Research Article

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Oscillatory bifurcation problems for ODEs with logarithmic nonlinearity

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Abstract: We study the global structure of the oscillatory perturbed bifurcation problem which comes from the stationary logarithmic Schrödinger equation

$$-u''(t) = \lambda(\log(1 + u(t)) + \sin u(t)), \quad u(t) > 0, \quad t \in I := (-1, 1), \quad u(\pm 1) = 0,$$

where $\lambda > 0$ is a parameter. It is known that λ is a continuous function of $\alpha > 0$, written as $\lambda(\alpha)$, where α is the maximum norm $\alpha = \|u_\lambda\|_\infty$ of the solution u_λ associated with λ . In the field of bifurcation theory, the study of global structures of bifurcation curves is one of the main subjects of research, and it is important to investigate the influence of the oscillatory term on the global structure of bifurcation curve. Due to the effect of $\sin u$, it is reasonable to expect that an oscillatory term appears in the second term of the asymptotic formula for $\lambda(\alpha)$ as $\alpha \rightarrow \infty$ (cf. [1]). Contrary to expectation, we show that the asymptotic formula for $\sqrt{\lambda(\alpha)}$ as $\alpha \rightarrow \infty$ does not contain oscillatory terms by the third term of $\sqrt{\lambda(\alpha)}$. This result implies that the oscillatory term has almost no influence on the global structure of $\lambda(\alpha)$. The result is, therefore, unexpected, new and novel, since such phenomenon as this is not known so far. For the proof, the involved time map method and stationary phase method are used.

Keywords: oscillatory bifurcation, logarithmic nonlinearity, stationary phase method

MSC 2020: 34B18, 34C23

1 Introduction

We study the global structure of the oscillatory perturbed bifurcation problem which comes from the stationary logarithmic Schrödinger equation

$$-u''(t) = \lambda f(u(t)), \quad t \in I := (-1, 1), \quad (1)$$

$$u(t) > 0, \quad t \in I, \quad (2)$$

$$u(-1) = u(1) = 0, \quad (3)$$

where $f(u)$ is continuous for $u \in [0, \infty)$ and $f(u) > 0$ for $u > 0$. Then we know from [2] that, for a given $\alpha > 0$, there exists a unique classical solution pair (λ, u_α) of (1)–(3) satisfying $\alpha = \|u_\alpha\|_\infty$ for any given $\alpha > 0$. Moreover, we know from [2, p. 2] that λ is a continuous function of $\alpha > 0$, we write as $\lambda = \lambda(\alpha)$ for $\alpha > 0$.

The study of global and local structures of bifurcation diagrams has a long history. Many topics considered there have the background in mathematical biology, engineering, and have been investigated intensively by many authors. To take an example, for the bifurcation problems related to logistic equations, we refer to [3–7] and references therein. For the readers who are interested in various types of global

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behavior of bifurcation curves, we refer to [8]. In particular, global shapes of bifurcation diagrams has been considered by many researchers. However, the precise asymptotic behavior of $\lambda(\alpha)$ as $\alpha \rightarrow \infty$ is not elucidated so much yet. In this paper, we study an interesting nonlinear eigenvalue problem in which both oscillatory term and logarithmic term are included to show that, surprisingly, the oscillatory term hardly gives effect to the asymptotic behavior of $\lambda(\alpha)$ as $\alpha \rightarrow \infty$.

To clarify our intention, we introduce the global behavior of bifurcation curves when $f(u) = u + \sin u$ and $f(u) = \log(1 + u)$. As for oscillatory bifurcation problems, the typical model problem $f(u) = u + \sin u$ in (1) with (2)–(3) was first considered in [9]. Besides, the asymptotic formula for $\lambda(\alpha)$ is known (see [10,11]).

Theorem 1.1. [10,11] *Let $f(u) = u + \sin u$ in (1) and consider (1) with (2) and (3). Then as $\alpha \rightarrow \infty$,*

$$\lambda(\alpha) = \frac{\pi^2}{4} - \frac{\pi^{3/2}}{\sqrt{2}} \alpha^{-3/2} \sin\left(\alpha - \frac{\pi}{4}\right) + o(\alpha^{-3/2}). \quad (4)$$

As expected, the oscillatory term appears as the second term in (4). For other oscillatory problems, we refer to [11–17]. Next, we introduce a recent result for the case $f(u) = \log(1 + u)$, which is motivated by the logarithmic Schrödinger equation and the Klein-Gordon equation with logarithmic potential (cf. [18]).

Theorem 1.2. [1] *Let $f(u) = \log(1 + u)$ ($u \geq 0$) and consider (1)–(3). Then as $\alpha \rightarrow \infty$,*

$$\sqrt{\lambda(\alpha)} = \sqrt{\frac{2\alpha}{\log(1 + \alpha)}} \left\{ 1 - \frac{1}{2}(4 \log 2 - 3) \frac{1}{\log(1 + \alpha)} + \frac{3}{8}(5 - 8 \log 2 + C_1) \frac{1}{(\log(1 + \alpha))^2} \right\} + R_3, \quad (5)$$

where C_1 is a constant specified in Section 3 and R_3 is the remainder term satisfying the following optimal estimate. Namely, there exists a constant $0 < b_3 < 1$ such that for $\alpha \gg 1$,

$$b_3(\log(1 + \alpha))^{-3} \leq |R_3| \leq b_3^{-1}(\log(1 + \alpha))^{-3}. \quad (6)$$

By Theorems 1.1 and 1.2, we expect that if $f(u) = \log(1 + u) + \sin u$, then the asymptotic behavior of $\lambda(\alpha)$ is composed of the mixture of (5) and (6).

Now we state our main result.

Theorem 1.3. *Let $f(u) = \log(1 + u) + \sin u$ ($u \geq 0$) and consider (1)–(3). Then as $\alpha \rightarrow \infty$,*

$$\begin{aligned} \sqrt{\lambda(\alpha)} &= \sqrt{\frac{2\alpha}{\log(1 + \alpha)}} \left[1 - \frac{1}{2}(4 \log 2 - 3) \frac{1}{\log(1 + \alpha)} + \frac{3}{8}(5 - 8 \log 2 + C_1) \frac{1}{(\log(1 + \alpha))^2} \right. \\ &\quad + O((\log(1 + \alpha))^{-3}) - \frac{1}{2 \log(1 + \alpha)} \sqrt{\frac{\pi}{\alpha}} \sin\left(\alpha - \frac{\pi}{4}\right) + \frac{4}{3} \sqrt{\frac{\pi}{\alpha}} \frac{1}{(\log(1 + \alpha))^2} \\ &\quad \left. \times \left\{ \cos \alpha \cos\left(\alpha - \frac{\pi}{4}\right) - \frac{1}{\sqrt{2}} \cos\left(2\alpha - \frac{\pi}{4}\right) - 3 \sin\left(\alpha - \frac{\pi}{4}\right) \right\} \right]. \end{aligned} \quad (7)$$

It should be mentioned that (7) is more precise than (5), and certainly, (7) contains the oscillatory terms. However, we note that (7) can be written as (5). Therefore, from a view point of asymptotic expansion formula for $\sqrt{\lambda(\alpha)}$, (5) and (7) are formally the same. In other words, the perturbed oscillatory term gives almost no influence on the global structure of $\lambda(\alpha)$. As far as the author knows, such phenomenon has not been obtained yet. Therefore, the result here is novel and gives the original contribution to the development of the study of global behavior of bifurcation curves. It should be emphasized that, by the precise and very involved calculations to obtain (7), we reach the new aspect of bifurcation theory.

