Research Article

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Some estimates for commutators of Littlewood-Paley g -functions

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Abstract: The aim of this paper is to establish the boundedness of commutator $[b, \dot{g}_r]$ generated by Littlewood-Paley g -functions $\dot{g}_r$ and $b \in \text{RBMO}(\mu)$ on non-homogeneous metric measure space. Under assumption that λ satisfies ε -weak reverse doubling condition, the author proves that $[b, \dot{g}_r]$ is bounded from Lebesgue spaces $L^p(\mu)$ into Lebesgue spaces $L^p(\mu)$ for $p \in (1, \infty)$ and also bounded from spaces $L^1(\mu)$ into spaces $L^{1,\infty}(\mu)$. Furthermore, the boundedness of $[b, \dot{g}_r]$ on Morrey space $M_q^p(\mu)$ and on generalized Morrey $L^{p,\phi}(\mu)$ is obtained.

Keywords: non-homogeneous metric measure space, Littlewood-Paley g -function, commutator, space $\text{RBMO}(\mu)$, generalized Morrey space

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1 Introduction

Although the metric measure spaces equipped with the polynomial growth conditions (see [1–5]) and spaces of homogeneous type in the sense of Coifman and Weiss [6,7] are two important classes of function spaces in harmonic analysis, there exist no relations between the non-doubling measure spaces and spaces of homogeneous type. To solve this problem, Hytönen [8] first introduced the non-homogeneous metric measure spaces satisfying the so-called geometrically doubling and upper doubling conditions. From then on, many papers focus on the properties of function spaces and operators over non-homogeneous metric measure spaces. For example, Cao and Zhou [9] obtained the definition of Morrey space on non-homogeneous metric measure space and also proved that Hardy-Littlewood maximal operator, Calderón-Zygmund operator and fractional integral are bounded on Morrey space. Fu and Zhao [10] proved that generalized homogeneous Littlewood-Paley g -function is bounded from atomic Hardy space $H_{\text{atb}}^1(\mu)$ into space $L^1(\mu)$ and also bounded from space $\text{RBMO}(\mu)$ into space $\text{RBLO}(\mu)$. For more development of harmonic analysis on non-homogeneous metric measure space the readers can see [11–19] and references therein.

Let (X, d, μ) be a non-homogeneous metric measure space in the sense of Hytönen [8]. In this setting, the author proves that the commutator $[b, \dot{g}_r]$ generated by $b \in \text{RBMO}(\mu)$ and generalized homogeneous Littlewood-Paley g -function $\dot{g}_r$ is bounded on Lebesgue spaces $L^p(\mu)$ for $p \in (1, \infty)$ and also bounded from spaces $L^1(\mu)$ into spaces $L^{1,\infty}(\mu)$. Furthermore, the boundedness of $[b, \dot{g}_r]$ on Morrey space $M_q^p(\mu)$ and on generalized Morrey space $L^{p,\phi}(\mu)$ is also established in this paper. In 2016, Fu and Zhao obtained the boundedness of $\dot{g}_r$ on atomic Hardy space $H_{\text{atb}}^1(\mu)$ and on space $\text{RBMO}(\mu)$ (see [10]). In 2021, Lu and Tao [11] proved that the $\dot{g}_r$ is bounded on Lipschitz space $\text{Lip}_\beta(\mu)$ and on generalized Morrey space $L^{p,\phi}(\mu)$.

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Before stating the main results of this paper, we first recall some necessary notions. The following definitions of geometrically doubling and upper doubling conditions are from [8].

Definition 1.1. A metric space (X, d) is said to be geometrically doubling if there exists some $N_0 \in \mathbb{N}$ such that, for any ball $B(x, r) \subset X$ with $x \in X$ and $r \in (0, \infty)$, there exists a finite ball covering $\{B(x_i, r/2)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most N_0 .

Remark 1.2. Let (X, d) be a metric measure. Hytönen [8] showed that the geometrically doubling (X, d) is equivalent to the following statement: for any $\varepsilon \in (0, 1)$ and any ball $B(x, r) \subset X$ with $x \in X$ and $r \in (0, \infty)$, there exists a finite ball covering $\{B(x_i, \varepsilon r)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most ε^{-n_0} , where $n_0 := \log_2 N_0$.

Definition 1.3. A metric measure space (X, d, μ) is said to be upper doubling if μ is a Borel measure on X and there exist a dominating function $\lambda : X \times (0, \infty) \rightarrow (0, \infty)$ and constant $C_{(\lambda)} > 0$, depending on λ , such that, for each $x \in X$, $r \rightarrow \lambda(x, r)$ is non-decreasing and for all $x \in X$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \leq \lambda(x, r) \leq C_{(\lambda)} \lambda(x, r/2). \quad (1.1)$$

Moreover, Hytönen et al. [12] showed that there exists another dominating function $\tilde{\lambda}$ such that $\tilde{\lambda} \leq \lambda$, $C_{(\tilde{\lambda})} \leq C_{(\lambda)}$ and, for all $x, y \in X$ with $d(x, y) \leq r$,

$$\tilde{\lambda}(x, r) \leq C_{(\lambda)} \tilde{\lambda}(y, r). \quad (1.2)$$

Here and in what follows, we always assume that λ satisfies (1.2).

