Research Article

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The mixed metric dimension of flower snarks and wheels

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Abstract: New graph invariant, which is called a mixed metric dimension, has been recently introduced. In this paper, exact results of the mixed metric dimension on two special classes of graphs are found: flower snarks $J_n$ and wheels $W_n$. It is proved that the mixed metric dimension for $J_n$ is equal to 5, while for higher dimensions it is constant and equal to 4. For $W_n$, the mixed metric dimension is not constant, but it is equal to $n$ when $n \geq 4$, while it is equal to 4, for $n = 3$.

Keywords: wheel graphs, mixed metric dimension, flower snarks, graph theory, discrete mathematics

MSC 2020: 05C12

1 Introduction

Let $G = (V, E)$ be a connected graph, where $V$ represent a set whose elements are called vertices and $E$ represent a set whose elements are called edges. The mixed metric dimension of graphs was introduced by Kelenc et al. (2017) in [1]. This dimension of graph $G$ is the mixed version of the metric dimension and edge metric dimension.

In the connected graph $G$, the distance between two vertices $u$ and $v$ is the length of a shortest $u - v$ path in $G$. The vertex $w$ resolves a pair $u, v \in V$ if $d(u, w) \neq d(v, w)$. The metric coordinates $r(v, S)$ of vertex $v$ with respect to an ordered set of vertices $S = \{w_1, w_2, \ldots, w_k\}$ are defined as $r(v, S) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$. The set $S$ is a resolving set if every two vertices $u$ and $v$ from $G$ are resolved by at least one vertex from the set $S$. The metric basis of graph $G$ is the resolving set of the minimum cardinality. The metric dimension of graph $G$ is the cardinality of the metric basis for graph $G$ and is denoted by $\beta(G)$. Slater in [2] and Harary and Melter in [3] independently of one another introduced resolving sets for graphs.

Also, several works are published about applications and some theoretical properties of this invariant. For instance, applications to the direction of robots in networks are analyzed in [4] and applications to chemistry in [5,6], among others. In general, there are several other variations of metric dimension in the literature: resolving dominating sets [7], strong metric dimension [8], local metric dimension [9], $k$-metric dimension [10], $k$-metric antidimension [11,12], etc.

The distance between a vertex $w$ and an edge $uv$ of graph $G$ is defined as $d(ww, w) = \min\{d(u, w), d(v, w)\}$. The vertex $w$ resolves a pair $e, f \in E$ if $d(w, e) \neq d(w, f)$. The metric coordinates $r(e, S)$ of edge $e$ with respect to an ordered set of vertices $S = \{w_1, w_2, \ldots, w_k\}$ are defined as $r(e, S) = (d(e, w_1), d(e, w_2), \ldots, d(e, w_k))$. The set $S$ is an edge resolving set if every two edges $e$ and $f$ from $G$ are resolved by at least one vertex from the set $S$. The edge metric basis of the graph $G$ is the edge resolving set of the minimum cardinality. The edge metric dimension of $G$ is the cardinality of the edge metric basis for graph $G$ and is denoted by $\beta_e(G)$. The concept of the edge metric dimension of graph $G$ was introduced by Kelenc et al. in [13].

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For a given graph $G$, since every vertex of the graph is uniquely determined by resolving set and every edge of the graph is uniquely determined by the edge resolving set, the logical question is: whether every edge resolving set of the graph $G$ is also the resolving set and vice versa? In [13], authors proved that there are several graph families for which the edge resolving set is also a resolving set, but in general case, it is not valid for every graph $G$. Similarly, for every graph $G$ resolving set is not necessarily the edge resolving set.

Let $V \cup E$ be a set of items, where each item is either a vertex or an edge. The vertex $v$ resolves a pair of items if $d(v, a) \neq d(v, b)$. The metric coordinates $r(a, S)$ of item $a$ with respect to an ordered set of vertices $S = \{w_1, w_2, ..., w_k\}$ are defined as $r(a, S) = (d(a, w_1), d(a, w_2), ..., d(a, w_k))$. The set $S$ is resolved set if every two items from $G$ are resolved by at least one vertex from the set $S$. The mixed metric basis of graph $G$ is the mixed resolving set of the minimum cardinality. The mixed metric dimension of $G$ is the cardinality of the mixed metric basis for graph $G$ and is denoted by $\beta_m(G)$.

Since concept of mixed metric dimension was introduced in [1], in this paper the mixed metric dimension for several well-known classes of graphs are founded: path, cycle, tree, grid, and complete bipartite graph. Exact values of the mixed metric dimension of unicyclic graphs and tight upper bounds for graphs which have unicicly subgraph are given in [14]. Extremal difference of mixed metric dimension and some graph invariants (cyclomatic number, edge, and strong metric dimension) are given in [15] and [16], respectively.

In this paper, this dimension will be studied for two special classes of graphs: flower snarks and wheels.

A flower snark is connected, bridgeless 3-regular graph. This graph is denoted by $J_n$ and have $4n$ vertices and $6n$ edges, where vertex-set is $V(J_n) = \{a_i, b_i, c_i, d_i \mid i = 0, ..., n - 1\}$. Vertices $\{a_i \mid 0 \leq i \leq n - 1\}$ are called central vertices, $\{b_i \mid 0 \leq i \leq n - 1\}$ is called a set of inner vertices and they induce the inner cycle, while set of vertices $\{c_i \mid 0 \leq i \leq n - 1\}$ and $\{d_i \mid 0 \leq i \leq n - 1\}$ are outer vertices and they induce the outer cycle. Let the edge set be $E(J_n) = \{a_i b_i, a_i c_i, a_i d_i, b_i b_{i+1}, c_i c_{i+1}, d_i d_{i+1} \mid i = 0, 1, ..., n - 1\}$, where the edges $a_{n-1}a_0$ and $b_{n-1}b_0$ are replaced by the edges $a_{n-1}b_0$ and $b_{n-1}a_0$. The indices are taken modulo $n$.

