Research Article

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Symmetric graphs of valency seven and their basic normal quotient graphs

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Abstract: We characterize seven valent symmetric graphs of order $2pq^n$ with $p < q$ odd primes, extending a few previous results. Moreover, a consequence partially generalizes the result of Conder, Li and Potočník [On the orders of arc-transitive graphs, J. Algebra **421** (2015), 167–186].

Keywords: symmetric graph, simple group, normal quotient graph

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1 Introduction

In this paper, graphs considered are undirected and have no loops and multiple edges. Let Γ be a graph. We denote by $V\Gamma$ and $\text{Aut}\Gamma$ the vertex set and the full automorphism group of Γ , respectively. The size $|V\Gamma|$ is called the *order* of Γ . For a nonnegative positive integer s , an s -arc of Γ is a sequence $v_0, v_1, \dots, v_s$ of $s + 1$ vertices of Γ such that v_{i-1}, v_i are adjacent for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s - 1$. If $G \leq \text{Aut}\Gamma$ is transitive on the set of s -arcs of Γ , then Γ is called (G, s) -arc-transitive; if Γ is (G, s) -arc-transitive but not $(G, s + 1)$ -arc-transitive, then Γ is called (G, s) -transitive. In particular, a 0-arc-transitive graph is called *vertex-transitive*, and a 1-arc-transitive graph is called *arc-transitive* or *symmetric*.

Let Γ be a G -arc-transitive graph. For a vertex intransitive normal subgroup N of G , the *normal quotient graph* of Γ with respect to N , denoted by Γ/N , is defined with all the N -orbits in $V\Gamma$ as vertex set, and two N -orbits B, C are adjacent if and only if some vertex in B is adjacent in Γ to some vertex in C . If Γ and Γ/N have the same valency, the original graph Γ is called a *normal cover* of the normal quotient graph Γ/N . In particular, Γ is called *basic* if $\text{Aut}\Gamma$ has no nontrivial normal subgroup N such that Γ and Γ/N have the same valency. By definition, there is a natural “two-step strategy” for studying symmetric graphs:

Step 1. Determine the normal quotient graphs of the original graphs.

Step 2. Reconstruct the original graphs from the normal quotient graphs by using covering techniques.

In the literature, there are a lot of contributions classifying symmetric graphs of order kp^n , where k is a small positive integer and p is a prime. For example, Chao [1], Cheng and Oxley [2] and Wang and Xu [3] classified symmetric graphs of order p , order $2p$ and order $3p$, respectively. For valency 3, 4 and 5 case, see [4–11] and references therein. Recently, a classification of 7-valent symmetric graphs of order $2pq$ with p, q distinct primes has been given in [12], and a characterization of 7-valent symmetric graphs of order $4p^n$ was obtained in [13].

This work aims to investigate 7-valent symmetric graphs of order $2pq^n$ with $p < q$ odd primes and $n \geq 2$ and in particular determines all basic normal quotient graphs of such graphs, extending the results in [10, 12, 13]. The proof depends on the classification of finite simple groups.

Our main result is as follows. For convenience, graphs appearing in Table 1 are introduced in Section 2.

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Theorem 1.1. Let Γ be a connected symmetric graph of valency 7 and order $2pq^n$, with $p < q$ odd primes and $n \geq 2$. Then Γ is s -transitive with $1 \leq s \leq 3$ and is a normal cover of one of the basic graphs Σ listed in Table 1.

For any given positive integer k , a result of Conder et al. [4] tells us that there are only finitely many connected 2-arc-transitive 7-valent graphs of order kp or kp^2 with p a prime. Theorem 1.1 together with [13, Theorem 1.1] (for case $p = 2$) has the following corollary.

Corollary 1.2.

- (1) For any given positive integer n , there are only finitely many connected 2-arc-transitive 7-valent graphs of order $2pq^n$ with $7 \neq p < q$ primes.
- (2) For any given positive integer n , there is no connected 2-arc-transitive 7-valent graphs of order $2pq^n$ with $7 < p < q$ primes.

2 Preliminaries

We introduce some examples and background results in this section.

2.1 Examples

As usual, for a positive integer n , denote by K_n , $K_{n,n}$ and $K_{n,n} - nK_2$ the complete graph of order n , the complete bipartite graph of order $2n$ and the graph deleted a 1-matching from $K_{n,n}$, respectively. Also, the Hoffman-Singleton graph of order 50 and valency 7 is denoted by $HS(50)$.

For a group G and a subset $S \subseteq G \setminus \{1\}$, with $S = S^{-1} := \{g^{-1} \mid g \in S\}$, the *Cayley graph* of group G with respect to S is with vertex set G and two vertices g and h are adjacent if and only if $hg^{-1} \in S$. This Cayley graph is denoted by $\text{Cay}(G, S)$.

Example 2.1. Let $G = \langle a, b \mid a^m = b^2 = 1, a^b = a^{-1} \rangle \cong D_{2m}$ be a dihedral group, with m a positive integer. Let k be a solution of the congruence equation

$$x^6 + x^5 + \cdots + x + 1 \equiv 0 \pmod{m}.$$

Define a Cayley graph

$$CD(2m, 7) = \text{Cay}(G, \{b, ab, a^{k+1}b, \dots, a^{k^5+k^4+\cdots+1}b\}).$$

Then $CD(2m, 7)$ is a connected arc-transitive graph of valency 7. In particular, if $m \geq 29$, then $CD(2m, 7)$ is arc-regular and $\text{Aut}(CD(2m, 7)) = D_{2m} : \mathbb{Z}_7$, see [14, Theorem 3.1].

The following are several specific examples, see [12, Section 3].

Table 1: Basic normal quotient graphs of symmetric graphs with order $2pq^n$

Σ	(p, q)	$\text{Aut } \Sigma$	s	Σ	(p, q)	$\text{Aut } \Sigma$	s
$K_{7,7}$	$p = 7$	$S_7 \wr \mathbb{Z}_2$	3	C_{30}	$(3, 5)$	S_8	2
$HS(50)$	$(3, 5)$	$PSU(3, 5). \mathbb{Z}_2$	2	C_{78}^1	$(3, 13)$	$PSL(2, 13)$	1
C_{78}^2	$(3, 13)$	$PGL(2, 13)$	1	C_{310}	$(5, 31)$	$\text{Aut}(PSL(5, 2))$	3
$CD(2p, 7)$	$7 \mid p - 1$	$D_{2p} \times \mathbb{Z}_7$	1				

Example 2.2.

- (i) There is unique connected symmetric 7-valent graph of order 30, denoted by C_{30} , and $\text{Aut}(C_{30}) = S_8$;
- (ii) There are exactly two connected symmetric 7-valent graphs of order 78, denoted by C_{78}^1 and C_{78}^2 , and $\text{Aut}(C_{78}^1) = \text{PSL}(2, 13)$ and $\text{Aut}(C_{78}^2) = \text{PGL}(2, 13)$;
- (iii) There is unique connected symmetric 7-valent graph of order 310, denoted by C_{310} , and $\text{Aut}(C_{310}) = \text{Aut}(\text{PSL}(5, 2))$.

