Research Article

Shibo Liu*

On quasilinear elliptic problems with finite or infinite potential wells

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Abstract: We consider quasilinear elliptic problems of the form

$$-\operatorname{div}(\phi(|\nabla u|)\nabla u) + V(x)\phi(|u|)u = f(u), \quad u \in W^{1,\Phi}(\mathbb{R}^N),$$

where ϕ and f satisfy suitable conditions. The positive potential $V \in C(\mathbb{R}^N)$ exhibits a finite or infinite potential well in the sense that $V(x)$ tends to its supremum $V_\infty \leq +\infty$ as $|x| \rightarrow \infty$. Nontrivial solutions are obtained by variational methods. When $V_\infty = +\infty$, a compact embedding from a suitable subspace of $W^{1,\Phi}(\mathbb{R}^N)$ into $L^\Phi(\mathbb{R}^N)$ is established, which enables us to get infinitely many solutions for the case that f is odd. For the case that $V(x) = \lambda a(x) + 1$ exhibits a steep potential well controlled by a positive parameter λ , we get nontrivial solutions for large λ .

Keywords: Orlicz-Sobolev space, Φ -Laplacian, critical points, $(PS)_c$ sequence**MSC 2020:** 35J20, 35J60, 35J70

1 Introduction

In this paper, we consider the following quasilinear elliptic problem in $\mathbb{R}^N$,

$$-\operatorname{div}(\phi(|\nabla u|)\nabla u) + V(x)\phi(|u|)u = f(u), \quad u \in W^{1,\Phi}(\mathbb{R}^N). \quad (1.1)$$

where $\phi : [0, \infty) \rightarrow [0, \infty)$ is a C^1 -function satisfying the following assumptions:

(ϕ_1) the function $t \mapsto \phi(t)t$ is increasing in $(0, \infty)$,

(ϕ_2) there exist $\ell, m \in (1, N)$ such that

$$\ell \leq \frac{\phi(|t|)t^2}{\Phi(t)} \leq m \quad \text{for all } t \neq 0, \quad (1.2)$$

where $\ell \leq m < \ell^*$ (note that for $p \in (1, N)$ we set $p^* = Np/(N - p)$),

$$\Phi(t) = \int_0^{|t|} \phi(s)s \, ds.$$

Nonlinear elliptic problems in $\mathbb{R}^N$ like (1.1) have been extensively studied. For example, if $\phi(t) \equiv 1$, then problem (1.1) reduces to the following stationary Schrödinger equation:

$$-\Delta u + V(x)u = f(u), \quad u \in H^1(\mathbb{R}^N), \quad (1.3)$$

* **Corresponding author: Shibo Liu**, Department of Mathematics, Xiamen University, Xiamen 361005, China, e-mail: liusb@xmu.edu.cn

which is a central topic in nonlinear analysis in the last decade, see [1–6] and references therein. If $\phi(t) = t^{p-2}$, then the leading term in (1.1) is the p -Laplacian operator $-\Delta_p$ and the corresponding problem has also been studied in many papers such as [7–10]. If $\phi(t) = t^{p-2} + t^{q-2}$, the leading term in (1.1) is the so-called (p, q) -Laplacian operator and results for the corresponding problems can be found in [11–13].

For general ϕ satisfying (ϕ_1) and (ϕ_2) , $-\Delta_\Phi u := -\operatorname{div}(\phi(|\nabla u|)\nabla u)$ is called the Φ -Laplacian of u . The Φ -Laplacian operator $-\Delta_\Phi$ arises in some applications such as nonlinear elasticity, plasticity and non-Newtonian fluids. Elliptic boundary value problems involving the Φ -Laplacian have been studied on a bounded domain $\Omega \subset \mathbb{R}^N$ in several recent papers, such as Clément et al. [14], Fukagai and Narukawa [15] and Carvalho et al. [16].

For unbounded domains such as $\mathbb{R}^N$, there are also some recent results on the quasilinear Φ -Laplacian problem (1.1). In Alves et al. [17], the authors studied problem (1.1) by variational methods under the following conditions on the potential V and the nonlinearity f :

(V_0) $V \in C(\mathbb{R}^N)$, $V_0 = \inf_{\mathbb{R}^N} V > 0$;

(f_1) $f \in C(\mathbb{R})$ satisfies

$$\lim_{|t| \rightarrow 0} \frac{f(t)}{\phi(|t|)t} = \lambda_0, \quad \lim_{|t| \rightarrow \infty} \frac{f(t)}{\phi_*(|t|)t} = 0, \quad (1.4)$$

where ϕ_* is related to Φ_* , the Sobolev conjugate function of Φ (see (2.4)), via

$$\Phi_*(t) = \int_0^{|t|} \phi_*(s) s \, ds;$$

(f_2) there exists $\theta > m$ such that for all $t \neq 0$,

$$0 < F(t) := \int_0^t f(s) \, ds \leq \frac{1}{\theta} t f(t).$$

Because problem (1.1) is settled on the unbounded domain $\mathbb{R}^N$, to overcome the lack of compactness of the relevant Sobolev embeddings, the authors considered the cases that V is radial or $\mathbb{Z}^N$ -periodic. Using a Strauss-type result and a Lions-type concentration lemma in Orlicz-Sobolev spaces established in the paper, they obtained nontrivial solutions for the problem via the mountain pass theorem [18].

Remark 1.1. If $\phi(t) = t^{p-2}$, then $-\Delta_\Phi$ is the p -Laplacian operator $-\Delta_p$ and (1.1) becomes the p -Laplacian equation

$$-\operatorname{div}(|\nabla u|^{p-2} \nabla u) + V(x)|u|^{p-2}u = f(u), \quad u \in W^{1,p}(\mathbb{R}^N),$$

which has been extensively studied in the last few decades, see e.g. [8,19]. In this case, if $p \in (1, N)$, then a direct computation using (2.4) gives

$$\Phi_*(t) = (p^*)^{-p^*} p^{N/(p-N)} |t|^{p^*}, \quad \phi_*(t) = (p^*)^{1-p^*} p^{N/(p-N)} t^{p^*-2}.$$

Consequently, the second limit in (1.4) is precisely the usual (asymptotically) subcritical growth condition

$$\lim_{|t| \rightarrow \infty} \frac{f(t)}{|t|^{p^*-2}t} = 0.$$

While if $p = N$, we could not determine the functions Φ_* and ϕ_* via (2.4). For this case to apply variational methods one usually requires that f behaves like $\exp(\alpha |u|^{N/(N-1)})$, see [19] for a typical reference.

For the autonomous case that $V(x) \equiv 0$ and $f(u) = |u|^{s-2} - |u|^{\alpha-2}$, nontrivial solutions for (1.1) have also been obtained in [20,21] via mountain pass theorem, thanks to the compact embeddings from the radial

Orlicz-Sobolev spaces to certain Lebesgue spaces $L^r(\mathbb{R}^N)$ established in these papers. The main difference of these two papers is in the assumptions on ϕ .

In [22], Chorfi and Rădulescu studied the following problem:

$$-\operatorname{div}(\phi(|\nabla u|)\nabla u) + a(x)|u|^{\alpha-2}u = f(x, u) \quad \text{in } \mathbb{R}^N, \quad (1.5)$$

where the function ϕ is the same as in [21] and a verifies

$$\lim_{|x| \rightarrow 0} a(x) = +\infty, \quad \lim_{|x| \rightarrow +\infty} a(x) = +\infty. \quad (1.6)$$

Note that the zero order term on the left-hand side of (1.5) is a power function of u , which is different than that of (1.1). Using the strategy initiated by Rabinowitz [6], condition (1.6) enables the authors to overcome the lack of compactness and obtain a nontrivial solution for problem (1.5).

There are also some papers for the case that there is a parameter $\varepsilon > 0$ in (1.1), existence and multiplicity of solutions for the equation were obtained for ε small, see [23–25].

Our results are closely related to those of Alves et al. [17]. As mentioned before, in their paper they studied the case that the potential V is radial or $\mathbb{Z}^N$ -periodic. In our first result, we consider the case that V satisfies the following condition due to Bartsch and Wang [26] in their study of (1.3):

$$(V_1) \text{ for all } M > 0, \mu(V^{-1}(-\infty, M]) < \infty.$$

Here μ is the Lebesgue measure on $\mathbb{R}^N$. Note that (V_1) is satisfied if V is coercive:

$$\lim_{|x| \rightarrow \infty} V(x) = +\infty. \quad (1.7)$$

To apply variational methods let X be a suitable subspace of the Orlicz-Sobolev space $W^{1,\Phi}(\mathbb{R}^N)$ that will be made clear in Section 2 and consider the C^1 -functional $\mathcal{J} : X \rightarrow \mathbb{R}$ given by

$$\mathcal{J}(u) = \int_{\mathbb{R}^N} \Phi(|\nabla u|) + \int_{\mathbb{R}^N} V(x)\Phi(|u|) - \int_{\mathbb{R}^N} F(u).$$

Then, solutions of (1.1) are critical points of $\mathcal{J}$.

Theorem 1.2. *Suppose ϕ satisfies $(\phi_1), (\phi_2)$; V satisfies $(V_0), (V_1)$; f satisfies $(f_1), (f_2)$.*

(1) *If $\lambda_0 = 0$, then (1.1) has a nontrivial solution.*

(2) *If $\lambda_0 \geq 0$ and f is odd, then (1.1) has a sequence of solutions $\{u_n\}$ such that $\mathcal{J}(u_n) \rightarrow +\infty$.*

Motivated by Bartsch and Wang [26], the assumption (V_1) enables us to establish a compact embedding result from our working space X into subcritical Orlicz spaces, see Lemma 2.3. With this result we can regain compactness for our functional $\mathcal{J}$ and get critical points.