Remark 1.4. It is natural to expect that the following asymptotic formulae for $\sqrt{\lambda(\alpha)}$ in (8) and (9) are valid.

$$\sqrt{\lambda(\alpha)} = \sqrt{\frac{2\alpha}{\log(1 + \alpha)}} \left\{ 1 + \sum_{n=1}^{\infty} b_n (\log(1 + \alpha))^{-n} \right\}, \quad (8)$$

$$\sqrt{\lambda(\alpha)} = \sqrt{\frac{2\alpha}{\log(1+\alpha)}} \left\{ 1 + \sum_{n=1}^{\infty} b_n (\log(1+\alpha))^{-n} + \sum_{n=1}^{\infty} d_n(\alpha) \alpha^{-1/2} (\log(1+\alpha))^{-n} \right\}. \quad (9)$$

Here, $\{b_n\}$ and $\{d_n(\alpha)\}$ ($n = 1, 2, \dots$) are expected to be constants and oscillatory functions, which are determined by induction, respectively. If (9) is valid, then the oscillatory term does not appear in *any* n th term of the asymptotic expansion of $\sqrt{\lambda(\alpha)}$ and it is a remarkable and significant result in the field of bifurcation theory. However, if the readers look at Section 3, then they understand immediately that it seems almost impossible to determine whether the conjecture (9) is valid or not, since the calculation is quite long and complicated to obtain even the third term of (5). One of the foreseeable extensions in this direction would be to present a computer-assisted analysis of (8) and (9).

The proof of Theorem 1.3 depends on the very involved time map method and stationary phase method. The essential point is that if the terms of time map (22) in Section 2 do not contain $\sin x$, then we are able to calculate them by Taylor expansion and by a direct calculation. However, since they contain $\sin x$, we need to calculate them by the involved stationary phase method. It seems to be new to apply stationary phase method to the calculation of the asymptotic behavior of bifurcation curve in such a complicated situation. The calculation is performed by the precise integration by parts again and again. It was possible to get all the Lemmas, especially in Section 3 by the very involved calculations.

2 Second term of $\lambda(\alpha)$ in Theorem 1.3

In this section, let $\alpha \gg 1$. In what follows, we denote by C the various positive constants independent of α . It is known from [8, p. 143] that if $(u_\alpha, \lambda(\alpha)) \in C^2(\bar{I}) \times \mathbb{R}_+$ satisfies (1)–(3), then

$$u_\alpha(t) = u_\alpha(-t), \quad 0 \leq t \leq 1, \quad (10)$$

$$u_\alpha(0) = \max_{-1 \leq t \leq 1} u_\alpha(t) = \alpha, \quad (11)$$

$$u'_\alpha(t) > 0, \quad -1 < t < 0. \quad (12)$$

We apply the standard time-map argument to (1) (cf. [1]). By (1), we have

$$\{u''_\alpha(t) + \lambda(\log(1 + u_\alpha(t)) + \sin u_\alpha(t))\} u'_\alpha(t) = 0.$$

By this, (11) and putting $t = 0$, we obtain

$$\begin{aligned} \frac{1}{2} u'_\alpha(t)^2 + \lambda \{u \log(1 + u_\alpha(t)) - u_\alpha(t) + \log(1 + u_\alpha(t)) - \cos u_\alpha(t)\} &= \text{const.} \\ &= \lambda \{\alpha \log(1 + \alpha) - \alpha + \log(1 + \alpha) - \cos \alpha\}. \end{aligned}$$

This along with (12) implies that for $-1 \leq t \leq 0$,

$$u'_\alpha(t) = \sqrt{2\lambda} \sqrt{\alpha \log(1 + \alpha) - u_\alpha(t) \log(1 + u_\alpha(t)) - (\cos \alpha - \cos u_\alpha(t)) + \xi(u_\alpha(t))}, \quad (13)$$

where

$$\xi(u) := \log(1 + \alpha) - \log(1 + u) - (\alpha - u). \quad (14)$$

By this and (13), we obtain

$$\sqrt{2\lambda} = \int_{-1}^0 \frac{u'_\alpha(t)}{\sqrt{\eta(t) + \xi(u_\alpha(t))}} dt,$$

where

$$\eta(t) := \alpha \log(1 + \alpha) - u_\alpha(t) \log(1 + u_\alpha(t)) - (\cos \alpha - \cos(u_\alpha(t))).$$

By this and putting $\theta := u_\alpha(t)$, we obtain

$$\sqrt{2\lambda} = \int_0^\alpha \frac{1}{\sqrt{\alpha \log(1+\alpha) - u_\alpha(t) \log(1+\theta) - (\cos \alpha - \cos(\theta) + \xi(\theta))}} d\theta.$$

By this and putting $\theta := \alpha s^2$, we obtain

$$\begin{aligned} \sqrt{2\lambda} &= \int_0^1 \frac{2as}{\sqrt{\alpha(1-s^2)\log(1+\alpha) + as^2A_\alpha(s) - B_\alpha(s) + \xi(as^2)}} ds \\ &:= \frac{2\sqrt{\alpha}}{\sqrt{\log(1+\alpha)}} \int_0^1 \frac{s}{\sqrt{1-s^2}} \frac{1}{\sqrt{1+g_\alpha(s)}} ds, \end{aligned} \quad (15)$$

where

$$A_\alpha(s) := \log(1+\alpha) - \log(1+as^2), \quad (16)$$

$$B_\alpha(s) := \cos \alpha - \cos(as^2), \quad (17)$$

$$g_\alpha(s) := \frac{1}{\log(1+\alpha)} \frac{s^2}{(1-s^2)} A_\alpha(s) - \frac{1}{\alpha(1-s^2)\log(1+\alpha)} B_\alpha(s) + \frac{\xi(as^2)}{\alpha(1-s^2)\log(1+\alpha)}. \quad (18)$$

We note that $B_\alpha(s)$ causes the difficulty to prove Theorem 1.3. For $0 \leq s \leq 1$, we have

$$\left| \frac{1}{\log(1+\alpha)} \frac{s^2}{(1-s^2)} A_\alpha(s) \right| \leq \frac{s^2}{1-s^2} \frac{1}{\log(1+\alpha)} \int_{as^2}^\alpha \frac{1}{1+x} dx \leq \frac{1}{\log(1+\alpha)} \frac{as^2}{1+as^2} \leq \frac{1}{\log(1+\alpha)} \ll 1, \quad (19)$$

$$\left| \frac{1}{\alpha(1-s^2)\log(1+\alpha)} B_\alpha(s) \right| \leq \frac{1}{\log(1+\alpha)} \ll 1, \quad (20)$$

$$\left| \frac{\xi(as^2)}{\alpha(1-s^2)\log(1+\alpha)} \right| \leq \frac{2}{\log(1+\alpha)} \ll 1. \quad (21)$$

By this, (15) and Taylor expansion, we obtain

$$\begin{aligned} \sqrt{2\lambda} &= \frac{2\sqrt{\alpha}}{\sqrt{\log(1+\alpha)}} \int_0^1 \frac{s}{\sqrt{1-s^2}} \left\{ 1 + \sum_{n=1}^\infty (-1)^n \frac{(2n-1)!!}{n!2^n} g_\alpha(s)^n \right\} ds \\ &= \frac{2\sqrt{\alpha}}{\sqrt{\log(1+\alpha)}} \int_0^1 \frac{s}{\sqrt{1-s^2}} \left\{ 1 - \frac{1}{2} g_\alpha(s) + \sum_{n=2}^\infty (-1)^n \frac{(2n-1)!!}{n!2^n} g_\alpha(s)^n \right\} ds, \end{aligned} \quad (22)$$

where $(2n-1)!! = (2n-1)(2n-3)\cdots 3 \cdot 1$, $(-1)!! = 1$. We see from (22) that the second term of $\lambda(\alpha)$ in Theorem 1.3 follows from Lemma 2.1.