**Property 1.** [17] Let $J_n$ be a flower snark graph and $j \in \{0, 1, ..., n - 1\}$ be an arbitrary number. Then the function $h_j : V(J_n) \rightarrow V(J_n)$ defined as:

\[
\begin{align*}
    h_j(a_i) &= \begin{cases} 
        a_{i-j}, & i \leq j \leq n - 1, \\
        a_{n+i-j}, & 0 \leq j < i,
    \end{cases} \\
    h_j(b_i) &= \begin{cases} 
        b_{i-j}, & i \leq j \leq n - 1, \\
        b_{n+i-j}, & 0 \leq j < i,
    \end{cases} \\
    h_j(c_i) &= \begin{cases} 
        c_{i-j}, & i \leq j \leq n - 1, \\
        d_{n+i-j}, & 0 \leq j < i,
    \end{cases} \\
    h_j(d_i) &= \begin{cases} 
        d_{i-j}, & i \leq j \leq n - 1, \\
        c_{n+i-j}, & 0 \leq j < i,
    \end{cases}
\end{align*}
\]

is an isomorphism of flower snark graph $J_n$.

A wheel graph is a cycle of length at least three, with a single vertex in the center connected to every vertex on the cycle. These graphs are denoted by $W_n$ and have $n + 1$ vertices and $2n$ edges, where the vertex-set is $V(W_n) = \{v_0, v_1, ..., v_n\}$ and the edge-set is $E(W_n) = \{v_0v_i \mid 1 \leq i \leq n\} \cup \{v_ii \mid 1 \leq i \leq n - 1\} \cup \{v_nv_0\}$. Vertex $v_0$ is called as an interior vertex of the graph, and all other vertices are called external vertices.

Figure 1 shows the flower snark graph $J_6$. Its mixed metric dimension is 4, which is obtained through total enumeration. The one mixed metric basis is $\{b_0, c_4, c_6, d_3\}$. In the below figure, the vertices that are elements of the basis are shown in larger circles, too.

Figure 2 shows the wheel graph $W_6$. Its mixed metric dimension is 8, which is obtained through total enumeration. The one mixed metric basis is $\{v_4, v_2, v_3, v_6, v_5, v_0, v_7, v_8\}$. In the below figure, also the vertices that are elements of the basis are shown in larger circles.
In the next part of the section, some theoretical properties of metric dimension, edge metric dimension, and mixed metric dimension will be presented following the literature.

**Property 2.** [1] For any graph $G$, it holds that

\[ \beta_m(G) \geq \max\{\beta(G), \beta_e(G)\}. \]

In the next theorem [18], for the flower snarks, metric dimension $J_n$ is given.

**Theorem 1.** [18] Let $J_n$ be the flower snark. Then for every odd positive integer $n \geq 5$, it holds that $\beta(J_n) = 3$.

**Proposition 1.** [1] Let $v$ be an arbitrary vertex in a graph $G$ and let $S = V(G) \setminus \{v\}$. If $(\forall w \in N(v)) (\exists x \in S) d(v, x) \neq d(w, x)$, then $S$ is a mixed resolving set for the graph $G$.

**Proposition 2.** [19] Let $G$ be a connected graph, then $\beta_e(G) \geq 1 + \lceil \log_2 \delta(G) \rceil$. 

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**Figure 1:** Graph $J_9$.  

**Figure 2:** Graph $W_6$. 

In the next part of the section, some theoretical properties of metric dimension, edge metric dimension, and mixed metric dimension will be presented following the literature.
2 New theoretical results

In this section, the mixed metric dimension of two important classes of graphs is considered: flower snarks and wheel graph.

2.1 Mixed metric dimension of flower snarks

The metric dimension for flower snarks is given in [18]. So it was interesting to determine the value for mixed metric dimension for these graphs. In Theorem 2, the result about the mixed metric dimension is given and here it is shown that the dimension is different for \( n = 5 \) and odd \( n \geq 7 \). It should be noted that we will omit all cases which are equivalent with regard to isomorphism given in Property 1.

First, let set of vertices \( V(J_n) \) be partitioned as follows: \( V_1 = \{a_i|0 \leq i \leq n - 1\} \), \( V_2 = \{b_i|0 \leq i \leq n - 1\} \), \( V_3 = \{c_i|0 \leq i \leq n - 1\} \), and \( V_4 = \{d_j|0 \leq i \leq n - 1\} \).

Next, several lemmas are proposed and proved, which help us to prove Theorem 2.

Lemma 1. If \( S \) is an arbitrary mixed resolving set of \( J_n \), then:
(a) \( S \cap (V_1 \cup V_2) \neq \emptyset \);
(b) \( S \cap (V_3 \cup V_4) \neq \emptyset \);
(c) \( (\forall i \in \{0, 1, \ldots, n - 1\}) S \cap \{a_i, b_j, c_j, d_j|i \leq j \leq i + k - 1\} \neq \emptyset \).