Lemma 2.3. Let Γ be a connected 7-valent symmetric graph. Then the following statements hold, where $p < q$ are primes.

- (1) [2, Table 1] If $|\text{VT}| = 2p$, then $\Gamma = K_{7,7}$ or $\text{CD}(2p, 7)$ with $7|p - 1$.
- (2) [12, Section 4] If $|\text{VT}| = 2pq$, then $\Gamma = C_{30}, C_{78}^1, C_{78}^2, C_{310}$ or $\text{CD}(2pq, 7)$ with $p = 7$ or $7|p - 1$ and $7|q - 1$.

2.2 Background results

For a positive integer m and a group T , denote by $\pi(m)$ the number of primes which divide m , and by $\pi(T)$ the set of primes dividing $|T|$. The group T is called a K_n -group if $|\pi(T)| = n$. The simple K_n -groups with $3 \leq n \leq 6$ are classified in [15] and [16].

Theorem 2.4. [16, Theorem A] Let T be a simple K_5 -group. Then one of the following holds:

- (a) $T = \text{PSL}(2, q)$ with $\pi(q^2 - 1) = 4$;
- (b) $T = \text{PSU}(3, q)$ with $\pi((q^2 - 1)(q^3 + 1)) = 4$;
- (c) $T = \text{PSL}(3, q)$ with $\pi((q^2 - 1)(q^3 - 1)) = 4$;
- (d) $T = \text{O}_5(q)$ with $\pi(q^4 - 1) = 4$;
- (e) $T = \text{Sz}(2^{2m+1})$ with $\pi((2^{2m+1} - 1)(2^{4m+2} + 1)) = 4$;
- (f) $T = \text{R}(3^{2m+1})$ with $\pi((3^{4m+2} - 1)) = 3$ and $\pi(3^{4m+2} - 3^{2m+1} + 1) = 1$;
- (g) $T = A_{11}, A_{12}, M_{22}, J_3, \text{HS}, \text{He}, \text{McL}, \text{PSL}(4, 4), \text{PSL}(4, 5), \text{PSL}(4, 7), \text{PSL}(5, 2), \text{PSL}(5, 3), \text{PSL}(6, 2), \text{O}_7(3), \text{O}_9(2), \text{PSp}(6, 3), \text{PSp}(8, 2), \text{PSU}(4, 4), \text{PSU}(4, 5), \text{PSU}(4, 7), \text{PSU}(4, 9), \text{PSU}(5, 3), \text{PSU}(6, 2), \text{O}^+(8, 3), \text{O}^-(8, 2), {}^3\text{D}_4(3), \text{G}_2(4), \text{G}_2(5), \text{G}_2(7)$ or $\text{G}_2(9)$.

The vertex stabilizers of connected 7-valent symmetric graphs are known, refer to [17, Theorem 3.4], where F_n with n a positive integer denotes the Frobenius group of order n .

Lemma 2.5. Let Γ be a connected 7-valent (G, s) -transitive graph, where $G \leq \text{Aut}\Gamma$ and $s \geq 1$. Then $s \leq 3$ and one of the following holds, where $\alpha \in \text{VT}$.

- (a) If G_α is soluble, then $|G_\alpha| \mid 2^2 \cdot 3^2 \cdot 7$. Furthermore, the couple (s, G_α) is listed in the following table.

s	1	2	3
G_α	$\mathbb{Z}_7, F_{14}, F_{21}, F_{14} \times \mathbb{Z}_2, F_{21} \times \mathbb{Z}_3$	$F_{42}, F_{42} \times \mathbb{Z}_2, F_{42} \times \mathbb{Z}_3$	$F_{42} \times \mathbb{Z}_6$

- (b) If G_α is insoluble, then $|G_\alpha| \mid 2^{24} \cdot 3^4 \cdot 5^2 \cdot 7$. Furthermore, the couple (s, G_α) is listed in the following table.

s	2	3
G_α	$\text{PSL}(3, 2), \text{ASL}(3, 2),$ $\text{ASL}(3, 2) \times \mathbb{Z}_2, A_7, S_7$	$\text{PSL}(3, 2) \times S_4, A_7 \times A_6, S_7 \times S_6, (A_7 \times A_6) : \mathbb{Z}_2,$ $\mathbb{Z}_2^6 : (\text{SL}(2, 2) \times \text{SL}(3, 2)), [2^{20}] : (\text{SL}(2, 2) \times \text{SL}(3, 2))$
$ G_\alpha $	$2^3 \cdot 3 \cdot 7, 2^6 \cdot 3 \cdot 7,$ $2^7 \cdot 3 \cdot 7, 2^3 \cdot 3^2 \cdot 5 \cdot 7, 2^4 \cdot 3^2 \cdot 5 \cdot 7$	$2^6 \cdot 3^2 \cdot 7, 2^6 \cdot 3^4 \cdot 5^2 \cdot 7, 2^8 \cdot 3^4 \cdot 5^2 \cdot 7, 2^7 \cdot 3^4 \cdot 5^2 \cdot 7,$ $2^{10} \cdot 3^2 \cdot 7, 2^{24} \cdot 3^2 \cdot 7$

In particular, if $5 \mid |G_\alpha|$, then $|G_\alpha| \mid 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$, and $G_\alpha^{\Gamma(\alpha)} \cong A_7$ or S_7 ; if $5 \nmid |G_\alpha|$, then $|G_\alpha| \mid 2^{24} \cdot 3^2 \cdot 7$.

The following theorem is a special case of [18, Lemma 2.5], which slightly improves a nice result of Praeger [19, Theorem 4.1].

Theorem 2.6. Let Γ be a connected G -arc-transitive graph of odd prime valency, and let $N \triangleleft G$ have more than two orbits on $V\Gamma$, where $G \leq \text{Aut}\Gamma$. Then the following statements hold.

- (1) N is semiregular on $V\Gamma$, $G/N \leq \text{Aut}\Gamma/N$, Γ/N is G/N -arc-transitive, and Γ is a normal N -cover of Γ/N ;
- (2) Γ is (G, s) -arc-transitive if and only if Γ/N is $(G/N, s)$ -arc-transitive, where $1 \leq s \leq 5$ or $s = 7$;
- (3) $G_\alpha \cong (G/N)_\delta$, for all $\alpha \in V\Gamma$ and $\delta \in V(\Gamma/N)$.

A transitive permutation group $X \leq \text{Sym}(\Omega)$ is called *quasiprimitive* if each minimal normal subgroup of X is transitive on Ω , while X is called *biquasiprimitive* if each of its minimal normal subgroups has at most two orbits and there exists one minimal normal subgroup which has exactly two orbits on Ω .

We have a next generalization of [6, Lemma 5.1].

Lemma 2.7. Let Γ be a connected G -arc-transitive r -valent graph of order $2q^n$, where $G \leq \text{Aut}\Gamma$, $n \geq 2$, and $r \geq 5$ and $q \geq 5$ are primes. Then either $\Gamma = K_{7,7}$ or $\text{HS}(50)$, or G has a minimal normal elementary abelian q -subgroup.

