Research Article

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Construction of a family of non-stationary combined ternary subdivision schemes reproducing exponential polynomials

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Abstract: In this paper, we propose a family of non-stationary combined ternary $(2m + 3)$ -point subdivision schemes, which possesses the property of generating/reproducing high-order exponential polynomials. This scheme is obtained by adding variable parameters on the generalized ternary subdivision scheme of order 4. For such a scheme, we investigate its support and exponential polynomial generation/reproduction and get that it can generate/reproduce certain exponential polynomials with suitable choices of the parameters and reach $2m + 3$ approximation order. Moreover, we discuss its smoothness and show that it can produce C^{2m+2} limit curves. Several numerical examples are given to show the performance of the schemes.

Keywords: non-stationary combined ternary scheme, exponential polynomial generation/reproduction, approximation order, smoothness

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1 Introduction

Subdivision schemes are an efficient tool to design smooth curves and surfaces from a given initial polyline/polyhedral mesh. Over the past two decades, subdivision schemes have shown their usefulness in several application contexts ranging from computer aided geometric design and signal/image processing to computer graphics and animation. Recently, subdivision schemes have become of interest also in biomedical imaging applications (see, e.g., [1–3]) and isogeometric analysis (IgA), a modern computational approach that integrates finite element analysis into conventional CAD systems (see, e.g., [4–7]). Generally, given a set of initial values on a coarse grid, a subdivision scheme is a set of rules that recursively define sets of values on finer grids. If the rules are the same at all refinement levels, then the scheme is stationary, otherwise it is non-stationary.

One of the important capabilities of non-stationary schemes (which stationary schemes do not have) is the reproduction of exponential polynomials. Such schemes may be useful for the processing of families of oscillatory signals that are well approximated by combinations of exponential polynomials, such as speech signals, and narrowband signals in general. Hence, there have been continuous works on non-stationary subdivision schemes generating exponential polynomials. Romani [8] converted three exponential B-spline schemes into interpolatory schemes without changing the generation property. Conti et al. [9] transformed the non-stationary approximating schemes into interpolatory ones with the same generation property.

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For other references on this method, see, e.g., [10–15]. All of the aforementioned works convert a known approximating subdivision scheme into a new interpolatory subdivision scheme by a suitable polynomial correction of the symbol of the approximating one. Another way to construct interpolatory schemes is to derive them from approximating ones using the push-back operation presented in [16]. With this method, Lin et al. [17] derived interpolatory surface subdivision from the approximating subdivision. Zhang and Wang [18] gave a semi-stationary subdivision scheme by this operation. Novara and Romani [19] used the push-back operation and constructed a combined ternary 4-point scheme which can unify quite a number of existing approximating and interpolatory schemes. However, previous works are restricted to the stationary schemes which can only generate/reproduce algebraic polynomials. Zheng and Zhang [20] first generalized the push-back operation to the non-stationary case and constructed a non-stationary combined subdivision scheme. As ternary subdivision schemes compare favorably with their binary analogues because of generating limit functions with the same (or higher) smoothness but smaller support (see, e.g., [21]), Zhang et al. [22] generalized the push-back operation to the non-stationary ternary case and presented a non-stationary combined ternary 5-point subdivision scheme. In this paper, to generate and reproduce more general exponential polynomials, we propose a family of non-stationary combined ternary $(2m + 3)$ -point subdivision schemes based on the generalized ternary scheme of order 4 proposed in [22]. For such a scheme, we investigate its properties, including the support, exponential polynomial generation/reproduction, approximation order and smoothness. Some examples are given to illustrate the feature of the proposed scheme.

The rest of the paper is organized as follows: Section 2 recalls some basic knowledge about subdivision schemes. Section 3 is devoted to the construction of the new family of non-stationary combined ternary $(2m + 3)$ -point subdivision schemes, which reproduces high-order exponential polynomials. The support, exponential polynomial generation/reproduction, approximation order and smoothness are investigated in Section 4. Section 5 concludes the paper with a short summary and further research work.

2 Background

In this section, we review a few basic definitions and results about ternary subdivision scheme which form the basis of the rest of this paper.

This paper is mainly concerned with non-stationary ternary subdivision schemes formalized as follows. Given a sequence of initial control points $\mathbf{P}^0 = \{P_i^0 \in \mathbb{R} : i \in \mathbb{Z}\}$, the control points $\mathbf{P}^k = \{P_j^k : j \in \mathbb{Z}\}$ are iteratively refined with the rule

$$(\mathbf{P}^{k+1})_i = (S_{\mathbf{a}^k} \mathbf{P}^k)_i := \sum_j a_{i-3j}^k P_j^k, \quad k \in \mathbb{N},$$

where $S_{\mathbf{a}^k}$ is the k -level subdivision operator mapping $\ell_0(\mathbb{Z})$ to $\ell_0(\mathbb{Z})$, and $\ell_0(\mathbb{Z})$ denotes the linear space of real sequences with finite support. The set of coefficients $\mathbf{a}^k = \{a_i^k : i \in \mathbb{Z}\}$ in the aforementioned equations is termed as the mask of the subdivision scheme at level k and the Laurent polynomial $a^k(z) = \sum_{i \in \mathbb{Z}} a_i^k z^i$, $z \in \mathbb{C} \setminus \{0\}$ is called the k -level symbol of the non-stationary ternary scheme. A non-stationary ternary subdivision scheme is denoted as $\{S_{\mathbf{a}^k}\}_{k \in \mathbb{N}}$, and when all subdivision operators are the same, the subdivision scheme is simply denoted as $\{S_{\mathbf{a}}\}$. We denote the subsymbols of the subdivision scheme $\{S_{\mathbf{a}^k}\}_{k \in \mathbb{N}}$ by $a_h^k(z) = \sum_{i \in \mathbb{Z}} a_{3i+h}^k z^{3i+h}$, $h = 0, 1, 2$, and remark that the j th derivative of a subsymbol is

$$D^j a_h^k(z) = \sum_{i \in \mathbb{Z}} \rho_{j,h}(i) a_{3i+h}^k z^{3i+h-j}, \quad \rho_{j,h}(x) = \prod_{n=0}^{j-1} (3x + h - n),$$

where D^j denotes the j th order differential operator.

In the following, we recall some knowledge about the generation/reproduction of exponential polynomials.

Definition 2.1. [8] Let $T \in \mathbb{N}$ and $\chi = \{\chi_0, \chi_1, \dots, \chi_T\}$ with $\chi_T \neq 0$ a finite set of real or imaginary numbers. The T -dimensional space of exponential polynomials $V_{T,\chi}$ is defined by

$$V_{T,\chi} := \left\{ f: \mathbb{R} \rightarrow \mathbb{C}, f \in C^T(\mathbb{R}) : \sum_{j=0}^T \chi_j D^j f = 0 \right\}.$$

Lemma 2.2. [8] Let $\chi(z) = \sum_{j=0}^T \chi_j z^j$ and denote by $\{(\theta_i, \tau_i)\}_{i=0, \dots, N}$ the set of zeros with multiplicity, satisfying

$$D^r \chi(\theta_i) = 0, \quad r = 0, \dots, \tau_i - 1, \quad i = 0, \dots, N.$$

Then $T = \sum_{i=0}^N \tau_i$, $V_{T,\chi} := \text{span}\{\chi^r e^{\theta_i x}, r = 0, \dots, \tau_i - 1, i = 0, \dots, N\}$.

Definition 2.3. [23] Let $\{a^k(z)\}_{k \in \mathbb{N}}$ be a sequence of subdivision symbols. The subdivision scheme associated with the sequence of symbols $\{a^k(z)\}_{k \in \mathbb{N}}$ is said to be $V_{T,\chi}$ -generating, if it is convergent and for $f \in V_{T,\chi}$ there exists an initial sequence $\mathbf{f}^{(0)}$ uniformly sampled from $\tilde{f} \in V_{T,\chi}$ such that

$$\lim_{k \rightarrow \infty} S_{a^{n+k}} S_{a^{n+k-1}} \cdots S_{a^n} \mathbf{f}^{(0)} = f, \quad \forall n \geq 0.$$

Definition 2.4. [23] Let $\{a^k(z)\}_{k \in \mathbb{N}}$ be a sequence of subdivision symbols. The subdivision scheme associated with the sequence of symbols $\{a^k(z)\}_{k \in \mathbb{N}}$ is said to be $V_{T,\chi}$ -reproducing, if it is convergent and for $f \in V_{T,\chi}$ and an initial sequence $\mathbf{f}^{(0)}$ uniformly sampled from f , it holds

$$\lim_{k \rightarrow \infty} S_{a^{n+k}} S_{a^{n+k-1}} \cdots S_{a^n} \mathbf{f}^{(0)} = f, \quad \forall n \geq 0.$$

The following results give conditions on $a^k(z)$ for $\{S_a^k\}_{k \in \mathbb{N}}$ to generate/reproduce $V_{T,\chi}$.

Theorem 2.5. [24] A non-stationary ternary subdivision scheme associated with symbols $\{a^k(z)\}_{k \in \mathbb{N}}$ generates $V_{T,\chi}$, if the following conditions are satisfied

$$D^r a^k(\mu) = 0, \quad r = 0, \dots, \tau_i - 1,$$

for all $\mu \in U_k := \{\varepsilon e^{-\theta_i 3^{-k-1}} : \varepsilon \in \{e^{2\pi i/3}, e^{4\pi i/3}\}, t^2 = -1, i = 1, \dots, N\}$.