Proof.
(a) Suppose the opposite, i.e., \( S \cap (V_1 \cup V_2) = \emptyset \), which means that \( S \subseteq (V_1 \cup V_2) \). For each \( i \) and \( j \), such that \( 0 \leq i, j \leq n - 1 \), it holds that \( d(b_j, c_i) = d(b_j, c_j) = d(a_i, c_i) + 1 \) so \( d(a_i, c_i) = d(a_i, c_i) = d(a_i, b_j, c_j) = d(a_i, b_j) \), which means that edge \( a_ib_j \) has the same metric coordinates as vertex \( a_i \) with respect to \( V_1 \cup V_2 \), i.e., \( r(a_i, b_j, V_1 \cup V_2) = r(a_i, V_1 \cup V_2) \). Since \( S \subseteq (V_1 \cup V_2) \), then \( r(a_i, b_j, S) = r(a_i, S) \) holds. Therefore, the set \( S \) is not the mixed resolving set of \( J_n \), which is a contradiction with a starting assumption.

(b) Suppose the opposite, i.e., \( S \cap (V_3 \cup V_4) = \emptyset \), which means that \( S \subseteq (V_3 \cup V_4) \). For each \( i \) and \( j \), such that \( 0 \leq i, j \leq n - 1 \), and \( i \neq j \), it holds that \( d(c_i, a_i) = d(d_i, a_i) = d(a_i, a_i) - 1 \) and \( d(c_i, b_i) = d(d_i, b_i) = d(a_i, b_i) + 1 \). For \( i = j \), similarly \( d(c_i, a_i) = d(d_i, a_i) = 1 \) and \( d(c_i, b_i) = d(d_i, b_i) = 2 = d(a_i, b_i) + 1 \) hold. Next, for each \( j \), such that \( 0 \leq j \leq n - 1 \) it holds that vertex \( c_j \) has the same metric coordinates as vertex \( d_j \) with respect to \( V_1 \cup V_2 \), i.e., \( r(c_j, V_1 \cup V_2) = r(d_j, V_1 \cup V_2) \). Since \( S \subseteq (V_1 \cup V_2) \), then \( r(c_j, S) = r(d_j, S) \) holds. Therefore, the set \( S \) is not the mixed resolving set of \( J_n \), which is a contradiction with a starting assumption.

(c) Suppose the opposite, i.e., \( S \cap \{a_i, b_j, c_j, d_j|i \leq j \leq i + k - 1\} = \emptyset \), which means that \( S \subseteq \{a_i, b_j, c_j, d_j|i \leq j \leq i + k - 1\} \), and \( d[i - k - 1 \leq j \leq i - 1] \). For only remained case, when \( j = i - k - 1 \), \( d(b_{i-k-1}, c_{i-k-1}) = d(b_i, a_i, a_{i-k-1}) - 1 \), \( d(b_{i-k-1}, b_{i-k-1}) = d(b_i, b_{i-k-1}) - 1 \), \( d(b_{i-k-1}, c_{i-k-1}) = d(b_i, c_{i-k-1}) - 1 \) and \( d(b_{i-k-1}, d_{i-k-1}) = d(b_i, d_{i-k-1}) - 1 \) hold, so \( d(b_ib_{i-k-1}, a_i) = d(b_i, a_i) \), \( d(b_ib_{i-k-1}, b_i) = d(b_i, b_i) \), \( d(b_ib_{i-k-1}, c_i) = d(b_i, c_i) \) and \( d(b_ib_{i-k-1}, d_i) = d(b_i, d_i) \), which means that edge \( b_ib_{i-k-1} \) has the same metric coordinates as vertex \( b_{i-k-1} \) with respect to \( \{a_i, b_j, c_j, d_j|i - k - 1 \leq j \leq i - 1\} \). Therefore, \( r(b_ib_{i-k-1}, \{a_i, b_j, c_j, d_j|i - k - 1 \leq j \leq i - 1\}) = r(b_{i-k-1}, \{a_i, b_j, c_j, d_j|i - k - 1 \leq j \leq i - 1\}) \). Having in mind that \( S \subseteq \{a_i, b_j, c_j, d_j|i - k - 1 \leq j \leq i - 1\} \), we have \( r(b_ib_{i-k-1}, S) = r(b_{i-k-1}, S) \), so that \( S \) is not the mixed resolving set of \( J_n \), which is a contradiction with a starting assumption.

It should be noted that, as is mentioned earlier, all indices in part (c) take modulo \( n \). Moreover, without loss of generality, one vertex from \( S \cap (V_3 \cup V_4) \), from Lemma 1 part (b), should be transformed into the vertex \( c_0 \) by isomorphism from Property 1.
Theorem 2. For odd \( n \geq 5 \), it holds that \( \beta_m(J_n) = \begin{cases} 
5, & n = 5, \\
4, & n \geq 7. 
\end{cases} \)

Proof. Step 1: Exact value for \( n \in \{5, 7, 9\} \)

By using total enumeration technique, it can be shown that:
- \( \beta_m(J_5) = 5 \), with mixed metric basis \( \{a_1, b_0, b_1, c_0, d_3\} \);
- \( \beta_m(J_7) = 4 \), with mixed metric basis \( S = \{b_0, c_1, c_5, d_3\} \);
- \( \beta_m(J_9) = 4 \), with mixed metric basis \( S = \{b_0, c_1, c_6, d_3\} \).

Step 2: Upper bound equals 4 for \( n \geq 11 \).

Let \( S = \{b_0, c_0, c_{k+2}, d_3\} \). It will be proved that \( S \) is the mixed resolving set. The representations of coordinates of each vertex and each edge, with respect to \( S \), are shown in Tables 1 and 2.

As it can be seen from Tables 1 and 2, all items have mutually different metric coordinates, so \( S \) is the mixed resolving set. Therefore, \( \beta_m(J_n) \leq 4 \).

Step 3: Lower bound equals 4 for \( n \geq 11 \).