Although the doubling measure condition is not assumed uniformly for all balls on (X, d, μ) , Hytönen [8] showed that there exist many balls satisfying the (α, β) -doubling condition, i.e., let $\alpha, \beta \in (1, \infty)$, a ball $B \subset X$ is said to be (α, β) -doubling if $\mu(\alpha B) \leq \beta \mu(B)$. Throughout this paper, for any $\alpha \in (1, \infty)$ and ball B , the smallest (α, β_α) -doubling ball of the form $\alpha^j B$ with $j \in \mathbb{N}$ is denoted by $\tilde{B}^\alpha$, where

$$\beta_\alpha := \alpha^{3(\max\{n, n_0\})} + 30^n + 30^{n_0}.$$

We always denote by $\tilde{B}$ the smallest $(6, \beta_6)$ -doubling ball of the form $\tilde{B}^6$ in this paper.

We now recall the definition of coefficient $K_{B,S}$ introduced in [8], which is very close to the quantity $K_{Q,R}$ introduced by Tolsa [3], that is, for any two balls $B \subset S$, define

$$K_{B,S} := 1 + \int_{(2S) \setminus B} \frac{1}{\lambda(c(B), d(x, c(B)))} d\mu(x), \quad (1.3)$$

where $c(B)$ represents the center of ball B . For more properties of the $K_{B,S}$, we can see [12, Lemma 2.1].

The following regularized bounded mean oscillation (RBMO) space was from [8].

Definition 1.4. Let $\rho > 1$. A function $f \in L^1_{\text{loc}}(\mu)$ is said to be in the space $\text{RBMO}(\mu)$ if there exist a positive constant C and, for any ball B , a number f_B such that

$$\frac{1}{\mu(\rho B)} \int_B |f(y) - f_B| d\mu(y) \leq C \quad (1.4)$$

and, for any two balls B and S such that $B \subset S$,

$$|f_B - f_S| \leq CK_{B,S}. \quad (1.5)$$

The infimum of the positive constants C satisfying both (1.4) and (1.5) is defined to be the $\text{RBMO}(\mu)$ norm of f and denoted by $\|f\|_{\text{RBMO}(\mu)}$. Moreover, Hytönen [8] also showed that the space $\text{RBMO}(\mu)$ is independent of the choice of the constant $\rho \in (1, \infty)$.

In 2015, Tan and Li [13] gave an approximation of the identity $S := \{S_k\}_{k \in \mathbb{Z}}$ associated with $(2, 2(C_\lambda + 1))$ -doubling balls $\{Q_{x,k}\}_{x \in \text{supp}(\mu), k \in \mathbb{Z}}$ on (X, d, μ) , which are integral operators associated with kernels $S_k(x, y)$ on $X \times X$ satisfying the following conditions:

(A-1) $S_k(x, y) = S_k(y, x)$ for all $x, y \in \text{supp}(\mu)$.

(A-2) For any $k \in \mathbb{Z}$ and $x \in \text{supp}(\mu)$, $S_k 1(x) = 1 = S_k^* 1(x)$, where S_k^* is the adjoint operator of S_k .

(A-3) For each $k \in \mathbb{Z}$ and $x \in \text{supp}(\mu)$, $\text{supp}(S_k(x, \cdot)) \subset Q_{x,k-1}$.

(A-4) For all $x, y \in X$ and $k \in \mathbb{Z}$, if $x \neq y$ or $Q_{x,k} \neq \{x\}$, then there exists a non-negative constant C such that

$$0 \leq S_k(x, y) \leq \frac{C}{\lambda(x, r(Q_{x,k}) + r(Q_{y,k}) + d(x, y))}.$$

(A-5) For all $x, \tilde{x}, y \in X$ and $k \in \mathbb{Z}$, if $x, \tilde{x} \in Q_{x_0,k}$ for some $x_0 \in \text{supp}(\mu)$ and $x \neq \tilde{x}$, then there exists a positive constant C such that

$$|S_k(x, y) - S_k(\tilde{x}, y)| \leq \left[\frac{d(x, \tilde{x})}{r(Q_{x_0,k})} \right]^\varepsilon \frac{C}{\lambda(x, r(Q_{x,k}) + r(Q_{y,k}) + d(x, y))},$$

where $\varepsilon \in (0, \infty)$ and $Q_{x,k}$ represents a fixed doubling ball center as x of generation k .

Moreover, Tan and Li [13] showed that the aforementioned results through (A-1) to (A-5) are still correct if the $(2, 2(C_\lambda + 1))$ -doubling balls are replaced by $(6, \beta_6)$ -doubling balls.

We now recall the definition of generalized homogeneous Littlewood-Paley g -function introduced in [10].