Lemma 2.1. As $\alpha \rightarrow \infty$,

$$L := \int_0^1 \frac{s}{\sqrt{1-s^2}} g_\alpha(s) ds = \frac{1}{\log(1+\alpha)} (4 \log 2 - 3) - \frac{1}{\log(1+\alpha)} \sqrt{\frac{\pi}{\alpha}} \sin\left(\alpha - \frac{\pi}{4}\right) + O\left(\frac{1}{\alpha \log(1+\alpha)}\right). \quad (23)$$

The proof of Lemma 2.1 is a combination of Lemmas 2.4 and 2.5. By (18), we have

$$L = L_1 + L_2 + L_3, \quad (24)$$

where

$$L_1 := -\frac{1}{\alpha \log(1+\alpha)} \int_0^1 \frac{s}{(1-s^2)^{3/2}} B_\alpha(s) ds, \quad (25)$$

$$L_2 := \frac{1}{\log(1+\alpha)} \int_0^1 \frac{s^3}{(1-s^2)^{3/2}} A_\alpha(s) ds, \quad (26)$$

$$L_3 := \frac{1}{\alpha \log(1+\alpha)} \int_0^1 \frac{s}{(1-s^2)^{3/2}} \xi(\alpha s^2) ds. \quad (27)$$

Since L_1 contains $B_\alpha(s)$, the most important point of Lemma 2.1 is to obtain the asymptotic behavior of L_1 . To do this, we apply the stationary phase method to L_1 .

Lemma 2.2. [17,19] Assume that $h \in C^2[0, 1]$. Consider

$$I(\mu) := \int_0^1 h(x) e^{i\mu w(x)} dx, \quad (28)$$

where $w(x) = \cos^2(\pi x/2)$. Then as $\mu \rightarrow \infty$,

$$I(\mu) = \sqrt{\frac{1}{\mu\pi}} h(0) e^{i(\mu - \pi/4)} + \sqrt{\frac{1}{\mu\pi}} h(1) e^{i\pi/4} + O(\mu^{-1}). \quad (29)$$

In particular,

$$I_s(\mu) := \operatorname{Im} I(\mu) = \int_0^1 h(x) \sin(\mu w(x)) dx = \sqrt{\frac{1}{\mu\pi}} \left(h(0) \sin\left(\mu - \frac{\pi}{4}\right) + \frac{1}{\sqrt{2}} h(1) \right) + O(\mu^{-1}). \quad (30)$$

Remark 2.3. By a slight modification in the proof of Lemma 2.2, we are able to apply Lemma 2.4 to $h(x, \mu)$ satisfying the condition in Lemma 2.2 and $|h(x, \mu)|, |(d/dx)h(x, \mu)| \leq C$ for $0 \leq x \leq 1$ and $\mu \geq \mu_0 > 0$ (cf. [10, Lemma 2.1]).

Lemma 2.4. As $\alpha \rightarrow \infty$,

$$L_1 = -\frac{1}{\log(1+\alpha)} \sqrt{\frac{\pi}{\alpha}} \sin\left(\alpha - \frac{\pi}{4}\right) + O\left(\frac{1}{\alpha \log(1+\alpha)}\right). \quad (31)$$

Proof. We put $s = \sin \theta$ in (25) and $\theta = (\pi/2)x$ in (32). Then by integration parts and Lemma 2.2, we have

$$\begin{aligned} L_1 &= -\frac{1}{\alpha \log(1+\alpha)} \int_0^{\pi/2} \frac{1}{\cos^2 \theta} \sin \theta B_\alpha(\sin \theta) d\theta \\ &= -\frac{1}{\alpha \log(1+\alpha)} [\tan \theta \sin \theta B_\alpha(\sin \theta)]_0^{\pi/2} + \frac{1}{\alpha \log(1+\alpha)} \int_0^{\pi/2} \sin \theta B_\alpha(\sin \theta) d\theta \\ &\quad + \frac{2}{\log(1+\alpha)} \int_0^{\pi/2} \sin^3 \theta \sin(\alpha \sin^2 \theta) d\theta \\ &= \frac{\pi}{\log(1+\alpha)} \int_0^1 \cos^3\left(\frac{\pi}{2}x\right) \sin\left(\alpha \cos^2\left(\frac{\pi}{2}x\right)\right) dx + O\left(\frac{1}{\alpha \log(1+\alpha)}\right) \\ &= \frac{1}{\log(1+\alpha)} \sqrt{\frac{\pi}{\alpha}} \sin\left(\alpha - \frac{\pi}{4}\right) + O\left(\frac{1}{\alpha \log(1+\alpha)}\right). \end{aligned} \quad (32)$$

Here, we use l'Hôpital's rule to obtain

$$\lim_{\theta \rightarrow \pi/2} \tan \theta \sin \theta B_\alpha(\sin \theta) = \lim_{\theta \rightarrow \pi/2} \frac{2\alpha \sin \theta \cos \theta \sin(\alpha \sin^2 \theta)}{-\sin \theta} = 0. \quad (33)$$

Thus, the proof is complete. $\square$

Since L_2 and L_3 do not contain $B_\alpha(s)$, the asymptotic formulae for L_2 and L_3 have been obtained in [1].

Lemma 2.5. [1] As $\alpha \rightarrow \infty$,

$$L_1 = (4 \log 2 - 2) \frac{1}{\log(1 + \alpha)} + O\left(\frac{1}{\alpha \log(1 + \alpha)}\right), \quad (34)$$

$$L_3 = -\frac{1}{\log(1 + \alpha)} + O\left(\frac{1}{\alpha \log(1 + \alpha)}\right). \quad (35)$$

Proof of Lemma 2.1. By Lemmas 2.4 and 2.5, we obtain Lemma 2.1. $\square$

3 The third term of $\lambda(\alpha)$ in Theorem 1.3

By (22), to obtain the third term of $\lambda(\alpha)$, we calculate

$$M := \int_0^1 \frac{s}{\sqrt{1-s^2}} g_\alpha(s)^2 ds. \quad (36)$$

We have

$$\begin{aligned} g_\alpha(s)^2 &= \frac{1}{(\log(1 + \alpha))^2} \frac{s^4}{(1-s^2)^2} A_\alpha(s)^2 + \frac{1}{\alpha^2 (\log(1 + \alpha))^2} \frac{B_\alpha(s)^2}{(1-s^2)^2} \\ &\quad + \frac{1}{\alpha^2 (\log(1 + \alpha))^2} \frac{\xi(as^2)^2}{(1-s^2)^2} - 2 \frac{1}{\alpha (\log(1 + \alpha))^2} \frac{s^2}{(1-s^2)^2} A_\alpha(s) B_\alpha(s) \\ &\quad + 2 \frac{1}{\alpha (\log(1 + \alpha))^2} \frac{s^2}{(1-s^2)^2} A_\alpha(s) \xi(as^2) - 2 \frac{1}{\alpha^2 (\log(1 + \alpha))^2} \frac{1}{(1-s^2)^2} B_\alpha(s) \xi(as^2). \end{aligned} \quad (37)$$