Theorem 2.6. [24] Let $z_i^k := e^{-\theta_i/3^{k+1}}$, $i = 1, \dots, N$. A non-stationary ternary subdivision scheme associated with symbols $\{a^k(z)\}_{k \in \mathbb{N}}$ reproduces $V_{T,\chi}$ if it generates $V_{T,\chi}$ and there exists a shift parameter p such that for each $k \in \mathbb{N}$

$$D^r a^k(z_i^k) = 3(z_i^k)^{2p-r} q_r(2p), \quad q_r(p) = \begin{cases} \prod_{j=0}^{r-1} (p-j), & r = 1, \dots, \tau_i - 1, \\ 1, & r = 0. \end{cases}$$

3 Family of non-stationary combined ternary $(2m + 3)$ -point subdivision schemes

In this section, our goal is to construct a family of non-stationary combined ternary $(2m + 3)$ -point subdivision schemes reproducing exponential polynomials, on the basis of the generalized ternary subdivision scheme of order 4 in [22].

For the reader's convenience, refinement rules of the generalized ternary subdivision scheme of order 4 are recalled here:

$$\begin{pmatrix} \bar{P}_{3i-1}^{k+1} \\ \bar{P}_{3i}^{k+1} \\ \bar{P}_{3i+1}^{k+1} \end{pmatrix} = \begin{pmatrix} \frac{4(v^k)^2 + 4v^k + 2}{3(1 + 2v^k)^2} & \frac{8(v^k)^2 + 8v^k}{3(1 + 2v^k)^2} & \frac{1}{3(1 + 2v^k)^2} \\ \frac{2v^k + 2}{3(1 + 2v^k)^2} & \frac{12(v^k)^2 + 8v^k - 1}{3(1 + 2v^k)^2} & \frac{2v^k + 2}{3(1 + 2v^k)^2} \\ \frac{1}{3(1 + 2v^k)^2} & \frac{8(v^k)^2 + 8v^k}{3(1 + 2v^k)^2} & \frac{4(v^k)^2 + 4v^k + 2}{3(1 + 2v^k)^2} \end{pmatrix} \begin{pmatrix} P_{i-1}^k \\ P_i^k \\ P_{i+1}^k \end{pmatrix},$$

where

$$v^k = \frac{1}{2} \left(e^{\frac{t}{3^{k+1}}} + e^{-\frac{t}{3^{k+1}}} \right), \quad t \in \{0, s, ts | s > 0\}, \quad k \in \mathbb{N}. \quad (1)$$

From Proposition 2 of [25], we know v^{k+1} and v^k indeed satisfy the following iteration

$$v^{k+1} = \frac{1}{2} \operatorname{Re}((v^k + \sqrt{(v^k)^2 - 1})^{1/3} + (v^k + \sqrt{(v^k)^2 - 1})^{-1/3}).$$

We construct the non-stationary combined ternary $(2m + 3)$ -point scheme from the generalized ternary subdivision scheme of order 4, by adding variable parameters ξ^k, α_i^k ($i = -m, \dots, m, m \in \mathbb{N}$) which depend on the subdivision level $k \in \mathbb{N}$. The subdivision rules are obtained as

$$\begin{aligned} P_{3i-1}^{k+1} &= \bar{P}_{3i-1}^{k+1} + \sum_{j=-m}^m \alpha_j^k \Delta P_{i-j}^k, \\ P_{3i}^{k+1} &= \bar{P}_{3i}^{k+1} + \xi^k \Delta P_i^k, \\ P_{3i+1}^{k+1} &= \bar{P}_{3i+1}^{k+1} + \sum_{j=-m}^m \alpha_j^k \Delta P_{i-j}^k, \end{aligned} \quad (2)$$

where $\Delta P_{i-j}^k = -(P_{i-j-1}^k - 2P_{i-j}^k + P_{i-j+1}^k)$, and the corresponding k -level symbol can be written as

$$a^k(z) = a_0^k(z) + a_1^k(z) + a_2^k(z), \quad (3)$$

where

$$\begin{aligned} a_0^k(z) &= \left(\frac{2v^k + 2}{3(1 + 2v^k)^2} - \xi^k \right) (z^3 + z^{-3}) + \frac{12(v^k)^2 + 8v^k - 1}{3(1 + 2v^k)^2} + 2\xi^k, \\ a_1^k(z) &= -\alpha_m^k z^{3m+4} + (2\alpha_m^k - \alpha_{m-1}^k) z^{3m+1} + \sum_{j=-m+1}^{m-1} (-\alpha_{j+1}^k + 2\alpha_j^k - \alpha_{j-1}^k) z^{1-3j} + (2\alpha_{-m}^k - \alpha_{-m+1}^k) z^{-3m+1} \\ &\quad - \alpha_{-m}^k z^{-3m-2} + \frac{1}{3(1 + 2v^k)^2} z^4 + \frac{8(v^k)^2 + 8v^k}{3(1 + 2v^k)^2} z + \frac{4(v^k)^2 + 4v^k + 3}{3(1 + 2v^k)^2} z^{-2}, \\ a_2^k(z) &= -\alpha_{-m}^k z^{3m+2} + (2\alpha_{-m}^k - \alpha_{-m+1}^k) z^{3m-1} + \sum_{j=-m+1}^{m-1} (-\alpha_{j-1}^k + 2\alpha_j^k - \alpha_{j+1}^k) z^{-1-3j} + (2\alpha_m^k - \alpha_{m-1}^k) z^{-1-3m} \\ &\quad - \alpha_m^k z^{-3m-4} + \frac{4(v^k)^2 + 4v^k + 3}{3(1 + 2v^k)^2} z^2 + \frac{8(v^k)^2 + 8v^k}{3(1 + 2v^k)^2} z^{-1} + \frac{1}{3(1 + 2v^k)^2} z^{-4}. \end{aligned} \quad (4)$$

Suppose the refinement rules in (2) do not depend on the level k , i.e., $\xi^k \equiv \xi$, $\alpha_i^k \equiv \alpha_i$ and $v^k \equiv 1$ ($t = 0$ in (1)), then the scheme (2) reduces to the stationary one

$$\begin{aligned} P_{3i-1}^{k+1} &= \frac{10}{27} P_{i-1}^k + \frac{16}{27} P_i^k + \frac{1}{27} P_{i+1}^k + \sum_{j=-m}^m \alpha_j P_{i-j}^k, \\ P_{3i}^{k+1} &= \frac{4}{27} P_{i-1}^k + \frac{19}{27} P_i^k + \frac{4}{27} P_{i+1}^k + \xi P_i^k, \\ P_{3i+1}^{k+1} &= \frac{1}{27} P_{i-1}^k + \frac{16}{27} P_i^k + \frac{10}{27} P_{i+1}^k + \sum_{j=-m}^m \alpha_j P_{i-j}^k. \end{aligned} \quad (5)$$

Remark 1. The new scheme (2) is a combined scheme, in the sense that it reduces to an approximating scheme when $\xi^k \neq \frac{2(v^k+1)}{3(1+2v^k)^2}$ and an interpolatory scheme when $\xi^k = \frac{2(v^k+1)}{3(1+2v^k)^2}$. Note that, when $m = 1$, it turns into the non-stationary combined ternary 5-point subdivision scheme in [22]. In particular, if $(\alpha_{-1}^k, \alpha_0^k, \alpha_1^k, \xi^k) = \left(-\omega^k + \frac{2v^k+2}{3(2v^k-1)(1+2v^k)^3}, 2v^k\omega^k(4(v^k)^2-3) + \frac{4v^k(1+v^k)}{3(1+2v^k)^3(2v^k-1)}, -\omega^k, \frac{2v^k+2}{3(1+2v^k)^3}\right)$, where $\omega^k \in \mathbb{R}$ is the free parameter, it actually becomes the non-stationary ternary 5-point interpolatory C^2 subdivision scheme in [26].

4 Properties of the non-stationary combined ternary $(2m+3)$ -point subdivision scheme

In this section, we discuss the properties of the proposed non-stationary combined ternary subdivision scheme (2), including the support, exponential polynomial generation/reproduction, approximation order and smoothness.

4.1 Support

The support of a subdivision scheme represents how far one vertex affects its neighboring points whose size directly influences local support property of the subdivision curve. In this subsection, we study the support of the proposed scheme (2).

Theorem 4.1. Let $\phi_n := \lim_{k \rightarrow \infty} S_{a^{n+k}} S_{a^{n+k-1}} \dots S_{a^n} \delta_0$, $n \geq 0$ be the family of basic limit functions generated by the non-stationary combined ternary subdivision scheme (2), where δ_0 is the delta-sequence, i.e., $\delta_0 = \{\delta_{\alpha,0}\}_{\alpha \in \mathbb{Z}}$. Then the scheme has support width $3m+4$, i.e., the basic limit functions vanish outside the interval $\left[-\frac{3m+4}{2}, \frac{3m+4}{2}\right]$.

Proof. Assume the supports of the k -level masks associated with a non-stationary subdivision scheme are $[l(k), r(k)]$, $k \in \mathbb{N}$. In fact, the supports of the limit basic functions ϕ_n are proved to be included in $[L_n, R_n] = [\sum_{k=n}^{\infty} 3^{n-k-1}l(k), \sum_{k=n}^{\infty} 3^{n-k-1}r(k)]$ (see, e.g., [22,27]).