Proof. If G is quasiprimitive or biquasiprimitive on $V\Gamma$, by [20, Theorem 1.2], Lemma 2.7 is true. Suppose that G is neither quasiprimitive nor biquasiprimitive on $V\Gamma$. Then G has a minimal normal subgroup N , which has at least three orbits on $V\Gamma$, by Theorem 2.6, N is semiregular on $V\Gamma$ and hence $|N| \mid 2q^n$. It follows that either $N = \mathbb{Z}_2$ or $N = \mathbb{Z}_q^d$ for some $d < n$. For the former case, the normal quotient graph Γ/N is arc-transitive of odd order q^n and odd valency r , a contradiction. Therefore, $N = \mathbb{Z}_q^d$, as required. $\square$

3 Technical lemmas

The two lemmas regarding simple groups in this section are based on the classifications of simple K_n -groups with $3 \leq n \leq 6$, obtained in [15] and [16].

Lemma 3.1. Let $r < s$ be odd primes, and let T be a nonabelian simple group such that $|T| \mid 2^{25} \cdot 3^2 \cdot 7rs^l$ and $7rs^l \mid |T|$ for $l \geq 1$. Then one of the following holds.

- (i) $|\pi(T)| = 4$, and T is isomorphic to one of the groups listed in Table 2.
- (ii) $|\pi(T)| = 5$, and T is isomorphic to one of the groups listed in Table 3. In particular, $l = 1$.

Proof. Clearly, $3 \leq |\pi(T)| \leq 5$. If $|\pi(T)| = 3$, by [15, Theorem I], there are exactly eight specific simple K_3 -groups listed in [15, Table 1], checking the orders, no group T exists in this case.

- (i) Suppose $|\pi(T)| = 4$. By [15, Theorem I], either
 - (a) T is isomorphic to one of the groups listed in [15, Table 2]; or
 - (b) $T = \text{PSL}(2, q)$ for some prime power q .

Table 2: The simple K_4 -groups in Lemma 3.1

T	$ T $	T	$ T $	T	$ T $
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$
$\text{PSL}(2, 13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$	$\text{PSL}(2, 27)$	$2^2 \cdot 3^3 \cdot 7 \cdot 13$	$\text{PSL}(2, 97)$	$2^5 \cdot 3 \cdot 7^2 \cdot 97$
$\text{PSL}(2, 127)$	$2^3 \cdot 3^2 \cdot 7 \cdot 127$	$\text{PSL}(3, 4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	$\text{PSL}(3, 8)$	$2^9 \cdot 3^2 \cdot 7^2 \cdot 73$
$\text{PSU}(3, 5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$				

Assume (a) occurs. Suppose $5 \in \pi(T)$. As $7 \in \pi(T)$, by checking [15, Table 2], T is a $\{2, 3, 5, 7\}$ -group, and so $r, s \in \{3, 5, 7\}$. If $s = 7$, then $r = 3$ or 5 , and one easily checks that no group T exists in the case. If $s = 5$, then $r = 3$, so $|T| \mid 2^{25} \cdot 3^3 \cdot 5^l \cdot 7$ and $7 \cdot 3 \cdot 5^l \mid |T|$, and one may derive that $T = J_2, A_7, A_8, PSL(3, 4)$ and $PSU(3, 5)$. Suppose now $5 \notin \pi(T)$. By [15, Table 2], $r = 3$ or 7 and $s > 7$, hence $|T| \mid 2^{25} \cdot 3^3 \cdot 7^2 \cdot s^l$, by checking the orders, we obtain $T = PSL(3, 8)$.

Now assume (b) occurs. If q is a power of 2, 3 or 7, by [15, Table 3], the only example is $T = PSL(2, 27)$. For the other cases, by [15, Theorem 3.2], $q \geq 11$ is a prime, note that $|PSL(2, q)|$ is always divisible by 3, we conclude that T is a $\{2, 3, 7, q\}$ -group, hence $s = q, l = 1$, and $r \in \{3, 7\}$. Furthermore, since $|T| \mid 2^{25} \cdot 3^2 \cdot 7rs^l$, we have $\frac{q-1}{2} \cdot \frac{q+1}{2} \mid 2^{24} \cdot 3^3 \cdot 7^2$, and as $\left(\frac{q-1}{2}, \frac{q+1}{2}\right) = 1$, it follows that either $q + 1 \mid 2 \cdot 3^3 \cdot 7^2$ if $2 \mid \frac{q-1}{2}$ or $q - 1 \mid 2 \cdot 3^3 \cdot 7^2$ if $2 \mid \frac{q+1}{2}$. Then a computation by [21] shows $q \in \{13, 17, 19, 41, 43, 53, 97, 127, 293, 379, 881, 883\}$. Checking the orders, we obtain $T = PSL(2, 13), PSL(2, 97)$ or $PSL(2, 127)$.

(ii) Suppose $|\pi(T)| = 5$. Then T is a $\{2, 3, 7, r, s\}$ -group and satisfies part (a)–(g) of Theorem 2.4. We analyze these cases one by one in the following. Note that $|T| \mid 2^{25} \cdot 3^2 \cdot 7rs^l$, we obtain

$$2^{26} \nmid |T|, \quad 3^3 \nmid |T|, \quad 7^2 \nmid |T|, \quad r^2 \nmid |T|. \quad (1)$$

Assume part (a) of Theorem 2.4 occurs. Then

$$|T| = |PSL(2, q)| = \frac{1}{(2, q-1)} q(q-1)(q+1). \quad (2)$$

Since $|T| \mid 2^{25} \cdot 3^2 \cdot 7rs^l$ and $7rs^l \mid |T|$, we have $q \neq 3, 3^2, 7$ or r , and then derive from equation (1) that $q = 2^i$ with $1 \leq i \leq 25$ or $q = s^l$. For the former case, since $\pi(q^2 - 1) = 4$, we get $q = 2^6, 2^8, 2^9, 2^{11}$ or 2^{23} , and by checking the orders, we obtain $T = PSL(2, 2^6)$. For the latter case, we have $\frac{q+1}{2} \cdot \frac{q-1}{2} \mid 2^{24} \cdot 3^2 \cdot 7r$, and as $\left(\frac{q+1}{2}, \frac{q-1}{2}\right) = 1$, it follows that either $q - 1 \mid 2^{25} \cdot 3^2 \cdot 7$ if $r \mid \frac{q+1}{2}$, or $q + 1 \mid 2^{25} \cdot 3^2 \cdot 7$ if $r \mid \frac{q-1}{2}$. Recall that $\pi(q^2 - 1) = 4$, $q = s^l$ with $s > 7$ and T satisfies equation (1), a direct computation by [21] shows that $q = 29, 41, 43, 71, 83, 113, 167, 223, 503, 673, 2017, 3583, 64513, 2752513$ or 16515073 , as in Table 3.