As a special case of (V_1) , (1.7) can be interpreted as V has an infinite potential well. In our next result, we investigate the case that V exhibits a finite potential well:

$$(V_2) \text{ for all } x \in \mathbb{R}^N, V(x) < V_\infty := \lim_{|x| \rightarrow \infty} V(x) < \infty.$$

Under the assumption (V_2) the above compact embedding is not valid anymore. Hence, to get critical points of $\mathcal{J}$, we need the following monotonicity assumptions on ϕ and f :

(ϕ_3^s) for some $s \geq 2$, the function $t \mapsto \phi(t)/t^{s-2}$ is nonincreasing on $(0, \infty)$,

(f_3^s) for some $s \geq 2$, the function $t \mapsto f(t)/|t|^{s-1}$ is strictly increasing on $(0, \infty)$ and $(-\infty, 0)$.

Note that (f_3^s) implies that for all $\xi \in \mathbb{R} \setminus \{0\}$, $t \mapsto f(t\xi)/t^{s-1}$ is strictly increasing on $(0, \infty)$. Our result reads as follows.

Theorem 1.3. *Suppose ϕ satisfies $(\phi_1), (\phi_2)$ and (ϕ_3^s) ; V satisfies $(V_0), (V_2)$; f satisfies (f_1) with $\lambda_0 = 0$, (f_2) and (f_3^s) . Then (1.1) has a nontrivial solution.*

For our last result, we consider the case that the potential $V(x)$ is of the form $\lambda a(x) + 1$ with $\lambda > 0$ and a satisfies

(a₁) $a \in C(\mathbb{R}^N)$, $a \geq 0$ and $a^{-1}(0)$ has nonempty interior;

(a₂) for some $M_0 > 0$ we have $\mu(a^{-1}(-\infty, M_0]) < \infty$.

For problem (1.3), these conditions are introduced by Bartsch and Wang [26] and characterize V as possessing a steep potential well whose height is controlled by the positive parameter λ . Our result for this case is the following theorem.

Theorem 1.4. Suppose (ϕ_1) , (ϕ_2) , (a_1) and (a_2) are satisfied, $f \in C(\mathbb{R})$ satisfies (f_1) with $\lambda_0 = 0$ and (f_2) . Then there exists $\lambda^* > 0$ such that for all $\lambda \geq \lambda^*$, the problem

$$-\operatorname{div}(\phi(|\nabla u|)\nabla u) + V_\lambda(x)\phi(|u|)u = f(u), \quad u \in W^{1,\Phi}(\mathbb{R}^N), \quad (1.8)$$

has a nontrivial solution, where $V_\lambda(x) = \lambda a(x) + 1$.

Our Theorems 1.2 and 1.4 are generalizations of Theorem 2.1 and part of Theorem 2.4 in Bartsch and Wang [26], respectively. However, even for the semilinear case that $\phi(t) \equiv 1$, our Theorem 1.4 is slightly general than the corresponding result in [26], because in (f_1) we only require f to be asymptotically subcritical, that is the second limit in (1.4) holds, while in [26, Theorem 2.4] the nonlinearity f is strictly subcritical, meaning that the growth of f at infinity is controlled by a subcritical power function $|t|^{q-2}t$ for some $q \in (2, 2^*)$. See Remark 3.6 for more details. Roughly speaking, Theorem 1.3 also generalizes Rabinowitz [6, Theorem 4.27].

Both [6] and [26] are concerned on the semilinear equation (1.3). Our Φ -Laplacian equation (1.1) is much more general.

The paper is organized as follows. In Section 2, we recall some concepts and results about Orlicz spaces and prove the compact embedding lemma (Lemma 2.3) mentioned before. The existence of nontrivial solutions is proved in Section 3. Finally, in Section 4 we deal with the multiplicity result stated in Theorem 1.2(2).

2 Orlicz-Sobolev spaces

In this section, we recall some results about Orlicz spaces and Orlicz-Sobolev spaces that we will use for proving our main results. The reader is referred to [17, 27] and references therein, in particular [28], for more details.

A convex, even continuous function $\Phi : \mathbb{R} \rightarrow [0, \infty)$ is called a nice Young function, $\mathcal{N}$ -function for short, if $\Phi(t) = 0$ is equivalent to $t = 0$, and

$$\lim_{t \rightarrow 0} \frac{\Phi(t)}{t} = 0, \quad \lim_{t \rightarrow +\infty} \frac{\Phi(t)}{t} = +\infty.$$

The $\mathcal{N}$ -function Φ satisfies the Δ_2 -condition if there is a constant $K > 0$ such that

$$\Phi(2t) \leq K\Phi(t) \quad \text{for all } t \geq 0.$$

Then, for an open subset Ω of $\mathbb{R}^N$, under the natural addition and scale multiplication,

$$L^\Phi(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} \mid u \text{ is measurable, } \int_\Omega \Phi(|u|) < \infty \right\}$$

is a vector space. Equipped with the Luxemburg norm

$$|u|_{\Phi} = \inf \left\{ \alpha > 1 \mid \int_{\Omega} \Phi \left(\frac{|u|}{\alpha} \right) \leq 1 \right\},$$

$L^{\Phi}(\Omega)$ is a Banach space, called Orlicz space. The Orlicz-Sobolev space $W^{1,\Phi}(\mathbb{R}^N)$ is the completion of $C_0^{\infty}(\mathbb{R}^N)$ under the norm

$$\|u\|_1 = |\nabla u|_{\Phi} + |u|_{\Phi}. \quad (2.1)$$

The complement function of Φ , denoted by $\check{\Phi}$, is given by the Legendre transformation

$$\check{\Phi}(s) = \max_{t \geq 0} \{st - \Phi(t)\} \quad \text{for } s \geq 0.$$

Then,

$$\check{\Phi}(\Phi'(t)) \leq \Phi(2t) \quad \text{for } t \geq 0, \quad (2.2)$$

and we have the Hölder inequality

$$\int_{\Omega} |uv| \leq 2|u|_{\Phi} |v|_{\check{\Phi}} \quad (2.3)$$

for $u \in L^{\Phi}(\Omega)$ and $v \in L^{\check{\Phi}}(\Omega)$.

When

$$\int_1^{+\infty} \frac{\Phi^{-1}(s)}{s^{(N+1)/N}} ds = +\infty,$$

the function Φ_* given by

$$\Phi_*^{-1}(t) = \int_0^t \frac{\Phi^{-1}(s)}{s^{(N+1)/N}} ds \quad (2.4)$$

is called the Sobolev conjugate function of Φ . It is known that similar to (1.2) we have

$$\varrho^* \leq \frac{\phi_*(|t|)t^2}{\Phi_*(t)} \leq m^* \quad \text{for all } t \neq 0. \quad (2.5)$$

It is also known that if Φ and $\check{\Phi}$ satisfy the Δ_2 -condition, then $L^{\Phi}(\Omega)$ and $W^{1,\Phi}(\mathbb{R}^N)$ are reflexive and separable. Moreover,

$$u_n \rightarrow u \text{ in } L^{\Phi}(\Omega) \Leftrightarrow \int_{\Omega} \Phi(|u_n - u|) \rightarrow 0, \quad (2.6)$$

$$u_n \rightarrow u \text{ in } W^{1,\Phi}(\mathbb{R}^N) \Leftrightarrow \int_{\mathbb{R}^N} (\Phi(|\nabla u_n - \nabla u|) + \Phi(|u_n - u|)) \rightarrow 0. \quad (2.7)$$

In addition, $\{u_n\}$ is bounded in $L^{\Phi}(\mathbb{R}^N)$ if and only if $\{\Phi(|u_n|)\}$ is bounded in $L^1(\mathbb{R}^N)$. This can be seen by setting $V = 1$ in (2.10) below.

Let Ψ be an $\mathcal{N}$ -function verifying Δ_2 -condition. It is well known that if

$$\overline{\lim}_{t \rightarrow 0} \frac{\Psi(t)}{\Phi(t)} < +\infty, \quad \overline{\lim}_{|t| \rightarrow +\infty} \frac{\Psi(t)}{\Phi_*(t)} < +\infty,$$

then we have a continuous embedding $W^{1,\Phi}(\mathbb{R}^N) \hookrightarrow L^\Psi(\mathbb{R}^N)$. Moreover, if

$$\lim_{|t| \rightarrow 0} \frac{\Psi(t)}{\Phi(t)} < +\infty, \quad \lim_{|t| \rightarrow \infty} \frac{\Psi(t)}{\Phi_s(t)} = 0, \quad (2.8)$$

then the embedding $W^{1,\Phi}(\mathbb{R}^N) \hookrightarrow L^\Psi_{\text{loc}}(\mathbb{R}^N)$ is compact, such Ψ is called subcritical.

For the study of problem (1.1), we introduce the following subspace of $W^{1,\Phi}(\mathbb{R}^N)$. Assuming (V_0) , (ϕ_1) , and (ϕ_2) on the linear subspace

$$X = \left\{ u \in W^{1,\Phi}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V(x)\Phi(|u|) < \infty \right\},$$

we equip the norm $\|u\| = |\nabla u|_\Phi + |u|_{\Phi,V}$, where

$$|u|_{\Phi,V} = \inf \left\{ \alpha > 0 \mid \int_{\Omega} V(x)\Phi\left(\frac{|u|}{\alpha}\right) \leq 1 \right\}.$$

Then $(X, \|\cdot\|)$ is a separable reflexive Banach space, which will be simply denoted by X . If V is bounded, then X is precisely the original Orlicz-Sobolev space $W^{1,\Phi}(\mathbb{R}^N)$, the norm $\|\cdot\|$ is equivalent to the one given in (2.1).