From the k -level subsymbols of the scheme (2) in (4), we have $l(k) = -(3m+4)$, $r(k) = 3m+4$, for $k \in \mathbb{N}$. Thus, for $n \geq 0$, the left endpoint

$$L_n = \sum_{k=n}^{\infty} 3^{n-k-1}l(k) = -(3m+4) \times 3^{n-1} \sum_{k=n}^{\infty} 3^{-k} = -\frac{3m+4}{2},$$

while the right endpoint

$$R_n = \sum_{k=n}^{\infty} 3^{n-k-1}r(k) = (3m+4) \times 3^{n-1} \sum_{k=n}^{\infty} 3^{-k} = \frac{3m+4}{2}.$$

Hence, the supports of the limit functions are $\left[-\frac{3m+4}{2}, \frac{3m+4}{2}\right]$, which completes the proof. $\square$

For $m = 1$, when

$$(\alpha_{-1}^k, \alpha_0^k, \alpha_1^k, \xi^k) = \left(\frac{-4v^k-3}{27(v^k+1)(2v^k+1)^2}, \frac{-4v^k(v^k+1)}{27(1+2v^k)^2}, \frac{1}{54(v^k+1)(2v^k+1)^2}, \frac{-(8v^k+7)(2v^k-1)}{54(v^k+1)(2v^k+1)}\right),$$

Figure 1: The basis limit functions of the approximating 5-point scheme (6).

the non-stationary combined ternary $(2m + 3)$ -point scheme (2) turns into an approximating 5-point subdivision scheme

$$\begin{aligned}
 P_{3i-1}^{k+1} &= \frac{4v^k + 3}{27(v^k + 1)(2v^k + 1)^2} P_{i-2}^k + \frac{40(v^k)^3 + 80(v^k)^2 + 50v^k + 12}{27(v^k + 1)(1 + 2v^k)^2} P_{i-1}^k + \frac{128(v^k)^3 + 256(v^k)^2 + 136v^k + 5}{54(1 + v^k)(1 + 2v^k)^2} P_i^k \\
 &\quad + \frac{4(v^k)^3 + 8(v^k)^2 + 13(v^k) + 10}{27(v^k + 1)(1 + 2v^k)^2} P_{i+1}^k - \frac{1}{54(v^k + 1)(2v^k + 1)^2} P_{i+2}^k, \\
 P_{3i}^{k+1} &= \frac{32(v^k)^3 + 64(v^k)^2 + 64v^k + 29}{54(1 + v^k)(1 + 2v^k)^2} P_{i-1}^k + \frac{76(v^k)^3 + 152(v^k)^2 + 71v^k - 2}{27(1 + v^k)(1 + 2v^k)^2} P_i^k \\
 &\quad + \frac{32(v^k)^3 + 64(v^k)^2 + 64v^k + 29}{54(1 + v^k)(1 + 2v^k)^2} P_{i+1}^k, \\
 P_{3i+1}^{k+1} &= -\frac{1}{54(v^k + 1)(2v^k + 1)^2} P_{i-2}^k + \frac{4(v^k)^3 + 8(v^k)^2 + 13(v^k) + 10}{27(v^k + 1)(1 + 2v^k)^2} P_{i-1}^k + \frac{128(v^k)^3 + 256(v^k)^2 + 136v^k + 5}{54(1 + v^k)(1 + 2v^k)^2} P_i^k \\
 &\quad + \frac{40(v^k)^3 + 80(v^k)^2 + 50v^k + 12}{27(v^k + 1)(1 + 2v^k)^2} P_{i+1}^k + \frac{4v^k + 3}{27(v^k + 1)(2v^k + 1)^2} P_{i+2}^k.
 \end{aligned} \tag{6}$$

The basic limit functions of the approximating scheme (6) are shown in Figure 1 with different parameters $v^0 = 0.01$ (solid line), 1 (dashed line), 100 (dotted line), respectively.

4.2 Exponential polynomial generation/reproduction property

The goal of this subsection is to study the exponential polynomial generation/reproduction property of the scheme (2). We start by defining the exponential polynomial spaces $EP_\Omega := \text{span}\{x^r e^{\theta x} : 0 \leq r < \tau, (\theta, \tau) \in \Omega\}$ for any finite $\Omega \subset \mathbb{C} \times \mathbb{N}_0$, with the convention that, for $\tau = 0$, no function is added to the generators of EP_Ω .

Now, given

$$\Gamma = (\gamma_0, \gamma_1, \dots, \gamma_\ell) \in \{1, 2\} \times \mathbb{N} \times \mathbb{N}_0^{\ell-1}, \quad \ell \in \mathbb{N},$$

one can define

$$m := \sum_{s=0}^{\ell} \gamma_s - 1, \quad m' := m - 1, \quad m \in \mathbb{N},$$

then, fixed $t \in \mathbb{R}^+ \cup [0, \pi)$,

$$\Gamma\Lambda_t := \{(0, 2\gamma_0 + 1)\} \cup \{(\pm st, \gamma_s)\}_{s=1}^\ell, \quad \Gamma\Lambda'_t := \{(0, 2\gamma_0)\} \cup \{(\pm st, \gamma_s)\}_{s=1}^\ell. \quad (7)$$

The exponential polynomial spaces $EP_{\Gamma\Lambda_t}$ and $EP_{\Gamma\Lambda'_t}$ are actually $2m + 3$ and $2m + 4$ dimensional spaces.

The following theorem shows that the proposed scheme can reproduce high-order exponential polynomials with suitable choices of parameters $\xi^k, \alpha_{-m}^k, \dots, \alpha_m^k$.

Theorem 4.2. *Given the sets $\Gamma\Lambda_t$ and $\Gamma\Lambda'_t$, $t \in \mathbb{R}^+ \cup [0, \pi)$ defined by (7), under suitable choices of the parameters $\xi^k, \alpha_{-m}^k, \dots, \alpha_m^k$, if the scheme (2) is convergent, then it can reproduce $EP_{\Gamma\Lambda_t}$ and generate $EP_{\Gamma\Lambda'_t}$. Moreover, the reproduction of exponential polynomial $e^{\pm tx}$ implies that the scheme (2) reduces to an interpolatory one.*

Proof. In view of Theorem 2.6, the scheme (2) can reproduce $EP_{\Gamma\Lambda_t}$, if for $p = 0, 1, \dots, 2\gamma_0$ and $q = 0, 1, \dots, \gamma_s$, $s = 1, \dots, \ell$, the corresponding k -level symbol satisfies

$$\begin{aligned} D^p a^k(\varepsilon) &= 0, \quad D^p a^k(1) = 3\delta_{p,0}, \quad \varepsilon \in \{e^{2\pi i/3}, e^{4\pi i/3}\}, \\ D^q a^k(\varepsilon e^{\pm st_{k+1}}) &= 0, \quad D^q a^k(e^{\pm st_{k+1}}) = 3\delta_{q,0}, \quad t_{k+1} = t/3^{k+1}. \end{aligned} \quad (8)$$

Note that

$$D^p a^k(\varepsilon) = \sum_{h=0}^2 D^p a_h^k(\varepsilon) = \sum_{h=0}^2 \varepsilon^{h-p} \sum_{i \in \mathbb{Z}} \rho_{p,h}(i) a_{3i+h}^k \varepsilon^{3i} = \sum_{h=0}^2 \varepsilon^{h-p} D^p a_h^k(1),$$

because $\varepsilon^{3i} = 1$ for all $i \in \mathbb{Z}$. Then the conditions $D^p a^k(\varepsilon) = 0$ and $D^p a^k(1) = 3\delta_{p,0}$ can be written as the linear system

$$\begin{cases} \varepsilon^{-p} D^p a_0^k(1) + \varepsilon^{1-p} D^p a_1^k(1) + \varepsilon^{2-p} D^p a_2^k(1) = 0, \\ D^p a_0^k(1) + D^p a_1^k(1) + D^p a_2^k(1) = 3\delta_{p,0}, \end{cases}$$

where ξ^k and $\alpha_i^k (i = -m, \dots, m)$ are the unknowns of the system. This leads to the result that $D^p a_0^k(1) = D^p a_1^k(1) = D^p a_2^k(1) = \delta_{p,0}$. The other two equations in (8) can be dealt with in a similar way. Thus, fixed p and q , (8) is actually equivalent to the following three linear systems:

$$\begin{cases} D^p a_0^k(1) = \delta_{p,0}, \\ D^q a_0^k(e^{\pm st_{k+1}}) = \delta_{q,0}, \end{cases} \quad \begin{cases} D^p a_1^k(1) = \delta_{p,0}, \\ D^q a_1^k(e^{\pm st_{k+1}}) = \delta_{q,0}, \end{cases} \quad \begin{cases} D^p a_2^k(1) = \delta_{p,0}, \\ D^q a_2^k(e^{\pm st_{k+1}}) = \delta_{q,0}. \end{cases} \quad (9)$$

Because $a_1^k(z) = a_2^k(z^{-1})$, we have $D^p a_1^k(e^{it_{k+1}}) = D^p a_2^k(e^{-it_{k+1}})$, which implies the latter two linear systems in (9) are equivalent.

Now, let us find the desired $\xi^k, \alpha_{-m}^k, \dots, \alpha_m^k$ by solving the linear systems in (9). For the first system in (9), we can easily get the result $\xi^k = \frac{2v^k + 2}{3(1 + v^k)^2}$. For the second linear system in (9), we make a substitution as follows:

$$\begin{aligned} t_0^k &= -\alpha_1^k + 2\alpha_0^k - \alpha_{-1}^k, \quad t_m^k = 2\alpha_m^k - \alpha_{m-1}^k, \quad t_{m+1}^k = -\alpha_m^k, \quad t_{2m+1}^k = 2\alpha_{-m}^k - \alpha_{-m+1}^k, \\ t_j^k &= -\alpha_{j+1}^k + 2\alpha_j^k - \alpha_{j-1}^k, \quad t_{m+j+1}^k = -\alpha_{j+1}^k + 2\alpha_j^k - \alpha_{j-1}^k, \quad j = 1, \dots, m-1, \\ t_{2m+2}^k &= -\alpha_{-m}^k, \quad t_{2m+3}^k = \frac{4(v^k)^2 + 4v^k + 2}{(1 + 2v^k)^2}, \quad t_{2m+4}^k = \frac{8(v^k)^2 + 8v^k}{(1 + 2v^k)^2}, \quad t_{2m+5}^k = \frac{1}{3(1 + 2v^k)^2}. \end{aligned} \quad (10)$$