We put

$$M_1 := \frac{1}{\alpha^2 (\log(1 + \alpha))^2} J_1 = \frac{1}{\alpha^2 (\log(1 + \alpha))^2} \int_0^1 \frac{s}{(1-s^2)^{5/2}} B_\alpha(s)^2 ds, \quad (38)$$

$$M_2 := \frac{-2}{\alpha (\log(1 + \alpha))^2} J_2 = \frac{-2}{\alpha (\log(1 + \alpha))^2} \int_0^1 \frac{s^3}{(1-s^2)^{5/2}} A_\alpha(s) B_\alpha(s) ds, \quad (39)$$

$$M_3 := \frac{-2}{\alpha^2 (\log(1 + \alpha))^2} J_3 = \frac{-2}{\alpha^2 (\log(1 + \alpha))^2} \int_0^1 \frac{s}{(1-s^2)^{5/2}} B_\alpha(s) \xi(as^2) ds, \quad (40)$$

$$M_4 := \frac{1}{(\log(1 + \alpha))^2} J_4 = \frac{1}{(\log(1 + \alpha))^2} \int_0^1 \frac{s^5}{(1-s^2)^{5/2}} A_\alpha(s)^2 ds, \quad (41)$$

$$M_5 := \frac{1}{\alpha^2 (\log(1 + \alpha))^2} J_5 = \frac{1}{\alpha^2 (\log(1 + \alpha))^2} \int_0^1 \frac{s \xi(as^2)^2}{(1-s^2)^{5/2}} ds, \quad (42)$$

$$M_6 := \frac{2}{\alpha (\log(1 + \alpha))^2} J_6 = \frac{2}{\alpha (\log(1 + \alpha))^2} \int_0^1 \frac{s^3}{(1-s^2)^{5/2}} A_\alpha(s) \xi(as^2) ds. \quad (43)$$

Then

$$M = M_1 + M_2 + \cdots + M_6. \quad (44)$$

The most important point of this section is to calculate M_1 , M_2 , M_3 , since these three terms contain $B_\alpha(s)$. Now we calculate M_1 , M_2 , M_3 by using the stationary phase method. To do this, we use the following equality:

$$\frac{1}{\cos^4 \theta} = \frac{d}{d\theta} \left(\frac{\sin \theta}{3 \cos^3 \theta} (2 \cos^2 \theta + 1) \right). \quad (45)$$

Lemma 3.1. As $\alpha \rightarrow \infty$,

$$M_1 = \frac{4}{3} \sqrt{\pi} \left(\frac{1}{\alpha^{1/2} (\log(1 + \alpha))^2} \right) \left\{ \cos \alpha \cos \left(\alpha - \frac{\pi}{4} \right) - \frac{1}{\sqrt{2}} \cos \left(2\alpha - \frac{\pi}{4} \right) \right\} + O \left(\frac{1}{\alpha (\log(1 + \alpha))^2} \right). \quad (46)$$

Proof. We put $s = \sin \theta$ in (38). By integration by parts and (45), we have

$$\begin{aligned} J_1 &= \int_0^{\pi/2} \frac{1}{\cos^4 \theta} \sin \theta B_\alpha(\sin \theta)^2 d\theta \\ &= \int_0^{\pi/2} \left(\frac{\sin \theta}{3 \cos^3 \theta} (2 \cos^2 \theta + 1) \right)' \sin \theta B_\alpha(\sin \theta)^2 d\theta \\ &= \left[\frac{\sin \theta}{3 \cos^3 \theta} (2 \cos^2 \theta + 1) \sin \theta B_\alpha(\sin \theta)^2 \right]_0^{\pi/2} \\ &\quad - \int_0^{\pi/2} \left(\frac{\sin \theta}{3 \cos^3 \theta} (2 \cos^2 \theta + 1) \right) \{ \cos \theta B_\alpha(\sin \theta)^2 + 4\alpha \sin^2 \theta \cos \theta \sin(\alpha \sin^2 \theta) B_\alpha(\sin \theta) \} d\theta \\ &:= J_{10} - \frac{1}{3} J_{11} - \frac{4}{3} J_{12}. \end{aligned} \quad (47)$$

By the same argument with l'Hôpital's rule as (33), we see that $J_{10} = 0$. By using integration by parts, we obtain

$$\begin{aligned} J_{11} &= \int_0^{\pi/2} \frac{1}{\cos^2 \theta} \sin \theta (2 \cos^2 \theta + 1) B_\alpha(\sin \theta)^2 d\theta \\ &= [\tan \theta \sin \theta (2 \cos^2 \theta + 1) B_\alpha(\sin \theta)^2]_0^{\pi/2} - \int_0^{\pi/2} \sin \theta (2 \cos^2 \theta + 1) B_\alpha(\sin \theta)^2 d\theta \\ &\quad + 4 \int_0^{\pi/2} \sin^3 \theta B_\alpha(\sin \theta)^2 d\theta - 4\alpha \int_0^{\pi/2} \sin^3 \theta (2 \cos^2 \theta + 1) B_\alpha(\sin \theta) \sin(\alpha \sin^2 \theta) d\theta \\ &= O(\alpha). \end{aligned} \quad (48)$$

By l'Hôpital's rule and integration by parts, we obtain

$$\begin{aligned} J_{12} &= \alpha \int_0^{\pi/2} \frac{1}{\cos^2 \theta} \{ \sin^3 \theta (2 \cos^2 \theta + 1) \sin(\alpha \sin^2 \theta) B_\alpha(\sin \theta) \} d\theta \\ &= \alpha [\tan \theta \{ \sin^3 \theta (2 \cos^2 \theta + 1) \sin(\alpha \sin^2 \theta) B_\alpha(\sin \theta) \}]_0^{\pi/2} - 3\alpha \int_0^{\pi/2} \sin^3 \theta (2 \cos^2 \theta + 1) \sin(\alpha \sin^2 \theta) B_\alpha(\sin \theta) d\theta \end{aligned} \quad (49)$$

$$\begin{aligned}
& + 4\alpha \int_0^{\pi/2} \sin^5 \theta \sin(\alpha \sin^2 \theta) B_\alpha(\sin \theta) d\theta - 2\alpha^2 \int_0^{\pi/2} \sin^5 \theta (2 \cos^2 \theta + 1) \cos(\alpha \sin^2 \theta) B_\alpha(\sin \theta) d\theta \\
& - 2\alpha^2 \int_0^{\pi/2} \sin^5 \theta (2 \cos^2 \theta + 1) \sin^2(\alpha \sin^2 \theta) d\theta \\
& = -2\alpha^2 \cos \alpha \int_0^{\pi/2} \sin^5 \theta (2 \cos^2 \theta + 1) \cos(\alpha \sin^2 \theta) d\theta \\
& + 2\alpha^2 \int_0^{\pi/2} \sin^5 \theta (2 \cos^2 \theta + 1) (\cos^2(\alpha \sin^2 \theta) - \sin^2(\alpha \sin^2 \theta)) d\theta + O(\alpha) \\
& := J_{121} + J_{122} + O(\alpha).
\end{aligned}$$

