following three-colored digraphs G and H where E_H contains two symmetric arrows $a \rightarrow b$ and $b \rightarrow a$ (Figure 1):

Figure 1: The edge-colored digraphs G and H , respectively.

We define $f_0 : G \rightarrow H$ on the appropriate set of vertices in the following way:

$$f_0(x) = a, \quad f_0(y) = b, \quad f_0(z) = c.$$

As for the $f_1 : G \rightarrow H$, set

$$f_1(x) = b, \quad f_1(y) = a, \quad f_1(z) = d.$$

It is easy to see that both f_0 and f_1 are colored morphisms which are in fact colored homomorphisms. Let $F : G \square I_1 \rightarrow H$ be the digraph homotopy given on the set of vertices in the following way:

$$\begin{aligned} F(x, 0) &= a, & F(y, 0) &= b, & F(z, 0) &= c, \\ F(x, 1) &= b, & F(y, 1) &= a, & F(z, 1) &= d. \end{aligned}$$

The image of restriction of the mapping F to the set of arrows

$$E_{[0,1]} = \{(x, 0) \rightarrow (x, 1), (y, 0) \rightarrow (y, 1), (z, 0) \rightarrow (z, 1)\}$$

consists of the blue-colored arrows and defines the constant coloring $\varphi_{[0,1]}$. Thus, we have a colored morphism $(F, \varphi_{[0,1]})$ and for any regular path v in G , $\kappa(v) \leq 2$. Moreover, it is easy to see that for every such path $\kappa(F_*(\hat{v}_k)) \leq 2$. In other words, the mappings f_0 and f_1 are two-colored homotopic. Note that $F : G \square I_1 \rightarrow H$ is in fact a homomorphism.

Now we turn our attention to $C^s\mathcal{ND}$ and provide an example of non trivial homotopy in this category for $s = 2$.

Example 3.7. Consider digraphs G and H given in Figure 2.

We define $f : G \rightarrow H$ on the appropriate set of vertices in the following way:

$$f(x) = a, \quad f(y) = b, \quad f(z) = c, \quad f(v) = d.$$

As for the $g : G \rightarrow H$, we set

$$g(x) = c, \quad g(y) = h, \quad g(z) = g, \quad g(v) = b.$$

It is easy to see that both f and g are colored homomorphisms. Denote by I_2 the segment digraph $0 \rightarrow 1 \rightarrow 2$ and let $F : G \square I_2 \rightarrow G$ be the digraph mapping given on the set of vertices of the digraph $G \square I_2$ in the following way:

$$\begin{aligned} F(x, 0) &= a, & F(y, 0) &= b, & F(z, 0) &= c, & F(v, 0) &= d, \\ F(x, 1) &= g, & F(y, 1) &= f, & F(z, 1) &= e, & F(v, 1) &= a, \\ F(x, 2) &= c, & F(y, 2) &= h, & F(z, 2) &= g, & F(v, 2) &= b. \end{aligned} \tag{3.1}$$

Figure 2: The edge-colored digraph G and H , respectively.

Note that $F|_{G \times \{0\}} = f$, $F|_{G \times \{2\}} = g$, whereas F is a colored homomorphism and the sequence of colored morphisms $f = f_0 = F|_{G \times \{0\}}$, $f_1 = F|_{G \times \{1\}}$, $f_2 = F|_{G \times \{2\}} = g$ defines a two-colored homotopy between f and g .

Proposition 3.8. *The functors $CS : CG \rightarrow CD$ and $CNS : CNG \rightarrow CND$ defined above induce functors*

$$C^s S' : C^s G' \rightarrow C^s D', \quad C^s NS' : C^s NG' \rightarrow C^s ND'.$$

Proof. It follows directly from [2, Proposition 6.5]. $\square$

Example 3.9. Now we give an example of a homotopy between colored morphisms which is not the s -colored homotopy for any $s \geq 0$. Consider the colored digraph G .

$$a \xrightarrow{\text{red}} b \xleftarrow{\text{green}} c \xleftarrow{\text{red}} d$$

Let $f_0 : G \rightarrow G$ be the identity morphism and f_1 be a morphism defined on the set of vertices in the following way: $f_1(a) = a$, $f_1(b) = f_1(c) = f_1(d) = b$. Denote by I_2 the segment digraph $0 \rightarrow 1 \rightarrow 2$ and let $F : G \square I_2 \rightarrow G$ be the digraph mapping given on the set of vertices as follows:

$$\begin{aligned} F(a, 0) &= a, & F(b, 0) &= b, & F(c, 0) &= c, & F(d, 0) &= d, \\ F(a, 1) &= a, & F(b, 1) &= b, & F(c, 1) &= b, & F(d, 1) &= c, \\ F(a, 2) &= a, & F(b, 2) &= b, & F(c, 2) &= b, & F(d, 2) &= b. \end{aligned} \quad (3.2)$$

It is easy to see that F is a digraph mapping as well as a homotopy. The restrictions $F|_{G \square \{0\}}$ and $F|_{G \square \{2\}}$ coincide, respectively, with the morphisms f_0 and f_1 . Nevertheless, the mapping F is not a colored homotopy since the mapping $F|_{G \square \{1\}} : G \square \{1\} \rightarrow G$ given on the set of vertices by the formula in the middle row of (3.2) is not a colored morphism.