Then, the second linear system (9) can be written as

$$\mathbf{A}^k \mathbf{t}^k = \mathbf{g}, \quad (11)$$

where $\mathbf{A}^k = (A_0^k, A_1^k, \overline{A_1^k}, \dots, A_\ell^k, \overline{A_\ell^k})^T$ is a $(2m+3) \times (2m+6)$ matrix with

$$A_0^k = \begin{pmatrix} D^0(z)(1) & 0 & \dots & 0 \\ D^0(z^4)(1) & D^1(z^4)(1) & \dots & D^{2\gamma_0}(z^4)(1) \\ \vdots & \vdots & \vdots & \vdots \\ D^0(z^{3m+4})(1) & D^1(z^{3m+4})(1) & \dots & D^{2\gamma_0}(z^{3m+4})(1) \\ D^0(z^{-2})(1) & D^1(z^{-2})(1) & \dots & D^{2\gamma_0}(z^{-2})(1) \\ D^0(z^{-5})(1) & D^1(z^{-5})(1) & \dots & D^{2\gamma_0}(z^{-5})(1) \\ \vdots & \vdots & \vdots & \vdots \\ D^0(z^{-3m-2})(1) & D^1(z^{-3m-2})(1) & \dots & D^{2\gamma_0}(z^{-3m-2})(1) \\ D^0(z^{-2})(1) & D^1(z^{-2})(1) & \dots & D^{2\gamma_0}(z^{-2})(1) \\ D^0(z)(1) & 0 & \dots & 0 \\ D^0(z^4)(1) & D^1(z^4)(1) & \dots & D^{2\gamma_0}(z^4)(1) \end{pmatrix},$$

$$A_s^k = \begin{pmatrix} D^0(z)(e^{st_{k+1}}) & 0 & \dots & 0 \\ D^0(z^4)(e^{st_{k+1}}) & D^1(z^4)(e^{st_{k+1}}) & \dots & D^{2\gamma_0}(z^4)(e^{st_{k+1}}) \\ \vdots & \vdots & \vdots & \vdots \\ D^0(z^{3m+4})(e^{st_{k+1}}) & D^1(z^{3m+4})(e^{st_{k+1}}) & \dots & D^{2\gamma_0}(z^{3m+4})(e^{st_{k+1}}) \\ D^0(z^{-2})(e^{st_{k+1}}) & D^1(z^{-2})(e^{st_{k+1}}) & \dots & D^{2\gamma_0}(z^{-2})(e^{st_{k+1}}) \\ D^0(z^{-5})(e^{st_{k+1}}) & D^1(z^{-5})(e^{st_{k+1}}) & \dots & D^{2\gamma_0}(z^{-5})(e^{st_{k+1}}) \\ \vdots & \vdots & \vdots & \vdots \\ D^0(z^{-3m-2})(e^{st_{k+1}}) & D^1(z^{-3m-2})(e^{st_{k+1}}) & \dots & D^{2\gamma_0}(z^{-3m-2})(e^{st_{k+1}}) \\ D^0(z^{-2})(e^{st_{k+1}}) & D^1(z^{-2})(e^{st_{k+1}}) & \dots & D^{2\gamma_0}(z^{-2})(e^{st_{k+1}}) \\ D^0(z)(e^{st_{k+1}}) & 0 & \dots & 0 \\ D^0(z^4)(e^{st_{k+1}}) & D^1(z^4)(e^{st_{k+1}}) & \dots & D^{2\gamma_0}(z^4)(e^{st_{k+1}}) \end{pmatrix}, \quad s = 1, \dots, \ell,$$

$$\overline{A_s^k} = \begin{pmatrix} D^0(z)(e^{-st_{k+1}}) & 0 & \dots & 0 \\ D^0(z^4)(e^{-st_{k+1}}) & D^1(z^4)(e^{-st_{k+1}}) & \dots & D^{2\gamma_0}(z^4)(e^{-st_{k+1}}) \\ \vdots & \vdots & \vdots & \vdots \\ D^0(z^{3m+4})(e^{-st_{k+1}}) & D^1(z^{3m+4})(e^{-st_{k+1}}) & \dots & D^{2\gamma_0}(z^{3m+4})(e^{-st_{k+1}}) \\ D^0(z^{-2})(e^{-st_{k+1}}) & D^1(z^{-2})(e^{-st_{k+1}}) & \dots & D^{2\gamma_0}(z^{-2})(e^{-st_{k+1}}) \\ D^0(z^{-5})(e^{-st_{k+1}}) & D^1(z^{-5})(e^{-st_{k+1}}) & \dots & D^{2\gamma_0}(z^{-5})(e^{-st_{k+1}}) \\ \vdots & \vdots & \vdots & \vdots \\ D^0(z^{-3m-2})(e^{-st_{k+1}}) & D^1(z^{-3m-2})(e^{-st_{k+1}}) & \dots & D^{2\gamma_0}(z^{-3m-2})(e^{-st_{k+1}}) \\ D^0(z^{-2})(e^{-st_{k+1}}) & D^1(z^{-2})(e^{-st_{k+1}}) & \dots & D^{2\gamma_0}(z^{-2})(e^{-st_{k+1}}) \\ D^0(z)(e^{-st_{k+1}}) & 0 & \dots & 0 \\ D^0(z^4)(e^{-st_{k+1}}) & D^1(z^4)(e^{-st_{k+1}}) & \dots & D^{2\gamma_0}(z^4)(e^{-st_{k+1}}) \end{pmatrix},$$

$\mathbf{t}^k = (t_0^k, t_1^k, \dots, t_{2m+5}^k)^T$ and $\mathbf{g}$ is the $(2m+3) \times 1$ vector, $\mathbf{g} = (1, \underbrace{0, \dots, 0}_{2\gamma_0}, 1, \underbrace{0, \dots, 0}_{\gamma_1-1}, 1, \underbrace{0, \dots, 0}_{\gamma_1-1}, \dots, 1, \underbrace{0, \dots, 0}_{\gamma_\ell-1}, 1, \underbrace{0, \dots, 0}_{\gamma_\ell-1})^T$.

Thus, by Gaussian elimination, it can be seen that the rank of $\mathbf{A}^k$ and $(\mathbf{A}^k | \mathbf{g})$ is $2\gamma_0 + 1 + 2\sum_{s=1}^{\ell} \gamma_s = 2m + 3$. Since the unknown $\mathbf{t}^k$ is a $(2m+6) \times 1$ vector which has $2m+3$ unknown elements $t_0^k, t_1^k, \dots, t_{2m+2}^k$, then the linear system (11) is uniquely solvable. Hence, with such chosen parameter, the scheme (2) reproduces $EP_{\Gamma\Lambda_\ell}$.

Similarly, using Theorem 2.5 we have the result for the generation of $EP_{\Gamma\Lambda'_\ell}$. By Theorem 2.5, the scheme (2) can generate $EP_{\Gamma\Lambda'_\ell}$, if for $p_1 = 0, 1, \dots, 2\gamma_0 - 1$ and $q_1 = 0, 1, \dots, \gamma_s, s = 1, \dots, \ell$, the corresponding k -level symbol satisfies

$$D^{p_1}a^k(\varepsilon) = 0, D^{q_1}a^k(\varepsilon e^{\pm st_{k+1}}) = 0. \quad (12)$$

Let

$$\begin{aligned} h_0^k &= -\alpha_1^k + 2\alpha_0^k - \alpha_{-1}^k, & h_m^k &= 2\alpha_m^k - \alpha_{m-1}^k, & h_{m+1}^k &= -\alpha_m^k, & h_{2m+1}^k &= 2\alpha_{-m}^k - \alpha_{-m+1}^k, \\ h_j^k &= -\alpha_{j+1}^k + 2\alpha_j^k - \alpha_{j-1}^k, & h_{m+j+1}^k &= -\alpha_{-j+1}^k + 2\alpha_{-j}^k - \alpha_{-j-1}^k, & j &= 1, \dots, m-1, \\ h_{2m+2}^k &= -\alpha_{-m}^k, & h_{2m+3}^k &= \frac{1}{3(1+2v^k)^2}, & h_{2m+4}^k &= \frac{2v^k+2}{3(1+2v^k)^2} - \xi^k, \\ h_{2m+5}^k &= \frac{4(v^k)^2+4v^k+2}{(1+2v^k)^2}, & h_{2m+6}^k &= \frac{8(v^k)^2+8v^k}{(1+2v^k)^2}, & h_{2m+7}^k &= \frac{12(v^k)^2+8v^k-1}{(1+2v^k)^2} + 2\xi^k, \end{aligned}$$

then it is not difficult to observe that equation (12) is equivalent to

$$\mathbf{B}^k \mathbf{h}^k = \mathbf{G}, \quad (13)$$

where $\mathbf{B}^k = (B_0^k, B_1^k, \overline{B_1^k}, \dots, B_\ell^k, \overline{B_\ell^k})^T$ is a $(2m+4) \times (2m+8)$ matrix with