Suppose the opposite, i.e., \( \beta_m(J_n) \leq 3 \). By Property 2 and Theorem 1, we have \( \beta_m(J_n) = 3 \). Let \( S \) be the mixed resolving set of \( J_n \). Then, from Lemma 1, parts (a) and (b), two members of set \( S \) could be derived, i.e., \( c_0 \in S \) and \( (\exists b)(a_i \in S \lor b_i \in S) \). Let the remaining third member of set \( S \) has index \( j \), i.e., \( (\exists j) (a_j \in S \lor b_j \in S \lor c_j \in S \lor d_j \in S) \).

From Lemma 1, part (c), it holds that indices \( i \) and \( j \) have not arbitrary values:
- (I) \( i \neq 0 \land j \neq 0 \);
- (II) \( 1 \leq i \leq k \Rightarrow k + 1 \leq j \leq 2k \);
- (III) \( k + 1 \leq i \leq 2k \Rightarrow 1 \leq j \leq k \).

Table 1: Metric coordinates of vertices of \( J_{3k+1} \)

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Cond.</th>
<th>( r(v, S) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_0 )</td>
<td>0 \leq i \leq 1</td>
<td>((i + 1, 2 - i, k + i, 4 - i))</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>3 \leq i \leq k</td>
<td>((i + 1, i, k + 3 - i, i - 2))</td>
</tr>
<tr>
<td>( a_0 )</td>
<td>( k + 2 \leq i \leq k + 3 )</td>
<td>((2k + 2 - i, 2k + 3 - i, i - 1 - k, i - 1))</td>
</tr>
<tr>
<td>( a_0 )</td>
<td>( k + 4 \leq i \leq 2k )</td>
<td>((2k + 2 - i, 2k + 3 - i, i - 1 - k, 2k + 5 - i))</td>
</tr>
<tr>
<td>( b_0 )</td>
<td>0 \leq i \leq 1</td>
<td>((i, 3 - i, k + i + 1, 5 - i))</td>
</tr>
<tr>
<td>( b_2 )</td>
<td>( 2 \leq i \leq k )</td>
<td>((2, 3 - i, k + 2, 3))</td>
</tr>
<tr>
<td>( b_0 )</td>
<td>( k + 2 \leq i \leq k + 3 )</td>
<td>((2k + 1 - i, 2k + 4 - i, i - k, i - 1))</td>
</tr>
<tr>
<td>( b_0 )</td>
<td>( k + 4 \leq i \leq 2k )</td>
<td>((2k + 1 - i, 2k + 4 - i, i - k, 2k + 6 - i))</td>
</tr>
<tr>
<td>( c_0 )</td>
<td>( 1 \leq i \leq 2 )</td>
<td>((i + 1, 1 - i, k + 2 - i, 5 - i))</td>
</tr>
<tr>
<td>( c_0 )</td>
<td>( 2 \leq i \leq k )</td>
<td>((i + 1, i, 3 - i, k + 2 - i, i - 1))</td>
</tr>
<tr>
<td>( c_0 )</td>
<td>( k + 1 \leq i \leq k + 2 )</td>
<td>((2k + 3 - i, i - 1, 2k + 2 - i, i - 1))</td>
</tr>
<tr>
<td>( c_0 )</td>
<td>( k + 3 \leq i \leq 2k )</td>
<td>((2k + 3 - i, 2k + 4 - i, i - 2 - k, 2k + 4 - i))</td>
</tr>
<tr>
<td>( d_0 )</td>
<td>( 1 \leq i \leq 2 )</td>
<td>((i + 1, i, 3 - i, k + 3 - i))</td>
</tr>
<tr>
<td>( d_0 )</td>
<td>( 3 \leq i \leq k )</td>
<td>((i + 1, 3 - i, k + 4 - i, i - 3))</td>
</tr>
<tr>
<td>( d_0 )</td>
<td>( k + 1 \leq i \leq k + 2 )</td>
<td>((2k + 3 - i, 2k + 2 - i, k + 4 - i, i - 3))</td>
</tr>
<tr>
<td>( d_0 )</td>
<td>( k + 3 \leq i \leq 2k )</td>
<td>((2k + 3 - i, 2k + 2 - i, i - k, i - 3))</td>
</tr>
<tr>
<td>( d_0 )</td>
<td>( k + 5 \leq i \leq 2k )</td>
<td>((2k + 3 - i, 2k + 2 - i, i - k, 2k + 6 - i))</td>
</tr>
</tbody>
</table>
Part (I) holds, because if \( (i = 0 \lor j = 0) \Rightarrow \)

\[
(S \cap \{a_p, b_p, c_p, d_p \mid 1 \leq p \leq k\} = \emptyset \lor S \cap \{a_p, b_p, c_p, d_p \mid k + 1 \leq p \leq 2k\} = \emptyset),
\]

right hand side of implication is in direct contradiction with Lemma 1, part (c).

If \( 1 \leq i \leq k \), then again by Lemma 1(c), \( S \cap \{a_p, b_p, c_p, d_p \mid 1 \leq p \leq k\} \neq \emptyset \) holds. Since \( i \notin \{p \mid k + 1 \leq p \leq 2k\} \), then it must be \( j \in \{p \mid k + 1 \leq p \leq 2k\} \), so part (II) holds.

Part (III) also follows from Lemma 1, in a similar way to part (II).

We have eight possible cases for mixed resolving set \( S \).