Assume part (b) occurs. Then

$$|T| = |PSU(3, q)| = \frac{1}{(3, q+1)} q^3(q-1)(q+1)^2(q^2 - q + 1). \quad (3)$$

Table 3: The simple K_5 -groups in Lemma 3.1

T	M_{22}	$PSL(5, 2)$
$ T $	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	$2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$
T	$PSL(2, 2^6)$	$PSL(2, 29)$
$ T $	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	$2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 29$
T	$PSL(2, 41)$	$PSL(2, 43)$
$ T $	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 41$	$2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$
T	$PSL(2, 71)$	$PSL(2, 83)$
$ T $	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 71$	$2^2 \cdot 3 \cdot 7 \cdot 41 \cdot 83$
T	$PSL(2, 113)$	$PSL(2, 167)$
$ T $	$2^4 \cdot 3 \cdot 7 \cdot 19 \cdot 113$	$2^3 \cdot 3 \cdot 7 \cdot 83 \cdot 167$
T	$PSL(2, 223)$	$PSL(2, 503)$
$ T $	$2^5 \cdot 3 \cdot 7 \cdot 37 \cdot 223$	$2^3 \cdot 3^2 \cdot 7 \cdot 251 \cdot 503$
T	$PSL(2, 673)$	$PSL(2, 2017)$
$ T $	$2^5 \cdot 3 \cdot 7 \cdot 337 \cdot 673$	$2^5 \cdot 3^2 \cdot 7 \cdot 1009 \cdot 2017$
T	$PSL(2, 3583)$	$PSL(2, 64513)$
$ T $	$2^9 \cdot 3^2 \cdot 7 \cdot 199 \cdot 3583$	$2^{10} \cdot 3^2 \cdot 7 \cdot 32257 \cdot 64513$
T	$PSL(2, 2752513)$	$PSL(2, 16515073)$
$ T $	$2^{17} \cdot 3 \cdot 7 \cdot 1376257 \cdot 2752513$	$2^{18} \cdot 3^2 \cdot 7 \cdot 8257537 \cdot 16515073$

By equation (1), q is a 2-power or an s -power. If q is a 2-power, then $q = 2^i$ with $1 \leq i \leq 8$, and as $\pi((q^2 - 1)(q^3 + 1)) = 4$, we get $q = 2^4, 2^5$ or 2^7 ; however, in these three cases, $|T|$ is always not divisible by 7, a contradiction. If q is an s -power, as $7^2 \nmid |T|$ and $r^2 \nmid |T|$, we obtain $(q + 1)^2 |2^{25} \cdot 3^2$, or equivalently $q + 1 | 2^{12} \cdot 3$. Since $\pi((q^2 - 1)(q^3 + 1)) = 4$, computation in [21] shows $q = 11$ and 23; however, in both cases, $|T|$ is not divisible by 7, also a contradiction.

Assume part (c) occurs. Then

$$|T| = |\text{PSL}(3, q)| = \frac{1}{(3, q - 1)} q^3 (q - 1)^2 (q + 1) (q^2 + q + 1). \quad (4)$$

By equation (1), one derives q is a 2-power or an s -power. Then with similar discussion to that in part (b) above, one may draw a contradiction.

Assume part (d) occurs. Then

$$|T| = |\text{O}_5(q)| = \frac{1}{2} q^4 (q^4 - 1) (q^3 - 1) (q^2 - 1). \quad (5)$$

By equation (1), q is a 2-power or an s -power. If q is a 2-power, then $q = 2^i$ with $1 \leq i \leq 6$. Since $\pi(q^4 - 1) = 4$, we have $q = 2^3$ or 2^4 , and $|T| = 2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$ or $2^{16} \cdot 3^2 \cdot 5^2 \cdot 17^2 \cdot 257$, respectively, contradicting equation (1). If q is an s -power, as $7^2 \nmid |T|$ and $r^2 \nmid |T|$, we have $(q^2 - 1)^2 |2^{25} \cdot 3^2$, and so $\frac{q+1}{2} \cdot \frac{q-1}{2} | 2^{11} \cdot 3$. Noting that $\left(\frac{q+1}{2}, \frac{q-1}{2}\right) = 1$, we conclude that either $\frac{q+1}{2} | 3$ or $\frac{q-1}{2} | 3$, implying $s \leq q \leq 7$, a contradiction.

Assume part (e) occurs. Then $|T| = |\text{Sz}(2^{2m+1})| = 2^{4m+2} (2^{4m+2} + 1) (2^{2m+1} - 1)$. Since $2^{26} \nmid |T|$ and $\pi((2^{2m+1} - 1)(2^{4m+2} + 1)) = 4$, we derive $m = 3$; however, $|\text{Sz}(2^7)| = 2^{14} \cdot 5 \cdot 29 \cdot 113 \cdot 127$ is not divisible by 7, a contradiction.

Assume part (f) occurs. Then $|T| = |\text{R}(3^{2m+1})| = 3^{6m+3} (3^{6m+3} + 1) (3^{2m+1} - 1)$, so $3^9 || T|$, contradicting $3^3 \nmid |T|$.

Finally, assume part (g) occurs. Checking the orders of the 30 specific simple groups there, we obtain $T = M_{22}$ and $\text{PSL}(5, 2)$. $\square$

Lemma 3.2. Let $r < s$ be odd primes, and let T be a nonabelian simple group such that $|T| | 2^9 \cdot 3^4 \cdot 5^2 \cdot 7rs^l$ and $35rs^l || T|$ with $l \geq 1$. Then one of the following holds.

- (i) $|\pi(T)| = 4$, and T is isomorphic to one of the groups listed in Table 4.
- (ii) $|\pi(T)| = 5$, $l = 1$ and T is isomorphic to one of the groups listed in Table 5.
- (iii) $|\pi(T)| = 6$, $l = 1$ and $T = J_1, M_{23}$, or $\text{PSL}(2, q)$ with $q = 139, 181, 211, 239, 281, 349, 379, 421, 601, 631, 701, 769, 811, 839, 1009, 1049, 1051, 1399, 1511, 1889, 2099, 2239, 2267, 2269, 2591, 2689, 2801, 3779, 4481, 6481, 6719, 7559, 10079, 12601, 15121, 21601, 26881, 28351, 30241, 37799, 53759, 56701, 69119, 96769, 172801, 201599, 453599, 483839$ or 907199.

Proof. Obviously, $3 \leq |\pi(T)| \leq 6$. If $|\pi(T)| = 3$, by [15, Theorem I], T is isomorphic to one of the eight groups listed in [15, Table 1]. However, the order of each group is not divisible by 35, a contradiction.

- (i) Assume $|\pi(T)| = 4$. By [15, Theorem I], T is isomorphic to one of the groups listed in [15, Table 3] or $T = \text{PSL}(2, q)$ for some prime power q . For the former case, since $35rs^l || T|$ and $|T| | 2^9 \cdot 3^4 \cdot 5^2 \cdot 7rs^l$, one easily derives that $T = J_2, A_{10}, \text{PSU}(3, 5)$ or $\text{PSP}(4, 7)$. For the latter case, note that $3 || \text{PSL}(2, q)|$ and $35 || T|$, T is a $\{2, 3, 5, 7\}$ -group, then by [15, Table 3], the only example is $T = \text{PSL}(2, 49)$.