$$\begin{aligned} B_0^k &= \begin{pmatrix} D^0(z+z^{-1})(\varepsilon) & D^1(z+z^{-1})(\varepsilon) & \dots & D^{2\gamma_0-1}(z+z^{-1})(\varepsilon) \\ D^0(z^2+z^{-2})(\varepsilon) & D^1(z^2+z^{-2})(\varepsilon) & \dots & D^{2\gamma_0-1}(z^2+z^{-2})(\varepsilon) \\ D^0(z^5+z^{-5})(\varepsilon) & D^1(z^5+z^{-5})(\varepsilon) & \dots & D^{2\gamma_0-1}(z^5+z^{-5})(\varepsilon) \\ \vdots & \vdots & \vdots & \vdots \\ D^0(z^{3m+2}+z^{-3m-2})(\varepsilon) & D^1(z^{3m+2}+z^{-3m-2})(\varepsilon) & \dots & D^{2\gamma_0-1}(z^{3m+2}+z^{-3m-2})(\varepsilon) \\ D^0(z^4+z^{-4})(\varepsilon) & D^1(z^4+z^{-4})(\varepsilon) & \dots & D^{2\gamma_0-1}(z^4+z^{-4})(\varepsilon) \\ D^0(z^7+z^{-7})(\varepsilon) & D^1(z^7+z^{-7})(\varepsilon) & \dots & D^{2\gamma_0-1}(z^7+z^{-7})(\varepsilon) \\ \vdots & \vdots & \vdots & \vdots \\ D^0(z^{3m+4}+z^{-3m-4})(\varepsilon) & D^1(z^{3m+4}+z^{-3m-4})(\varepsilon) & \dots & D^{2\gamma_0-1}(z^{3m+4}+z^{-3m-4})(\varepsilon) \\ D^0(z^4+z^{-4})(\varepsilon) & D^1(z^4+z^{-4})(\varepsilon) & \dots & D^{2\gamma_0-1}(z^4+z^{-4})(\varepsilon) \\ D^0(z^3+z^{-3})(\varepsilon) & D^1(z^3+z^{-3})(\varepsilon) & \dots & D^{2\gamma_0-1}(z^3+z^{-3})(\varepsilon) \\ \vdots & \vdots & \vdots & \vdots \\ D^0(z+z^{-1})(\varepsilon) & D^1(z+z^{-1})(\varepsilon) & \dots & D^{2\gamma_0-1}(z+z^{-1})(\varepsilon) \\ D^0(1) & 0 & \dots & 0 \end{pmatrix}, \\ B_s^k &= \begin{pmatrix} D^0(z+z^{-1})(\varepsilon e^{st_{k+1}}) & \dots & D^{2\gamma_0-1}(z+z^{-1})(\varepsilon e^{st_{k+1}}) \\ D^0(z^2+z^{-2})(\varepsilon e^{st_{k+1}}) & \dots & D^{2\gamma_0-1}(z^2+z^{-2})(\varepsilon e^{st_{k+1}}) \\ D^0(z^5+z^{-5})(\varepsilon e^{st_{k+1}}) & \dots & D^{2\gamma_0-1}(z^5+z^{-5})(\varepsilon e^{st_{k+1}}) \\ \vdots & \vdots & \vdots \\ D^0(z^{3m+2}+z^{-3m-2})(\varepsilon e^{st_{k+1}}) & \dots & D^{2\gamma_0-1}(z^{3m+2}+z^{-3m-2})(\varepsilon e^{st_{k+1}}) \\ D^0(z^4+z^{-4})(\varepsilon e^{st_{k+1}}) & \dots & D^{2\gamma_0-1}(z^4+z^{-4})(\varepsilon e^{st_{k+1}}) \\ D^0(z^7+z^{-7})(\varepsilon e^{st_{k+1}}) & \dots & D^{2\gamma_0-1}(z^7+z^{-7})(\varepsilon e^{st_{k+1}}) \\ \vdots & \vdots & \vdots \\ D^0(z^{3m+4}+z^{-3m-4})(\varepsilon e^{st_{k+1}}) & \dots & D^{2\gamma_0-1}(z^{3m+4}+z^{-3m-4})(\varepsilon e^{st_{k+1}}) \\ D^0(z^4+z^{-4})(\varepsilon e^{st_{k+1}}) & \dots & D^{2\gamma_0-1}(z^4+z^{-4})(\varepsilon e^{st_{k+1}}) \\ D^0(z^3+z^{-3})(\varepsilon e^{st_{k+1}}) & \dots & D^{2\gamma_0-1}(z^3+z^{-3})(\varepsilon e^{st_{k+1}}) \\ \vdots & \vdots & \vdots \\ D^0(z+z^{-1})(\varepsilon e^{st_{k+1}}) & \dots & D^{2\gamma_0-1}(z+z^{-1})(\varepsilon e^{st_{k+1}}) \\ D^0(1) & \dots & 0 \end{pmatrix}, \quad s = 1, \dots, \ell, \end{aligned}$$

$$\overline{B}_s^k = \begin{pmatrix} D^0(z^{3m+2} + z^{-3m-2})(\varepsilon e^{-st_{k+1}}) & \dots & D^{2\gamma_0-1}(z^{3m+2} + z^{-3m-2})(\varepsilon e^{-st_{k+1}}) \\ D^0(z^4 + z^{-4})(\varepsilon e^{-st_{k+1}}) & \dots & D^{2\gamma_0-1}(z^4 + z^{-4})(\varepsilon e^{-st_{k+1}}) \\ D^0(z^7 + z^{-7})(\varepsilon e^{-st_{k+1}}) & \dots & D^{2\gamma_0-1}(z^7 + z^{-7})(\varepsilon e^{-st_{k+1}}) \\ \vdots & \vdots & \vdots \\ D^0(z^{3m+4} + z^{-3m-4})(\varepsilon e^{-st_{k+1}}) & \dots & D^{2\gamma_0-1}(z^{3m+4} + z^{-3m-4})(\varepsilon e^{-st_{k+1}}) \\ D^0(z^4 + z^{-4})(\varepsilon e^{-st_{k+1}}) & \dots & D^{2\gamma_0-1}(z^4 + z^{-4})(\varepsilon e^{-st_{k+1}}) \\ D^0(z^3 + z^{-3})(\varepsilon e^{-st_{k+1}}) & \dots & D^{2\gamma_0-1}(z^3 + z^{-3})(\varepsilon e^{-st_{k+1}}) \\ \vdots & \vdots & \vdots \\ D^0(z + z^{-1})(\varepsilon e^{-st_{k+1}}) & \dots & D^{2\gamma_0-1}(z + z^{-1})(\varepsilon e^{-st_{k+1}}) \\ D^0(1) & \dots & 0 \end{pmatrix},$$

$\mathbf{h}^k = (h_0^k, h_1^k, \dots, h_{2m+7}^k)^T$ and $\mathbf{G}$ is the $(2m+4) \times 1$ vector, $\mathbf{G} = (1, \underbrace{0, \dots, 0}_{2\gamma_0-1}, 1, \underbrace{0, \dots, 0}_{\gamma_1-1}, 1, \underbrace{0, \dots, 0}_{\gamma_1-1}, \dots, 1, \underbrace{0, \dots, 0}_{\gamma_\ell-1}, 1, \underbrace{0, \dots, 0}_{\gamma_\ell-1})^T$. By Gaussian elimination, we conclude that the rank of $\mathbf{B}^k$ and $(\mathbf{B}^k|\mathbf{G})$ is $2m+4$. Since the

$(2m+8) \times 1$ vector $\mathbf{h}^k$ has $2m+5$ unknown variables, the linear system (13) can be solved containing a free parameter. Since the scheme (2) is assumed to be convergent, we have $h_0^k + h_1^k + \dots + h_{2m+7}^k = 1$, which implies parameters $\xi^k, \alpha_{-m}^k, \dots, \alpha_m^k$ to be unique. Hence, the scheme (2) can generate $EP_{\Gamma\Lambda_t^1}$.

To conclude, we show that the scheme (2) turns into an interpolatory scheme, if the scheme reproduces $e^{\pm tx}$. In view of Theorem 2.6, it suffices to show that

$$a^k(e^{\pm t_{k+1}}) = 3, \quad a^k(\varepsilon e^{\pm t_{k+1}}) = 0,$$

if the scheme (2) reproduces $e^{\pm tx}$. Then a direct calculation gives the result $\xi^k = \frac{2v^k+2}{3(1+v^k)^2}$, which implies that the scheme (2) reduces to an interpolatory one. $\square$

As an example, we take $m=1$ in (2) and obtain the non-stationary combined ternary 5-point subdivision scheme in [22]

$$\begin{aligned} P_{3i-1}^{k+1} &= \overline{P}_{3i-1}^{k+1} + \alpha_{-1}^k \Delta P_{i-1}^k + \alpha_0^k \Delta P_i^k + \alpha_1^k \Delta P_{i+1}^k, \\ P_{3i}^{k+1} &= \overline{P}_{3i}^{k+1} + \xi^k \Delta P_i^k, \\ P_{3i+1}^{k+1} &= \overline{P}_{3i+1}^{k+1} + \alpha_1^k \Delta P_{i-1}^k + \alpha_0^k \Delta P_i^k + \alpha_{-1}^k \Delta P_{i+1}^k. \end{aligned} \quad (14)$$

The scheme (14) can reproduce $EP_{\Gamma\Lambda_t^1}^1 := \{1, x, x^2, e^{\pm tx}\}$, when

$$\begin{aligned} \alpha_{-1}^k &= \frac{8(v^k)^2 + 12v^k + 1}{9(2v^k - 1)(2v^k + 1)^4}, \quad \alpha_0^k = \frac{2v^k(8(v^k)^3 + 16(v^k)^2 + 18v^k + 9)}{9(2v^k + 1)^4}, \\ \alpha_1^k &= -\frac{4(v^k)^2 + 6v^k + 5}{9(2v^k - 1)(2v^k + 1)^4}, \quad \xi^k = \frac{2(v^k + 1)}{3(2v^k + 1)^2}. \end{aligned} \quad (15)$$

As a consequence, the conic sections such as circles, parabolas and hyperbolas can be exactly reproduced by the scheme (14), but not the curves like the cardioid. To reproduce more exponential polynomials, we take $m=2$ in (2). The scheme (2) becomes the non-stationary combined ternary 7-point scheme

$$\begin{aligned} P_{3i-1}^{k+1} &= \overline{P}_{3i-1}^{k+1} + \alpha_{-2}^k \Delta P_{i-2}^k + \alpha_{-1}^k \Delta P_{i-1}^k + \alpha_0^k \Delta P_i^k + \alpha_1^k \Delta P_{i+1}^k + \alpha_2^k \Delta P_{i+2}^k, \\ P_{3i}^{k+1} &= \overline{P}_{3i}^{k+1} + \xi^k \Delta P_i^k, \\ P_{3i+1}^{k+1} &= \overline{P}_{3i+1}^{k+1} + \alpha_2^k \Delta P_{i-2}^k + \alpha_1^k \Delta P_{i-1}^k + \alpha_0^k \Delta P_i^k + \alpha_{-1}^k \Delta P_{i+1}^k + \alpha_{-2}^k \Delta P_{i+2}^k, \end{aligned} \quad (16)$$