4 Path homology of colored graphs and digraphs

In this section, we construct a collection of path homology theories defined on various categories of colored graphs and digraphs and describe their basic properties. To illustrate the definitions thus introduced, we also provide several examples. Let (G, φ) be a colored digraph. Recall that for every nonempty set of arrows $A \subset E_G$ we have defined the function κ which returns the number of different colors that are being used in the coloring of arrows from the set A .

Definition 4.1. An allowed regular elementary path $e_{i_0 \dots i_p}$ ($p \geq 1$) is called s -colored if $\kappa(A) = s$, where A consists of arrows $i_0 \rightarrow i_1, \dots, i_{p-1} \rightarrow i_p$ of the path $e_{i_0 \dots i_p}$. Every elementary path e_i is called 0-colored.

Note that the set of all the elementary 0-colored paths coincides with the set of vertices V_G . For $k, p \geq 1$, we define a free R -module $\mathcal{L}_p^k = \mathcal{L}_p^k(G, R) = \mathcal{L}_p^k(G, \varphi)$ as a submodule of $\mathcal{A}_p(G, R)$ generated by all the allowed regular elementary s -colored paths with $1 \leq s \leq k$. Let $\mathcal{L}_p^0 = \mathcal{L}_p^0(G) = 0$ for $p \geq 1$ and $\mathcal{L}_{-1}^k = 0$,

$$\mathcal{L}_0^k = \mathcal{L}_0^k(G) = \langle e_i | e_i \in V_G \rangle$$

for $k \geq 0$. Note that $\mathcal{L}_0^k$ is the free R -module generated by all the elementary 0-colored paths. Thus, we obtain submodules $\mathcal{L}_p^k \subset \mathcal{A}_p(G, R) \subset \mathcal{R}_p(V_G, R)$ for $p \geq -1$ and $k \geq 0$.

For $p \geq 1$, let

$$\Theta_p^k = \Theta_p^k(G, R) := \{v \in \mathcal{L}_p^k : \partial v \in \mathcal{L}_{p-1}^k\} \quad (4.1)$$

be a submodule of $\mathcal{L}_p^k$ and let $\Theta_p^k = \mathcal{L}_p^k$ for $p = -1, 0$. The following result follows directly from the definition in (4.1).

Proposition 4.2. For $p \geq 0$ and any $k \geq 0$, we have $\partial(\Theta_p^k) \subset \Theta_{p-1}^k$. Hence, the elements of Θ_*^k are ∂ -invariant and there is a chain complex

$$0 \leftarrow \Theta_0^k \xleftarrow{\partial} \Theta_1^k \xleftarrow{\partial} \dots \xleftarrow{\partial} \Theta_{p-1}^k \xleftarrow{\partial} \Theta_p^k \xleftarrow{\partial} \dots \quad (4.2)$$

with the differential that is induced by the differential ∂ of the chain complex $\mathcal{R}_*(V_G, R)$.

The homology groups of chain complex (4.2) will be referred to as k -colored path homology groups and denoted by $\mathcal{H}_p^k(G, R)$.

Proposition 4.3. For any colored digraph (G, φ) , we have a filtration

$$\Theta_*^1(G, R) \subset \Theta_*^2(G, R) \subset \dots \subset \Omega_*(G, R). \quad (4.3)$$

Moreover, for a k -colored digraph G this filtration is finite and $\Theta_*^k(G, R) = \Omega_*(G, R)$.

Proof. For $k \geq 1$, we have natural inclusions $\mathcal{L}_p^k(G) \subset \mathcal{L}_p^{k+1}(G) \subset \mathcal{A}_p(G)$ and the statement follows from (4.1). $\square$

Example 4.4. Consider the cubic digraph $G = (V, E)$ in Figure 3 with proper coloring. We compute all colored homology groups of this digraph for $R = \mathbb{R}$ to illustrate the definition.

The module $\Theta_0^1(G)$ is generated by elements e_i where $i \in V_G$, hence $\text{rank } \Theta_0^1(G) = 8$. On the other hand, the module $\Theta_1^1(G)$ is generated by all the possible edges, so $\text{rank } \Theta_1^1(G) = 12$. Since G is a proper colored digraph, we deduce that $\Theta_i^1(G) = 0$ for $i \geq 2$. Having in mind all the facts stated above, by direct computation, we obtain $\mathcal{H}_0^1(G) = \mathbb{R}$, $\mathcal{H}_1^1(G) = \mathbb{R}^5$ and $\mathcal{H}_i^1(G) = 0$ for $i \geq 2$.

The modules $\Theta_0^2(G)$ and $\Theta_1^2(G)$ are isomorphic to $\Theta_0^1(G)$ and $\Theta_1^1(G)$, respectively. The module $\Theta_2^2(G)$ is generated by the elements

$$e_{013} - e_{023}, \quad e_{015} - e_{045}, \quad e_{026} - e_{046}, \quad e_{137} - e_{157}, \quad e_{237} - e_{267}, \quad e_{457} - e_{467}$$

and thus it follows that $\text{rank } \Theta_2^2(G) = 6$. The module $\mathcal{L}_3^2(G)$ is generated by the elements e_{0137} and e_{0467} and in consequence, the module $\Theta_3^2(G)$ is trivial since no linear combination of the paths ∂e_{0137} and ∂e_{0467} is in $\mathcal{L}_2^2(G)$. The module $\Theta_i^2(G)$ is trivial for $i \geq 4$ since in this case the module $\mathcal{L}_i^2(G)$ already is trivial. We have the commutative diagram

$$\begin{array}{ccc} \Theta_1^1(G) & \xrightarrow{\partial} & \Theta_0^1(G) \\ \downarrow = & & \downarrow = \\ \Theta_1^2(G) & \xrightarrow{\partial} & \Theta_0^2(G). \end{array}$$

Figure 3: The proper edge-colored digraph cube.