Table 4: The simple K_4 -groups in Lemma 3.2

T	$ T $	T	$ T $	T	$ T $
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	$\text{PSU}(3, 5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$
$\text{PSp}(4, 7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	$\text{PSL}(2, 49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$		

(ii) Assume $|\pi(T)| = 5$. Then $s > 7$ and T satisfies parts (a)–(g) of Theorem 2.4. Since $|T| \mid 2^9 \cdot 3^4 \cdot 5^2 \cdot 7s^l$, we have

$$2^{10} \nmid |T|, \quad q \nmid 3^6 \nmid |T|, \quad 5^4 \nmid |T|, \quad 7^3 \nmid |T|. \quad (6)$$

Suppose $T = \text{PSL}(2, q)$, as in part (a) of Theorem 2.4. Then T is a $\{2, 3, 5, 7, s\}$ -group as $3 \mid |\text{PSL}(2, q)|$. If q is a 2-power, then $q = 2^6, 2^8$ or 2^9 since $2^{10} \nmid |T|$ and $\pi(q^2 - 1) = 4$, by checking the orders, we obtain $T = \text{PSL}(2, 2^6)$. If q is a 3-power, then $q \in \{3, 3^2, 3^3, 3^4, 3^5\}$ since $3^6 \nmid |T|$, it follows that $\pi(q^2 - 1) \neq 4$, a contradiction. If q is a 5-power, then $q = 5^3$ because $5^4 \nmid |T|$ and $\pi(q^2 - 1) = 4$, which gives rise to an example $T = \text{PSL}(2, 5^3)$. If q is a 7-power, then $q = 7$ or 7^2 since $7^3 \nmid |T|$, contradicting $\pi(q^2 - 1) = 4$. Now, assume that q is an s -power. Then $\frac{q+1}{2} \cdot \frac{q-1}{2} \mid 2^9 \cdot 3^5 \cdot 5^3 \cdot 7^2$. Since $\left(\frac{q+1}{2}, \frac{q-1}{2}\right) = 1$, we have $\frac{q-1}{2} \mid 3^5 \cdot 5^3 \cdot 7^2$ or $\frac{q+1}{2} \mid 3^5 \cdot 5^3 \cdot 7^2$. Recall that $\pi(q^2 - 1) = 4$, computation in [21] shows $q \in \{29, 41, 43, 71, 89, 149, 151, 251, 269, 271, 293, 449, 751, 809, 2251, 2647, 4051, 7937, 12149, 20249, 23813\}$. Checking the orders, we obtain the examples $T = \text{PSL}(2, 29)$, $\text{PSL}(2, 41)$ and $\text{PSL}(2, 449)$.

Suppose $T = \text{PSU}(3, q)$, as in part (b). Since $\pi((q^2 - 1)(q^3 + 1)) = 4$, by equations (3) and (6), we derive that q is an s -power, and $(q + 1)^2 \mid 2^{10} \cdot 3^5 \cdot 5^3 \cdot 7^2$, so $q + 1 \mid 2^5 \cdot 3^2 \cdot 5 \cdot 7$. Since $\pi((q^2 - 1)(q^3 + 1)) = 4$, a computation by [21] shows that $q \in \{11, 13, 17, 19, 23\}$; however, by checking the orders, no group T exists in the case. Similarly, one may exclude part (c), namely $T = \text{PSL}(3, q)$.

Suppose $T = \text{Sz}(2^{2m+1})$ or $R(3^{2m+1})$, as in part (d) or (e). Then $|T| = 2^{4m+2}(2^{4m+2} + 1)(2^{2m+1} - 1)$ or $3^{6m+3}(3^{6m+3} + 1)(3^{2m+1} - 1)$, respectively. Since $|\pi(\text{Sz}(2^3))| = 4$, $T \neq \text{Sz}(2^3)$, and hence $2^{10} \nmid |T|$ or $3^9 \nmid |T|$, contradicting equation (6).

Suppose $T = O_5(q)$, as in part (f). Since $\pi(q^4 - 1) = 4$, by equations (5) and (6), we conclude that q is an s -power, and $(q - 1)^3 \mid 2^{10} \cdot 3^5 \cdot 5^3$, hence $q - 1 \mid 2^3 \cdot 3 \cdot 5$. It follows $q = 11$ and $|T| = |O_5(11)| = 2^8 \cdot 3^2 \cdot 5^2 \cdot 11^4 \cdot 61$ is not divisible by 7, a contradiction.

Finally, suppose T lies in the groups listed in part (g). Checking the orders, we obtain $T = A_{11}, A_{12}, M_{22}$, or HS .

(iii) Assume that $|\pi(T)| = 6$. Then $7 < r < s$ and $s > 11$. By [16, Theorem B], one of the following holds:

- (a) $T = \text{PSL}(2, q)$ where $\pi(q^2 - 1) = 5$;
- (b) $T = \text{PSL}(3, q)$ where $\pi((q^2 - 1)(q^3 - 1)) = 5$;
- (c) $T = \text{PSL}(4, q)$ where $\pi((q^2 - 1)(q^3 - 1)(q^4 - 1)) = 5$;
- (d) $T = \text{PSU}(3, q)$ where $\pi((q^2 - 1)(q^3 + 1)) = 5$;
- (e) $T = \text{PSU}(4, q)$ where $\pi((q^2 - 1)(q^3 + 1)(q^4 - 1)) = 5$;
- (f) $T = O_5(q)$ where $\pi(q^4 - 1) = 5$;
- (g) $T = G_2(q)$ where $\pi(q^6 - 1) = 5$;
- (h) $T = \text{Sz}(2^{2m+1})$ where $\pi((2^{2m+1} - 1)(2^{4m+2} + 1)) = 5$;
- (i) $T = R(3^{2m+1})$ where $\pi((3^{2m+1} - 1)(3^{6m+3} + 1)) = 5$;
- (j) T is one of the 38 groups listed in [16, Theorem B].

Table 5: The simple K_5 -groups in Lemma 3.2

T	$ T $	T	$ T $
A_{11}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	A_{12}	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$
$\text{PSL}(2, 2^6)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	$\text{PSL}(2, 5^3)$	$2^3 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 31$
$\text{PSL}(2, 29)$	$2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 29$	$\text{PSL}(2, 41)$	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 41$
$\text{PSL}(2, 71)$	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 71$	$\text{PSL}(2, 251)$	$2^2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 251$
$\text{PSL}(2, 449)$	$2^6 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 449$		

Recall that $|T| \mid 2^9 \cdot 3^4 \cdot 5^2 \cdot 7rs^l$, then we have

$$2^{10} \nmid |T|, 3^5 \nmid |T|, 5^3 \nmid |T|, 7^2 \nmid |T|, r^2 \nmid |T|. \quad (7)$$

Since $|Sz(2^{2m+1})| = 2^{4m+2}(2^{4m+2} + 1)(2^{2m+1} - 1)$ and $2^{10} \nmid |T|$, we have $m = 1$ and $|\pi(T)| = |\pi(Sz(8))| = 4 \neq 6$, this contradiction excludes case (h). Since $|R(3^{2m+1})| = 3^{6m+3}(3^{6m+3} + 1)(3^{2m+1} - 1)$, $3^9 \nmid |T|$, contradicting equation (7), this excludes case (i). For case (j), by checking the orders, we have $T = J_1$ or M_{23} .