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**Table 2:** Metric coordinates of edges of \( J_{2k+1} \)

<table>
<thead>
<tr>
<th>Edge</th>
<th>Cond.</th>
<th>( r(e, S) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_i b_0 )</td>
<td>( 1 \leq i \leq 2 )</td>
<td>( (0, 2, k, 4) )</td>
</tr>
<tr>
<td>( a_i b_i )</td>
<td>( 3 \leq i \leq k )</td>
<td>( (i, i, k + 1, 4 - i) )</td>
</tr>
<tr>
<td>( a_i, b_{k+1} )</td>
<td>( k + 2 \leq i \leq k + 3 )</td>
<td>( (2k + 1 - i, 2k + 3 - i, i - 1 - k, i - 2) )</td>
</tr>
<tr>
<td>( a_i b_i )</td>
<td>( k + 4 \leq i \leq 2k )</td>
<td>( (2k + 1 - i, 2k + 3 - i, i - 1 - k, 2k + 5 - i) )</td>
</tr>
<tr>
<td>( a_i c_0 )</td>
<td>( 1 \leq i \leq 2 )</td>
<td>( (1, 1, k, 4) )</td>
</tr>
<tr>
<td>( a_i c_i )</td>
<td>( 3 \leq i \leq k )</td>
<td>( (i + 1, i - 1, k + 2 - i, 4 - i) )</td>
</tr>
<tr>
<td>( a_i, d_{i+1} )</td>
<td>( k + 1 \leq i \leq k + 2 )</td>
<td>( (2k + 2 - i, i - 1, k + 2 - i, i - 2) )</td>
</tr>
<tr>
<td>( a_i d_i )</td>
<td>( k + 3 \leq i \leq 2k )</td>
<td>( (2k + 2 - i, 2k + 3 - i, i - 1 - k, 2k + 5 - i) )</td>
</tr>
<tr>
<td>( b_i b_1 )</td>
<td>( 1 \leq i \leq 2 )</td>
<td>( (0, 2, k + 1, 4) )</td>
</tr>
<tr>
<td>( b_i b_{i+1} )</td>
<td>( 3 \leq i \leq k )</td>
<td>( (i, i + 1, k + 3 - i, 4 - i) )</td>
</tr>
<tr>
<td>( c_i c_0 )</td>
<td>( 1 \leq i \leq 2 )</td>
<td>( (i + 1, i - 1, k + 3 - i, 4 - i) )</td>
</tr>
<tr>
<td>( c_i, c_{i+1} )</td>
<td>( 3 \leq i \leq k )</td>
<td>( (i + 1, i, k + 3 - i, i - 3) )</td>
</tr>
<tr>
<td>( c_i, c_{i+2} )</td>
<td>( k + 2 \leq i \leq 2k - 1 )</td>
<td>( (2k - i, 2k + 3 - i, 2, i - 1) )</td>
</tr>
<tr>
<td>( c_i d_0 )</td>
<td>( 2 \leq i \leq 2, 3 )</td>
<td>( (2, 0, k + 1, 4) )</td>
</tr>
<tr>
<td>( c_i d_1 )</td>
<td>( 2, 1, k, 5 )</td>
<td>( (2, 0, k + 1, 4) )</td>
</tr>
<tr>
<td>( d_i d_1 )</td>
<td>( 1 \leq i \leq 2 )</td>
<td>( (2, 2, k - 1, 2) )</td>
</tr>
<tr>
<td>( d_i, d_{i+1} )</td>
<td>( 3 \leq i \leq k )</td>
<td>( (i + 2, i + 1, k - 1 + i, 2 - i) )</td>
</tr>
<tr>
<td>( d_i, d_{i+2} )</td>
<td>( k + 2 \leq i \leq k + 3 )</td>
<td>( (2k - 2 - i, 2k + 1 - i, i - k, i - 3) )</td>
</tr>
<tr>
<td>( d_i d_{i+1} )</td>
<td>( k + 4 \leq i \leq 2k - 1 )</td>
<td>( (2k - 2 - i, 2k + 1 - i, i - k, 2k + 5 - i) )</td>
</tr>
</tbody>
</table>
Case 1. \( S = \{c_0, a_i, a_j\} \).

Since \( i \neq 0 \land j \neq 0 \), then
\[
d(a_0c_0, c_0) = d(c_0, c_0) = 0, \quad d(a_0c_0, a_i) = d(a_0, a_i) - 1 = d(c_0, a_i), \quad \text{and} \quad d(a_0c_0, a_j) = d(a_0, a_j) - 1 = d(c_0, a_j).
\]
Therefore, \( r(a_0c_0, S) = r(c_0, S) \), which means that \( S \) is not the mixed resolving set.

Case 2. \( S = \{c_0, a_i, b_j\} \)

Subcase 1. \( 1 \leq i \leq k \)

From part (II), it holds that \( k + 1 \leq j \leq 2k \). Then
\[
d(a_0c_0, c_0) = d(c_0d_{2k}, c_0) = 0 \quad \text{and} \quad d(a_0c_0, a_i) = d(c_0, a_i) = d(c_0d_{2k}, a_i).
\]

Also, since
\[
d(a_0c_0, b_j) = d(a_0, b_j) = d(b_0, b_j) + 1, \quad d(c_0d_{2k}, b_j) = d(d_{2k}, b_j) = d(b_2k, b_j) + 2, \quad \text{and} \quad d(b_0, b_j) = d(b_1, b_j) + 1,
\]
it follows that \( d(a_0c_0, b_j) = d(c_0d_{2k}, b_j) \).
Therefore, \( r(a_0c_0, S) = r(c_0d_{2k}, S) \), which means that \( S \) is not the mixed resolving set.