Hence,

$$\text{rank}\{\ker [\partial : \Theta_1^2(G) \rightarrow \Theta_0^2(G)]\} = 5.$$

We can see directly that the differential $\partial : \Theta_2^2(G) \rightarrow \Theta_1^2(G)$ has the kernel $\mathbb{R}$. Direct computations in which all gathered information is used give us desired homology groups, namely $\mathcal{H}_0^2(G) = \mathbb{R}$, $\mathcal{H}_1^2(G) = 0$, $\mathcal{H}_2^2(G) = \mathbb{R}$, and $\mathcal{H}_i^2(G) = 0$ for $i \geq 3$.

In the case of $k \geq 3$, the modules $\Theta_0^3(G)$, $\Theta_1^3(G)$, and $\Theta_2^3(G)$ are isomorphic to $\Theta_0^2(G)$, $\Theta_1^2(G)$, and $\Theta_2^2(G)$, respectively. The module $\Theta_3^3(G)$ is generated by the element

$$e_{0137} - e_{0157} + e_{0457} - e_{0467} + e_{0267} - e_{0237}$$

and the differential $\partial : \Theta_3^3(G) \rightarrow \Theta_2^3(G)$ is a monomorphism. The considerations like in the case of $\Theta_1^2(G)$ give $\mathcal{H}_0^3(G) = \mathbb{R}$ and $\mathcal{H}_i^3(G) = 0$ for $i \geq 1$.

Theorem 4.5. *Let $f : (G, \varphi) \rightarrow (H, \psi)$ be a morphism of colored digraphs. For $k \geq 1$, the morphism f_* in (2.2) induces a morphism of chain complexes*

$$\Theta_*^k(G, R) \rightarrow \Theta_*^k(H, R) \quad (4.4)$$

and, hence, a homomorphism of k -colored path homology groups

$$f_* : \mathcal{H}_*^k(G, R) \rightarrow \mathcal{H}_*^k(H, R).$$

Proof. It follows from Definition 3.1 and (2.2) that $f_*(\mathcal{L}_p^k(G, R)) \subset \mathcal{L}_p^k(H, R)$. Thus, it is sufficient to prove that $f_*(\Theta_p^k(G, R)) \subset \Theta_p^k(H, R)$. By (4.1), for any $v \in \Theta_p^k(G, R)$ we have $v \in \mathcal{L}_p^k(G)$ and $\partial v \in \mathcal{L}_{p-1}^k(G)$. Hence, $f_*(v) \in \mathcal{L}_p^k(H, R)$ and $\partial(f_*(v)) = f_*(\partial v) \in \mathcal{L}_{p-1}^k(H, R)$, which implies $f_*(v) \in \Theta_p^k(H, R)$. $\square$

Now we state the k -colored homotopy invariance of k -colored path homology groups of digraphs.

Theorem 4.6. *Let $f \simeq g : (G, \varphi) \rightarrow (H, \psi)$ be k -colored homotopic digraph morphisms for $k \geq 1$. Then f and g induce the identical homomorphisms*

$$f_* = g_* : \mathcal{H}_*^k(G, R) \rightarrow \mathcal{H}_*^k(H, R)$$

of k -colored homology groups.

Proof. The colored k -homotopy is a special case of a homotopy, and we give only the sketch of proof (see [2, §2.5 and §3.2] for details). It follows from Definitions 3.3 and 3.4 that it is sufficient to consider the case of one-step k -colored homotopy. By Theorem 4.5, the colored morphisms f and g induce morphisms of chain complexes

$$f_*, g_* : \Theta_*^k(G, R) \rightarrow \Theta_*^k(H, R).$$

For any regular path $v = e_{i_0 \dots i_p} \in \Theta_p^k(G, R)$ define a path $\hat{v} \in \Theta_{p+1}^k(G \square I_1, R)$ by

$$\hat{v} = \sum_{k=0}^p \hat{v}_k = \sum_{k=0}^p (-1)^p e_{i_0 \dots i_k i'_k \dots i'_p},$$

where i_j denotes the vertex $(i_j, 0) \in G \square \{0\}$ and i'_j denotes the vertex $(i_j, 1) \in G \square \{1\}$. By Definition 3.3, the colored morphism F induces a morphism of chain complexes

$$F_* : \Theta_*^k(G \square I_1, R) \rightarrow \Theta_*^k(H, R)$$

and we define homomorphisms $L_p : \Theta_p^k(G, R) \rightarrow \Theta_{p+1}^k(H, R)$ by $L_p(v) = F_*(\hat{v})$. Then, $\partial L_p + L_{p-1} \partial = g_* - f_*$. Hence, the collection of homomorphisms L_p is a chain homotopy (see [14, Chapter 2.2]) between the chain mappings f_* and g_* and the theorem follows. $\square$