Suppose case (a) occurs. Since $\pi(q^2 - 1) = 5$, $q \neq 3, 3^2, 3^3, 3^4, 5, 5^2, 7$, and by equation (7), we have that either $q = 2^i (1 \leq i \leq 9)$ or s^l . The former case does not give examples by checking the orders. For the latter case, by equation (2), we have $\frac{q-1}{2} \cdot \frac{q+1}{2} \mid 2^8 \cdot 3^4 \cdot 5^2 \cdot 7r$, and as $\left(\frac{q-1}{2}, \frac{q+1}{2}\right) = 1$, it follows that either $\frac{q+1}{2} \mid 2^9 \cdot 3^4 \cdot 5^2 \cdot 7$ or $\frac{q-1}{2} \mid 2^9 \cdot 3^4 \cdot 5^2 \cdot 7$. Recall that $\pi(q^2 - 1) = 5$ and $s > 11$, computation in [21] shows that q lies in part (iii) of Lemma 3.2.

Suppose case (b) occurs. Since $\pi((q^2 - 1)(q^3 - 1)) = 5$, $q \neq 2, 2^2, 2^3$ or 3 , and by equations (4) and (7), we derive that q is an s -power, and $(q - 1)^2 \mid 2^9 \cdot 3^4 \cdot 5^2$, implying $q - 1 \mid 2^4 \cdot 3^2 \cdot 5$. Since $\pi((q^2 - 1)(q^3 - 1)) = 5$, a computation by [21] shows that $q = 37, 41$ or 241 , which does not give rise to examples by checking the orders. Similarly, one may exclude case (d).

Suppose case (e) occurs. Then

$$|T| = |PSU(4, q)| = \frac{1}{(4, q+1)} q^6 (q^2 - 1)^2 (q^2 + 1)(q^3 + 1).$$

By equation (7), one may conclude that q is an s -power, and $(q^2 - 1)^2 \mid 2^9 \cdot 3^4 \cdot 5^2$, hence $\frac{q-1}{2} \cdot \frac{q+1}{2} \mid 2^2 \cdot 3^2 \cdot 5$. As $\left(\frac{q-1}{2}, \frac{q+1}{2}\right) = 1$, we obtain $\frac{q-1}{2} \mid 2^2 \cdot 5$ or $\frac{q+1}{2} \mid 2^2 \cdot 5$. It follows that $q = 19$ since $\pi((q^2 - 1)(q^3 + 1)(q^4 - 1)) = 5$, and $|T| = |PSU(4, 19)| = 2^7 \cdot 3^4 \cdot 5^3 \cdot 7^3 \cdot 19^6 \cdot 181$, contradiction equation (7). For case (c), then

$$|T| = |PSL(4, q)| = \frac{1}{(4, q-1)} q^6 (q^2 - 1)(q^3 - 1)(q^4 - 1),$$

a similar argument to that in case (e) may draw a contradiction.

Suppose case (f) occurs. Since $\pi(q^4 - 1) = 5$, $q \neq 2, 2^2$ and 3 , by equations (5) and (7), we conclude that q is an s -power, and $(q^2 + 1)(q^3 - 1)(q^2 - 1)^2 \mid 2^{10} \cdot 3^4 \cdot 5^2 \cdot 7r$, hence $(q^2 - 1)^2 \mid 2^{10} \cdot 3^4 \cdot 5^2$, or equivalently $q^2 - 1 \mid 2^5 \cdot 3^2 \cdot 5$. As discussed in the above paragraph, one may derive that $q = 17, 19, 29, 31$ or 89 , which does not give rise to example by checking the orders. Finally, for case (g), $|T| = |G_2(q)| = q^6(q^6 - 1)(q^2 - 1)$, a similar discussion may draw a contradiction. $\square$

4 Vertex quasiprimitive and vertex biquasiprimitive cases

Let Γ be a connected G -arc-transitive 7-valent graph of order $2pq^n$, where $G \leq \text{Aut} \Gamma$, $p < q$ are odd primes and $n \geq 2$. Let N be a minimal normal subgroup of G . Then $N = T^d$, with T a simple group and $d \geq 1$. Let $\alpha \in V\Gamma$.

Lemma 4.1. *If N is nonabelian, then $d = 1$.*

Proof. Suppose for a contradiction that N is nonabelian and $d \geq 2$. Then $|N| \nmid 2pq^n$, $N_\alpha \neq 1$ and N has at most two orbits on $V\Gamma$ by Theorem 2.6. Set $N = T_1 \times T_2 \times \cdots \times T_d$ with each $T_i \cong T$.

Assume first N is transitive on $V\Gamma$. Since $1 \neq N_\alpha \triangleleft G_\alpha$ and Γ is connected, we have $1 \neq N_\alpha^{\Gamma(\alpha)} \triangleleft G_\alpha^{\Gamma(\alpha)}$. It follows that $N_\alpha^{\Gamma(\alpha)}$ is transitive, and Γ is N -arc-transitive. If T_1 is transitive on $V\Gamma$, then the centralizer $C_N(T_1)$ is semiregular on $V\Gamma$ (see [22, Theorem 4.2A]), so is T_2 , which is a contradiction as $|T_2|$ does not divide $|V\Gamma| = 2pq^n$; if T_1 has at least three orbits on $V\Gamma$, by Theorem 2.6, T_1 is semiregular, again a contradiction. Therefore, T_1 has exactly two orbits, say U and W , on $V\Gamma$. Since $T_1 \triangleleft N$, U and W form an N -block system on $V\Gamma$. It follows that the set stabilizer N_U is of index 2 in N , which is a contradiction because $N = T^d$ has no subgroup with index 2.

Assume now N has exactly two orbits, say Δ_1 and Δ_2 , on $V\Gamma$. Then Γ is a bipartite graph with bipartitions Δ_1 and Δ_2 . Let $G^+ = G_{\Delta_1} = G_{\Delta_2}$, the stabilizer on the bipartitions. If G^+ acts unfaithfully on Δ_1 , by [23, Lemma 5.2], Γ is a complete bipartite graph, so $\Gamma = K_{7,7}$ as $\text{val}(\Gamma) = 7$ and hence $|V\Gamma| = 14$, a contradiction. Suppose G^+ acts faithfully on Δ_1 . Then $N \leq G^+$ can be viewed as a transitive permutation group on Δ_1 . If T_1 is transitive on Δ_1 , then [22, Theorem 4.2A] implies T_2 is semiregular on Δ_1 , hence $|T_2| \mid pq^n$, a contradiction. Thus, T_1 has at least two orbits on Δ_1 . It then follows from [7, Lemma 3.2] that T_1 is semiregular on Δ_1 , also a contradiction. $\square$

The next two lemmas exclude the vertex quasiprimitive and vertex biquasiprimitive cases.