Definition 1.5. Let $k \in \mathbb{Z}$, $r \in [2, \infty)$, $D_k(x, y) := S_k(x, y) - S_{k-1}(x, y)$ for all $x, y \in X$, and D_k be the corresponding integral operator associated with the kernel $D_k(x, y)$. Then, the generalized homogeneous Littlewood-Paley g -function $\dot{g}_r$ is defined by

$$\dot{g}_r(f)(x) := \left(\sum_{k \in \mathbb{Z}} |D_k(x, \cdot) f(x)|^r \right)^{\frac{1}{r}}, \quad \text{for any } x \in X. \quad (1.6)$$

Given a function $b \in \text{RBMO}(\mu)$, the commutator $[b, \dot{g}_r]$, which is generated by b and $\dot{g}_r$ as in (1.6), is defined by

$$[b, \dot{g}_r](x) := \left(\sum_{k \in \mathbb{Z}} |D_k(x, \cdot) (b(x) - b(\cdot)) f(x)|^r \right)^{\frac{1}{r}}, \quad (1.7)$$

where $x \in X$ and $f \in L_b^\infty(\mu)$ being the space of all $L^\infty(\mu)$ functions with bounded support.

The following definition of generalized Morrey space is from [14].

Definition 1.6. Let $k > 1$ and $1 < p < \infty$. Suppose that ϕ is an increasing function on $(0, \infty)$. Then, the generalized Morrey space $L^{p,\phi}(\mu)$ is defined by

$$L^{p,\phi}(\mu) := \{f \in L_{\text{loc}}^p(\mu) : \|f\|_{L^{p,\phi}(\mu)} < \infty\},$$

where

$$\|f\|_{L^{p,\phi}(\mu)} = \sup_B [\phi(\mu(kB))]^{-\frac{1}{p}} \left(\int_B |f(x)|^p d\mu(x) \right)^{\frac{1}{p}}. \quad (1.8)$$

Remark 1.7.

(1) From [14], Lu and Tao showed that the norm $\|f\|_{M_\phi^p(\mu)}$ is independent of the choice of parameter k for $k > 1$.

- (2) If we take $\phi(t) = t^{1-\frac{q}{p}}$ with $t > 0$ and $1 < q \leq p < \infty$, then $L^{p,\phi}(\mu)$ defined as in (1.8) is just the Morrey space $M_q^p(\mu)$, which was introduced by Cao and Zhou [9], that is, let $k > 1$ and $1 < q \leq p < \infty$, then Morrey space $M_q^p(\mu)$ is defined by

$$M_q^p(\mu) = \{f \in L_{\text{loc}}^q(\mu) : \|f\|_{M_q^p(\mu)} < \infty\},$$

where

$$\|f\|_{M_q^p(\mu)} := \sup_B [\mu(kB)]^{\frac{1}{p}-\frac{1}{q}} \left(\int_B |f(y)|^q d\mu(y) \right)^{\frac{1}{q}}. \quad (1.9)$$

We now recall the following ε -weak reverse doubling condition introduced in [16].

Definition 1.8. Let $\varepsilon \in (0, \infty)$. A dominating function λ is said to satisfy the ε -weak reverse doubling condition if, for all $r \in (0, 2\text{diam}(X))$ and $a \in (1, 2\text{diam}(X)/r)$, there exists a number $C(a) \in [1, \infty)$, depending only on a and X , such that, for all $x \in X$,

$$\lambda(x, ar) \geq C(a)\lambda(x, r)$$

and, moreover,

$$\sum_{k=1}^{\infty} \frac{1}{[C(a^k)]^\varepsilon} < \infty. \quad (1.10)$$

The main theorems of this paper are stated as follows:

Theorem 1.9. Let $b \in \text{RBMO}(\mu)$ and $r \in [2, \infty)$. Suppose that λ satisfies the ε -weak reverse doubling condition defined as in Definition 1.8. Then, there exists a constant $C > 0$ such that, for all $f \in L^p(\mu)$ with $p \in (1, \infty)$,

$$\|[b, \dot{g}_r](f)\|_{L^p(\mu)} \leq C\|b\|_{\text{RBMO}(\mu)} \|f\|_{L^p(\mu)}.$$

Theorem 1.10. Let $b \in \text{RBMO}(\mu)$ and $r \in [2, \infty)$. Suppose that λ satisfies the ε -weak reverse doubling condition defined as in Definition 1.8. Then, there exists a constant $C > 0$ such that, for all $f \in L^1(\mu)$,

$$\|[b, \dot{g}_r](f)\|_{L^{1,\infty}(\mu)} \leq C\|b\|_{\text{RBMO}(\mu)} \|f\|_{L^1(\mu)}.$$

Theorem 1.11. Let $b \in \text{RBMO}(\mu)$, $r \in [2, \infty)$ and $1 < q \leq p < \infty$. Suppose that λ satisfies the weak reverse doubling condition defined as in Definition 1.8. Then, there exists a constant $C > 0$ such that, for all $f \in M_q^p(\mu)$,

$$\|[b, \dot{g}_r](f)\|_{M_q^p(\mu)} \leq C\|b\|_{\text{RBMO}(\mu)} \|f\|_{M_q^p(\mu)}.$$