We put $\theta = \pi(1-x)/2$. By Lemma 2.2, we obtain

$$\begin{aligned}
J_{121} &= -2\alpha^2 \cos \alpha \int_0^{\pi/2} \sin^5 \theta (2 \cos^2 \theta + 1) \cos(\alpha \sin^2 \theta) d\theta \\
&= -\pi\alpha^2 \cos \alpha \int_0^1 \cos^5 \left(\frac{\pi}{2}x \right) \left(2 \sin^2 \left(\frac{\pi}{2}x \right) + 1 \right) \cos \left(\alpha \cos^2 \left(\frac{\pi}{2}x \right) \right) dx \\
&= -\pi\alpha^2 \cos \alpha \left(\sqrt{\frac{1}{\alpha\pi}} \cos \left(\alpha - \frac{\pi}{4} \right) + O(\alpha^{-1}) \right),
\end{aligned} \tag{50}$$

$$\begin{aligned}
J_{122} &= 2\alpha^2 \int_0^{\pi/2} \sin^5 \theta (2 \cos^2 \theta + 1) \cos(2\alpha \sin^2 \theta) d\theta \\
&= \pi\alpha^2 \int_0^{\pi/2} \cos^5 \left(\frac{\pi}{2}x \right) \left(2 \sin^2 \left(\frac{\pi}{2}x \right) + 1 \right) \cos \left(2\alpha \cos^2 \left(\frac{\pi}{2}x \right) \right) dx \\
&= \pi\alpha^2 \left(\sqrt{\frac{1}{2\alpha\pi}} \cos \left(2\alpha - \frac{\pi}{4} \right) + O(\alpha^{-1}) \right).
\end{aligned} \tag{51}$$

By (47)–(51), we obtain (46). Thus, the proof is complete. $\square$

Lemma 3.2. As $\alpha \rightarrow \infty$,

$$M_2 = -\frac{2\sqrt{\pi}}{\alpha^{1/2}(\log(1+\alpha))^2} \sin \left(\alpha - \frac{\pi}{4} \right) + O \left(\frac{1}{\alpha(\log(1+\alpha))^2} \right). \tag{52}$$

Proof. We put $s = \sin \theta$. By (39), (45) and integration by parts, we have

$$\begin{aligned}
J_2 &= \int_0^{\pi/2} \frac{1}{\cos^4 \theta} \sin^3 \theta A_\alpha(\sin \theta) B_\alpha(\sin \theta) ds \\
&= \left[\left(\frac{\sin \theta}{3 \cos^3 \theta} (2 \cos^2 \theta + 1) \right) \sin^3 \theta A_\alpha(\sin \theta) B_\alpha(\sin \theta) \right]_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin \theta}{3 \cos^3 \theta} (2 \cos^2 \theta + 1) \\
&\quad \times \{ 3 \sin^2 \theta \cos \theta A_\alpha(\sin \theta) B_\alpha(\sin \theta) + \sin^3 \theta A'_\alpha(\sin \theta) B_\alpha(\sin \theta) + \sin^3 \theta A_\alpha(\sin \theta) B'_\alpha(\sin \theta) \} ds \\
&:= J_{20} - (J_{21} + J_{22} + J_{23}).
\end{aligned} \tag{53}$$

By l'Hopital's rule, we see that $J_{20} = 0$. We have

$$\begin{aligned}
 J_{21} &= \int_0^{\pi/2} \frac{1}{\cos^2 \theta} \sin^3 \theta (2 \cos^2 \theta + 1) A_\alpha(\sin \theta) B_\alpha(\sin \theta) d\theta \\
 &= [\tan \theta \sin^3 \theta (2 \cos^2 \theta + 1) A_\alpha(\sin \theta) B_\alpha(\sin \theta)]_0^{\pi/2} - \int_0^{\pi/2} \tan \theta \{ \sin^3 \theta (2 \cos^2 \theta + 1) A_\alpha(\sin \theta) B_\alpha(\sin \theta) \}' d\theta \\
 &= J_{210} - (J_{211} + J_{212} + J_{213} + J_{214}) \\
 &:= J_{210} - 3 \int_0^{\pi/2} \sin^3 \theta (2 \cos^2 \theta + 1) A_\alpha(\sin \theta) B_\alpha(\sin \theta) d\theta + 4 \int_0^{\pi/2} \sin^5 \theta A_\alpha(\sin \theta) B_\alpha(\sin \theta) d\theta \\
 &\quad - \int_0^{\pi/2} \tan \theta \sin^3 \theta (2 \cos^2 \theta + 1) A_\alpha'(\sin \theta) B_\alpha(\sin \theta) d\theta \\
 &\quad - \int_0^{\pi/2} \tan \theta \sin^3 \theta (2 \cos^2 \theta + 1) A_\alpha(\sin \theta) B_\alpha'(\sin \theta) d\theta.
 \end{aligned} \tag{54}$$

By l'Hopital's rule, we see that $J_{210} = 0$. It was shown in [19] that for $\alpha \gg 1$,

$$0 \leq \sin \theta A_\alpha(\sin \theta) = -2 \sin \theta \log(\sin \theta) + O\left(\frac{1}{\sqrt{\alpha}}\right), \tag{55}$$

$$0 \leq \sin^2 \theta A_\alpha(\sin \theta) = -2 \sin^2 \theta \log(\sin \theta) + O\left(\frac{1}{\alpha}\right). \tag{56}$$

Moreover, it is clear that for $0 \leq \theta \leq \pi/2$,

$$\sin^2 \theta - \frac{1}{\alpha} \leq \sin^2 \theta \left(\frac{\alpha \sin^2 \theta}{1 + \alpha \sin^2 \theta} \right) \leq \sin^2 \theta. \tag{57}$$

By (56), as $\alpha \rightarrow \infty$, we obtain

$$\begin{aligned}
 |J_{211}| &= 3 \left| \int_0^{\pi/2} \sin \theta (2 \cos^2 \theta + 1) (\sin^2 \theta A_\alpha(\sin \theta)) (\cos \theta - \cos(\alpha \sin^2 \theta)) d\theta \right| \\
 &\leq C \left| \int_0^{\pi/2} \sin \theta (2 \cos^2 \theta + 1) (-\sin^2 \theta \log(\sin^2 \theta)) d\theta \right| + O(\alpha^{-1}) \leq C.
 \end{aligned} \tag{58}$$

By the same argument as that in (58), we see that $|J_{212}| \leq C$. By (17), we have

$$|J_{213}| = 2 \int_0^{\pi/2} \sin^3 \theta (2 \cos^2 \theta + 1) \frac{\alpha \sin^2 \theta}{1 + \alpha \sin^2 \theta} B_\alpha(\sin \theta) d\theta \leq C. \tag{59}$$

By Lemma 2.2 and (56), we obtain

$$\begin{aligned}
 J_{214} &= 2\alpha \int_0^{\pi} \sin^5 \theta (2 \cos^2 \theta + 1) A_\alpha(\sin \theta) \sin(\alpha \sin^2 \theta) d\theta \\
 &= 2\alpha \int_0^{\pi/2} \sin^3 \theta (2 \cos^2 \theta + 1) (\sin^2 \theta A_\alpha(\sin \theta)) \sin(\alpha \sin^2 \theta) d\theta \\
 &= 2\alpha \int_0^{\pi/2} \sin^3 \theta (2 \cos^2 \theta + 1) (-\sin^2 \theta \log(\sin^2 \theta)) \sin(\alpha \sin^2 \theta) d\theta + O(1)
 \end{aligned} \tag{60}$$

$$\begin{aligned}
&= -\pi\alpha \int_0^1 \cos^5\left(\frac{\pi x}{2}\right) \left(2\sin^2\left(\frac{\pi x}{2}\right) + 1\right) \log\left(\cos^2\left(\frac{\pi x}{2}\right)\right) \sin\left(\alpha \cos^2\left(\frac{\pi x}{2}\right)\right) dx + O(1) \\
&= -\pi\alpha O(\alpha^{-1}) + O(1) = O(1).
\end{aligned}$$