Lemma 2.1. Assume that (V_0) , (ϕ_1) and (ϕ_2) hold and for $t \geq 0$ let

$$\xi_0(t) = \min\{t^\ell, t^m\}, \quad \xi_1(t) = \max\{t^\ell, t^m\}. \quad (2.9)$$

Then for all $u \in X$ we have

$$\xi_0(|u|_{\Phi,V}) \leq \int_{\mathbb{R}^N} V(x)\Phi(|u|) \leq \xi_1(|u|_{\Phi,V}). \quad (2.10)$$

Proof. According to [27, Lemma 2.1], we have

$$\xi_0(\rho)\Phi(t) \leq \Phi(\rho t) \leq \xi_1(\rho)\Phi(t) \quad \text{for } \rho, t \geq 0. \quad (2.11)$$

Taking $\rho = |u|_{\Phi,V}$ and $t = |u(x)|/|u|_{\Phi,V}$, we get

$$\int_{\mathbb{R}^N} V(x)\Phi(|u|) = \int_{\mathbb{R}^N} V(x)\Phi\left(|u|_{\Phi,V} \frac{|u|}{|u|_{\Phi,V}}\right) \leq \xi_1(|u|_{\Phi,V}) \int_{\mathbb{R}^N} V(x)\Phi\left(\frac{|u|}{|u|_{\Phi,V}}\right) \leq \xi_1(|u|_{\Phi,V})$$

because by the definition of $|\cdot|_{\Phi,V}$, the integral in the last line is not greater than 1. The first inequality in (2.10) can be proved similarly. $\square$

Remark 2.2. Similar to (2.11), because of (2.5), for

$$\xi_0^*(t) = \min\{t^{\ell^*}, t^{m^*}\}, \quad \xi_1^*(t) = \max\{t^{\ell^*}, t^{m^*}\},$$

we have

$$\xi_0^*(\rho)\Phi_*(t) \leq \Phi_*(\rho t) \leq \xi_1^*(\rho)\Phi_*(t) \quad \text{for } \rho, t \geq 0.$$

Because $m < \ell^*$, using this and (2.11) we have

$$0 < \frac{\Phi(t)}{\Phi_s(t)} \leq \frac{\Phi(1)\xi_1(t)}{\Phi_s(1)\xi_0^*(t)} \leq \frac{\Phi(1)}{\Phi_s(1)} \frac{t^m}{t^{\ell^*}} \rightarrow 0 \quad \text{as } |t| \rightarrow \infty.$$

Therefore, Φ is an $\mathcal{N}$ -function verifying Δ_2 -condition and (2.8).

Lemma 2.3. Suppose ϕ satisfies $(\phi_1), (\phi_2)$; V satisfies $(V_0), (V_1)$. Then for any $\mathcal{N}$ -function Ψ verifying Δ_2 -condition and (2.8), the embedding $X \hookrightarrow L^\Psi(\mathbb{R}^N)$ is compact. In particular, $X \hookrightarrow L^\Phi(\mathbb{R}^N)$ is compact.

Proof. Assume that $\{u_n\}$ is a sequence in X such that $u_n \rightarrow 0$ in X , we want to show that $u_n \rightarrow 0$ in $L^\Psi(\mathbb{R}^N)$. First, we have $\|u_n\| \leq C_1$ for some $C_1 > 0$.

For any $\varepsilon > 0$, by (2.8) there is $k > 0$ such that

$$\Psi(t) = \Psi(|t|) \leq \varepsilon \Phi_*(|t|), \quad \text{for } |t| > k.$$

Since the embedding $X \hookrightarrow L^\Phi(\mathbb{R}^N)$ is continuous and $\|u_n\| \leq C_1$, we deduce that $\{u_n\}$ is bounded in $L^\Phi(\mathbb{R}^N)$. Hence,

$$\int_{|u_n| > k} \Psi(|u_n|) \leq \varepsilon \int_{\mathbb{R}^N} \Phi_*(|u_n|) \leq \varepsilon \xi_1^*(|u_n|_{\Phi_*}) \leq C_2 \varepsilon, \quad (2.12)$$

for some $C_2 > 0$, where we have used an inequality for Φ_* similar to (2.10).

Given $M > 0$ and $R > 0$, set

$$A_R = \{x \in \mathbb{R}^N \mid |x| \geq R, V(x) \geq M\}, \quad B_R = \{x \in \mathbb{R}^N \mid |x| \geq R, V(x) < M\}.$$

By the first limit in (2.8), there is $\kappa > 0$ such that for $t \in [0, k]$ we have $\Psi(t) \leq \kappa \Phi(t)$. Because $\{u_n\}_{\Phi, V}$ is bounded, we can take $M > 0$ large enough such that

$$\frac{\kappa}{M} \xi_1(|u_n|_{\Phi, V}) < \varepsilon.$$

It follows from (2.8) that Ψ is bounded in $[0, k]$. By (V_1) , we have $\mu(B_R) \rightarrow 0$ as $R \rightarrow \infty$, thus we can choose $R > 0$ such that

$$\mu(B_R) \cdot \sup_{[0, k]} \Psi < \varepsilon.$$

Using the above inequalities and Lemma 2.1 we have

$$\begin{aligned} \int_{A_R \cap \{|u_n| \leq k\}} \Psi(|u_n|) &\leq \kappa \int_{A_R \cap \{|u_n| \leq k\}} \Phi(|u_n|) \leq \frac{\kappa}{M} \int_{\mathbb{R}^N} V(x) \Phi(|u_n|) \leq \frac{\kappa}{M} \xi_1(|u_n|_{\Phi, V}) \leq \varepsilon, \\ \int_{B_R \cap \{|u_n| \leq k\}} \Psi(|u_n|) &\leq \mu(B_R) \cdot \sup_{[0, k]} \Psi < \varepsilon. \end{aligned}$$

Consequently,

$$\int_{|u_n| \leq k} \Psi(|u_n|) \leq \left(\int_{|x| \leq R} + \int_{A_R \cap \{|u_n| \leq k\}} + \int_{B_R \cap \{|u_n| \leq k\}} \right) \Psi(|u_n|) \leq \int_{|x| \leq R} \Psi(|u_n|) + 2\varepsilon.$$

Now using (2.12) we get

$$\int_{\mathbb{R}^N} \Psi(|u_n|) = \int_{|u_n| > k} \Psi(|u_n|) + \int_{|u_n| \leq k} \Psi(|u_n|) \leq (C_2 + 2)\varepsilon + \int_{|x| \leq R} \Psi(|u_n|). \quad (2.13)$$

Since the embedding $X \hookrightarrow L_{\text{loc}}^\Psi(\mathbb{R}^N)$ is compact, from $u_n \rightarrow 0$ in X and (2.13) we have

$$\overline{\lim}_{n \rightarrow \infty} \int_{\mathbb{R}^N} \Psi(|u_n|) \leq (C_2 + 2)\varepsilon.$$

Because ε is arbitrary, this implies

$$\int_{\mathbb{R}^N} \Psi(|u_n|) \rightarrow 0$$

and $u_n \rightarrow 0$ in $L^\Psi(\mathbb{R}^N)$. □

3 Nontrivial solutions

From now on, we assume the conditions (ϕ_1) , (ϕ_2) , (V_0) and (f_1) . Then, the functional $\mathcal{J} : X \rightarrow \mathbb{R}$ given by

$$\mathcal{J}(u) = \int_{\mathbb{R}^N} \Phi(|\nabla u|) + \int_{\mathbb{R}^N} V(x)\Phi(|u|) - \int_{\mathbb{R}^N} F(u) \quad (3.1)$$

is of class C^1 . The derivative of $\mathcal{J}$ is given by

$$\langle \mathcal{J}'(u), v \rangle = \int_{\mathbb{R}^N} \phi(|\nabla u|) \nabla u \cdot \nabla v + \int_{\mathbb{R}^N} V(x) \phi(|u|) uv - \int_{\mathbb{R}^N} f(u) v \quad u, v \in X.$$

Thus, critical points of $\mathcal{J}$ are precisely weak solutions of our problem (1.1).

Under the assumptions (ϕ_1) , (ϕ_2) , (V_0) , (f_1) and (f_2) , it has also been proved in [17, Lemma 4.1] that $\mathcal{J}$ satisfies the mountain pass geometry: for some $\rho > 0$ and $\varphi \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}$,

$$\inf_{\|u\|=\rho} \mathcal{J}(u) = \eta > 0, \quad \lim_{t \rightarrow +\infty} \mathcal{J}(t\varphi) = -\infty. \quad (3.2)$$

Remark 3.1. In [17], $\mathcal{J}(t\varphi) \rightarrow -\infty$ as $t \rightarrow +\infty$ is only verified for $\varphi \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}$. But we can prove that $\mathcal{J}$ is anti-coercive on any finite dimensional subspace, see the verification of condition (2) of Proposition 4.3 in the proof of Theorem 1.2(2). Therefore, the limit in (3.2) is in fact valid for any $\varphi \in X$.

Denote $I = [0, 1]$ and set

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} \mathcal{J}(\gamma(t)) \quad (3.3)$$

being $\Gamma = \{\gamma \in C(I, X) | \gamma(0) = 0, \mathcal{J}(\gamma(1)) < 0\}$. Note that $c \geq \eta > 0$.

According to the mountain pass theorem [18, 29], there is a sequence $\{u_n\} \subset X$ such that

$$\mathcal{J}(u_n) \rightarrow c, \quad \mathcal{J}'(u_n) \rightarrow 0. \quad (3.4)$$

Such sequence is called a $(PS)_c$ sequence (named after R. Palais and S. Smale). Under the assumptions (ϕ_1) , (ϕ_2) , (V_0) , (f_1) and (f_2) , it has been shown in [17, Lemma 4.2] that the $(PS)_c$ sequence $\{u_n\}$ we just obtained is bounded in X .

The following result has been established in [17, Lemma 4.3].

Lemma 3.2. Suppose (ϕ_1) , (ϕ_2) , (V_0) and (f_1) hold. Let $\{u_n\}$ be a $(PS)_c$ sequence of $\mathcal{J}$. If $u_n \rightharpoonup u$ in X , then $\nabla u_n \rightarrow \nabla u$ a.e. in $\mathbb{R}^N$ and $\mathcal{J}'(u) = 0$.

3.1 Proof of Theorem 1.2(1)

By the above arguments, we know that $\mathcal{J}$ has a bounded $(PS)_c$ sequence $\{u_n\}$. Since X is reflexive, we may assume that $u_n \rightharpoonup u$ in X . By Lemma 3.2, u is a critical point of $\mathcal{J}$. We need to show that $u \neq 0$. Thanks to the compact embedding established in Lemma 2.3, this can be achieved as in [17, p. 454]. For the reader's convenience, we include the argument below.