Theorem 1.12. Let $b \in \text{RBMO}(\mu)$, $r \in [2, \infty)$ and $p \in (1, \infty)$. Suppose that $\phi : (0, \infty) \rightarrow (0, \infty)$ is an increasing function and the mapping $s \mapsto \frac{\phi(s)}{s}$ is almost decreasing, namely, there is a constant $C > 0$ such that, the following inequality

$$\frac{\phi(w)}{w} \leq C \frac{\phi(s)}{s} \quad (1.11)$$

holds for $s \geq w$. Then, there exists a positive constant C such that, for all $f \in L^{p,\phi}(\mu)$,

$$\|[b, \dot{g}_r](f)\|_{L^{p,\phi}(\mu)} \leq C\|b\|_{\text{RBMO}(\mu)} \|f\|_{L^{p,\phi}(\mu)}.$$

Finally, we make some conventions on notation. Throughout the paper, C represents a positive constant, which is independent of the main parameters. For any subset E of X , we use χ_E to denote its characteristic function. Given any $q \in (1, \infty)$, let $q' := q/(q-1)$ denote its conjugate index. For any ball B , $c(B)$ and $r(B)$ represent the center and radius of ball B , respectively. Furthermore, $m_B(f)$ denotes the mean value of function f over ball B , that is, $m_B(f) = \frac{1}{\mu(B)} \int_B f(y) d\mu(y)$.

2 Preliminaries

To prove the main theorems of this paper, we should recall some necessary results in this section (see [10,14,16,17]).

Lemma 2.1. *Let (X, d, μ) be a non-homogeneous metric measure space satisfying the weak reverse doubling condition, $r \in [2, \infty)$ and $p \in (1, \infty)$. Then, there exists a constant $C > 0$ such that, for all $f \in L^p(\mu)$,*

$$\|\dot{g}_r(f)\|_{L^p(\mu)} \leq C\|f\|_{L^p(\mu)}. \quad (2.1)$$

Corollary 2.2. *If $f \in \text{RBMO}(\mu)$, then there exists a constant $C > 0$ such that, for any ball B , $\tau \in (1, \infty)$ and $s \in [1, \infty)$,*

$$\left(\frac{1}{\mu(\tau B)} \int_B |f(y) - f_B|^s d\mu(y) \right)^{\frac{1}{s}} \leq C\|f\|_{\text{RBMO}(\mu)}. \quad (2.2)$$

Lemma 2.3.

(1) *Let $p \in (1, \infty)$, $t \in (1, p)$ and $\rho \in [5, \infty)$. The following maximal operators, respectively, defined for all $f \in L^1_{\text{loc}}(\mu)$ and $x \in X$,*

$$M_{t,\rho}f(x) := \sup_{B \ni x} \left(\frac{1}{\mu(\rho B)} \int_B |f(y)|^t d\mu(y) \right)^{\frac{1}{t}} \quad (2.3)$$

and

$$Nf(x) := \sup_{\substack{B \ni x \\ B \text{ doubling}}} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y)$$

are bounded on $L^p(\mu)$ and also bounded from $L^1(\mu)$ into $L^{1,\infty}(\mu)$.

(2) *For all $f \in L^1_{\text{loc}}(\mu)$, it holds true that $|f(x)| \leq Nf(x)$ for μ -a.e $x \in X$.*

Lemma 2.4. *Let $f \in L^1_{\text{loc}}(\mu)$ satisfy $\int_X f(x) d\mu(x) = 0$ when $\|\mu\| := \mu(X) < \infty$. Assume that, for some p, q satisfying $1 < q \leq p < \infty$, $\inf\{1, Nf\} \in \dot{M}^p_q(\mu)$. Then, there exists a constant $C > 0$,*

$$\|Nf\|_{M^p_q(\mu)} \leq C\|M^\sharp f\|_{M^p_q(\mu)}, \quad (2.4)$$

where the sharp maximal operator $M^\sharp$ is defined by, for all $f \in L^1_{\text{loc}}(\mu)$ and $x \in X$,

$$M^\sharp f(x) := \sup_{B \ni x} \frac{1}{\mu(6B)} \int_B |f(y) - m_B(f)| d\mu(y) + \sup_{\substack{x \in B \subset S \\ B, S \text{ } (6, \beta_6)\text{-doubling}}} \frac{|m_B(f) - m_S(f)|}{K_{B,S}}. \quad (2.5)$$

Lemma 2.5. *Let $p > 1$, $1 < t < p < \infty$ and $\phi : (0, \infty) \rightarrow (0, \infty)$ be an increasing function. Suppose that $M_{t,\rho}$ is defined as in (2.3) and the mapping $s \mapsto \frac{\phi(s)}{s}$ satisfies the condition (1.11). Then, there exists a positive constant $C > 0$ such that, for all $f \in L^{p,\phi}(\mu)$,*

$$\|M_{t,\rho}f\|_{L^{p,\phi}(\mu)} \leq C\|f\|_{L^{p,\phi}(\mu)}.$$

3 Proof of Theorems 1.9–1.12

Proof of Theorem 1.9. For any $r \in [2, \infty)$, $p \in (1, \infty)$, $f \in L^p(\mu)$ and $x \in \mathcal{X}$, we first claim that

$$M^\sharp([b, \dot{g}_r](f))(x) \leq C \|b\|_{\text{RBMO}(\mu)} \{M_{r,6}(f)(x) + M_{r,6}(\dot{g}_r(f))(x)\}. \quad (3.1)$$