Corollary 4.7. *If colored digraphs (G, φ) and (H, ψ) are k -colored homotopy equivalent for $k \geq 1$, then colored homology groups $\mathcal{H}_*^k(G, R)$ and $\mathcal{H}_*^k(H, R)$ are isomorphic and mutually inverse isomorphisms of these groups are induced by the k -colored homotopy inverse colored morphisms.*

Corollary 4.8. *For $k \geq 1$, the k -colored homology groups $\mathcal{H}_*^k(\cdot, R)$ provide a collection of functors from the k -colored homotopy category $C^k\mathcal{D}'$ to the category of R -modules and homomorphisms.*

Let (G, φ) be a colored graph. For $k \geq 1$, define the k -colored path homology groups of G as the k -colored path homology groups of the corresponding symmetric digraph

$$\mathcal{H}_n^k(G, R) := \mathcal{H}_n^k(\tilde{G}, R). \quad (4.5)$$

Corollary 4.9. *For $n \geq 0$, the k -colored path homology groups $\mathcal{H}_n^k(\cdot, R)$ defined in (4.5) give a collection of functors from the homotopy category $C^k\mathcal{G}'$ to the category of R -modules and homomorphisms.*

5 Algebraic properties of colored path homology

In this section, we describe the basic algebraic properties of the colored path homology groups. In particular, we introduce a notion of the *relative colored path homology* and construct various diagrams of exact sequences that give effective methods of computation. Then we construct a spectral sequence of colored homology groups following [15, Chapter 7] and present several examples.

Let (H, φ_H) with $H = (V_H, E_H)$ be an edge-colored (di)graph. A colored (di)graph $G = (V_G, E_G)$ with the coloring φ_G is a *colored sub-(di)graph* of H if $V_G \subset V_H$, $E_G \subset E_H$ and $\varphi_G = \varphi_H|_{E_G}$. In what follows, we shall denote the functions φ_G and φ_H simply by φ since this cannot lead to confusion. In this case, we write $(H, G; \varphi)$ or $(G \subset H, \varphi)$ and call the pair (H, G) a *colored pair of (di)graphs*.

Proposition 5.1. *Let $(G \subset H, \varphi)$ be a pair of colored digraphs. The natural inclusion homomorphism $i : (G, \varphi) \rightarrow (H, \varphi)$ induces a monomorphism of chain complexes $i_* : \Theta_*^k(G, R) \rightarrow \Theta_*^k(H, R)$ and hence provides a short exact sequence*

$$0 \longrightarrow \Theta_*^k(G, R) \longrightarrow \Theta_*^k(H, R) \longrightarrow \Theta_*^k(H, R)/\Theta_*^k(G, R) \longrightarrow 0$$

of chain complexes.

Proof. The inclusion i induces an inclusion $i_* : \mathcal{L}_n^k(G, R) \rightarrow \mathcal{L}_n^k(H, R)$ of modules for any $k \geq 0$ and $n \geq -1$. For the pair $G \subset H$ by (4.4), we have

$$v \in \Theta_n^k(G) \subset \mathcal{L}_n^k(G) \subset \mathcal{L}_n^k(H) \Rightarrow \partial v \in \Theta_{n-1}^k(G) \subset \mathcal{L}_{n-1}^k(G) \subset \mathcal{L}_{n-1}^k(H).$$

Hence, $\Theta_*^k(G, R) \subset \Theta_*^k(H, R)$ and the result follows. $\square$

Now for any $k \geq 0$, we define a chain complex $\Theta_*^k(H, G) = \Theta_*^k(H, G; R)$ as a factor-complex $\Theta_*^k(H)/\Theta_*^k(G)$. We denote homology groups of this complex by $\mathcal{H}_*^k(H, G) = \mathcal{H}_*^k(H, G; R)$ and call these groups the *relative colored path homology groups*.

Corollary 5.2. *Under the assumption of Proposition 5.1, for any $k \geq 0$ there is a homology long exact sequence*

$$\cdots \longrightarrow \mathcal{H}_n^k(G, R) \longrightarrow \mathcal{H}_n^k(H, R) \longrightarrow \mathcal{H}_n^k(H, G; R) \longrightarrow \mathcal{H}_{n-1}^k(G, R) \longrightarrow \cdots$$

Proof. See [16, Chapter 6.5, Theorem 4]. $\square$

The homology long exact sequence of a pair of colored digraphs $G \subset H$ provides algebraic relations between colored path homology groups of digraphs G and H and is an effective computing tool in many cases.

Example 5.3. Consider the three-colored digraph pair $G \subset H$, where G is the induced sub-digraph with the set of vertices $V_G = \{0, 1, 2\}$ in Figure 4. We compute all colored homology groups of this pair of digraphs for $R = \mathbb{R}$ to illustrate the definition.