Assume that $u = 0$. By Lemma 2.3, the embedding $X \hookrightarrow L^\Phi(\mathbb{R}^N)$ is compact. Thus, $u_n \rightarrow 0$ in $L^\Phi(\mathbb{R}^N)$ and we get

$$\int_{\mathbb{R}^N} \Phi(|u_n|) \rightarrow 0. \quad (3.5)$$

By (f_1) , for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(t)t| \leq \varepsilon \Phi_*(|t|) + C_\varepsilon \Phi(|t|).$$

Using this inequality and (3.5), and the boundedness of $\{u_n\}$ in $L^\Phi(\mathbb{R}^N)$, we deduce

$$\int_{\mathbb{R}^N} f(u_n)u_n \rightarrow 0.$$

Now, because $\langle \mathcal{J}'(u_n), u_n \rangle \rightarrow 0$, we obtain

$$\int_{\mathbb{R}^N} \phi(|\nabla u_n|)|\nabla u_n|^2 + \int_{\mathbb{R}^N} V(x)\phi(|u_n|)u_n^2 \rightarrow 0.$$

From this and (ϕ_2) we get

$$\int_{\mathbb{R}^N} \Phi(|\nabla u_n|) + \int_{\mathbb{R}^N} V(x)\Phi(|u_n|) \rightarrow 0.$$

That is $u_n \rightarrow 0$ in X . But $\mathcal{J}(u_n) \rightarrow c > 0$, this is a contradiction.

Remark 3.3. In Lemma 4.1, we will show that $\mathcal{J}$ satisfies the (PS) condition. Hence, the $(PS)_c$ sequence $\{u_n\}$ has a subsequence that converges to a nonzero critical point u at the level $c > 0$. We include the above argument for its simplicity.

3.2 Proof of Theorem 1.3

For convenience, in this subsection we assume all the conditions on ϕ , V and f required in Theorem 1.3. As before, $\mathcal{J}$ has a bounded $(PS)_c$ sequence $\{u_n\}$, $u_n \rightharpoonup u$ in $X = W^{1,\Phi}(\mathbb{R}^N)$ and u is a critical point of $\mathcal{J}$. To show that $u \neq 0$, we need to consider the limiting functional $\mathcal{J}_\infty : X \rightarrow \mathbb{R}$,

$$\mathcal{J}_\infty(u) = \int_{\mathbb{R}^N} \Phi(|\nabla u|) + \int_{\mathbb{R}^N} V_\infty \Phi(|u|) - \int_{\mathbb{R}^N} F(u).$$

Lemma 3.4. *If $u = 0$, then $\{u_n\}$ is also a $(PS)_c$ sequence of $\mathcal{J}_\infty$.*

Proof. By condition (V_2) , for any $\varepsilon > 0$, there is $R > 0$ such that

$$|V(x) - V_\infty| < \varepsilon \quad \text{for } |x| \geq R.$$

If $u = 0$, then $u_n \rightarrow 0$ in X . By the compactness of the embedding $X \hookrightarrow L^\Phi_{\text{loc}}(\mathbb{R}^N)$, we have

$$\int_{|x|<R} \Phi(|u_n|) \rightarrow 0.$$

Consequently,

$$\begin{aligned} |\mathcal{J}_\infty(u_n) - \mathcal{J}(u_n)| &= \int_{\mathbb{R}^N} (V_\infty - V(x))\Phi(|u_n|) \\ &= \left(\int_{|x|<R} + \int_{|x|\geq R} \right) (V_\infty - V(x))\Phi(|u_n|) \\ &\leq V_\infty \int_{|x|<R} \Phi(|u_n|) + \varepsilon \int_{|x|\geq R} \Phi(|u_n|) \\ &\leq V_\infty \int_{|x|<R} \Phi(|u_n|) + C\varepsilon, \end{aligned}$$

because $\{u_n\}$ is bounded in $L^\Phi(\mathbb{R}^N)$. It follows that

$$\overline{\lim}_{n \rightarrow \infty} |\mathcal{J}_\infty(u_n) - \mathcal{J}(u_n)| \leq C\varepsilon,$$

which implies $\mathcal{J}_\infty(u_n) - \mathcal{J}(u_n) \rightarrow 0$.

In a similar manner, we can prove

$$\|\mathcal{J}'_\infty(u_n) - \mathcal{J}'(u_n)\| = \sup_{h \in X, \|h\|=1} \left| \int_{\mathbb{R}^N} (V_\infty - V(x))u_n h \right| \rightarrow 0.$$

Thus, $\mathcal{J}_\infty(u_n) \rightarrow c$ and $\mathcal{J}'_\infty(u_n) \rightarrow 0$. $\square$

Considering the constant potential V_∞ as a $\mathbb{Z}^N$ -periodic function, it has been shown in the proof of [17, Theorem 1.8(b)] that for $\{u_n\}$, the bounded $(PS)_c$ sequence of $\mathcal{J}_\infty$ obtained in Lemma 3.4, there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ such that setting $v_n(x) = u_n(x - y_n)$ for $x \in \mathbb{R}^N$, then $v_n \rightharpoonup v$ in X , and v is a nonzero critical point of $\mathcal{J}_\infty$.

We claim that $\mathcal{J}_\infty(v) \leq c$. In addition to the obvious fact that $v_n \rightarrow v$ a.e. in $\mathbb{R}^N$, by applying Lemma 3.2 to $\mathcal{J}_\infty$ we also have $\nabla v_n \rightarrow \nabla v$ a.e. in $\mathbb{R}^N$. By the assumptions (ϕ_2) , (f_2) , and $\theta > m$, we have

$$\Phi(|t|) - \frac{1}{\theta}\phi(|t|)t^2 \geq \left(1 - \frac{m}{\theta}\right)\Phi(|t|) \geq 0, \quad \frac{1}{\theta}f(t)t - F(t) \geq 0 \quad (3.6)$$

for $t \geq 0$. Hence, we may apply the Fatou lemma to get

$$\begin{aligned} c &= \lim_{n \rightarrow \infty} \left\{ \mathcal{J}_\infty(v_n) - \frac{1}{\theta} \langle \mathcal{J}'_\infty(v_n), v_n \rangle \right\} \\ &= \lim_{n \rightarrow \infty} \left\{ \int_{\mathbb{R}^N} \left(\Phi(|\nabla v_n|) - \frac{1}{\theta}\phi(|\nabla v_n|)|\nabla v_n|^2 \right) + \int_{\mathbb{R}^N} V_\infty \left(\Phi(|v_n|) - \frac{1}{\theta}\phi(|v_n|)|v_n|^2 \right) + \int_{\mathbb{R}^N} \left(\frac{1}{\theta}f(v_n)v_n - F(v_n) \right) \right\} \\ &\geq \int_{\mathbb{R}^N} \left(\Phi(|\nabla v|) - \frac{1}{\theta}\phi(|\nabla v|)|\nabla v|^2 \right) + \int_{\mathbb{R}^N} V_\infty \left(\Phi(|v|) - \frac{1}{\theta}\phi(|v|)|v|^2 \right) + \int_{\mathbb{R}^N} \left(\frac{1}{\theta}f(v)v - F(v) \right) \\ &= \mathcal{J}_\infty(v) - \frac{1}{\theta} \langle \mathcal{J}'_\infty(v), v \rangle = \mathcal{J}_\infty(v). \end{aligned} \quad (3.7)$$

Define a C^1 -function $\varrho : [0, \infty) \rightarrow \mathbb{R}$ by

$$\varrho(t) = \mathcal{J}_\infty(tv) = \int_{\mathbb{R}^N} \Phi(t|\nabla v|) + \int_{\mathbb{R}^N} V_\infty \Phi(t|v|) - \int_{\mathbb{R}^N} F(tv).$$

Then for $t > 0$,

$$\varrho'(t) = \langle \mathcal{J}'_\infty(tv), v \rangle = t \int_{\mathbb{R}^N} (\phi(t|\nabla v|)|\nabla v|^2 + V_\infty \phi(t|v|)v^2) - \int_{\mathbb{R}^N} f(tv)v.$$

Hence, for the $s \geq 2$ in assumptions (ϕ_3^s) and (f_3^s) we have

$$\begin{aligned} \varrho'(t) &> 0 \Leftrightarrow \frac{1}{t^{s-2}} \int_{\mathbb{R}^N} (\phi(t|\nabla u|)|\nabla u|^2 + V_\infty \phi(t|v|)v^2) > \int_{\mathbb{R}^N} \frac{f(tv)v}{t^{s-1}}, \\ \varrho'(t) &= 0 \Leftrightarrow \frac{1}{t^{s-2}} \int_{\mathbb{R}^N} (\phi(t|\nabla u|)|\nabla u|^2 + V_\infty \phi(t|v|)v^2) = \int_{\mathbb{R}^N} \frac{f(tv)v}{t^{s-1}}, \\ \varrho'(t) &< 0 \Leftrightarrow \frac{1}{t^{s-2}} \int_{\mathbb{R}^N} (\phi(t|\nabla u|)|\nabla u|^2 + V_\infty \phi(t|v|)v^2) < \int_{\mathbb{R}^N} \frac{f(tv)v}{t^{s-1}}. \end{aligned}$$

Since $\varrho'(1) = \langle \mathcal{J}'_\infty(v), v \rangle = 0$, by (ϕ_3^s) and (f_3^s) and a monotonicity argument we see that

$$\varrho'(t) > 0 \quad \text{for } t \in (0, 1), \quad \varrho'(t) < 0 \quad \text{for } t \in (1, \infty).$$

Hence,

$$\mathcal{J}_\infty(v) = \varrho(1) = \max_{t \geq 0} \varrho(t) = \max_{t \geq 0} \mathcal{J}_\infty(tv). \quad (3.8)$$

Now, we are ready to conclude the proof of Theorem 1.3. For the bounded $(PS)_c$ sequence $\{u_n\}$ given in (3.4), we know that the weak limit u of a subsequence is a critical point of $\mathcal{J}$. If $u = 0$, by Lemma 3.4, this $\{u_n\}$ is also a $(PS)_c$ sequence of the limiting functional $\mathcal{J}_\infty$, which will produce a nonzero critical point v of $\mathcal{J}_\infty$ satisfying $\mathcal{J}_\infty(v) \leq c$, see (3.7).