Once (3.1) is proved, by applying Lemma 2.3 and Theorem 4.2 in [15], we have

$$\begin{aligned} \|[b, \dot{g}_r](f)\|_{L^p(\mu)} &\leq \|N([b, \dot{g}_r](f))\|_{L^p(\mu)} \\ &\leq C \|M^\sharp([b, \dot{g}_r](f))\|_{L^p(\mu)} \\ &\leq C \|b\|_{\text{RBMO}(\mu)} \{\|M_{r,6}(\dot{g}_r(f))\|_{L^p(\mu)} + \|M_{r,6}(f)\|_{L^p(\mu)} + \|M_{r,6}(f)\|_{L^p(\mu)}\} \\ &\leq C \|b\|_{\text{RBMO}(\mu)} \|f\|_{L^p(\mu)}, \end{aligned}$$

which is just the desired conclusion.

To show (3.1), by the definition of sharp maximal function $M^\sharp$, there exists a family of numbers $\{b_B\}_B$ such that, for all x and B with $B \ni x$,

$$\frac{1}{\mu(6B)} \int_B |\dot{g}_{r,b}(f)(x) - h_B| d\mu(x) \leq C \|b\|_{\text{RBMO}(\mu)} \{M_{r,6}(f)(x) + M_{r,6}(\dot{g}_r(f))(x)\} \quad (3.2)$$

and, for all balls B, S satisfying $B \subset S$ and $B \ni x$,

$$|h_B - h_S| \leq CK_{B,S} \|b\|_{\text{RBMO}(\mu)} \{M_{r,6}(\dot{g}_r(f))(x) + M_{r,6}(f)(x) + M_{r,6}(f)(x)\}, \quad (3.3)$$

where

$$h_B := m_B(\dot{g}_r((b - b_B)f\chi_{\mathcal{X} \setminus 6B})) \quad \text{and} \quad h_S := m_S(\dot{g}_r((b - b_S)f\chi_{\mathcal{X} \setminus 6S})).$$

To prove (3.2), for a fixed ball B and x with $x \in B$, write

$$\begin{aligned} [b, \dot{g}_r](f)(x) &\leq C |b(x) - b_B| \left(\sum_{k \in \mathbb{Z}} |D_k f(x)|^r \right)^{\frac{1}{r}} + C \left(\sum_{k \in \mathbb{Z}} |D_k(b(\cdot) - b_B)f_1(x)|^r \right)^{\frac{1}{r}} + C \left(\sum_{k \in \mathbb{Z}} |D_k(b(\cdot) - b_B)f_2(x)|^r \right)^{\frac{1}{r}} \\ &\leq C \{|b(x) - b_B| \dot{g}_r(f)(x) + \dot{g}_r((b(\cdot) - b_B)f_1)(x) + \dot{g}_r((b(\cdot) - b_B)f_2)(x)\}, \end{aligned}$$

where $f_1 := f\chi_{6B}$ and $f_2 := f - f_1$.

By applying Hölder inequality, (2.1) and (2.2), we obtain that

$$\begin{aligned} \frac{1}{\mu(6B)} \int_B |b(x) - b_B| \dot{g}_r(f)(x) d\mu(x) &\leq \left(\frac{1}{\mu(6B)} \int_B |\dot{g}_r(f)(x)|^r d\mu(x) \right)^{\frac{1}{r}} \left(\frac{1}{\mu(6B)} \int_B |b(x) - b_B|^r d\mu(x) \right)^{\frac{1}{r'}} \\ &\leq C \|b\|_{\text{RBMO}(\mu)} M_{r,6}(\dot{g}_r(f))(x). \end{aligned} \quad (3.4)$$

To estimate $\dot{g}_r((b(\cdot) - b_B)f_1)$, take $s := \sqrt{r}$. From Hölder inequality, (2.1) and (2.2), it follows that

$$\begin{aligned} \frac{1}{\mu(6B)} \int_B |\dot{g}_r((b(\cdot) - b_B)f_1)(x)| d\mu(x) &\leq \frac{1}{[\mu(6B)]^{\frac{1}{s}}} \left(\int_B |\dot{g}_r((b(\cdot) - b_B)f_1)(x)|^s d\mu(x) \right)^{\frac{1}{s}} \\ &\leq \frac{C}{[\mu(6B)]^{\frac{1}{s}}} \|(b(\cdot) - b_B)f_1\|_{L^s(\mu)} \\ &\leq C \left(\frac{1}{\mu(6B)} \int_{6B} |b(y) - b_B|^{ss'} d\mu(y) \right)^{\frac{1}{ss'}} \left(\frac{1}{\mu(6B)} \int_{6B} |f(y)|^{ss} d\mu(y) \right)^{\frac{1}{ss}} \\ &\leq C \|b\|_{\text{RBMO}(\mu)} M_{r,6}(f)(x). \end{aligned} \quad (3.5)$$