Once again, we apply direct techniques for homology groups computation only to obtain that in the case of G and H , the colored path homology groups look as follows:

$$\begin{aligned}\mathcal{H}_0^1(G) &= \mathcal{H}_1^1(G) = \mathbb{R}, & \mathcal{H}_i^1(G) &= 0 \quad \text{for } i \geq 2, \\ \mathcal{H}_0^k(G) &= \mathbb{R}, & \mathcal{H}_i^k(G) &= 0 \quad \text{for } k \geq 2, i \geq 1, \\ \mathcal{H}_0^1(H) &= \mathbb{R}, & \mathcal{H}_1^1(H) &= \mathbb{R}^2, & \mathcal{H}_i^1(H) &= 0 \quad \text{for } i \geq 2, \\ \mathcal{H}_0^k(H) &= \mathbb{R}, & \mathcal{H}_1^k(H) &= \mathbb{R}, & \mathcal{H}_i^k(H) &= 0 \quad \text{for } k \geq 2, i \geq 2.\end{aligned}$$

Having in mind all the computations for G and H we can provide the homology groups for the chain complex $\Theta_*^1(H, G)$, namely

$$\mathcal{H}_0^1(H, G) = 0, \quad \mathcal{H}_1^1(H, G) = \mathbb{R}, \quad \mathcal{H}_i^1(H, G) = 0 \quad \text{for } i \geq 2.$$

All the above leads us to the conclusion that in the case of $k \geq 2$ the homology long exact sequence comes down to the following:

$$\begin{aligned}\cdots \longrightarrow 0 \longrightarrow \mathcal{H}_1^k(G) = 0 \longrightarrow \mathcal{H}_1^k(H) = \mathbb{R} \longrightarrow \mathcal{H}_1^k(H, G) = \mathbb{R} \longrightarrow \\ \mathcal{H}_0^k(G) = \mathbb{R} \longrightarrow \mathcal{H}_0^k(H) = \mathbb{R} \longrightarrow \mathcal{H}_0^k(H, G) = 0.\end{aligned}$$

Let (G, φ) be an edge-colored digraph. By Proposition 4.3, we have a filtration (4.3) of the chain complex $\Omega_*(G, R)$. From now on we write this filtration in the form

$$\cdots \subset \Theta_*^{-1}(G) \subset \Theta_*^0(G) \subset \Theta_*^1(G) \subset \cdots \subset \Omega_*(G), \quad (5.1)$$

where $\Theta_*^k(G) = 0$, for $k \leq -1$ and we put $\Theta_*(G) = \Omega_*(G, R)$. We denote $\mathcal{H}_*(G) := H_*(\Theta_*(G))$.

Recall that any inclusion of chain complexes $C_* \rightarrow \mathcal{D}_*$ gives a short exact sequence of chain complexes

$$0 \longrightarrow C_* \longrightarrow \mathcal{D}_* \longrightarrow \mathcal{D}_*/C_* \longrightarrow 0,$$

where $\mathcal{D}_*/C_*$ is the quotient chain complex with boundary operator induced by that of $\mathcal{D}_*$ (see [16, Chapter 4] for details). Hence, any pair of consequent chain complexes $\Theta_*^{p-1} \subset \Theta_*^p$ in (5.1) defines a chain complex $\Theta_*^p/\Theta_*^{p-1}$ and we denote $\mathcal{H}_*^{p/(p-1)}(G) := H_*(\Theta_*^p/\Theta_*^{p-1})$ its homology groups.

Figure 4: The colored pair of digraphs (H, G) .

Theorem 5.4. For every colored digraph (G, φ) , filtration (5.1) has the following properties.

1. $\Theta_*^p = 0$ for $p < 0$.
2. There is a short exact sequence of chain complexes

$$0 \rightarrow \Theta_*^{p-1} \rightarrow \Theta_*^p \rightarrow \Theta_*^p / \Theta_*^{p-1} \rightarrow 0 \quad (5.2)$$

with $H_{p+q}(\Theta_*^p / \Theta_*^{p-1}) = \mathcal{H}_{p+q}^{p/(p-1)}(G) = 0$ for $q < 0$.

3.

$$\bigcup_{p \geq 0} \Theta_*^p(G) = \Theta_*(G).$$

Proof. The first statement follows from the definition of filtration. The elements of the module $\Theta_*^p / \Theta_*^{p-1}$ are given by linear combinations of allowed paths of length p in which the arrows are colored exactly by p different colors. Consider an elementary allowed path $e_{i_0 \dots i_s}$. The arrows of this path can be colored at most by s colors. Hence, $e_{i_0 \dots i_s} \in \Theta_s^{p-1}$ for $s < p$, that is the module $\Theta_s^p / \Theta_s^{p-1}$ is trivial for $s < p$. Hence, $\mathcal{H}_s^{p/(p-1)}(G) = H_s(\Theta_*^p / \Theta_*^{p-1}) = H_s(0) = 0$ for $s < p$ and the second statement follows. Any path has a finite number of edges that are colored by a finite number of colors. Now the third statement follows. $\square$

Corollary 5.5. The exact sequence (5.2) induces a homology long exact sequence of colored homology groups

$$\dots \xrightarrow{\partial} \mathcal{H}_n^{p-1}(G) \xrightarrow{i} \mathcal{H}_n^p(G) \xrightarrow{j} \mathcal{H}_n^{p/(p-1)}(G) \xrightarrow{\partial} \mathcal{H}_{n-1}^{p-1}(G) \rightarrow \dots$$

Proof. We have $\mathcal{H}_*^{p/(p-1)}(G) = H_*(\Theta_*^p / \Theta_*^{p-1})$ and $\mathcal{H}_*^p(G) = H_*(\Theta_*^p)$ for $p \geq 0$. $\square$

Now we construct a spectral sequence of colored homology groups using the filtration (5.1) (see [15, Chapter 7]).