By (58)–(60), we obtain that $|J_{21}| \leq C$. By integration by parts, we have

$$\begin{aligned}
J_{22} &= -\frac{2}{3}\alpha \int_0^{\pi/2} \frac{1}{\cos^2\theta} \sin^5\theta (2\cos^2\theta + 1) \frac{1}{1 + \alpha \sin^2\theta} B_\alpha(\sin\theta) d\theta \\
&= -\frac{2}{3}\alpha \left[\tan\theta \sin^5\theta (2\cos^2\theta + 1) \frac{1}{1 + \alpha \sin^2\theta} B_\alpha(\sin\theta) \right]_0^{\pi/2} \\
&\quad + \frac{2}{3} \int_0^{\pi/2} (10\sin^3\theta \cos^2\theta - 4\sin^5\theta + 5\sin^3\theta) \frac{\alpha \sin^2\theta}{1 + \alpha \sin^2\theta} B_\alpha(\sin\theta) d\theta \\
&\quad - \frac{4}{3} \int_0^{\pi/2} \sin^3\theta (2\cos^2\theta + 1) \left(\frac{\alpha \sin^2\theta}{1 + \alpha \sin^2\theta} \right)^2 B_\alpha(\sin\theta) d\theta \\
&\quad + \frac{4}{3} \alpha \int_0^{\pi/2} \sin^5\theta (2\cos^2\theta + 1) \frac{\alpha \sin^2\theta}{1 + \alpha \sin^2\theta} \sin(\alpha \sin^2\theta) d\theta \\
&= J_{220} + J_{221} + J_{222} + J_{223}.
\end{aligned} \tag{61}$$

By l'Hôpital's rule, we see that $J_{220} = 0$. Obviously, we have $|J_{221}| \leq C$, $|J_{222}| \leq C$. By putting $\theta = \pi(1-x)/2$, Lemma 2.2 and (57), we obtain

$$\begin{aligned}
J_{223} &= \frac{4}{3}\alpha \int_0^{\pi/2} \sin^5\theta (2\cos^2\theta + 1) (1 + O(\alpha^{-1})) \sin(\alpha \sin^2\theta) d\theta \\
&= \frac{2\pi}{3}\alpha \int_0^1 \cos^5\left(\frac{\pi x}{2}\right) \left(2\sin^2\left(\frac{\pi x}{2}\right) + 1\right) \sin\left(\alpha \cos^2\left(\frac{\pi x}{2}\right)\right) dx + O(1) \\
&= \frac{2\pi}{3}\alpha \left\{ \sqrt{\frac{1}{\alpha\pi}} \sin\left(\alpha - \frac{\pi}{4}\right) + O(\alpha^{-1}) \right\} + O(1).
\end{aligned} \tag{62}$$

By this and (61), we obtain

$$J_{22} = \frac{2\sqrt{\pi}}{3} \sqrt{\alpha} \sin\left(\alpha - \frac{\pi}{4}\right) + O(1). \tag{63}$$

We calculate J_{23} . We put $\theta = \frac{\pi}{2}(1-x)$. Then

$$J_{23} = \frac{\pi}{3}\alpha \int_0^1 h(x) \sin\left(\alpha \cos^2\left(\frac{\pi x}{2}\right)\right) dx, \tag{64}$$

where

$$h(x) := \frac{1}{\sin^2(\pi x/2)} \cos^5\left(\frac{\pi x}{2}\right) \left(2\sin^2\left(\frac{\pi x}{2}\right) + 1\right) A_\alpha\left(\cos\left(\frac{\pi x}{2}\right)\right). \tag{65}$$

By direct calculation, we see that $h(x)$ satisfies the conditions in Remark 2.3. For completeness, the proof will be given in Appendix. By (65) and l'Hôpital's rule, we have

$$h(0) = \frac{\alpha}{1 + \alpha}, \quad h(1) = 0. \tag{66}$$

We apply Lemma 2.2 to (64) to obtain

$$J_{23} = \frac{\sqrt{\pi}}{3} \sqrt{\alpha} \sin\left(\alpha - \frac{\pi}{4}\right) + O(1). \quad (67)$$

By this, (54) and (63), we obtain

$$J_2 = \sqrt{\pi\alpha} \sin\left(\alpha - \frac{\pi}{4}\right) + O(1). \quad (68)$$

By this and (39), we obtain (52). Thus, the proof is complete. $\square$

Lemma 3.3. As $\alpha \rightarrow \infty$,

$$M_3 = -\frac{2\sqrt{\pi}}{\alpha^{1/2}(\log(1+\alpha))^2} \sin\left(\alpha - \frac{\pi}{4}\right) + O\left(\frac{1}{\alpha(\log(1+\alpha))^2}\right). \quad (69)$$

Proof. By putting $s = \sin \theta$ in (40), we obtain

$$\begin{aligned} J_3 &= \int_0^1 \frac{s}{(1-s^2)^{5/2}} B_\alpha(s) \xi(\alpha s^2) ds \\ &= \int_0^{\pi/2} \frac{1}{\cos^4 \theta} \sin \theta B_\alpha(\sin \theta) (A_\alpha(\sin \theta) - \alpha \cos^2 \theta) d\theta \\ &= \int_0^{\pi/2} \frac{1}{\cos^4 \theta} \sin \theta B_\alpha(\sin \theta) A_\alpha(\sin \theta) d\theta - \alpha \int_0^{\pi/2} \frac{1}{\cos^2 \theta} B_\alpha(\sin \theta) \sin \theta d\theta \\ &:= J_{31} + J_{32}. \end{aligned} \quad (70)$$

By (45) and integration by parts, we have

$$\begin{aligned} J_{31} &= J_{310} + J_{311} + J_{312} + J_{313} \\ &:= \left[\left(\frac{\sin \theta}{3 \cos^3 \theta} (2 \cos^2 \theta + 1) \right) \sin \theta B_\alpha(\sin \theta) A_\alpha(\sin \theta) \right]_0^{\pi/2} \\ &\quad - \frac{1}{3} \int_0^{\pi/2} \frac{1}{\cos^2 \theta} (2 \sin \theta \cos^2 \theta + \sin \theta) B_\alpha(\sin \theta) A_\alpha(\sin \theta) d\theta \\ &\quad - \frac{2}{3} \alpha \int_0^{\pi/2} \frac{1}{\cos^2 \theta} \sin^3 \theta (2 \cos^2 \theta + 1) \sin(\alpha \sin^2 \theta) A_\alpha(\sin \theta) d\theta \\ &\quad + \frac{2}{3} \alpha \int_0^{\pi/2} \frac{1}{\cos^2 \theta} \sin^3 \theta (2 \cos^2 \theta + 1) B_\alpha(\sin \theta) \frac{1}{1 + \alpha \sin^2 \theta} d\theta. \end{aligned} \quad (71)$$