Set

$$\begin{aligned}
 \alpha_{-2}^k &= -\frac{128(v^k)^6 + 128(v^k)^5 - 144(v^k)^4 - 56(v^k)^3 + 64(v^k)^2 - 12v^k - 3}{9(2v^k - 1)^4(1 + 2v^k)^6(8(v^k)^3 - 6v^k + 1)(4(v^k)^2 - 3)}, \\
 \alpha_{-1}^k &= \frac{1024(v^k)^9 + 1536(v^k)^8 - 1152(v^k)^7 - 1664(v^k)^6 + 672(v^k)^5 + 576(v^k)^4 - 160(v^k)^3 - 36(v^k)^2 + 2}{9(2v^k - 1)^3(1 + 2v^k)^6(8(v^k)^3 - 6v^k + 2)}, \\
 \alpha_0^k &= \frac{1024(v^k)^{10} + 1024(v^k)^9 - 1152(v^k)^7 - 768(v^k)^6 + 672(v^k)^5 + 304(v^k)^4 - 200(v^k)^3 - 12(v^k)^2 + 4v^k + 1}{9(1 + 2v^k)^6(2v^k - 1)^4}, \\
 \alpha_1^k &= -\frac{512(v^k)^9 + 768(v^k)^8 - 832(v^k)^6 - 384(v^k)^5 + 432(v^k)^4 + 160(v^k)^3 - 108(v^k)^2 + 4}{9(2v^k - 1)^3(1 + 2v^k)^6(8(v^k)^3 - 6v^k + 1)}, \\
 \alpha_2^k &= \frac{64(v^k)^6 + 64(v^k)^5 - 64(v^k)^3 - 4(v^k)^2 + 30v^k - 6}{9(2v^k - 1)^4(1 + 2v^k)^6(8(v^k)^3 - 6v^k + 1)(4(v^k)^2 - 3)}, \\
 \xi^k &= \frac{2(v^k + 1)}{3(1 + 2v^k)^2},
 \end{aligned} \tag{17}$$

the scheme (16) reproduces the exponential polynomial space $EP_{\Gamma_{\Lambda_t}}^{2,1} := \{1, x, x^2, e^{\pm tx}, e^{\pm 2tx}\}$ which contains the exponential polynomial space $EP_{\Gamma_{\Lambda_t}}^1$. Similarly, with a suitable choice of $(\alpha_{-2}^k, \alpha_{-1}^k, \alpha_0^k, \alpha_1^k, \alpha_2^k, \xi^k)$, the scheme (16) can reproduce the exponential polynomial space $EP_{\Gamma_{\Lambda_t}}^{2,2} := \{1, x, x^2, e^{\pm tx}, e^{\pm 3tx}\}$, when

$$\begin{aligned}
 \alpha_{-2}^k &= -\frac{2(2v^k)^{14} + 6(2v^k)^{13} - 21(2v^k)^{12} - 60(2v^k)^{11} + 99(2v^k)^{10} + 231(2v^k)^9 - 268(2v^k)^8 - 402(2v^k)^7 + 414(2v^k)^6}{9(2v^k - 1)^3(2v^k + 1)^6(8(v^k)^3 - 6v^k + 1)^4(8(v^k)^3 - 6v^k - 1)(4(v^k)^2 - 3)} \\
 &\quad - \frac{258(2v^k)^5 - 297(2v^k)^4 + 12(2v^k)^3 + 32(2v^k)^2 + 3}{9(2v^k - 1)^3(2v^k + 1)^6(8(v^k)^3 - 6v^k + 1)^4(8(v^k)^3 - 6v^k - 1)(4(v^k)^2 - 3)}, \\
 \alpha_{-1}^k &= \frac{2(2v^k)^{23} + 6(2v^k)^{22} - 39(2v^k)^{21} - 104(2v^k)^{20} + 342(2v^k)^{19} + 933(2v^k)^{18} - 1784(2v^k)^{17} - 4275(2v^k)^{16}}{9(2v^k + 1)^6(2v^k - 1)^3(4(v^k)^2 - 3)(8(v^k)^3 - 6v^k + 1)^4(8(v^k)^3 - 6v^k - 1)} \\
 &\quad + \frac{6129(2v^k)^{15} + 11904(2v^k)^{14} - \frac{2871}{2}(2v^k)^{13} - 20367(2v^k)^{12} + 22699(2v^k)^{11} + 20415(2v^k)^{10} - 23025(2v^k)^9}{9(2v^k + 1)^6(2v^k - 1)^3(4(v^k)^2 - 3)(8(v^k)^3 - 6v^k + 1)^4(8(v^k)^3 - 6v^k - 1)} \\
 &\quad + \frac{-10374(2v^k)^8 + 13536(2v^k)^7 + 1641(2v^k)^6 - 3970(2v^k)^5 + 198(2v^k)^4 + 654(2v^k)^3 - 156(2v^k)^2 - 72v^k + 18}{9(2v^k + 1)^6(2v^k - 1)^3(4(v^k)^2 - 3)(8(v^k)^3 - 6v^k + 1)^4(8(v^k)^3 - 6v^k - 1)}, \\
 \alpha_0^k &= \frac{(2v^k)^{19} + 4(2v^k)^{18} - 5(2v^k)^{17} - 44(2v^k)^{16} - 35(2v^k)^{15} + 184(2v^k)^{14} + 381(2v^k)^{13} - 348(2v^k)^{12} - 1443(2v^k)^{11}}{9(8(v^k)^3 - 6v^k + 1)^3(2v^k - 1)^2(2v^k + 1)^6(4(v^k)^2 - 3)} \\
 &\quad + \frac{255(2v^k)^{10} + 2901(2v^k)^9 - 48(2v^k)^8 - 3268(2v^k)^7 + 230(2v^k)^6 + 1886(2v^k)^5 - 385(2v^k)^4 - 403(2v^k)^3}{9(8(v^k)^3 - 6v^k + 1)^3(2v^k - 1)^2(2v^k + 1)^6(4(v^k)^2 - 3)} \\
 &\quad + \frac{122(2v^k)^2 + 42v^k - 15}{9(8(v^k)^3 - 6v^k + 1)^3(2v^k - 1)^2(2v^k + 1)^6(4(v^k)^2 - 3)}, \\
 \alpha_1^k &= -\frac{(2v^k)^{23} + 3(2v^k)^{22} - 15(2v^k)^{21} - 57(2v^k)^{20} + 81(2v^k)^{19} + 480(2v^k)^{18} - 118(2v^k)^{17} - 2367(2v^k)^{16} - 648(2v^k)^{15}}{9(2v^k - 1)^3(2v^k + 1)^6(8(v^k)^3 - 6v^k + 1)^4(8(v^k)^3 - 6v^k - 1)(4(v^k)^2 - 3)} \\
 &\quad - \frac{7584(2v^k)^{14} + 3564(2v^k)^{13} - 16434(2v^k)^{12} - 7465(2v^k)^{11} + 23982(2v^k)^{10} + 7140(2v^k)^9 - 22329(2v^k)^8}{9(2v^k - 1)^3(2v^k + 1)^6(8(v^k)^3 - 6v^k + 1)^4(8(v^k)^3 - 6v^k - 1)(4(v^k)^2 - 3)} \\
 &\quad - \frac{-1584(2v^k)^7 + 11607(2v^k)^6 - 1805(2v^k)^5 - 2466(2v^k)^4 + 750(2v^k)^3 + 150(2v^k)^2 - 144v^k}{9(2v^k - 1)^3(2v^k + 1)^6(8(v^k)^3 - 6v^k + 1)^4(8(v^k)^3 - 6v^k - 1)(4(v^k)^2 - 3)}, \\
 \alpha_2^k &= \frac{(2v^k)^{14} + 3(2v^k)^{13} - 6(2v^k)^{12} - 30(2v^k)^{11} + 129(2v^k)^9 + 73(2v^k)^8 - 309(2v^k)^7 - 171(2v^k)^6 + 426(2v^k)^5}{9(2v^k - 1)^3(2v^k + 1)^6(8(v^k)^3 - 6v^k + 1)^4(8(v^k)^3 - 6v^k - 1)(4(v^k)^2 - 3)} \\
 &\quad + \frac{81(2v^k)^4 - 273(2v^k)^3 + 79(2v^k)^2 + 36v^k - 12}{9(2v^k - 1)^3(2v^k + 1)^6(8(v^k)^3 - 6v^k + 1)^4(8(v^k)^3 - 6v^k - 1)(4(v^k)^2 - 3)}, \\
 \xi^k &= \frac{2(v^k + 1)}{3(1 + 2v^k)^2}.
 \end{aligned} \tag{18}$$

Figure 2 shows the limit curves generated by the subdivision scheme (16) with parameters in (17) and $v^0 = \cos \frac{\pi}{4}, 1, 10$, respectively. Figure 3 illustrates visual comparison of the limit curves generated by the schemes in [22,26] and the scheme (16) from the same control polygons with $v^0 = \cos \left(\frac{\pi}{4} \right)$. From Figure 3, we can see that only the scheme (16) with parameters in (17) can exactly reproduce the cardioid and pascal's

limaçon. Figure 4 displays the reproduction of the exponential polynomial space $EP_{\Gamma\Lambda_t}^{2,2}$ by the subdivision scheme (16) with parameters in (18).

Figure 2: Limit curves generated by the subdivision scheme (16) from the same control polygon with parameters in (17) and $v^0 = \cos \frac{\pi}{4}$ (solid line), 1 (dashed line), 10 (dash dotted line), respectively.