Let

$$D_{p,q} = \mathcal{H}_{p+q}^p(G) = H_{p+q}(\Theta_*^p), \quad E_{p,q} = \mathcal{H}_{p+q}^{p/(p-1)}(G) = H_{p+q}(\Theta_*^p / \Theta_*^{p-1}),$$

and $D_* = \{D_{p,q}\}$, $E_* = \{E_{p,q}\}$ be the corresponding bigraded R -modules. The exact sequence of Corollary 5.5 gives homomorphisms of colored homology groups:

$$\begin{aligned} i_{p,q} : D_{p,q} &= \mathcal{H}_{p+q}^p(G) \longrightarrow \mathcal{H}_{p+q}^{p+1}(G) = D_{p+1,q-1}, \\ j_{p,q} : D_{p,q} &= \mathcal{H}_{p+q}^p(G) \longrightarrow \mathcal{H}_{p+q}^{p/(p-1)}(G) = E_{p,q}, \\ k_{p,q} = \partial_{p,q} : E_{p,q} &= \mathcal{H}_{p+q}^{p/(p-1)}(G) \longrightarrow \mathcal{H}_{p+q-1}^{p-1}(G) = D_{p-1,q}, \end{aligned}$$

which define bigraded homomorphisms

$$\begin{aligned} i_* : D_* &\longrightarrow D_*, \\ j_* : D_* &\longrightarrow E_*, \\ k_* : E_* &\longrightarrow D_* \end{aligned} \quad (5.3)$$

of bidegree $(+1, -1)$, $(0, 0)$, and $(-1, 0)$, respectively. Now like in [15, Chapter 7], we obtain the following results.

Proposition 5.6. The homomorphisms in (5.3) fit into the commutative diagram

$$\begin{array}{ccc} D_* & \xrightarrow{i_*} & D_* \\ k_* \swarrow & & \swarrow j_* \\ & E_* & \end{array} \quad (5.4)$$

which is exact in each vertex. Thus, we have an exact couple of modules in the sense of [15].

Corollary 5.7. *The exact couple in (5.4) defines a spectral sequence with the first differential $d^1 = \{d_{p,q}\}$ where $d_{p,q} : E_{p,q}^1 \rightarrow E_{p-1,q}^1$ is given by*

$$j_{p-1,q} k_{p,q} : E_{p,q}^1 = \mathcal{H}_{p+q}^{p/(p-1)}(G) \longrightarrow \mathcal{H}_{p+q}^{(p-1)/(p-2)}(G) = E_{p-1,q}^1$$

of bidegree $(-1, 0)$. The group $E_{p,q}^r$ is isomorphic to the quotient group

$$\frac{\text{im}(H_{p+q}(\Theta_*^p/\Theta_*^{p-r}) \longrightarrow H_{p+q}(\Theta_*^p/\Theta_*^{p-1}))}{\text{im}(\partial : H_{p+q+1}(\Theta_*^{p+r-1}/\Theta_*^p) \longrightarrow H_{p+q}(\Theta_*^p/\Theta_*^{p-1}))}.$$

The differential d^{r+1} coincides with the composition $j_*(i_*)^{-r}k_*$.

We shall call this spectral sequence the *colored spectral sequence* of path homology groups of the colored digraph (G, φ) . Let

$$F_{p,q} := \text{im}(\mathcal{H}_{p+q}^p(G) \rightarrow H_{p+q}(\Theta_*)).$$

Then we have a natural inclusion $F_{p-1,q+1} \rightarrow F_{p,q}$, and hence, we can define a module

$$E_{p,q}^\infty := F_{p,q}/F_{p-1,q+1}.$$

Theorem 5.8. *The colored spectral sequence of a colored digraph (G, φ) converges, that is*

- (i) $E_{p,q}^r = E_{p,q}^{r+1}$ for $r > \max\{p, q + 1\}$, and
- (ii) $E_{p,q}^r \cong E_{p,q}^\infty$ for $r > \max\{p, q + 1\}$.

Proof. It follows from conditions 1 and 2 of Theorem 5.4 that $E_{p,q}^1 = 0$ for $p < 0$ or $q < 0$ and, hence $E_{p,q}^r = 0$ for $r \geq 1$. The group $E_{p,q}^{r+1}$ is the homology group of the chain complex

$$\cdots \rightarrow E_{p+r,q-r+1}^r \longrightarrow E_{p,q}^r \longrightarrow E_{p-r,q+r-1}^r \rightarrow \cdots$$

Thus for $r > \max\{p, q + 1\}$, we have $E_{p+r,q-r+1}^r = E_{p-r,q+r-1}^r = 0$, that is, $E_{p,q}^r = E_{p,q}^{r+1}$. By similar reasoning, using conditions 1 and 2 of Theorem 5.4, we can prove the isomorphism in (ii). See the detailed proof in [15, Chapter 7, Proposition 5] and [15, Chapter 7, Theorem 1]. $\square$