Together with the aforementioned estimates, to get (3.2), we still need to estimate the difference $|\dot{g}_r((b(\cdot) - b_B)f_2) - h_B|$. For all $y_1, y_2 \in B$, write

$$\begin{aligned} |\dot{g}_r((b(\cdot) - b_B)f_2)(y_1) - h_B| &\leq \frac{1}{\mu(B)} \int_B \left| \left(\sum_{k \in \mathbb{Z}} \left| \int_X D_k(y_1, z)(b(z) - b_B)f_2(z) d\mu(z) \right|^r \right)^{\frac{1}{r}} \right. \\ &\quad \left. - \left(\sum_{k \in \mathbb{Z}} \left| \int_X D_k(y_2, z)(b(z) - b_B)f_2(z) d\mu(z) \right|^r \right)^{\frac{1}{r}} \right| d\mu(y_2) \\ &\leq \frac{1}{\mu(B)} \int_B \left| \left(\sum_{k \in \mathbb{Z}} \left| \int_X (D_k(y_1, z) - D_k(y_2, z))(b(z) - b_B)f_2(z) d\mu(z) \right|^r \right)^{\frac{1}{r}} \right| d\mu(y_2), \end{aligned}$$

therefore, we only consider

$$\left(\sum_{k \in \mathbb{Z}} \left| \int_X (D_k(y_1, z) - D_k(y_2, z))(b(z) - b_B)f_2(z) d\mu(z) \right|^r \right)^{\frac{1}{r}}.$$

From Fubini-Tonelli theorem, Hölder inequality, (1.1), (1.6) and (2.2), it follows that

$$\begin{aligned} &\left(\sum_{k \in \mathbb{Z}} \left| \int_X (D_k(y_1, z) - D_k(y_2, z))(b(z) - b_B)f_2(z) d\mu(z) \right|^r \right)^{\frac{1}{r}} \\ &\leq \left[\int_{X \setminus 6B} \left(\sum_{k \in \mathbb{Z}} |D_k(y_1, z) - D_k(y_2, z)|^r \right) |b(z) - b_B|^r |f(z)|^r d\mu(z) \right]^{\frac{1}{r}} \\ &\leq C \left\{ \sum_{i=-\infty}^{H_B^{y_1}-3} \int_{Q_{y_1, i} \setminus Q_{y_1, i+1}} \left[\frac{d(y_1, y_2)}{r(Q_{y_1, i})} \right]^{re} \frac{|b(z) - b_B|^r |f(z)|^r}{[\lambda(y_1, d(y_1, z))]^r} d\mu(z) \right\}^{\frac{1}{r}} \\ &\quad + C \left\{ \int_{Q_{y_1, H_B^{y_1}-2} \setminus 6B} \frac{|b(z) - b_B|^r |f(z)|^r}{[\lambda(y_1, d(y_1, z))]^r} d\mu(z) \right\}^{\frac{1}{r}} \\ &\leq C \left\{ \sum_{i=3}^{\infty} \int_{Q_{y_1, H_B^{y_1}-i} \setminus Q_{y_1, H_B^{y_1}-i+1}} \left[\frac{r(B)}{r(Q_{y_1, H_B^{y_1}-i+3})} \right]^{re} \frac{|b(z) - b_B|^r |f(z)|^r}{[\lambda(y_1, d(y_1, z))]^r} d\mu(z) \right\}^{\frac{1}{r}} \\ &\quad + C \left\{ \int_{Q_{y_1, H_B^{y_1}-2} \setminus 6B} \frac{|b(z) - b_B|^r |f(z)|^r}{[\lambda(y_1, d(y_1, z))]^r} d\mu(z) \right\}^{\frac{1}{r}} \\ &\leq C \left\{ \sum_{i=3}^{\infty} \left[\frac{r(B)}{r(Q_{y_1, H_B^{y_1}-i+3})} \right]^{re} \frac{1}{[\lambda(y_1, r(Q_{y_1, H_B^{y_1}-i+1}))]^r} \int_{Q_{y_1, H_B^{y_1}-i}} |b(z) - b_B|^r |f(z)|^r d\mu(z) \right\}^{\frac{1}{r}} \\ &\quad + \frac{C}{\lambda(y_1, r(6B))} \left\{ \int_{Q_{y_1, H_B^{y_1}-2}} |b(z) - b_B|^r |f(z)|^r d\mu(z) \right\}^{\frac{1}{r}} \end{aligned} \tag{3.6}$$