We also know that $\mathcal{J}_\infty(tv) \rightarrow -\infty$ as $t \rightarrow +\infty$, see Remark 3.1. Choose $T > 0$ such that $\mathcal{J}_\infty(Tv) < 0$ and define $\gamma : [0, 1] \rightarrow X$, $\gamma(t) = tTv$. Then $\gamma(0) = 0$,

$$\mathcal{J}(\gamma(1)) < \mathcal{J}_\infty(\gamma(1)) = \mathcal{J}_\infty(Tv) < 0$$

because $V(x) < V_\infty$ for all $x \in \mathbb{R}^N$, hence $\gamma \in \Gamma$.

By assumption (V_2) , (3.7) and (3.8), we see that for $t \in (0, 1]$,

$$\mathcal{J}(\gamma(t)) < \mathcal{J}_\infty(\gamma(t)) \leq \mathcal{J}_\infty(v) \leq c. \quad (3.9)$$

Because $\gamma(0) = 0$, $\mathcal{J}(0) = 0$ and $c > 0$, (3.9) is also true at $t = 0$. Hence,

$$\max_{t \in [0, 1]} \mathcal{J}(\gamma(t)) = \mathcal{J}(\gamma(t_0)) < \mathcal{J}_\infty(v) \leq c$$

for some $t_0 \in [0, 1]$, contradicting the definition of c given in (3.3). Therefore, $u \neq 0$ and it is a nonzero critical point of $\mathcal{J}$.

3.3 Proof of Theorem 1.4

On the subspace

$$E = \left\{ u \in W^{1, \Phi}(\mathbb{R}^N) \left| \int_{\mathbb{R}^N} a(x) \Phi(|u|) < \infty \right. \right\}$$

of $W^{1, \Phi}(\mathbb{R}^N)$, we equip the norm

$$\|u\| = |\nabla u|_\Phi + |u|_{\Phi, (a+1)}.$$

Then E becomes a Banach space. To prove Theorem 1.4, we only need to find nonzero critical points of $\mathcal{J}_\lambda : E \rightarrow \mathbb{R}$,

$$\mathcal{J}_\lambda(u) = \int_{\mathbb{R}^N} \Phi(|\nabla u|) + \int_{\mathbb{R}^N} (\lambda a(x) + 1) \Phi(|u|) - \int_{\mathbb{R}^N} F(u).$$

As before, $\mathcal{J}_\lambda$ verifies the assumptions of the mountain pass theorem thus has a $(PS)_{c_\lambda}$ sequence $\{u_n^\lambda\}_{n=1}^\infty$ satisfying

$$\mathcal{J}_\lambda(u_n^\lambda) \rightarrow c_\lambda > 0, \quad \mathcal{J}'_\lambda(u_n^\lambda) \rightarrow 0, \quad (3.10)$$

where

$$c_\lambda = \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0, 1]} \mathcal{J}_\lambda(\gamma(t)) \quad (3.11)$$

being $\Gamma_\lambda = \{\gamma \in C(I, E) \mid \gamma(0) = 0, \mathcal{J}_\lambda(\gamma(1)) < 0\}$.

Moreover, the sequence $\{u_n^{\lambda}\}_{n=1}^{\infty}$ is bounded in E . Going to a subsequence if necessary, $u_n^{\lambda} \rightharpoonup u^{\lambda}$ in E , and u^{λ} is a critical point of $\mathcal{J}_{\lambda}$ due to Lemma 3.2. We need to show that $u^{\lambda} \neq 0$. Although the basic idea can be traced back to [26, §5], we need to create the required estimates more carefully because our differential operator $-\Delta_{\Phi}$ is much more complicated than in [26].

Take an $\mathcal{N}$ -function satisfying (2.8). From (f_1) with $\lambda_0 = 0$, for any $\varepsilon > 0$, there exists $C_{\varepsilon} > 0$ such that

$$\frac{1}{\ell}f(t)t - F(t) \leq \varepsilon(\Phi(|t|) + \Phi_*(|t|)) + C_{\varepsilon}\Psi(|t|). \quad (3.12)$$

By assumption (a_1) , we can take $v \in E \setminus \{0\}$, such that $\text{supp } v$ is contained in the interior of $a^{-1}(0)$. Then, by the mountain pass characterization of c_{λ} in (3.11), we have

$$c_{\lambda} \leq \max_{t \geq 0} \mathcal{J}_{\lambda}(tv) = \max_{t \geq 0} \left\{ \int \Phi(t|\nabla v|) + \int \Phi(t|v|) - \int F(tv) \right\} = \tilde{c} < +\infty, \quad (3.13)$$

see the proof of [17, Lemma 4.1(b)] for more details.

Lemma 3.5. *There exists $\alpha > 0$ such that for all $\lambda \geq 1$,*

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} \Psi(|u_n^{\lambda}|) \geq \alpha.$$

Proof. It has been shown in [17, Lemma 4.1(a)] that $u = 0$ is a strict local minimizer of $\mathcal{J}_0$. Since for $\lambda > 0$ we have $\mathcal{J}_{\lambda} \geq \mathcal{J}_0$, it follows that $\Gamma_{\lambda} \subset \Gamma_0$. By (3.11), it is easy to see that $c_{\lambda} \geq c_0 > 0$.

For simplicity of notation, in the proof of Lemmas 3.5 and 3.7 we drop the superscript λ and write u_n for u_n^{λ} . From (3.10), using (3.6), we have (note that $\lambda \geq 1$)

$$\begin{aligned} c_{\lambda} &= \lim_{n \rightarrow \infty} \left\{ \mathcal{J}_{\lambda}(u_n) - \frac{1}{\theta} \langle \mathcal{J}'_{\lambda}(u_n), u_n \rangle \right\} \\ &\geq \overline{\lim}_{n \rightarrow \infty} \left\{ \int_{\mathbb{R}^N} \left(\Phi(|\nabla u_n|) - \frac{1}{\theta} \phi(|\nabla u_n|) |\nabla u_n|^2 \right) + \int_{\mathbb{R}^N} (\lambda a(x) + 1) \left(\Phi(|u_n|) - \frac{1}{\theta} \phi(|u_n|) |u_n|^2 \right) \right\} \\ &\geq \left(1 - \frac{m}{\theta} \right) \overline{\lim}_{n \rightarrow \infty} \left\{ \int_{\mathbb{R}^N} \Phi(|\nabla u_n|) + \int_{\mathbb{R}^N} (\lambda a(x) + 1) \Phi(|u_n|) \right\}. \end{aligned} \quad (3.14)$$

Since the first integral in the last line is nonnegative, it follows that

$$\overline{\lim}_{n \rightarrow \infty} \int_{\mathbb{R}^N} (\lambda a(x) + 1) \Phi(|u_n|) \leq \frac{\theta m}{\theta - m} c_{\lambda}. \quad (3.15)$$

Moreover, as indicated in (3.13), $\{c_{\lambda}\}_{\lambda \geq 1}$ is bounded above by $\tilde{c}$, it follows from (3.14) that $\{u_n\}$ is bounded in E by a positive constant, which is independent of λ . Therefore, by the continuous embedding $E \hookrightarrow L^{\Phi}$, there exists $d > 0$ such that

$$\int_{\mathbb{R}^N} \Phi_*(|u_n|) \leq d. \quad (3.16)$$

Using (ϕ_2) we have $\Phi(t) \leq \ell^{-1} \phi(t) t^2$ for $t \geq 0$, then using (3.12), (3.16) and (3.15) we get

$$\begin{aligned} c_{\lambda} &= \lim_{n \rightarrow \infty} \left\{ \mathcal{J}_{\lambda}(u_n) - \frac{1}{\ell} \langle \mathcal{J}'_{\lambda}(u_n), u_n \rangle \right\} \\ &= \lim_{n \rightarrow \infty} \left\{ \int_{\mathbb{R}^N} \left(\Phi(|\nabla u_n|) - \frac{1}{\ell} \phi(|\nabla u_n|) |\nabla u_n|^2 \right) + \int_{\mathbb{R}^N} (\lambda a(x) + 1) \left(\Phi(|u_n|) - \frac{1}{\ell} \phi(|u_n|) |u_n|^2 \right) + \int_{\mathbb{R}^N} \left(\frac{1}{\ell} f(u_n) u_n - F(u_n) \right) \right\} \\ &\leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} \left(\frac{1}{\ell} f(u_n) u_n - F(u_n) \right) \end{aligned} \quad (3.17)$$

$$\begin{aligned}
&\leq \varlimsup_{n \rightarrow \infty} \left(\varepsilon \left(\int_{\mathbb{R}^N} \Phi(|u_n|) + \int_{\mathbb{R}^N} \Phi_*(|u_n|) \right) + C_\varepsilon \int_{\mathbb{R}^N} \Psi(|u_n|) \right) \\
&\leq \varepsilon \varlimsup_{n \rightarrow \infty} \left(d + \int_{\mathbb{R}^N} (\lambda a(x) + 1) \Phi(|u_n|) \right) + C_\varepsilon \varlimsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \Psi(|u_n|) \\
&\leq \varepsilon d + \frac{\theta m}{\theta - m} c_\lambda \varepsilon + C_\varepsilon \varlimsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \Psi(|u_n|).
\end{aligned}$$

Noting that $c_\lambda \geq c_0 > 0$, choosing ε small enough at the very beginning, the desired conclusion follows from (3.17). $\square$

Remark 3.6. Comparing with the argument in [26] for the case $\phi(t) \equiv 1$, because instead of being *strictly subcritical* (meaning that $f(t)$ is controlled by some subcritical $\mathcal{N}$ -function Ψ), our nonlinearity $f(t)$ is only *asymptotically subcritical*, in the estimate (3.17) the term involving Φ_* presents. Hence, we need the uniform bound (3.16), which in turn is ensured by the upper bound of $\{c_\lambda\}$ given in (3.13).