Lemma 4.2. *If G is quasiprimitive on $V\Gamma$, then no graph Γ exists.*

Proof. Since G is quasiprimitive on $V\Gamma$, N is transitive on $V\Gamma$. If N is abelian, then N is regular on $V\Gamma$ and so $|T|^d = |N| = 2pq^n$, a contradiction. Thus, N is nonabelian, and then by Lemma 4.1, we have $d = 1$ and $N = T$. Furthermore, since $T_\alpha \neq 1$, we conclude that Γ is T -arc-transitive, and hence T_α satisfies Lemma 2.5. We divide our proof into two cases depending on whether 5 divides $|T_\alpha|$ or not.

Case 1. Assume $5 \nmid |T_\alpha|$.

By Lemma 2.5, $|T_\alpha| \mid 2^{24} \cdot 3^2 \cdot 7$, and by the transitivity of T , we have $|T| = |V\Gamma||T_\alpha|$ divides $2^{25} \cdot 3^2 \cdot 7pq^n$; on the other hand, since Γ is T -arc-transitive, we have $7 \mid |T_\alpha|$, and so $7pq^n \mid |T|$. Therefore, T satisfies Lemma 3.1; in particular, $|\pi(T)| = 4$ or 5. If $|\pi(T)| = 5$, by Lemma 3.1(ii), $n = 1$, a contradiction.

Suppose $|\pi(T)| = 4$. Noting that $n \geq 2$, by Lemma 3.1(i), we easily conclude that the triple $(T, p, q^n) = (J_2, 3, 5^2)$ or $(PSU(3, 5), 3, 5^3)$. For the former case, $|V\Gamma| = 2 \cdot 3 \cdot 5^2$, so $|T_\alpha| = \frac{|T|}{|V\Gamma|} = 2^6 \cdot 3^2 \cdot 7$; however, by [24], J_2 has no subgroup with order $2^6 \cdot 3^2 \cdot 7$, a contradiction. For the latter case, $|V\Gamma| = 2 \cdot 3 \cdot 5^3$, hence $|T_\alpha| = \frac{|T|}{|V\Gamma|} = 2^3 \cdot 3 \cdot 7$, then by Lemma 2.5, we obtain $T_\alpha = \text{PSL}(3, 2)$; however, computation in [21] shows that no graph Γ exists in this case.

Case 2. Assume $5 \mid |T_\alpha|$.

By Lemma 2.5, $|T_\alpha| \mid 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$, and $T_\alpha^{\Gamma(\alpha)} \cong A_7$ or S_7 . It follows that $|T| = |V\Gamma||T_\alpha|$ divides $2^9 \cdot 3^4 \cdot 5^2 \cdot 7pq^n$; moreover, as $35 \mid |T_\alpha|$, we have $35pq^n$ divides $|T|$. Therefore, T satisfies Lemma 3.2; in particular $4 \leq \pi(T) \leq 6$. If $|\pi(T)| = 5$ or 6, by Lemma 3.2, we have $n = 1$, a contradiction.

Suppose $|\pi(T)| = 4$. Since $n \geq 2$, by Lemma 3.2(i), the only possibility is $T = \text{PSp}(4, 7)$ and $(p, q) = (3, 7^3)$ or $(5, 7^3)$. Consequently, $|V\Gamma| = 2 \cdot 3 \cdot 7^3$ or $2 \cdot 5 \cdot 7^3$, and $|T_\alpha| = \frac{|T|}{|V\Gamma|} = 2^7 \cdot 3 \cdot 5^2 \cdot 7$ or $2^7 \cdot 3^2 \cdot 5 \cdot 7$, respectively. By Lemma 2.5, it is a contradiction. $\square$

Lemma 4.3. *If G is biquasiprimitive on $V\Gamma$, then no graph Γ exists.*

Proof. Since G is biquasiprimitive on $V\Gamma$, G has a minimal normal subgroup $N = T^d$ which has exactly two orbits (say Δ_1 and Δ_2) on $V\Gamma$. Then Γ is a bipartite graph with bipartition Δ_1 and Δ_2 . Let $G^+ = G_{\Delta_1} = G_{\Delta_2}$. Then $N \leq G^+$, $|G : G^+| = 2$ and $G_\alpha = G_\alpha^+$. If N is abelian, then N is regular on Δ_1 and so $|T|^d = |N| = pq^n$, a contradiction. Hence, N is nonabelian, and by Lemma 4.1, we further conclude that $N = T$ is a nonabelian simple group.

If G^+ acts unfaithfully on Δ_1 or Δ_2 , by [23, Lemma 5.2], Γ is a complete bipartite graph, so $\Gamma = K_{7,7}$ and $|V\Gamma| = 14$, a contradiction.

Assume now G^+ acts faithfully on Δ_1 and Δ_2 . Then by [25, Theorem 1.5], either

- (1) G^+ is quasiprimitive on Δ_i ; or
- (2) G^+ has two normal subgroups M_1 and M_2 such that $M_1 \cong M_2$ are semiregular on $V\Gamma$. Furthermore, the group $M_1 \times M_2$ is regular on Δ_i .

For case (2), we have $|M_1|^2 = |\Delta_i| = pq^n$, a contradiction.

Suppose case (1) occurs. Since G^+ is quasiprimitive on Δ_i and has a simple minimal normal subgroup T , by O’Nan-Scott-Praeger theorem ([19]), $\text{soc}(G^+) = T$ or T^2 . For the latter case, G^+ is of holomorph type and T is regular on Δ_i , so $|T| = pq^n$, a contradiction. Therefore, $\text{soc}(G^+) = T$. Furthermore, if T is not the unique minimal normal subgroup of G , since $G = G^+ \cdot \mathbb{Z}_2$, one easily derives $G = G^+ \times \mathbb{Z}_2$, hence the normal subgroup $\mathbb{Z}_2$ has pq^n orbits on $V\Gamma$, contradicting the biquasiprimitivity of G . Thus, G is almost simple with socle T , and we may set $G = T \cdot o$, and $G^+ = T \cdot o'$ with $\mathbb{Z}_2 \leq o \leq \text{Out}(T)$ and $|o : o'| = 2$.

Case 1. Assume $5 \nmid |T_\alpha|$.

Since $T_\alpha \leq G_\alpha$, by Lemma 2.5, $|T_\alpha| \mid 2^{24} \cdot 3^2 \cdot 7$, and hence $|T| = |\Delta_i| |T_\alpha|$ divides $2^{24} \cdot 3^2 \cdot 7pq^n$; on the other hand, noting that $T_\alpha \neq 1$, we obtain $7 \mid |T_\alpha|$, and so $7pq^n \mid |T|$. Therefore, T satisfies Lemma 3.1, and $\pi(T) = 4$ or 5 .

If $\pi(T) = 5$, by Lemma 3.1(ii), we have $n = 1$, a contradiction.