Subcase 2. \( k + 1 \leq i \leq 2k \)

From part (III), it holds that \( 1 \leq j \leq k \). Then
\[
d(a_0c_0, c_0) = d(c_0d_{2k}, c_0) = 0 \quad \text{and} \quad d(a_0c_0, a_i) = d(c_0, a_i) = d(c_0d_{2k}, a_i).
\]

Also, since
\[
d(a_0c_0, b_j) = d(a_0, b_j) = d(b_0b_j) + 1, \quad d(c_0d_{2k}, b_j) = d(c_1, b_j) = d(d_{2k}, b_j) + 2, \quad \text{and} \quad d(b_0, b_j) = d(b_1, b_j) + 1,
\]
it follows that \( d(a_0c_0, b_j) = d(c_0d_{2k}, b_j) \).
Therefore, \( r(a_0c_0, S) = r(c_0d_{2k}, S) \), which means that \( S \) is not the mixed resolving set.

Case 3. \( S = \{c_0, a_i, c_j\} \)

Subcase 1. \( 1 \leq i \leq k \)

From part (II), it holds that \( k + 1 \leq j \leq 2k \). Then
\[
d(a_0c_0, c_0) = d(c_0d_{2k}, c_0) = 0 \quad \text{and} \quad d(a_0c_0, a_i) = d(c_0, a_i) = d(c_0d_{2k}, a_i).
\]

Also, since
\[
d(a_0c_0, c_j) = d(a_0, c_j) = d(c_2k, c_j) + d(c_2k, a_0) = d(c_2k, c_j) + 2 \quad \text{and} \quad d(c_0d_{2k}, c_j) = d(d_{2k}, c_j) = d(c_2k, c_j) + 2,
\]
it follows that \( d(a_0c_0, c_j) = d(c_0d_{2k}, c_j) \).
Therefore, \( r(a_0c_0, S) = r(c_0d_{2k}, S) \), which means that \( S \) is not the mixed resolving set.

Subcase 2. \( k + 1 \leq i \leq 2k \)

From part (III), it holds that \( 1 \leq j \leq k \). Then
\[
d(a_0c_0, c_0) = d(c_0, c_0) = 0, \quad d(a_0c_0, a_i) = d(a_0, a_i) - 1 = d(c_0, a_i), \quad \text{and} \quad d(a_0c_0, c_j) = d(c_0, c_j).
\]
Therefore, \( r(a_0c_0, S) = r(c_0, S) \), which means that \( S \) is not the mixed metric resolving set.

Case 4. \( S = \{c_0, a_i, d_j\} \)

Subcase 1. \( 1 \leq i \leq k \)

From part (II), it holds that \( k + 1 \leq j \leq 2k \). Then
\[
d(a_0c_0, c_0) = d(c_0, c_0) = 0, \quad d(a_0c_0, a_i) = d(a_0, a_i) - 1 = d(c_0, a_i), \quad \text{and} \quad d(a_0c_0, d_j) = d(a_0, d_j) - 1 = d(c_0, d_j).
\]
Therefore, \( r(a_0c_0, S) = r(c_0, S) \), which means that \( S \) is not the mixed resolving set.
Subcase 2. $k + 1 \leq i \leq 2k$

From part (III), it holds that $1 \leq j \leq k$. Then

\[ d(c_0a_0, c_0) = d(c_0a_0, c_0) = 0, \quad d(c_0a_0, a_i) = d(c_0a_0, a_i), \quad \text{and} \quad d(c_0a_0, d_i) = d(a_0, d_i) = d(c_0a_0, d_i). \]

Therefore, $r(c_0a_0, S) = r(c_0a_0, S)$ which means that $S$ is not the mixed resolving set.

Case 5. $S = \{c_0, b_i, a_i\}$ Reduce to Case 2. By substitution $j' = i, i' = j$.

Case 6. $S = \{c_0, b_i, b_j\}$

Subcase 1. $1 \leq i \leq k$

From part (II), it holds that $k + 1 \leq j \leq 2k$. Then

\[
\begin{align*}
d(a_0, c_0) &= 1 = d(a_0d_0, c_0), \\
d(a_0, b_i) &= d(b_0, b_i) + 1 = d(d_0, b_i) = d(a_0d_0, b_i), \quad \text{and} \\
d(a_0, b_i) &= d(b_0, b_i) + 1 = d(d_0, b_i) - 1 = d(a_0d_0, b_i).
\end{align*}
\]

Therefore, $r(a_0, S) = r(a_0d_0, S)$, which means that $S$ is not a mixed resolving set.

Subcase 2. $k + 1 \leq i \leq 2k$

Reduce to Subcase 2. By substitution $j' = i, i' = j$.

Case 7. $S = \{c_0, b_i, c_j\}$

Subcase 1. $1 \leq i \leq k$

From part (II), it holds that $k + 1 \leq j \leq 2k$. Then

\[
\begin{align*}
d(b_0b_1, c_0) &= d(b_0, c_0) = 2 = d(a_1, c_0) = d(a_1b_1, c_0), \\
d(b_0b_1, b_i) &= d(b_1, b_i) = d(a_1, b_i) - 1 = d(a_1b_1, b_i), \quad \text{and} \\
d(b_0b_1, c_j) &= d(b_0, c_j) = d(b_0, b_i) + 2 = d(c_0, c_j) + d(c_0, b_i) + d(d_0, c_j) + d(d_0, a_i) = d(a_1, c_j) = d(a_1b_1, c_j), \quad \text{where} \quad d(d_0, c_j) = d(d_0, a_i) = 2.
\end{align*}
\]

Therefore, $r(b_0b_1, S) = r(a_1b_1, S)$, which means that $S$ is not a mixed resolving set.