By using l'Hôpital's rule, we see that $J_{310} = 0$. By using integration by parts, (56) and l'Hôpital's rule, we obtain

$$\begin{aligned} J_{311} &= \left[-\frac{1}{3} \tan \theta (2 \sin \theta \cos^2 \theta + \sin \theta) B_\alpha(\sin \theta) A_\alpha(\sin \theta) \right]_0^{\pi/2} \\ &\quad + \frac{1}{3} \int_0^{\pi/2} (2 \cos^2 \theta - 4 \sin^2 \theta + 1) (\sin \theta A_\alpha(\sin \theta)) B_\alpha(\sin \theta) d\theta \end{aligned} \quad (72)$$

$$\begin{aligned}
& - \frac{2}{3} \int_0^{\pi/2} \sin \theta (2 \cos^2 \theta + 1) \frac{\alpha \sin^2 \theta}{1 + \alpha \sin^2 \theta} B_\alpha(\sin \theta) d\theta \\
& + \frac{2}{3} \alpha \int_0^{\pi/2} \sin \theta (2 \cos^2 \theta + 1) (\sin^2 \theta A_\alpha(\sin \theta)) \sin(\alpha \sin^2 \theta) d\theta \\
& = 0 + O(1) + O(1) + O(\alpha) = O(\alpha).
\end{aligned}$$

By using integration by parts, (56) and l'Hôpital's rule, Lemma 2.2 and putting $\theta = \pi x/2$, we obtain

$$\begin{aligned}
J_{312} &= \left[-\frac{2}{3} \alpha \tan \theta \sin^3 \theta (2 \cos^2 \theta + 1) \sin(\alpha \sin^2 \theta) A_\alpha(\sin \theta) \right]_0^{\pi/2} \\
&+ \frac{2}{3} \alpha \int_0^{\pi/2} (6 \cos^2 \theta - 4 \sin^2 \theta + 3) \sin^3 \theta A_\alpha(\sin \theta) \sin(\alpha \sin^2 \theta) d\theta \\
&+ \frac{4}{3} \alpha^2 \int_0^{\pi/2} \sin^3 \theta (2 \cos^2 \theta + 1) (\sin^2 \theta A_\alpha(\sin \theta)) \cos(\alpha \sin^2 \theta) d\theta \\
&- \frac{4}{3} \alpha \int_0^{\pi/2} \sin^3 \theta (2 \cos^2 \theta + 1) \sin(\alpha \sin^2 \theta) \frac{\alpha \sin^2 \theta}{1 + \alpha \sin^2 \theta} d\theta \\
&= -\frac{8}{3} \alpha^2 \int_0^{\pi/2} (2 \cos^2 \theta + 1) (\sin^5 \theta \log(\sin \theta)) \cos(\alpha \sin^2 \theta) d\theta + O(\alpha) \\
&= -\frac{4\pi}{3} \alpha^2 \int_0^1 \left(2 \sin^2 \left(\frac{\pi}{2} x \right) + 1 \right) \cos^5 \left(\frac{\pi}{2} x \right) \log \left(\cos \left(\frac{\pi}{2} x \right) \right) \cos \left(\alpha \cos^2 \left(\frac{\pi}{2} x \right) \right) dx + O(\alpha) \\
&= O(\alpha).
\end{aligned} \tag{73}$$

By using integration by parts, (56) and l'Hôpital's rule, we obtain

$$\begin{aligned}
J_{313} &= \frac{2}{3} \alpha \left[\tan \theta \sin^3 \theta (2 \cos^2 \theta + 1) B_\alpha(\sin \theta) \frac{1}{1 + \alpha \sin^2 \theta} \right]_0^{\pi/2} \\
&- \frac{2}{3} \int_0^{\pi/2} (6 \sin \theta \cos^2 \theta - 4 \sin^3 \theta + 3 \sin \theta) B_\alpha(\sin \theta) \frac{\alpha \sin^2 \theta}{1 + \alpha \sin^2 \theta} d\theta \\
&- \frac{4}{3} \alpha \int_0^{\pi/2} \sin^3 \theta (2 \cos^2 \theta + 1) \sin(\alpha \sin^2 \theta) \frac{\alpha \sin^2 \theta}{1 + \alpha \sin^2 \theta} d\theta \\
&+ \frac{4}{3} \int_0^{\pi/2} \sin \theta (2 \cos^2 \theta + 1) B_\alpha(\sin \theta) \left(\frac{\alpha \sin^2 \theta}{1 + \alpha \sin^2 \theta} \right)^2 d\theta \\
&= O(\alpha).
\end{aligned} \tag{74}$$

By using integration by parts, l'Hôpital's rule, Lemma 2.2 and putting $\theta = \pi(1-x)/2$, we obtain

$$\begin{aligned}
J_{32} &= -\alpha [\tan \theta B_\alpha(\sin \theta) \sin \theta]_0^{\pi/2} + 2\alpha^2 \int_0^{\pi/2} \sin^3 \theta \sin(\alpha \sin^2 \theta) d\theta + \alpha \int_0^{\pi/2} B_\alpha(\sin \theta) \sin \theta d\theta \\
&= \pi \alpha^2 \int_0^1 \cos^3 \left(\frac{\pi}{2} x \right) \sin \left(\alpha \cos^2 \left(\frac{\pi}{2} x \right) \right) dx + O(\alpha) \\
&= \sqrt{\pi} \alpha^{3/2} \sin \left(\alpha - \frac{\pi}{4} \right) + O(\alpha).
\end{aligned} \tag{75}$$

By (71)–(75), we see that

$$J_3 = \sqrt{\pi} \alpha^{3/2} \sin\left(\alpha - \frac{\pi}{4}\right) + O(\alpha). \quad (76)$$

By this and (40), we obtain (69). Thus, the proof is complete. $\square$

The asymptotic behaviors of M_4, M_5, M_6 , in which $B_\alpha(s)$ are not contained, have already been obtained in [1].

Lemma 3.4. [1] As $\alpha \rightarrow \infty$,

$$M_4 = C_1 \frac{1}{(\log(1 + \alpha))^2} + O\left(\frac{1}{\alpha(\log(1 + \alpha))^2}\right), \quad (77)$$

where

$$C_1 := -\frac{4}{3}C_{11} + \frac{1}{3}(C_{12} + C_{13} + C_{14} + C_{15} + C_{16}), \quad (78)$$

$$C_{11} := 20 \int_0^{\pi/2} \sin^3 \theta (\log(\sin \theta))^2 d\theta - 8 \int_0^{\pi/2} \sin^7 \theta \log(\sin \theta) d\theta, \quad (79)$$

$$C_{12} := 100 \int_0^{\pi/2} \sin^5 \theta (2 \cos^2 \theta + 1) (\log(\sin \theta))^2 d\theta, \quad (80)$$

$$C_{13} := 40 \int_0^{\pi/2} \sin^7 \theta (2 \cos^2 \theta + 1) \log(\sin \theta) d\theta, \quad (81)$$

$$C_{14} := 8 \int_0^{\pi/2} \sin^5 \theta (2 \cos^2 \theta + 1) d\theta, \quad (82)$$

$$C_{15} := 56 \int_0^{\pi/2} \sin^7 \theta (2 \cos^2 \theta + 1) \log(\sin \theta) d\theta, \quad (83)$$

$$C_{16} := -16 \int_0^{\pi/2} \sin^5 \theta (2 \cos^2 \theta + 1) \log(\sin \theta) d\theta, \quad (84)$$

$$M_5 = \frac{1}{(\log(1 + \alpha))^2} + O\left(\frac{1}{\alpha(\log(1 + \alpha))^2}\right), \quad (85)$$

$$M_6 = \frac{4}{(\log(1 + \alpha))^2} (-2 \log 2 + 1) + O\left(\frac{1}{\alpha(\log(1 + \alpha))^2}\right). \quad (86)$$