Suppose now $|\pi(T)| = 4$. Then by Lemma 3.1(i) and note that $n \geq 2$, we have $(T, p, q^n) = (J_2, 3, 5^2)$ or $(PSU(3, 5), 3, 5^3)$. For the former case, as $\text{Out}(J_2) \cong \mathbb{Z}_2$, we have $o = \mathbb{Z}_2$, $o' = 1$ and $G^+ = T$. It follows $|G_\alpha| = |G_\alpha^+| = |T_\alpha| = \frac{|T|}{|\Delta_i|} = 2^7 \cdot 3^2 \cdot 7$, which is a contradiction by Lemma 2.5. For the latter case, $|T_\alpha| = \frac{|T|}{|\Delta_i|} = 2^4 \cdot 3 \cdot 5 \cdot 7$. Since $\text{Out}(PSU(3, 5)) \cong S_3$, we have that either $o = S_3$ and $o' = \mathbb{Z}_3$, or $o = \mathbb{Z}_2$ and $o' = 1$. Thus, $|G_\alpha| = |G_\alpha^+| = |T_\alpha| \cdot |o'| = 2^4 \cdot 3 \cdot 5 \cdot 7$ or $2^4 \cdot 3^2 \cdot 5 \cdot 7$. By Lemma 2.5, the only possibility is $|G_\alpha| = 2^4 \cdot 3^2 \cdot 5 \cdot 7$, and $G_\alpha \cong S_7$; however, computation in [21] shows that no graph Γ exists in the case.

Case 2. Assume $5 \mid |T_\alpha|$.

As $T_\alpha \triangleleft G_\alpha$, by Lemma 2.5, $|T_\alpha| \mid 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$, and $T_\alpha^{\Gamma(\alpha)} \cong A_7$ or S_7 . It follows that $|T| = |V\Gamma| |T_\alpha|$ divides $2^8 \cdot 3^4 \cdot 5^2 \cdot 7pq^n$; moreover, as $35 \mid |T_\alpha|$, we have $35pq^n$ divides $|T|$. Hence, T satisfies Lemma 3.2.

If $|\pi(T)| = 5$ or 6 , by Lemma 3.2, $n = 1$, a contradiction.

Suppose $|\pi(T)| = 4$. By Lemma 3.2(i), we have $T = PSp(4, 7)$, and $(p, q^n) = (3, 7^3)$ or $(5, 7^3)$, so $|T_\alpha| = \frac{|T|}{|\Delta_i|} = 2^8 \cdot 3 \cdot 5^2 \cdot 7$ or $2^8 \cdot 3^2 \cdot 5^2 \cdot 7$, respectively. Furthermore, as $\text{Out}(PSp(4, 7)) \cong \mathbb{Z}_2$, we have $o = \mathbb{Z}_2$, $o' = 1$ and $G^+ = T$. Thus, $|G_\alpha| = |G_\alpha^+| = |T_\alpha| = 2^8 \cdot 3 \cdot 5^2 \cdot 7$ or $2^8 \cdot 3^2 \cdot 5^2 \cdot 7$, by Lemma 2.5, which is a contradiction. $\square$

5 Proofs of Theorem 1.1

We will complete the proof of Theorem 1.1 in this final section.

Lemma 5.1. *Let $p < q$ be odd primes, and let Γ be a connected G -arc-transitive 7-valent graph of order $2pq^m$, where $G \leq \text{Aut}\Gamma$ and $m \geq 2$. Then either Γ is a $\mathbb{Z}_3$ -cover of $HS(50)$ or G has a normal elementary abelian q -subgroup.*

Proof. By Lemmas 4.2 and 4.3, G is neither quasiprimitive nor biquasiprimitive on $V\Gamma$. Hence, G has a minimal normal subgroup, say N , which has at least three orbits on $V\Gamma$. By Theorem 2.6, N is semiregular, and hence $|N|$ divides $|V\Gamma| = 2pq^m$. It follows that N is soluble and $N \cong \mathbb{Z}_2, \mathbb{Z}_p$ or $\mathbb{Z}_q^s$ with $s \leq n$. For the last case, we are done. For the first case, by Theorem 2.6, Γ/N is connected arc-transitive of odd order pq^n and odd valency 7, a contradiction. For the second case, again by Theorem 2.6, Γ/N is G/N -arc-transitive of order $2q^m$ and valency 7. Clearly, $\Gamma/N \neq K_{7,7}$. It then follows from Lemma 2.7 that either $\Gamma/N = HS(50)$, or G/N has a minimal normal subgroup $M/N \cong \mathbb{Z}_q^k$ for some positive integer k . For the former case, $(p, q) = (3, 5)$, and Γ is a $\mathbb{Z}_3$ -cover of $HS(50)$. For the latter case, $M = \mathbb{Z}_p \cdot \mathbb{Z}_q^k$, and as $p < q$, the Sylow q -subgroup $\mathbb{Z}_q^k$ of M is characteristic in M , and hence normal in G . $\square$

Now, we are ready to prove Theorems 1.1 and Corollary 1.2.

Proof of Theorems 1.1. Suppose Γ is G -arc-transitive with $G \leq \text{Aut}\Gamma$. Let M be a maximal normal q -subgroup of G . Clearly, M has at least $2p \geq 6$ orbits on $V\Gamma$. Then by Theorem 1.1, Γ is a normal cover of Γ/M , and Γ/M is a connected G/M -arc-transitive graph of order $2pq^m$ with $0 \leq m \leq n-1$.

If $m = 0$, then Γ/M is of order $2p$, by Lemma 2.3(1), $\Gamma/M = K_{7,7}$ or $CD(2p, 7)$ with $7|p-1$.

If $m = 1$, then Γ/M is of order $2pq$, by Lemma 2.3(2), $\Gamma/M = K_8, C_{30}, C_{78}^1, C_{78}^2, C_{310}$, or $CD(2pq, 7)$ with $7|q-1$. For the last case, as $2pq > 31$, by Example 2.1, $\Gamma/M = CD(2pq, 7)$ is arc-regular, so $G/M = \text{Aut}(\Gamma/M) \cong D_{2pq} : \mathbb{Z}_7$. Now, G has a normal subgroup $H = M \cdot \mathbb{Z}_q$, which has $2p$ orbits on $V\Gamma$, and Γ is a normal cover of $\Gamma/H = CD(2p, 7)$ by Theorem 1.1.

Now assume $m \geq 2$. By Lemma 5.1, either Γ/M is a $\mathbb{Z}_3$ -cover of $HS(50)$, or G has a normal elementary abelian q -subgroup, say X/M . For the former case, Γ is a normal $M \cdot \mathbb{Z}_3$ -cover of $HS(50)$. For the latter case, X is a normal q -subgroup of G , and by the maximality of M , X has at most two orbits on $V\Gamma$. It follows that $pq^n || X|$, a contradiction.

Finally, one easily verifies that the graphs in Table 1 are basic graphs. This completes the proof of Theorems 1.1. $\square$

Proof of Corollary 1.2. Let Γ be a connected 2-arc-transitive 7-valent graphs of order $2pq^n$ with $q > p$ primes.

If $n = 1$, note that $CD(2pq, 7)$ is not 2-arc-transitive, Corollary 1.2 is true by Lemma 2.3(2).

Assume $n \geq 2$. If $p \neq 7$, by Theorems 1.1 and 2.6, Γ is a normal cover of C_{30} , $HS(50)$ or C_{310} , hence $(p, q) = (3, 5)$ or $(5, 31)$. Now Corollary 1.2 easily follows. $\square$

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