Subcase 2. $k + 1 \leq i \leq 2k$

From part (III), it holds that $1 \leq j \leq k$. Then

\[
\begin{align*}
d(b_0, c_0) &= 2 = d(d_{2k-1}, c_0), \\
d(b_0, b_i) &= d(b_2b_{k-1}, b_i) + 2 = d(d_{2k-1}, b_i), \quad \text{and} \\
d(b_0, c_j) &= d(b_0, b_i) + 2 = d(c_0, c_j) + d(d_{2k-1}, c_0) = d(d_{2k-1}, c_j), \quad \text{where} \quad d(d_{2k-1}, c_0) = 2.
\end{align*}
\]

Therefore, $r(b_0, S) = r(d_{2k-1}, S)$, which means that $S$ is not a mixed resolving set.

Case 8. $S = \{c_0, b_i, d_j\}$

Subcase 1. $1 \leq i \leq k$

From part (II), it holds that $k + 1 \leq j \leq 2k$. Then

\[
\begin{align*}
d(b_0b_1, c_0) &= d(b_0, c_0) = 2 = d(a_1, c_0) = d(a_1b_1, c_0), \\
d(b_0b_1, b_i) &= d(b_1, b_i) = d(a_1, b_i) - 1 = d(a_1b_1, b_i) \\
d(b_0b_1, d_j) &= d(b_0, d_j) = d(b_0, b_i) + 2 = d(d_0, d_j) = d(c_0, d_j) + d(c_0, a_i) = d(c_0, d_j) = d(c_0, d_j) = d(b_1, d_j) - 1 = d(a_1b_1, d_j),
\end{align*}
\]

where $d(c_0, a_i) = d(c_0, c_j) + d(c_0, a_i) = 2.$

Therefore, $r(b_0b_1, S) = r(a_1b_1, S)$, which means that $S$ is not a mixed resolving set.

Subcase 2. $k + 1 \leq i \leq 2k$

From part (III), it holds that $1 \leq j \leq k$. Then

\[
\begin{align*}
d(b_0, c_0) &= 2 = d(a_2, c_0), \quad \text{and} \quad d(b_0, b_i) = d(a_2, b_i), \quad \text{and} \\
d(b_0, d_j) &= d(b_0, d_j) + 2 = d(d_0, d_j) + 2 = d(d_0, d_j) + d(a_2, d_j) = d(a_2, d_j).
\end{align*}
\]

Therefore, $r(b_0, S) = r(a_2, S)$, which means that $S$ is not a mixed resolving set.
Since $S$ is not the mixed resolving set in all eight cases, which is a contradiction with starting assumption, so $\beta_M(J_n) \geq 4$. Therefore, from the previous three steps, the proof of theorem is completed.

It would be interesting to make comparison between mixed metric dimension and metric dimension for flower snarks. For $n = 5$ situation is easy since all three dimensions can be obtained by a total enumeration, so $\beta(J_n) = 3 < \beta_E(J_n) = 4 < \beta_M(J_n) = 5$. For odd $n \geq 7$, mixed metric dimension is larger than metric dimension, i.e., $\beta(J_n) = 3 < \beta_M(J_n) = 4$. From Theorem 2 it is easy to see that for $n \geq 7$, similar to metric dimension, mixed metric dimension for flower snarks is constant, i.e., it does not depend on $n$.

When we consider edge metric dimension, situation is not so clear. Theorem 2 has obvious corollary, that for $n \geq 7$ it is $\beta_E(J_n) \leq 4$. From Theorem 2 of [1], lower bound of edge metric dimension applied to flower snarks is $\beta_E(J_n) \geq 3$, so it holds $\forall n \geq 7) \beta_E(J_n) \in \{3, 4\}$. It would be interesting to find the exact value.

### 2.2 Mixed metric dimension of wheel graphs

In the following, the mixed metric dimension of wheel graphs is obtained. In the next theorem, the mixed metric dimension of these graphs is determined.

**Theorem 3.**

$$\beta_M(W_n) = \begin{cases} 4, & n = 3, \\ n, & n \geq 4. \end{cases}$$

**Proof.** For $n = 3$ using total enumeration, we have $\beta_M(W_3) = 4 = |W_3|$ so that $S = V(W_3)$.

Therefore, the mixed metric dimension of wheels will be considered only for case when $n \geq 4$. In order to present it more clearly, the representations of the mixed metric coordinates of each vertex and each edge with respect to $V(W_n)$ are given in Table 3.

Two steps will be considered.

**Step 1 :** Upper bound for $n \geq 4$.

Let $S = \{v|i≤ i≤ n\}$. It will be proved that $S$ is the mixed resolving set of $W_n$, for $n \geq 4$. Since $S = V\setminus \{v_0\}$, then for each item, vectors of metric coordinates with respect to $S$ are presented in Table 4.