Proof of Theorem 1.3. By Lemmas 3.1–3.4, we obtain

$$\begin{aligned} M &= (C_1 + 5 - 8 \log 2) \frac{1}{(\log(1 + \alpha))^2} + \frac{4\sqrt{\pi}}{3} \left(\frac{1}{\alpha^{1/2}(\log(1 + \alpha))^2} \right) \\ &\quad \times \left\{ \cos \alpha \cos\left(\alpha - \frac{1}{4}\pi\right) - \frac{1}{\sqrt{2}} \cos\left(2\alpha - \frac{1}{4}\pi\right) - 3 \sin\left(\alpha - \frac{\pi}{4}\right) \right\} + O\left(\frac{1}{\alpha(\log(1 + \alpha))^2}\right). \end{aligned} \quad (87)$$

Thus, along with (22) and Lemma 2.1, we obtain Theorem 1.3. $\square$

4 Conclusion

In this paper, we study the global structure of logarithmic bifurcation problem which is perturbed by the oscillatory term $\sin u$. We show that, although the nonlinear term $f(u) = \log(1 + u) + \sin u$ contains an oscillatory term, the asymptotic formula for $\sqrt{\lambda(\alpha)}$ as $\alpha \rightarrow \infty$ does not include an oscillatory term up to the third term, and it is the same as that for the case $f(u) = \log(1 + u)$. It should be emphasized that this sort of asymptotic formula has not been known yet and seems to be new. The result of this paper indicates the possibility that there are nonlinear oscillatory terms $f(u)$ such that the oscillatory terms do not appear in any n th term of the asymptotic expansion of $\sqrt{\lambda(\alpha)}$. This information is novel and gives us a new aspect for the global structures of bifurcation diagrams.

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Appendix

For $0 \leq x \leq 1$, let

$$h(x) := \frac{1}{\sin^2(\pi x/2)} \cos^5\left(\frac{\pi}{2}x\right) \left(2 \sin^2\left(\frac{\pi}{2}x\right) + 1\right) A_\alpha\left(\cos\left(\frac{\pi}{2}x\right)\right). \quad (88)$$

We show that $h(x)$ satisfies the conditions in Remark 2.3. Indeed, the essential point is as follows. For $0 \leq \theta \leq \pi/2$, we put

$$f(\theta) := \frac{A_\alpha(\sin \theta)}{\cos^2 \theta}. \quad (89)$$

By l'Hôpital's rule, we see that $f(\theta)$ is well defined at $\theta = \pi/2$. If $f(\theta)$ satisfies the conditions in Lemma 2.2 and Remark 2.3 at $\theta = \pi/2$, then so does $f(\theta) \sin^5 \theta (2 \cos^2 \theta + 1)$ at $\theta = \pi/2$. By putting $\theta = \pi(1-x)/2$, then we see that $h(x)$ satisfies the condition in Remark 2.3. By direct calculation, we have

$$\begin{aligned} f'(\theta) &= 2 \sin \theta \left(\frac{A_\alpha(\sin \theta)}{\cos^3 \theta} - \frac{\alpha}{(1 + \alpha \sin^2 \theta) \cos \theta} \right) \\ &= 2 \sin \theta \left[\frac{A_\alpha(\sin \theta)}{\cos^3 \theta} - \frac{\alpha}{1 + \alpha \cos \theta} + \frac{\alpha}{\cos \theta} \left(\frac{1}{1 + \alpha} - \frac{1}{1 + \alpha \sin^2 \theta} \right) \right] \\ &= 2 \sin \theta \left[\frac{A_\alpha(\sin \theta)}{\cos^3 \theta} - \frac{\alpha}{1 + \alpha \cos \theta} - \frac{\alpha^2}{(1 + \alpha)(1 + \alpha \sin^2 \theta)} \cos \theta \right]. \end{aligned} \quad (90)$$

By (90), we study the regularity of $\tilde{K}(\theta)$ at $\theta = \pi/2$, where

$$\tilde{K}(\theta) := \frac{A_\alpha(\sin \theta)}{\cos^3 \theta} - \frac{\alpha}{1 + \alpha \cos \theta}. \quad (91)$$

We put $\theta = \pi/2 - x$ in (4.4). Then we have

$$\tilde{K}(\theta) = \frac{1}{\sin^3 x} K(x) := \frac{1}{\sin^3 x} \left(A_\alpha(\cos^2 x) - \frac{\alpha}{1 + \alpha} \sin^2 x \right). \quad (92)$$

We have

$$\frac{d}{dx} A_\alpha(\cos^2 x) = \frac{\alpha \sin 2x}{1 + \alpha \cos^2 x}, \quad (93)$$

$$\frac{d^2}{dx^2} A_\alpha(\cos^2 x) = \frac{2\alpha \cos 2x}{1 + \alpha \cos^2 x} + \frac{\alpha^2 \sin^2 2x}{(1 + \alpha \cos^2 x)^2}, \quad (94)$$

$$\frac{d^3}{dx^3} A_\alpha(\cos^2 x) = -\frac{4\alpha \sin 2x}{1 + \alpha \cos^2 x} + \frac{3\alpha^2 \sin 4x}{(1 + \alpha \cos^2 x)^2} + \frac{2\alpha^3 \sin^3 2x}{(1 + \alpha \cos^2 x)^3}, \quad (95)$$

$$\begin{aligned} \frac{d^4}{dx^4} A_\alpha(\cos^2 x) &= \frac{-8\alpha \cos 2x}{1 + \alpha \cos^2 x} - \frac{4\alpha^2 \sin^2 2x}{(1 + \alpha \cos^2 x)^2} + \frac{12\alpha^2 \cos 4x}{(1 + \alpha \cos^2 x)^2} + \frac{6\alpha^3 \sin 4x \sin 2x}{(1 + \alpha \cos^2 x)^3} \\ &\quad + \frac{12\alpha^3 \sin^2 2x \cos 2x}{(1 + \alpha \cos^2 x)^3} + \frac{12\alpha^4 \sin^4 2x}{(1 + \alpha \cos^2 x)^4}. \end{aligned} \quad (96)$$

By (93)–(96), the Taylor expansion of $A_\alpha(\cos^2 x)$ at $x = 0$ is

$$A_\alpha(\cos^2 x) = \frac{\alpha}{1 + \alpha} x^2 + \frac{\alpha^2 - 2\alpha}{6(1 + \alpha)^2} x^4 + O(x^5). \quad (97)$$

By this, (92) and Taylor expansion, we obtain

$$\frac{1}{\sin^3 x} K(x) = \left(\frac{\alpha^2}{2(1 + \alpha)^2} x^4 + O(x^5) \right) \left(x - \frac{1}{6} x^3 + O(x^5) \right)^{-3} = \frac{\alpha^2}{2(1 + \alpha)^2} x + O(x^2). \quad (98)$$

By this and (90), near $\theta = \pi/2$, we obtain

$$f'(\theta) = \sin \theta \left\{ \frac{\alpha^2}{(1 + \alpha)^2} \left(\frac{\pi}{2} - \theta \right) + O \left(\left(\frac{\pi}{2} - \theta \right)^2 \right) \right\}. \quad (99)$$

By this, (88) and (89), we see that $h \in C^2[0, 1]$. Moreover, $|h(x, \alpha)|, |h'(x, \alpha)| \leq C$ for any $[0, 1]$ and $\alpha \gg 1$. Thus, the proof is complete. $\square$