$$\begin{aligned}
&\leq C \left\{ \sum_{i=3}^{\infty} \left[\frac{r(B)}{r(Q_{y_1, H_B^{y_1-i+3}})} \right]^{r\varepsilon} \frac{1}{[\lambda(y_1, r(Q_{y_1, H_B^{y_1-i+1}}))]^r} \right. \\
&\quad \times \left. \left[\int_{Q_{y_1, H_B^{y_1-i}}} |b(z) - b_{Q_{y_1, H_B^{y_1-i}}}|^r |f(z)|^r d\mu(z) + |b_B - b_{Q_{y_1, H_B^{y_1-i}}}|^r \int_{Q_{y_1, H_B^{y_1-i}}} |f(z)|^r d\mu(z) \right]^{\frac{1}{r}} \right. \\
&\quad \left. + \frac{C}{\lambda(y_1, r(6B))} \left\{ |b_{Q_{y_1, H_B^{y_1-2}}} - b_B|^r \int_{Q_{y_1, H_B^{y_1-2}}} |f(z)|^r d\mu(z) + \int_{Q_{y_1, H_B^{y_1-2}}} |b(z) - b_{Q_{y_1, H_B^{y_1-2}}}|^r |f(z)|^r d\mu(z) \right\}^{\frac{1}{r}} \right. \\
&\leq C \left\{ \sum_{i=3}^{\infty} \frac{1}{10^{(r(i-3)\varepsilon)}} \frac{1}{[\lambda(y_1, r(Q_{y_1, H_B^{y_1-i+1}}))]^r} \left[\|b\|_{\text{RBMO}(\mu)}^r [\mu(6Q_{y_1, H_B^{y_1-i}})] M_{r,6}^r(f)(x) \right. \right. \\
&\quad \left. \left. + i^r \|b\|_{\text{RBMO}(\mu)}^r [\mu(6Q_{y_1, H_B^{y_1-i}})] M_{r,6}^r(f)(x) \right]^{\frac{1}{r}} \right. \\
&\quad \left. + \frac{C}{\lambda(y_1, r(6B))} \{ \|b\|_{\text{RBMO}(\mu)}^r [\mu(6Q_{y_1, H_B^{y_1-2}})] M_{r,6}^r(f)(x) + \|b\|_{\text{RBMO}(\mu)}^r [\mu(6Q_{y_1, H_B^{y_1-i}})] M_{r,6}^r(f)(x) \}^{\frac{1}{r}} \right. \\
&\leq C \|b\|_{\text{RBMO}(\mu)} \{ M_{r^2,6}(f)(x) + M_{r,6}(f)(x) \},
\end{aligned}$$

where $H_B^{y_1}$ represents the largest integer k satisfying $B \subset B_{y_1,k}$ for any ball B and $y_1 \in B \cap \text{supp}(\mu)$. Moreover, we also need some known facts proved in [10].

$$\mathcal{X} \setminus 6B = \bigcup_{i=3}^{\infty} (Q_{y_2, H_B^{y_1-i}} \setminus Q_{y_2, H_B^{y_1-i+1}}) \cup (Q_{y_2, H_B^{y_1-i+1}} \setminus 6B)$$

and, for all $y_1, y_2 \in B$ and $z \in \mathcal{X} \setminus Q_{y_2, H_B^{y_1-2}}$,

$$\sum_{k \in \mathbb{Z}} |D_k(y_1, z) - D_k(y_2, z)|^r \leq C \left[\frac{d(y_1, y_2)}{r(Q_{y_1, H_B^{y_1-i+3}})} \right]^{r\varepsilon} \frac{1}{[\lambda(y_1, d(y_1, z))]^r}.$$

Combining the estimates for (3.4), (3.5) and (3.6), we get (3.2).

Now, we show the condition (3.3). Consider two balls $B, S \subset \mathcal{X}$ satisfying $B \subset S$ and $x \in B$, and let $N := N_{B,S} + 1$. Write

$$\begin{aligned}
|h_B - h_S| &\leq |m_B(\dot{g}_r((b - b_S)f\chi_{6^N B \setminus 6B}))| + |m_S(\dot{g}_r((b - b_S)f\chi_{6^N B \setminus 6S}))| + |m_B(\dot{g}_r((b - b_S)f\chi_{\mathcal{X} \setminus 6^N B}))| \\
&\quad - |m_S(\dot{g}_r((b - b_S)f\chi_{\mathcal{X} \setminus 6^N B}))| + |m_B(\dot{g}_r((b_S - b_B)f\chi_{\mathcal{X} \setminus 6B}))| \\
&=: E_1 + E_2 + E_3 + E_4.
\end{aligned}$$

With an argument similar to that used in the estimate of (3.6), it is easy to see that

$$E_3 \leq C \|b\|_{\text{RBMO}(\mu)} \{ M_{r^2,6}(f)(x) + M_{r,6}(f)(x) \}.$$

For $y \in B$, by applying Fubini-Tonelli Theorem, Minkowski inequality, Hölder inequality and (2.2), we can deduce that