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Cond.</th>
<th>$r(v, V)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_0$</td>
<td></td>
<td>$(0, 1, ..., 1)$</td>
</tr>
<tr>
<td>$v_1$</td>
<td></td>
<td>$(1, 0, 1, 2, ..., 2, 1)$</td>
</tr>
<tr>
<td>$v_2$</td>
<td></td>
<td>$(1, 1, 0, 1, 2, ..., 2)$</td>
</tr>
<tr>
<td>$v_3$</td>
<td>$3 ≤ i ≤ n - 2$</td>
<td>$(1, 2, ..., 2, 1, 0, 1, 2, ..., 2)$</td>
</tr>
<tr>
<td>$v_{n-1}$</td>
<td></td>
<td>$(1, 2, ..., 2, 1, 0, 1)$</td>
</tr>
<tr>
<td>$v_n$</td>
<td></td>
<td>$(1, 1, 2, ..., 2, 1, 0)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Edge</th>
<th>Cond.</th>
<th>$r(e, V)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_0v_1$</td>
<td>$1 ≤ i ≤ n$</td>
<td>$(0, 1, ..., 1, 0, 1, 2, ..., 2, 1)$</td>
</tr>
<tr>
<td>$v_0v_2$</td>
<td></td>
<td>$(1, 0, 0, 1, 2, ..., 2, 1)$</td>
</tr>
<tr>
<td>$v_0v_3$</td>
<td></td>
<td>$(1, 0, 1, 2, ..., 2, 1, 0)$</td>
</tr>
<tr>
<td>$v_0v_{n-1}$</td>
<td>$3 ≤ i ≤ n - 3$</td>
<td>$(1, 1, 0, 0, 1, 2, ..., 2, 1)$</td>
</tr>
<tr>
<td>$v_0v_n$</td>
<td></td>
<td>$(1, 1, 2, ..., 2, 1, 0, 1)$</td>
</tr>
<tr>
<td>$v_1v_{n-1}$</td>
<td></td>
<td>$(1, 2, ..., 2, 1, 0, 0, 1, 2, ..., 2)$</td>
</tr>
<tr>
<td>$v_{n-2}v_{n-1}$</td>
<td></td>
<td>$(1, 2, ..., 2, 1, 0, 0, 1)$</td>
</tr>
<tr>
<td>$v_{n-1}v_0$</td>
<td></td>
<td>$(1, 1, 2, ..., 2, 1, 0, 0)$</td>
</tr>
</tbody>
</table>
As it can be seen from Table 4, edge $v_i v_j$ has two zeros in the vector of metric coordinates $(d_{v_i v_j}, 0)$, while the vertex $v_i$ and the edge $v_i v_0$ have one zero in the vector of metric coordinates $(d_{v_i v_0}, 0)$.

**Step 2:** Lower bound for $n \geq 4$.

Assume the opposite that it is $\beta(W_n) = n - 1$. Then mixed resolving set $S$ exists, so that $|S| \leq n - 1$.

**Case 1.** $v_0 \notin S$.

Since $|V| = n + 1$ and $|S| \leq n - 1 \Rightarrow (3i) 1 \leq i \leq n$, it holds that $r(v_0, S) = r(v_0 v_i, S) = (1, \ldots, 1)$. It will be concluded that $S$ is not the mixed resolving set, which is a contradiction with a starting assumption.

**Case 2.** $v_0 \in S$.

Since $|V| = n + 1$ and $|S| \leq n - 1 \Rightarrow (3i, j) 1 \leq i, j \leq n$, it holds that $r(v_0, S) = r(v_0 v_{ij}, S) = (0, 1, \ldots, 1)$. It will be concluded that $S$ is not the mixed resolving set, which is a contradiction with a starting assumption.

Since $S$ is not the mixed resolving set in both cases, it follows that $\beta_m(W_n) \geq n$.

Therefore, from Steps 1 and 2, it follows that $\beta_m(W_n) = n$.

Also, it should be noted that Step 1 could be indirectly proved using Proposition 1 from [1]. In order to give constructive proof, we have decided on the proof presented above.

All three previously mentioned metric invariants could be compared. It is already known from [20] that

$$\beta(W_n) = \begin{cases} n = 3, 6, \\ n = 4, 5, \\ \frac{2n + 2}{5}, \ n \geq 7, \end{cases}$$

and from [13], it follows that

$$\beta_m(W_n) = \begin{cases} n, \ n = 3, 4, \\ n - 1, \ n \geq 5. \end{cases}$$

From the previous, it would be interesting to make comparison between mixed metric dimension, metric dimension and edge metric dimension, for wheel graphs. For $n = 3$ mixed metric dimension is
larger than metric and edge metric dimension, i.e., $\beta(W_2) = \beta_E(W_2) = 3 < \beta_M(W_2) = 4$. For $n = 4$ edge metric dimension and mixed metric dimension are larger then metric dimension, i.e., $\beta(W_n) = 2 < \beta_E(W_n) = 5 < \beta_M(W_n) = 6$. Then, for $n = 5$ and $n = 6$ holds $\beta(W_n) = 2 < \beta_E(W_n) = 4 < \beta_M(W_n) = 5$ and $\beta(W_n) = 3 < \beta_E(W_n) = 5 < \beta_M(W_n) = 6$, respectively. And, for $n \geq 7$ edge metric dimension is smaller than mixed metric dimension, and larger than the metric dimension, i.e., $\beta(W_n) = \left\lfloor \frac{2n+2}{5} \right\rfloor < \beta_E(W_n) = n - 1 < \beta_M(W_n) = n$. Unlike the flower snarks graphs described above, for wheel graphs, the mixed metric dimension depends on $n$.

3 Conclusions

In this paper, the mixed metric dimension for flower snarks and wheels graphs is considered. First, it is given lemma about some properties of the mixed resolving sets of flower snark graphs. Next, this lemma is used for obtaining exact value of the mixed metric dimension for flower snarks. It is proved that it is constant and equal to 4, for odd $n \geq 7$, while $\beta_M(f_6) = 5$. Last, it is present the mixed metric dimension for wheels, and it is proved to be equal to $n$, for $n \geq 4$, while $\beta_M(W_4) = 4$.

Further work can be directed in finding the mixed metric dimension of some other interesting classes of graphs. Other direction could be finding the exact value of the edge metric dimension of flower snarks.

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References