Lemma 3.7. For any $\varepsilon > 0$, there exists $\lambda^* \geq 1$ and $R > 0$, such that for $\lambda \geq \lambda^*$ we have

$$\varlimsup_{n \rightarrow \infty} \int_{|x| \geq R} \Psi(|u_n^\lambda|) < \varepsilon.$$

Proof. For $R > 0$, we set

$$A_R = \{x \in \mathbb{R}^N \mid |x| \geq R, a(x) \geq M_0\}, \quad B_R = \{x \in \mathbb{R}^N \mid |x| \geq R, a(x) < M_0\}.$$

As in (2.12), because of (2.8) there exists $k > 0$ such that

$$\int_{|u_n| > k} \Psi(|u_n|) \leq \frac{\varepsilon}{4}. \quad (3.18)$$

Using assumption (a_2) , as $R \rightarrow \infty$ we have $\mu(B_R) \rightarrow 0$; therefore, we can fix $R > 0$ such that

$$\int_{B_R \cap \{|u_n| \leq k\}} \Psi(|u_n|) \leq \mu(B_R) \cdot \sup_{t \in [0, k]} \Psi(t) < \frac{\varepsilon}{4}. \quad (3.19)$$

By the first limit from (2.8), there is $\kappa > 0$ such that $\Psi(t) \leq \kappa \Phi(t)$ for $t \in [0, k]$. Thus, using (3.15) and (3.13) we see that if λ is large enough,

$$\begin{aligned}
\int_{A_R \cap \{|u_n| \leq k\}} \Psi(|u_n|) &\leq \kappa \int_{A_R} \Phi(|u_n|) \\
&\leq \frac{\kappa}{\lambda M_0 + 1} \int_{A_R} (\lambda a(x) + 1) \Phi(|u_n|) \\
&\leq \frac{\kappa}{\lambda M_0 + 1} \frac{2\theta m}{\theta - m} c_\lambda \\
&\leq \frac{\kappa}{\lambda M_0 + 1} \frac{2\theta m}{\theta - m} \tilde{c} < \frac{\varepsilon}{4}.
\end{aligned} \quad (3.20)$$

For such large λ , combining (3.18), (3.19) and (3.20), we see that for the chosen $R > 0$,

$$\varlimsup_{n \rightarrow \infty} \int_{|x| \geq R} \Psi(|u_n|) \leq \varlimsup_{n \rightarrow \infty} \left(\int_{|u_n| > k} + \int_{B_R \cap \{|u_n| \leq k\}} + \int_{A_R \cap \{|u_n| \leq k\}} \right) \Psi(|u_n|) < \varepsilon. \quad \square$$

Having proven Lemmas 3.5 and 3.7, we are ready to complete the proof of Theorem 1.4. Set $\varepsilon = \alpha/2$ in Lemma 3.7 and fix $\lambda^* > 0$ and $R > 0$ as in the lemma. If $\lambda \geq \lambda^*$, for $\{u_n^\lambda\}$, the $(PS)_{c_\lambda}$ sequence of $\mathcal{J}_\lambda$, we have $u_n^\lambda \rightharpoonup u^\lambda$ and u^λ is a critical point of $\mathcal{J}_\lambda$. Since the embedding $E \hookrightarrow L^\Psi(B_R)$ is compact,

$$\int_{|x|<R} \Psi(|u^\lambda|) = \lim_{n \rightarrow \infty} \int_{|x|<R} \Psi(|u_n^\lambda|) \geq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} \Psi(|u_n^\lambda|) - \overline{\lim}_{n \rightarrow \infty} \int_{|x| \geq R} \Psi(|u_n^\lambda|) \geq \frac{\varepsilon}{2}.$$

Therefore, u^λ is a nonzero critical point of $\mathcal{J}_\lambda$.

4 Multiple solutions

Lemma 4.1. *Under the assumptions of Theorem 1.2, $\mathcal{J}$ satisfies the (PS) condition, that is, for any $c \in \mathbb{R}$, all $(PS)_c$ sequences of $\mathcal{J}$ have convergent subsequence.*

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence of $\mathcal{J}$. Then $\{u_n\}$ is bounded and we may assume that $u_n \rightharpoonup u$ in X . First, we show that up to a subsequence

$$\int_{\mathbb{R}^N} f(u_n)(u_n - u) \rightarrow 0. \quad (4.1)$$

By assumption (f_1) , for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$|f(t)| \leq \varepsilon \phi_*(|t|)|t| + C_\varepsilon \phi(|t|)|t| = \varepsilon \Phi'_*(|t|) + C_\varepsilon \Phi'(|t|). \quad (4.2)$$

For $u \in X$, by Hölder inequality (2.3) we have

$$\int_{\mathbb{R}^N} \Phi'(|u|)|v| \leq 2|\Phi'(|u|)|_{\tilde{\Phi}}|v|_{\Phi}.$$

Note that from (2.2), (2.11) and Lemma 2.1 with $V \equiv 1$, we have

$$\int_{\mathbb{R}^N} \tilde{\Phi}(|\Phi'(|u|)|) \leq \int_{\mathbb{R}^N} \Phi(2|u|) \leq \xi_1(2) \int_{\mathbb{R}^N} \Phi(|u|) \leq 2^m \xi_1(|u|_{\Phi}).$$

Therefore, since $\{u_n\}$ is bounded in $L^\Phi(\mathbb{R}^N)$, $\{\Phi'(|u_n|)\}$ is also bounded in $L^{\tilde{\Phi}}(\mathbb{R}^N)$. Similarly, $\{\Phi'_*(|u_n|)\}$ is bounded in $L^{\tilde{\Phi}_*}(\mathbb{R}^N)$. (We remind the reader that instead of the Sobolev conjugate function of $\tilde{\Phi}$, here $\tilde{\Phi}_*$ is the complement function of Φ_* .) Therefore,

$$M := 2 \sup_n |\Phi'_*(|u_n|)|_{\tilde{\Phi}_*} |u_n - u|_{\Phi_*} < +\infty. \quad (4.3)$$

Because $u_n \rightharpoonup u$ in X , by Lemma 2.3 we have $u_n \rightarrow u$ in $L^\Phi(\mathbb{R}^N)$. Now, using (4.2) and Hölder inequality we get

$$\begin{aligned} \left| \int_{\mathbb{R}^N} f(u_n)(u_n - u) \right| &\leq \varepsilon \int_{\mathbb{R}^N} \Phi'_*(|u_n|)|u_n - u| + C_\varepsilon \int_{\mathbb{R}^N} \Phi'(|u_n|)|u_n - u| \\ &\leq 2\varepsilon |\Phi'_*(|u_n|)|_{\tilde{\Phi}_*} |u_n - u|_{\Phi_*} + 2C_\varepsilon |\Phi'(|u_n|)|_{\tilde{\Phi}} |u_n - u|_{\Phi}. \end{aligned}$$

Since $u_n \rightarrow u$ in $L^\Phi(\mathbb{R}^N)$, using (4.3) and the boundedness of $\{\Phi'(|u_n|)\}$ in $L^{\tilde{\Phi}}(\mathbb{R}^N)$, it follows that

$$\overline{\lim}_{n \rightarrow \infty} \left| \int_{\mathbb{R}^N} f(u_n)(u_n - u) \right| \leq M\varepsilon,$$

which implies (4.1).

To prove that $u_n \rightarrow u$ in X , we adapt the argument of [16, Appendix A], where for $V(x) \equiv 0$, a Φ -Laplacian problem on a bounded domain is considered. Let $\mathcal{A} : X \rightarrow X^*$ be defined by

$$\langle \mathcal{A}(u), v \rangle = \int_{\mathbb{R}^N} \phi(|\nabla u|) \nabla u \cdot \nabla v + \int_{\mathbb{R}^N} V(x) \phi(|u|) uv.$$

Then it is well known that

- $\mathcal{A}$ is hemicontinuous, i.e., for all $u, v, w \in X$, the function

$$t \mapsto \langle \mathcal{A}(u + tv), w \rangle$$

is continuous on $[0, 1]$.

- $\mathcal{A}$ is strictly monotone: $\langle \mathcal{A}(u) - \mathcal{A}(v), u - v \rangle > 0$ for $u, v \in X$ with $u \neq v$.

By [30, Lemma 2.98], we know that $\mathcal{A}$ is pseudomonotone, i.e., for $\{u_n\} \subset X$,

$$u_n \rightharpoonup u \text{ in } X, \quad \overline{\lim}_{n \rightarrow \infty} \langle \mathcal{A}(u_n), u_n - u \rangle \leq 0 \quad (4.4)$$

together imply $\mathcal{A}(u_n) \rightharpoonup \mathcal{A}(u)$ in X^* and $\langle \mathcal{A}(u_n), u_n \rangle \rightarrow \langle \mathcal{A}u, u \rangle$.