Figure 3: Comparison between the limit curves obtained by the scheme in [26] with $\omega^k = \frac{1}{32(8(v^k)^3 - 2v^k + 2)}$ (left), the scheme in [22] with parameters in (15)(center) and the scheme (16) with parameters in (17)(right) from the same control polygons. The corresponding initial parameter is chosen as $v^0 = \cos\left(\frac{\pi}{4}\right)$.

Figure 4: Reproduction of the epicycloid, epitrochoid and astroid by the subdivision scheme (16) with parameters in (18) and $v^0 = \cos \frac{\pi}{4}$.

4.3 Approximation order

This subsection is devoted to the analysis of approximation order of the proposed scheme (2). For this purpose, first we recall the following definitions.

Definition 4.3. [28] Suppose that $\{S_{\mathbf{a}^k}\}_{k \in \mathbb{N}}$ be a non-stationary ternary scheme and $f^0 = \{f_i^0 : i \in \mathbb{Z}\}$ is sampled from an underlying function f with density 3^{-k_0} for some $k_0 \in \mathbb{N}$. If there exists the largest exponent $d > 0$ such that

$$\|S_{\mathbf{a}^k}^{\circ} f^0 - f\|_{L_{\infty}(\mathbb{R})} \leq C 3^{-k_0 d}$$

with a constant $C > 0$ independent of k_0 , then the exponent d is called the approximation order of the subdivision scheme.

Definition 4.4. [29] A non-stationary subdivision scheme $\{S_{\mathbf{a}^k}\}_{k \in \mathbb{N}}$ is said to be asymptotically similar to a stationary subdivision scheme $S_{\mathbf{a}}$ if the masks $\{\mathbf{a}^k\}_{k \in \mathbb{N}}$ and $\{\mathbf{a}\}$ have the same support $\mathcal{D}$ (i.e., $a_i^k = a_i = 0$ for $i \notin \mathcal{D}$) and satisfy

$$\lim_{k \rightarrow \infty} a_{\alpha}^k = a_{\alpha}, \quad \alpha \in \mathcal{D}.$$

The study of the approximation order for non-stationary binary schemes was carried out in [30], and we extend it here to the case of non-stationary ternary case. Theorem 4.5 can appear trivial after reading the proof of Theorem 21 in [30].

Theorem 4.5. Assume that a non-stationary ternary scheme $\{S_{\mathbf{a}^k}\}_{k \in \mathbb{N}}$ is V_T -reproducing where $V_T := \text{span}\{\varphi_0(x), \dots, \varphi_{T-1}(x)\}$ and is asymptotically similar to a convergent stationary ternary scheme $\{S_{\mathbf{a}}\}$. Assume further that the initial data are of the form $f^d := \{f_i^d = f(3^{-d}i), i \in \mathbb{Z}\}$ for some fixed $d \in \mathbb{N}$ and for some function $f \in W_{\infty}^{\gamma}(\mathbb{R})$ where $\gamma \in \mathbb{N}$, $\gamma \leq T$. If the Wronskian matrix $W_{V_{\gamma}}(0)$ of $V_{\gamma} \subseteq V_T$ is invertible, where the $T \times T$ Wronskian matrix $W_{V_T}(x)$ is defined by

$$W_{V_T}(x) := \left(\frac{1}{\beta!} \frac{d^{\beta} \varphi_{\alpha}(x)}{dx^{\beta}} \right)_{\alpha, \beta=0, \dots, T-1}, \quad (19)$$

then

$$\|g_{f^d} - f\|_{L_{\infty}(\mathbb{R})} \leq C_f 3^{-\gamma d}, \quad d \in \mathbb{N},$$

with a constant $C_f > 0$ depending only on f , where g_{f^d} is the limit of the subdivision scheme obtained from the initial data f^d .

In the following corollary, we give the approximation order of the non-stationary combined ternary subdivision scheme (2) by Theorem 4.5.

Corollary 4.6. If the stationary scheme (5) is convergent, then the non-stationary ternary combined subdivision scheme (2) reproducing the space $EP_{\Gamma_{\Lambda_t}}$ has approximation order $2m + 3$.

Proof. From Definition 4.4, we know that the non-stationary ternary scheme (2) is asymptotically similar to the ternary stationary scheme (5). Set $EP_{\Gamma_{\Lambda_t}} := V_{2m+3} = \{\varphi_0(x), \dots, \varphi_{2m+2}(x)\}$. Since the non-stationary ternary scheme (2) reproduces the exponential space $EP_{\Gamma_{\Lambda_t}}$, from (19) we get the Wronskian matrix of V_{2m+3} as follows:

$$W_{V_{2m+3}}(x) = (C_0(x), C_1(x), \overline{C_1(x)}, \dots, C_t(x), \overline{C_t(x)})^T \quad (20)$$

with

$$C_0(x) = \begin{pmatrix} 1 & x & \cdots & x^{2\gamma_0} \\ 0 & 1 & \cdots & D^1(x^{2\gamma_0}) \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & D^{2m+2}(x^{2\gamma_0}) \end{pmatrix},$$

$$C_s(x) = \begin{pmatrix} e^{stx} & xe^{stx} & \cdots & x^{\gamma_s-1}e^{stx} \\ D^1(e^{stx}) & D^1(xe^{stx}) & \cdots & D^1(x^{\gamma_s-1}e^{stx}) \\ \vdots & \vdots & \vdots & \vdots \\ D^{2m+2}(e^{stx}) & D^{2m+2}(xe^{stx}) & \cdots & D^{2m+2}(x^{\gamma_s-1}e^{stx}) \end{pmatrix}, \quad s = 1, \dots, \ell,$$

$$\overline{C_s}(x) = \begin{pmatrix} e^{-stx} & xe^{-stx} & \cdots & x^{\gamma_s-1}e^{-stx} \\ D^1(e^{-stx}) & D^1(xe^{-stx}) & \cdots & D^1(x^{\gamma_s-1}e^{-stx}) \\ \vdots & \vdots & \vdots & \vdots \\ D^{2m+2}(e^{-stx}) & D^{2m+2}(xe^{-stx}) & \cdots & D^{2m+2}(x^{\gamma_s-1}e^{-stx}) \end{pmatrix}.$$

From (20), we get $W_{T_\gamma}(0)$, $\gamma \leq 2m + 3$ is invertible.

Moreover, the non-stationary ternary combined subdivision scheme (2) is asymptotically similar to the convergent stationary scheme (5), hence the non-stationary ternary combined subdivision scheme (2) has approximation order $2m + 3$ from Theorem 4.5. $\square$

4.4 Smoothness

In this subsection, we discuss the smoothness of the non-stationary ternary scheme (2) and get that the scheme (2) reproducing the exponential space $EP_{\Gamma_{\Lambda_t}}$ can reach C^r convergence where $r \leq 2m + 2$, when the stationary counterpart (5) of the scheme (2) is C^r convergent. Before investigating the smoothness of the scheme (2), we recall the following results.

Theorem 4.7. [21] Let S be a stationary ternary subdivision scheme with the symbol $a^{(0)}(z)$ and its j th order difference scheme S_j ($j = 1, \dots, n + 1$) exist with symbol $a^{(j)}(z) = \left(\frac{3z}{1+z+z^2}\right)^j a^{(0)}(z)$. Suppose the symbol $a^{(l)}(z)$ satisfies

$$\sum_{i \in \mathbb{Z}} a_{3i}^{(l)} = \sum_{i \in \mathbb{Z}} a_{3i+1}^{(l)} = \sum_{i \in \mathbb{Z}} a_{3i+2}^{(l)}, \quad l = 0, \dots, n.$$

If there exists an $L \geq 1$ such that $\left\| \left(\frac{1}{3} S_{n+1} \right)^L \right\|_{\infty} < 1$, then the scheme S is C^n convergent, where

$$\left\| \left(\frac{1}{3} S_{n+1} \right)^L \right\|_{\infty} = \max \left\{ \sum_j |b_{i-3^L j}^L| : 0 \leq i < 3^L \right\}, \quad b^L(z) = \prod_{l=0}^{L-1} b(z^{3^l}), \quad b(z) = \frac{1}{3} a^{(n+1)}(z).$$

Definition 4.8. [31] A non-stationary ternary subdivision scheme $\{S_{\mathbf{a}^k}\}_{k \in \mathbb{N}}$ is said to satisfy approximate sum rules of order N , $N \in \mathbb{N}$, if, given

$$\mu^k := |a^k(1) - 3|, \quad \delta^k := \max_{\eta=0, \dots, N-1} |3^{-k\eta} D^\eta a^k(\varepsilon)|, \quad \varepsilon \in \{e^{2\pi i/3}, e^{4\pi i/3}\},$$

we have

$$\sum_{k \in \mathbb{N}} \mu_k < \infty, \quad \sum_{k \in \mathbb{N}} 3^{k(N-1)} \delta^k < \infty.$$

In the non-stationary case, approximate sum rules of order N and asymptotical similarity to a stationary C^{N-1} subdivision scheme are sufficient conditions for C^{N-1} convergence of non-stationary schemes (see [31]). By Corollary 1 of [31], we have the following result.

Theorem 4.9. *Assume that the non-stationary ternary subdivision scheme $\{S_{\mathbf{a}^k}\}_{k \in \mathbb{N}}$ is said to satisfy approximate sum rules of order N , $N \in \mathbb{N}$ and is asymptotic similar to a C^{N-1} convergent stationary scheme $S_{\mathbf{a}}$. Then, the non-stationary ternary scheme $\{S_{\mathbf{a}^k}\}_{k \in \mathbb{N}}$ is C^{N-1} convergent.*

In the following corollary, we study the approximate sum rules of the scheme (2).