$$\begin{aligned}
E_1 &\leq \frac{1}{\mu(B)} \int_B |\dot{g}_r((b - b_S)f\chi_{6^N B \setminus 6B})(y)| d\mu(y) \\
&\leq \frac{1}{\mu(B)} \int_B \left\{ \int_{6^N B \setminus 6B} \left[\sum_{k \in \mathbb{Z}} |D(y, z)|^r \right] |b(z) - b_S|^r |f(z)|^r d\mu(z) \right\}^{\frac{1}{r}} d\mu(y)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\mu(B)} \int_{6^N B \setminus 6B} \left(\int_B \left[\sum_{k \in \mathbb{Z}} |D(y, z)|^r \right]^{\frac{1}{r}} d\mu(y) \right) |b(z) - b_S| f(z) d\mu(z) \\
&\leq C \sum_{j=1}^{N-1} \int_{6^{j+1} B \setminus 6^j B} \frac{|b(z) - b_S|}{\lambda(c(B), d(c(B), z))} |f(z)| d\mu(z) \\
&\leq C \sum_{j=1}^{N-1} \frac{1}{\lambda(c(B), 6^j r_B)} \left[\int_{6^{j+1} B} |b(z) - b_{6^{j+1} B}| |f(z)| d\mu(z) + |b_{6^{j+1} B} - b_S| \int_{6^{j+1} B} |f(z)| d\mu(z) \right] \\
&\leq CK_{B,S} \|b\|_{\text{RBMO}(\mu)} M_{r,6}(f)(x) \sum_{j=1}^{N-1} \frac{\mu(6^{j+1} B)}{\lambda(c(B), 6^j r(B))} \\
&\leq CK_{B,S} \|b\|_{\text{RBMO}(\mu)} M_{r,6}(f)(x).
\end{aligned}$$

Similarly, it follows that $E_2 \leq CK_{B,S} \|b\|_{\text{RBMO}(\mu)} M_{r,6}(f)(x)$.

Now, let us estimate E_4 . For any $y \in B$, from (1.5), Fubini-Tonelli Theorem, Minkowski inequality and Hölder inequality, we have

$$\begin{aligned}
&\frac{1}{\mu(B)} \int_B |\dot{g}_r((b_B - b_S) f \chi_{X \setminus 6B})(y)| d\mu(y) \\
&\leq CK_{B,S} \|b\|_{\text{RBMO}(\mu)} \frac{1}{\mu(B)} \left[\int_B |\dot{g}_r(f \chi_{X \setminus 6B})(y)| d\mu(y) + \int_B |\dot{g}_r(f \chi_{6B})(y)| d\mu(y) \right] \\
&\leq CK_{B,S} \|b\|_{\text{RBMO}(\mu)} \{M_{r,6}(\dot{g}(f))(x) + M_{r,6}(f)(x)\}.
\end{aligned}$$

□

Proof of Theorem 1.10. By applying Lemma 2.3 and (3.1), it is obvious to see that

$$\begin{aligned}
\|[b, \dot{g}_r](f)\|_{L^{1,\infty}(\mu)} &\leq \|N([b, \dot{g}_r](f))\|_{L^{1,\infty}(\mu)} \\
&\leq C \|M^\sharp([b, \dot{g}_r](f))\|_{L^{1,\infty}(\mu)} \\
&\leq C \|b\|_{\text{RBMO}(\mu)} \{ \|M_{r,6}(\dot{g}_r(f))\|_{L^{1,\infty}(\mu)} + \|M_{r^2,6}(f)\|_{L^{1,\infty}(\mu)} + \|M_{r,6}(f)\|_{L^{1,\infty}(\mu)} \} \\
&\leq C \|b\|_{\text{RBMO}(\mu)} \|f\|_{L^1(\mu)},
\end{aligned}$$

which is the desired result. □

Proof of Theorem 1.11. By applying (2.4), (3.1) and Theorem 14 (see [9]), we have

$$\begin{aligned}
\|[b, \dot{g}_r](f)\|_{M_q^p(\mu)} &\leq \|N([b, \dot{g}_r](f))\|_{M_q^p(\mu)} \\
&\leq C \|M^\sharp([b, \dot{g}_r](f))\|_{M_q^p(\mu)} \\
&\leq C \|b\|_{\text{RBMO}(\mu)} \{ \|M_{r,6}(\dot{g}_r(f))\|_{M_q^p(\mu)} + \|M_{r^2,6}(f)\|_{M_q^p(\mu)} + \|M_{r,6}(f)\|_{M_q^p(\mu)} \} \\
&\leq C \|b\|_{\text{RBMO}(\mu)} \|f\|_{M_q^p(\mu)}.
\end{aligned}$$

□

Proof of Theorem 1.12. From Remark 1.7 (1), we assume $k = 6$ in (1.8). By applying Definition 1.6, Lemma 2.3 (2), Theorem 4.2 in [15] and (3.1), we can deduce that

$$\begin{aligned}
\|[b, \dot{g}_r](f)\|_{L^{p,\phi}(\mu)} &\leq \sup_B \frac{C}{[\phi(\mu(6B))]^{\frac{1}{p}}} \|N([b, \dot{g}_r](f))\|_{L^p(\mu,B)} \\
&\leq \sup_B \frac{C}{[\phi(\mu(6B))]^{\frac{1}{p}}} \|M^\sharp([b, \dot{g}_r](f))\|_{L^p(\mu,B)} \\
&\leq C \|b\|_{\text{RBMO}(\mu)} \{ \|M_{r,6}(\dot{g}_r(f))\|_{L^{p,\phi}(\mu)} + \|M_{r^2,6}(f)\|_{L^{p,\phi}(\mu)} + \|M_{r,6}(f)\|_{L^{p,\phi}(\mu)} \} \\
&\leq C \|b\|_{\text{RBMO}(\mu)} \|f\|_{L^{p,\phi}(\mu)}.
\end{aligned}$$

□

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