For our bounded $(PS)_c$ sequence $\{u_n\}$, (4.1) implies that (4.4) holds up to a subsequence. Therefore,

$$\langle \mathcal{A}(u_n), u_n \rangle \rightarrow \langle \mathcal{A}u, u \rangle. \quad (4.5)$$

According to Lemma 3.2, in addition to the well known $u_n \rightarrow u$ a.e. in $\mathbb{R}^N$ we also have $\nabla u_n \rightarrow \nabla u$ a.e. in $\mathbb{R}^N$. By the continuity of Φ we get

$$f_n := \Phi(|\nabla u_n - \nabla u|) + V(x)\Phi(|u_n - u|) \rightarrow 0 \text{ a.e. in } \mathbb{R}^N. \quad (4.6)$$

Let $g_n : \mathbb{R}^N \rightarrow \mathbb{R}$ be defined by

$$g_n = \frac{2^{m-1}}{\ell} \{ \phi(|\nabla u_n|) |\nabla u_n|^2 + V(x) \phi(|u_n|) |u_n|^2 + \Phi(|\nabla u|) + V(x) \Phi(|u|) \}.$$

Then by the monotonicity and convexity of Φ , using (2.11) and (ϕ_2) we get

$$\begin{aligned} |f_n| &\leq \Phi\left(\frac{2|\nabla u_n| + 2|\nabla u|}{2}\right) + V(x)\Phi\left(\frac{2|u_n| + 2|u|}{2}\right) \\ &\leq \frac{\Phi(2|\nabla u_n|) + \Phi(2|\nabla u|)}{2} + V(x) \frac{\Phi(2|u_n|) + \Phi(2|u|)}{2} \\ &\leq 2^{m-1} \{ \Phi(|\nabla u_n|) + \Phi(|\nabla u|) \} + V(x) \{ \Phi(|u_n|) + \Phi(|u|) \} \\ &\leq \frac{2^{m-1}}{\ell} \{ \phi(|\nabla u_n|) |\nabla u_n|^2 + V(x) \phi(|u_n|) |u_n|^2 + \Phi(|\nabla u|) + V(x) \Phi(|u|) \} = g_n. \end{aligned}$$

We have

$$g_n \rightarrow g := \frac{2^{m-1}}{\ell} \{ \phi(|\nabla u|) |\nabla u|^2 + V(x) \phi(|u|) |u|^2 + \Phi(|\nabla u|) + V(x) \Phi(|u|) \} \quad (4.7)$$

a.e. in $\mathbb{R}^N$, and $g \in L^1(\mathbb{R}^N)$. Moreover, using (4.5) we get

$$\int_{\mathbb{R}^N} (\phi(|\nabla u_n|) |\nabla u_n|^2 + V(x) \phi(|u_n|) |u_n|^2) = \langle \mathcal{A}(u_n), u_n \rangle \rightarrow \langle \mathcal{A}(u), u \rangle = \int_{\mathbb{R}^N} (\phi(|\nabla u|) |\nabla u|^2 + V(x) \phi(|u|) |u|^2),$$

which implies

$$\int_{\mathbb{R}^N} g_n \rightarrow \int_{\mathbb{R}^N} g. \quad (4.8)$$

Now, (4.6), (4.7), (4.8) and the generalized Lebesgue dominated theorem give

$$\int_{\mathbb{R}^N} [\Phi(|\nabla u_n - \nabla u|) + V(x)\Phi(|u_n - u|)] = \int_{\mathbb{R}^N} f_n \rightarrow 0,$$

that is $u_n \rightarrow u$ in X . □

Lemma 4.2. *The functional $\mathcal{F} : X \rightarrow \mathbb{R}$ defined by*

$$\mathcal{F}(u) = \int_{\mathbb{R}^N} F(u)$$

is weakly-strongly continuous, that is, if $u_n \rightharpoonup u$ in X , then $\mathcal{F}(u_n) \rightarrow \mathcal{F}(u)$.

Proof. Suppose $\{u_n\} \subset X$ satisfies $u_n \rightharpoonup u$ in X . Then $u_n \rightarrow u$ in $L^\Phi(\mathbb{R}^N)$, by Lemma 2.3. Thus, $\Phi(|u_n - u|) \rightarrow 0$ in $L^1(\mathbb{R}^N)$, which also implies

$$\Phi(2|u_n - u|) \rightarrow 0 \quad \text{in } L^1(\mathbb{R}^N).$$

By [31, Theorem 4.9], there exists $k \in L^1(\mathbb{R}^N)$ such that up to a subsequence,

$$\Phi(2|u_n - u|) \rightarrow 0 \quad \text{a.e. in } \mathbb{R}^N, \quad \Phi(2|u_n - u|) \leq k \quad \text{a.e. in } \mathbb{R}^N.$$

By the monotonicity and convexity of Φ ,

$$\Phi(|u_n|) \leq \Phi(|u_n - u| + |u|) \leq \frac{1}{2}\Phi(2|u_n - u|) + \frac{1}{2}\Phi(2|u|) \leq \frac{1}{2}(k + \Phi(2|u|)).$$

Since $k + \Phi(2|u|) \in L^1(\mathbb{R}^N)$, and $\Phi(|u_n|) \rightarrow \Phi(|u|)$ a.e. in $\mathbb{R}^N$, we deduce

$$\int_{\mathbb{R}^N} \Phi(|u_n|) \rightarrow \int_{\mathbb{R}^N} \Phi(|u|). \quad (4.9)$$

For any $\varepsilon > 0$, choose $C_\varepsilon > 0$ such that

$$|F(t)| \leq \varepsilon \Phi_*(|t|) + C_\varepsilon \Phi(|t|).$$

Then we have

$$|\mathcal{F}(u_n) - \mathcal{F}(u)| = \left| \int_{\mathbb{R}^N} F(u_n) - \int_{\mathbb{R}^N} F(u) \right| \leq \varepsilon \left(\int_{\mathbb{R}^N} \Phi_*(|u_n|) + \int_{\mathbb{R}^N} \Phi_*(|u|) \right) + C_\varepsilon \left(\int_{\mathbb{R}^N} \Phi(|u_n|) - \int_{\mathbb{R}^N} \Phi(|u|) \right).$$

Using (4.9) we get

$$\overline{\lim}_{n \rightarrow \infty} |\mathcal{F}(u_n) - \mathcal{F}(u)| \leq \varepsilon \left(\int_{\mathbb{R}^N} \Phi_*(|u_n|) + \int_{\mathbb{R}^N} \Phi_*(|u|) \right). \quad (4.10)$$

Since $\{u_n\}$ is bounded in X , by the continuous embedding $X \hookrightarrow L^\Phi(\mathbb{R}^N)$ we see that $\{u_n\}$ is bounded in $L^\Phi(\mathbb{R}^N)$, letting $\varepsilon \rightarrow 0$ in (4.10) we deduce $\mathcal{F}(u_n) \rightarrow \mathcal{F}(u)$. □

Now we are ready to prove the second part of Theorem 1.2. We need the following symmetric mountain pass theorem due to Ambrosetti and Rabinowitz [18].

Proposition 4.3. [32, Theorem 9.12] *Let $X = Y \oplus Z$ be an infinite dimensional Banach space with $\dim Y < \infty$. Suppose $\mathcal{J} \in C^1(X)$ is even, satisfies (PS), $\mathcal{J}(0) = 0$ and*

- (1) *for some $\rho > 0$, $\inf_{\partial B_\rho \cap Z} \mathcal{J} > 0$, where $B_\rho = \{u \in X \mid \|u\| < \rho\}$;*
 - (2) *for any finite dimensional subspace $W \subset X$, there is an $R = R(W)$ such that $\mathcal{J} \leq 0$ on $W \setminus B_{R(W)}$,*
- then $\mathcal{J}$ has a sequence of critical values $c_j \rightarrow +\infty$.*

4.1 Proof of Theorem 1.2(2)

We know that the C^1 -functional $\mathcal{J}$ given in (3.1) is even, satisfies (PS) and $\mathcal{J}(0) = 0$. To get an unbounded sequence of critical values of $\mathcal{J}$, it suffices to verify the two assumptions in Proposition 4.3.

Verification of (1). Since X is separable reflexive Banach space, there exist $\{e_i\}_1^\infty \subset X$ and $\{f^i\}_1^\infty \subset X^*$ such that $\langle f^i, e_j \rangle = \delta_j^i$ and

$$X = \overline{\text{span}}\{e_i \mid i \geq 1\}, \quad X^* = \overline{\text{span}}^{w^*}\{f^i \mid i \geq 1\}.$$

Let

$$Y_k = \overline{\text{span}}\{e_i \mid i < k\}, \quad Z_k = \overline{\text{span}}\{e_i \mid i \geq k\}.$$

In Lemma 4.2, we have proved that the functional $\mathcal{F}$ is weakly-strongly continuous. Therefore, by [33, Lemma 3.3] we see that

$$\beta_k = \sup_{u \in \partial B_1 \cap Z_k} |\mathcal{F}(u)| \rightarrow 0. \quad (4.11)$$

For $u \in \partial B_1$, we have $|\nabla u|_\Phi \leq 1$ and $|u|_{\Phi, V} \leq 1$. Hence, there exists a constant $c > 0$ such that

$$\int_{\mathbb{R}^N} \Phi(|\nabla u|) + \int_{\mathbb{R}^N} V(x)\Phi(|u|) \geq |\nabla u|_\Phi^\ell + |u|_{\Phi, V}^\ell \geq c.$$

Using (4.11), we can choose k such that $\beta_k < c$. Set $Z = Z_k$ and $Y = Y_k$. Then $\dim Y < \infty$, $X = Y \oplus Z$, for $u \in \partial B_1 \cap Z$ we have

$$\mathcal{J}(u) = \int_{\mathbb{R}^N} \Phi(|\nabla u|) + \int_{\mathbb{R}^N} V(x)\Phi(|u|) - \mathcal{F}(u) \geq c - \beta_k > 0.$$

This verifies condition (1) of Proposition 4.3.