Note that $F_{p,q} = \text{im}[\mathcal{H}_{p+q}^p(G) \rightarrow \mathcal{H}_{p+q}(G)]$ consists of the image in $\mathcal{H}_{p+q}(G)$ of colored path homology groups given by paths that are colored with no more than p colors. It follows from Theorem 5.8 that we have a finite filtration

$$F_{0,n} \subset F_{1,n-1} \subset \cdots \subset F_{n-1,1} \subset F_{n,0} = \mathcal{H}_n(G).$$

The successive quotients $E_{i,n-i}^\infty = F_{i,n-i}/F_{i-1,n-i+1}$ of this filtration have a description on the level of the sub-filtration

$$\Theta_*^p \subset \Theta_*^{p+1} \subset \Theta_*$$

of (5.1) in which the modules of two left complexes have a clear description as colored path homology groups. Thus, we can apply results about colored path homology groups to computations of path homology groups of a digraph.

Let (G, φ) be a three-colored digraph. Then the filtration in (5.1) gives a finite filtration

$$\Theta_*^1(G) \subset \Theta_*^2(G) \subset \Theta_*^3(G) = \Omega_*(G). \quad (5.5)$$

Theorem 5.9. *The colored homology groups of filtration (5.5) fit into the commutative braid of the exact sequence*

$$\begin{array}{ccccccc}
 \rightarrow \mathcal{H}_{n+1}^3(G) & \rightarrow & \mathcal{H}_{n+1}^{3/2}(G) & \rightarrow & \mathcal{H}_n^{2/1}(G) & \rightarrow & \\
 & \searrow & \nearrow & \searrow & \nearrow & \searrow & \\
 & & \mathcal{H}_{n+1}^{3/1}(G) & & \mathcal{H}_n^2(G) & & \\
 & \nearrow & \searrow & \nearrow & \searrow & \nearrow & \\
 \rightarrow \mathcal{H}_{n+1}^{2/1}(G) & \rightarrow & \mathcal{H}_n^1(G) & \rightarrow & \mathcal{H}_n^3(G) & \rightarrow &
 \end{array} \quad (5.6)$$

which consists of the relative colored exact sequences for the following pairs of chain complexes $\Theta_*^1 \subset \Theta_*^2$, $\Theta_*^1 \subset \Theta_*^3$, $\Theta_*^2 \subset \Theta_*^3$, and $(\Theta_*^2/\Theta_*^1) \subset (\Theta_*^3/\Theta_*^1)$.

Proof. By [16, Chapter 4], inclusions in (5.5) induce a short exact sequence

$$0 \rightarrow \Theta_*^2/\Theta_*^1 \rightarrow \Theta_*^3/\Theta_*^1 \rightarrow \Theta_*^3/\Theta_*^2 \rightarrow 0$$

and we can write down the commutative diagram of chain complexes

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \rightarrow & \Theta_*^1 & \rightarrow & \Theta_*^2 & \rightarrow & \Theta_*^2/\Theta_*^1 \rightarrow 0 \\
 & & \downarrow \cong & & \downarrow & & \downarrow \\
 0 & \rightarrow & \Theta_*^1 & \rightarrow & \Theta_*^3 & \rightarrow & \Theta_*^3/\Theta_*^1 \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & \rightarrow & \Theta_*^3/\Theta_*^2 & \xrightarrow{\cong} & \Theta_*^3/\Theta_*^2 \rightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

in which the rows and columns are short exact sequences. Passing to the homology long exact sequences of the rows and columns of this diagram we obtain the commutative braid of exact sequences. $\square$

Let (G, φ) be a k -colored digraph with $k \geq 3$. Then for any sub-filtration $\Theta_*^m \subset \Theta_*^l \subset \Theta_*^k$ of filtration (5.1) with $m < l < k$ there exists a braid of exact sequences that is similar to (5.6) for filtration (5.5).

We also remark that, for a filtration consisting of three chain complexes as in (5.5), the braid of exact sequences (5.6) gives more information about homology groups of these chain complexes than the spectral sequence constructed above.

Figure 5: An edge-colored digraph.

Example 5.10. Consider an edge-colored digraph (G, φ) in Figure 5 in which the vertices that are denoted by equal numbers and the arrows between pairs of such vertices are identified naturally. Note that the underlying non-directed graph is a one-dimensional skeleton of the minimal triangulation of the projective plane. Now we compute all colored homology groups in the braid of exact sequence (5.6) for $R = \mathbb{R}$. We denote $\langle a_1, \dots, a_n \rangle$ the free $\mathbb{R}$ -module generated by elements $a_1, \dots, a_n$. For $n = 1, 2, 3$, we have $\Theta_0^n = \langle e_1, \dots, e_6 \rangle$ and

$$\Theta_1^n = \langle e_{21}, e_{31}, e_{16}, e_{26}, e_{36}, e_{23}, e_{24}, e_{35}, e_{46}, e_{56}, e_{34}, e_{45}, e_{25}, e_{41}, e_{51} \rangle.$$

Thus, in these cases, $\text{rank } \Theta_0^n = 6$ and $\text{rank } \Theta_1^n = 15$. We can compute directly that the image of the differential $\partial : \Theta_1^n \rightarrow \Theta_0^n$ has rank 5. Hence, the rank of the kernel of this differential equals $15 - 5 = 10$ for $n = 1, 2, 3$.