Corollary 4.10. *If the non-stationary ternary subdivision scheme (2) reproduces the exponential space $EP_{\Gamma_{\Lambda_t}}$, then it satisfies approximate sum rules of order $2m + 3$.*

The proof of Corollary 4.10 will be given in Appendix, which is in analogue to the proof of Theorem 10 in [30]. And as a direct consequence of Theorem 4.9 and Corollary 4.10, we have the following result.

Theorem 4.11. *For the non-stationary ternary subdivision scheme (2) reproducing the exponential space $EP_{\Gamma_{\Lambda_t}}$, it is C^r convergent with $r \leq 2m + 2$, if the asymptotical similar counterpart (5) of the scheme (2) is C^r convergent.*

5 Conclusion

In this paper, we construct a family of non-stationary combined ternary $(2m + 3)$ -point subdivision schemes which can reproduce high-order exponential polynomials. The construction of the family is based on the generalized ternary scheme of order 4 introduced in [22] by adding variable parameters. The proposed schemes are able to generate/reproduce various exponential polynomial spaces $EP_{\Gamma_{\Lambda_t}}/EP_{\Gamma_{\Lambda_t}}$ and have approximation order of $2m + 3$. Moreover, they are C^{2m+2} convergent for a suitable choice of the parameters. In the future, we may focus on the analysis of other property of the scheme, such as the shape-preserving property, and the construction of non-stationary combined ternary even-point schemes which have high-order exponential polynomial generation/reproduction.

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Appendix

Proof of Corollary 4.10

Proof. Let $EP_{\Gamma\Lambda_t} := V_{2m+3} = \{\varphi_0(x), \dots, \varphi_{2m+2}(x)\}$ and define the functions

$$\mathcal{F}_{j,\beta}(x) := \sum_{n=0}^{2m+2} \mathbf{m}_{j,\beta,n}^k \varphi_n(x), \quad j = 0, 1, 2, \quad \beta = 0, 1, \dots, 2m+2, \quad (\text{A1})$$

where the coefficient vector $\mathbf{m}_{j,\beta}^k = (\mathbf{m}_{j,\beta,n}^k)_{n=0,1,\dots,2m+2}$ is the unique solution of the Hermite interpolation problem

$$D^\lambda \mathcal{F}_{j,\beta}(j \cdot 3^{-k-1}) = \delta_{\beta,\lambda} (-1)^\lambda \lambda!, \quad \lambda = 0, 1, \dots, 2m+2, \quad j = 0, 1, 2, \quad (\text{A2})$$

where $\delta_{\beta,\lambda}$ is the Kronecker delta symbol. Indeed, by (A1), (A2) is equivalent to

$$W_{V_{2m+3}}(j \cdot 3^{-k-1}) (\mathbf{m}_{j,\beta}^k)^T = \mathbf{c}_\beta^T, \quad j = 0, 1, 2 \quad (\text{A3})$$

with $\mathbf{c}_\beta := \delta_{\beta,\lambda} (-1)^\lambda \lambda! (\lambda = 0, \dots, 2m+2)$. From Corollary 4.6, we get that $W_{V_{2m+3}}(0)$ is invertible. Furthermore, there exists a neighborhood δ of zero such that $W_{V_{2m+3}}(x)$ is invertible for all $x \in \delta$. Hence, the linear system (A3) clearly has a unique solution.

Next, we define

$$Q_{j,\beta} := \sum_{i \in \mathbb{Z}} a_{j-3i}^k ((j \cdot 3^{-1} - i) 3^{-k})^\beta - \mathcal{F}_{j,\beta}(j \cdot 3^{-k-1}), \quad j = 0, 1, 2. \quad (\text{A4})$$

By (A2), we get that

$$\mathcal{F}_{0,\beta}(0) = \mathcal{F}_{1,\beta}(3^{-k-1}) = \mathcal{F}_{2,\beta}(2 \cdot 3^{-k-1}) = \delta_{\beta,0}. \quad (\text{A5})$$

Consequently, $\mathcal{F}_{0,0}(0) = \mathcal{F}_{1,0}(3^{-k-1}) = \mathcal{F}_{2,0}(2 \cdot 3^{-k-1}) = 1$, and

$$\sum_{i \in \mathbb{Z}} a_i^k - 3 = Q_{0,0} + Q_{1,0} + Q_{2,0}. \quad (\text{A6})$$

Moreover, due to (A5), we have

$$\begin{aligned} 3^{-\beta(k+1)} \sum_{i \in \mathbb{Z}} \varepsilon^{i\beta} a_i^k &= 3^{-\beta(k+1)} \left(\sum_{i \in \mathbb{Z}} \varepsilon^{-3i} (-3i)^\beta a_{-3i}^k + \sum_{i \in \mathbb{Z}} \varepsilon^{1-3i} (1-3i)^\beta a_{1-3i}^k + \sum_{i \in \mathbb{Z}} \varepsilon^{2-3i} (2-3i)^\beta a_{2-3i}^k \right) \\ &= \sum_{i \in \mathbb{Z}} (-i \cdot 3^{-k})^\beta a_{-3i}^k + \varepsilon \sum_{i \in \mathbb{Z}} ((3^{-1} - i) 3^{-k})^\beta a_{1-3i}^k + \varepsilon^2 \sum_{i \in \mathbb{Z}} ((2 \cdot 3^{-1} - i) 3^{-k})^\beta a_{2-3i}^k \\ &= Q_{0,\beta} + \varepsilon Q_{1,\beta} + \varepsilon^2 Q_{2,\beta}. \end{aligned} \quad (\text{A7})$$

We only estimate $Q_{0,\beta}$, $Q_{1,\beta}$ and $Q_{2,\beta}$ can be estimated analogously. Since $\mathcal{F}_{0,\beta}$ is a linear combination of exponential polynomials in V_{2m+3} and the non-stationary ternary scheme (2) reproduces V_{2m+3} , we obtain the identity $\mathcal{F}_{0,\beta}(0) = \sum_{i \in \mathbb{Z}} a_{-3i}^k \mathcal{F}_{0,\beta}(i 3^{-k})$. Plugging this into (A4) for $j = 0$ leads to

$$Q_{0,\beta} := \sum_{i \in \mathbb{Z}} a_{-3i}^k [(-i 3^{-k})^\beta - \mathcal{F}_{0,\beta}(i 3^{-k})]. \quad (\text{A8})$$

Now, we exploit the arguments of Taylor expansion for $\mathcal{F}_{0,\beta}$. Precisely, let $\mathcal{T}_{\mathcal{F}_{0,\beta}}^{2m+3}$ be the degree $(2m+2)$ Taylor polynomial of the function $\mathcal{F}_{0,\beta}$ around zero, that is,

$$\mathcal{T}_{\mathcal{F}_{0,\beta}}^{2m+3}(x) := \sum_{\lambda=0}^{2m+2} \frac{x^\lambda}{\lambda!} D^\lambda \mathcal{F}_{0,\beta}(0).$$

Then we replace $\mathcal{F}_{0,\beta}$ in $Q_{0,\beta}$ by its Taylor polynomial $\mathcal{T}_{\mathcal{F}_{0,\beta}}^{2m+3}$ plus the remainder term $\mathcal{R}_{\mathcal{F}_{0,\beta}}^{2m+3}$, such that we have the form $\mathcal{F}_{0,\beta}(i \cdot 3^{-k}) = \mathcal{T}_{\mathcal{F}_{0,\beta}}^{2m+3}(i \cdot 3^{-k}) + \mathcal{R}_{\mathcal{F}_{0,\beta}}^{2m+3}(i \cdot 3^{-k})$. Thus, from the Hermite interpolation conditions in (A2), we have $\mathcal{T}_{\mathcal{F}_{0,\beta}}^{2m+3}(i \cdot 3^{-k}) = (-i \cdot 3^k)^\beta$, which implies that

$$Q_{0,\beta} = - \sum_{i \in \mathbb{Z}} a_{-3i}^k \mathcal{R}_{\mathcal{F}_{0,\beta}}^{2m+3}(i \cdot 3^{-k}). \quad (\text{A9})$$

Moreover, by (A3), $(\mathbf{m}_{0,\beta}^k)^T = W_{V_{2m+3}}(0)^{-1} \mathbf{c}_\beta^T$ such that for a given β , and the coefficient vector $\mathbf{m}_{0,\beta}^k$ for $\mathcal{F}_{0,\beta}$ in (A1) can be bounded independently of $k \in \mathbb{N}$, which implies that the α th derivative of $\mathcal{F}_{0,\beta}$ for each $\alpha = 0, 1, \dots, 2m+2$ is uniformly bound around 0. Accordingly, $|\mathcal{R}_{\mathcal{F}_{0,\beta}}^{2m+3}(i3^{-k})| = O(3^{-k(2m+3)})$ and together with (A8) we have $|Q_{0,\beta}| = O(3^{-k(m+3)})$ as $k \rightarrow \infty$. Similarly, $|Q_{1,\beta}| = O(3^{-k(m+3)})$ and $|Q_{2,\beta}| = O(3^{-k(m+3)})$ as $k \rightarrow \infty$. Therefore, combining (A6) and (A7), we arrive at

$$\left| \sum_{i \in \mathbb{Z}} a_i^k - 3 \right| = O(3^{-k(2m+3)}), \quad \left| \sum_{i \in \mathbb{Z}} \varepsilon^i i^\beta a_i^k \right| = O(3^{-k(2m+3-\beta)}),$$

which leads to

$$\sum_{k=0}^{\infty} |a^k(1) - 3| < \infty, \quad \sum_{k=0}^{\infty} 3^{k(2m+2)} \delta_k < \infty,$$

for $\delta_k := \max_{\beta=0, \dots, 2m+2} 3^{-k\beta} |D^\beta a^k(\varepsilon)|$. Hence, the non-stationary ternary subdivision scheme (2) satisfies approximate sum rules of order $2m+3$. $\square$