Verification of (2). Because $\theta > m$, condition (f_2) implies

$$\lim_{|t| \rightarrow \infty} \frac{F(t)}{|t|^m} = +\infty.$$

Let W be any given finite dimensional subspace of X and $\{u_n\}$ be a sequence in W such that $\|u_n\| \rightarrow \infty$. Then

$$v_n = \frac{u_n}{\|u_n\|} \rightarrow v$$

for some $v \in W \cap \partial B_1$. For $x \in \{v \neq 0\}$, we have

$$|u_n(x)| = \|u_n\| |v_n(x)| \rightarrow +\infty.$$

Applying the Fatou lemma and noting $F \geq 0$, we deduce

$$\frac{1}{\|u_n\|^m} \int_{\mathbb{R}^N} F(u_n) \geq \frac{1}{\|u_n\|^m} \int_{v \neq 0} F(u_n) = \int_{v \neq 0} \frac{F(u_n)}{|u_n|} |v_n| \rightarrow +\infty.$$

Consequently, applying Lemma 2.1 we get

$$\begin{aligned} \mathcal{J}(u_n) &= \int_{\mathbb{R}^N} \Phi(|\nabla u_n|) + \int_{\mathbb{R}^N} V(x)\Phi(|u_n|) - \int_{\mathbb{R}^N} F(u_n) \leq \xi_1(|\nabla u_n|_\Phi) + \xi_1(|u_n|_{\Phi, V}) - \int_{\mathbb{R}^N} F(u_n) \\ &\leq |\nabla u_n|_\Phi^m + |\nabla u_n|_\Phi^\ell + |u_n|_{\Phi, V}^m + |u_n|_{\Phi, V}^\ell - \int_{\mathbb{R}^N} F(u_n) \leq 2(\|u_n\|^m + \|u_n\|^\ell) - \int_{\mathbb{R}^N} F(u_n) \\ &= 2 \|u_n\|^m \left(1 + \|u_n\|^{\ell-m} - \frac{1}{\|u_n\|^m} \int_{\mathbb{R}^N} F(u_n) \right) \rightarrow -\infty \end{aligned}$$

because $\ell \leq m$. Hence, condition (2) of Proposition 4.3 is verified, and the proof of Theorem 1.2(2) is completed.

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References

- [1] V. Coti Zelati Sissa and P. H. Rabinowitz, *Homoclinic type solutions for a semilinear elliptic PDE on $\mathbb{R}^N$* , Comm. Pure Appl. Math. **45** (1992), no. 10, 1217–1269, DOI: <https://doi.org/10.1002/cpa.3160451002>.
- [2] A. Szulkin and T. Weth, *Ground state solutions for some indefinite variational problems*, J. Funct. Anal. **257** (2009), no. 12, 3802–3822, DOI: <https://doi.org/10.1016/j.jfa.2009.09.013>.
- [3] W. Kryszewski and A. Szulkin, *Generalized linking theorem with an application to a semilinear Schrödinger equation*, Adv. Differ. Equ. **3** (1998), 441–472.
- [4] Y. Li, Z.-Q. Wang, and J. Zeng, *Ground states of nonlinear Schrödinger equations with potentials*, Ann. Inst. H. Poincaré Anal. Non Linéaire **23** (2006), no. 6, 829–837, DOI: <https://doi.org/10.1016/j.anihpc.2006.01.003>.
- [5] S. Liu, *On superlinear Schrödinger equations with periodic potential*, Calc. Var. Partial Differ. Equ. **45** (2012), 1–9, DOI: <https://doi.org/10.1007/s00526-011-0447-2>.
- [6] P. H. Rabinowitz, *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys. **43** (1992), 270–291, DOI: <https://doi.org/10.1007/BF00946631>.
- [7] C. Liu and Y. Zheng, *Existence of nontrivial solutions for p -Laplacian equations in $\mathbb{R}^N$* , J. Math. Anal. Appl. **380** (2011), no. 2, 669–679, DOI: <https://doi.org/10.1016/j.jmaa.2011.02.064>.
- [8] G. Li and C. Wang, *The existence of a nontrivial solution to p -Laplacian equations in $\mathbb{R}^N$ with supercritical growth*, Math. Methods Appl. Sci. **36** (2013), no. 1, 69–79, DOI: <https://doi.org/10.1002/mma.2570>.
- [9] R. Bartolo, A. M. Candela, and A. Salvatore, *Multiplicity results for a class of asymptotically p -linear equations on $\mathbb{R}^N$* , Commun. Contemp. Math. **18** (2016), 1550031, DOI: <https://doi.org/10.1142/S0219199715500315>.
- [10] M. Badiale, M. Guida, and S. Rolando, *Compactness and existence results for the p -Laplace equation*, J. Math. Anal. Appl. **451** (2017), 345–370, DOI: <https://doi.org/10.1016/j.jmaa.2017.02.011>.
- [11] G. Li and X. Liang, *The existence of nontrivial solutions to nonlinear elliptic equation of p - q -Laplacian type on $\mathbb{R}^N$* , Nonlinear Anal. **71** (2009), no. 5–6, 2316–2334, DOI: <https://doi.org/10.1016/j.na.2009.01.066>.
- [12] M. F. Chaves, G. Ercole, and O. H. Miyagaki, *Existence of a nontrivial solution for the (p, q) -Laplacian in $\mathbb{R}^N$ without the Ambrosetti-Rabinowitz condition*, Nonlinear Anal. **114** (2015), 133–141, DOI: <https://doi.org/10.1016/j.na.2014.11.010>.
- [13] R. Bartolo, A. M. Candela, and A. Salvatore, *On a class of superlinear (p, q) -Laplacian type equations on $\mathbb{R}^N$* , J. Math. Anal. Appl. **438** (2016), 29–41, DOI: <https://doi.org/10.1016/j.jmaa.2016.01.049>.
- [14] P. Clément, M. García-Huidobro, R. Manásevich, and K. Schmitt, *Mountain pass type solutions for quasilinear elliptic equations*, Calc. Var. Partial Differ. Equ. **11** (2000), 33–62, DOI: <https://doi.org/10.1007/s005260050002>.
- [15] N. Fukagai and K. Narukawa, *On the existence of multiple positive solutions of quasilinear elliptic eigenvalue problems*, Ann. Mat. Pura Appl. **186** (2007), 539–564, DOI: <https://doi.org/10.1007/s10231-006-0018-x>.
- [16] M. L. M. Carvalho, J. V. A. Gonçalves, and E. D. da Silva, *On quasilinear elliptic problems without the Ambrosetti-Rabinowitz condition*, J. Math. Anal. Appl. **426** (2015), 466–483, DOI: <https://doi.org/10.1016/j.jmaa.2015.01.023>.
- [17] C. O. Alves, G. M. Figueiredo, and J. A. Santos, *Strauss and Lions type results for a class of Orlicz-Sobolev spaces and applications*, Topol. Methods Nonlinear Anal. **44** (2014), no. 2, 435–456, DOI: <https://doi.org/10.12775/TMNA.2014.055>.
- [18] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.
- [19] J. A. Marcos do Ó, E. Medeiros, and U. Severo, *On a quasilinear nonhomogeneous elliptic equation with critical growth in $\mathbb{R}^N$* , J. Differ. Equ. **246** (2009), no. 4, 1363–1386, DOI: <https://doi.org/10.1016/j.jde.2008.11.020>.
- [20] J. A. Santos and S. H. M. Soares, *Radial solutions of quasilinear equations in Orlicz-Sobolev type spaces*, J. Math. Anal. Appl. **428** (2015), 1035–1053, DOI: <https://doi.org/10.1016/j.jmaa.2015.03.030>.
- [21] A. Azzollini, P. d’Avenia, and A. Pomponio, *Quasilinear elliptic equations in $\mathbb{R}^N$ via variational methods and Orlicz-Sobolev embeddings*, Calc. Var. Partial Differ. Equ. **49** (2014), 197–213, DOI: <https://doi.org/10.1007/s00526-012-0578-0>.
- [22] N. Chorfi and V. D. Rădulescu, *Standing wave solutions of a quasilinear degenerate Schrödinger equation with unbounded potential*, Electron. J. Qual. Theory Differ. Equ. **2016** (2016), 37, DOI: <https://doi.org/10.14232/ejqtde.2016.1.37>.

- [23] K. Ait-Mahiout and C. O. Alves, *Existence and multiplicity of solutions for a class of quasilinear problems in Orlicz-Sobolev spaces without Ambrosetti-Rabinowitz condition*, J. Elliptic Parabol. Equ. **4** (2018), 389–416, DOI: <https://doi.org/10.1007/s41808-018-0026-1>.
- [24] K. Ait-Mahiout and C. O. Alves, *Multiple solutions for a class of quasilinear problems in Orlicz-Sobolev spaces*, Asymptot. Anal. **104** (2017), no. 1–2, 49–66, DOI: <https://doi.org/10.3233/ASY-171428>.
- [25] C. O. Alves and A. R. da Silva, *Existence of multi-peak solutions for a class of quasilinear problems in Orlicz-Sobolev spaces*, Acta Appl. Math. **151** (2017), 171–198, DOI: <https://doi.org/10.1007/s10440-017-0107-4>.
- [26] T. Bartsch and Z. Q. Wang, *Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^N$* , Comm. Part. Differ. Equ. **20** (1995), no. 9–10, 1725–1741, DOI: <https://doi.org/10.1080/03605309508821149>.
- [27] N. Fukagai, M. Ito, and K. Narukawa, *Positive solutions of quasilinear elliptic equations with critical Orlicz-Sobolev nonlinearity on $\mathbb{R}^N$* , Funkcial. Ekvac. **49** (2006), no. 2, 235–267, DOI: <https://doi.org/10.1619/fesi.49.235>.
- [28] R. A. Adams and J. J. F. Fournier, *Sobolev Spaces*, in: Pure and Applied Mathematics (Amsterdam), vol. 140, second edition, Elsevier/Academic Press, Amsterdam, second edition, 2003.
- [29] H. Brezis and L. Nirenberg, *Remarks on finding critical points*, Comm. Pure Appl. Math. **44** (1991), no. 8–9, 939–963, DOI: <https://doi.org/10.1002/cpa.3160440808>.
- [30] S. Carl, V. K. Le, and D. Motreanu, *Nonsmooth Variational Problems and Their Inequalities, Comparison Principles and Applications*, Springer Monographs in Mathematics, Springer, New York, 2007.
- [31] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, Universitext, New York, 2011.
- [32] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, vol. 65 of CBMS Regional Conference Series in Mathematics, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1986.
- [33] X. Fan and X. Han, *Existence and multiplicity of solutions for $p(x)$ -Laplacian equations in $\mathbb{R}^N$* , Nonlinear Anal. **59** (2004), no. 1–2, 173–188, DOI: <https://doi.org/10.1016/j.na.2004.07.009>.