The module Θ_2^1 is generated by all one-colored paths of length 2, namely $\Theta_2^1 = \langle e_{456}, e_{451}, e_{416}, e_{516} \rangle$ and $\text{rank } \Theta_2^1 = 4$. The module Θ_3^1 is generated by the one-colored path e_{4516} of length 3. The modules Θ_n^1 are trivial for $n \geq 4$. We can compute the differentials $\partial : \Theta_3^1 \rightarrow \Theta_2^1$ and $\partial : \Theta_2^1 \rightarrow \Theta_1^1$ directly. We obtain $\text{rank}\{\partial(\Theta_3^1)\} = 1$ and $\text{rank}\{\partial(\Theta_2^1)\} = 3$. Hence,

$$\mathcal{H}_n^1(G, \mathbb{R}) = H_n(\Theta_*^1) = \begin{cases} \mathbb{R} & \text{for } n = 0, \\ \mathbb{R}^7 & \text{for } n = 1, \\ 0 & \text{for } n \geq 2. \end{cases}$$

To finish describing the chain complex Θ_*^2 , we must compute the modules and differentials in the dimensions greater or equal to 2. We must consider only one- or two-colored paths. We have

$$\Theta_2^2 = \langle e_{216}, e_{231}, e_{234}, e_{235}, e_{236}, e_{241}, e_{245}, e_{246}, e_{251}, e_{256}, e_{316}, e_{341}, e_{345}, e_{346}, e_{351}, e_{356}, e_{416}, e_{451}, e_{516}, e_{456} \rangle,$$

with $\text{rank } \Theta_2^2 = 20$ and

$$\Theta_3^2 = \langle e_{2316}, e_{2341}, e_{2345}, e_{2346}, e_{2351}, e_{2356}, e_{2416}, e_{2451}, e_{2456}, e_{2516}, e_{3416}, e_{3451}, e_{3456}, e_{3516}, e_{4516} \rangle$$

with $\text{rank } \Theta_3^2 = 15$. Furthermore, we have

$$\Theta_4^2 = \langle e_{23416}, e_{23451}, e_{23456}, e_{23516}, e_{24516}, e_{34516} \rangle$$

with $\text{rank } \Theta_4^2 = 6$ and $\Theta_5^2 = \langle e_{234516} \rangle$ with $\text{rank } \Theta_5^2 = 1$, and $\Theta_n^2 = 0$ for $n \geq 6$. Thus, we obtain

$$\text{rank}(\Theta_n^2) = \begin{cases} 6 & \text{for } n = 0, \\ 15 & \text{for } n = 1, \\ 20 & \text{for } n = 2, \\ 15 & \text{for } n = 3, \\ 6 & \text{for } n = 4, \\ 1 & \text{for } n = 5, \\ 0 & \text{for } n \geq 6. \end{cases}$$

Once again by direct computations, we obtain the following information:

$$\text{rank}\{\partial(\Theta_n^2)\} = \begin{cases} 10 & \text{for } n = 2, \\ 10 & \text{for } n = 3, \\ 5 & \text{for } n = 4, \\ 1 & \text{for } n = 5. \end{cases}$$

All the above enable us in turn to provide homology groups of the chain complex Θ_*^2 , namely

$$\mathcal{H}_n^2(G, \mathbb{R}) = H_n(\Theta_*^2) = \begin{cases} \mathbb{R} & \text{for } n = 0, \\ 0 & \text{for } n \geq 1. \end{cases}$$

Note that the analysis of paths for which at most three colors are used in the graph gives also $\Theta_*^3 = \Theta_*^2$ because there are no paths with three differently colored edges in G . Now, using the diagram chasing in

the diagram (5.6) (see the explanation of this method in [14, Chapter XII.3] for details), we find other homology groups fitting in this diagram: $\mathcal{H}_n^{3/2}(G, \mathbb{R}) = 0$ for all $n \geq 0$ and

$$\mathcal{H}_n^{2/1}(G, \mathbb{R}) = \mathcal{H}_n^{3/1}(G, \mathbb{R}) = \begin{cases} 0 & \text{for } n = 0, 1, \\ \mathbb{R}^7 & \text{for } n = 2, \\ 0 & \text{for } n \geq 3. \end{cases}$$

6 Discussion

One of the basic applications of homology theory is given by persistent homologies (see, e.g., [17–21]). The main constructions of this theory are based on the natural filtration of the space on which the homology theory is defined. Usually, this filtration is given by the n -skeleton of a cell complex or by metric conditions on the singular cells. The path homology groups of an arbitrary digraph or a path complex in the general case do not have any similar structure. It follows directly from the results above and from [4] that a vertex (and edge) coloring of a digraph gives a functorial filtration that induces the functorial filtration of path homology groups. Thus, by coloring a digraph in some way, we obtain an effective method to construct the persistence homology theory of a given digraph. It is necessary to remark here that vertex (or edge) weighted digraphs can be considered as colored digraphs and the results above apply to this case as well.

There is a natural way to apply the methods developed in this paper to the cohomology theory that is dual to the path homology theory. This is possible since any chain complex defined in this paper for a colored digraph defines also the corresponding colored cohomology groups of this digraph. Thus, we obtain new methods of computing cohomology groups that are used in physical applications of discrete mathematics (see, e.g., [5–7,22]).

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