Research Article

Sanbiao Tan, Guiyun Chen, and Yanxiong Yan*

Finite groups with $4p^2q$ elements of maximal order

<https://doi.org/10.1515/math-2021-0066>

received January 23, 2021; accepted June 24, 2021

Abstract: It is an interesting and difficult topic to determine the structure of a finite group by the number of elements of maximal order. This topic is related to Thompson's conjecture, that is, if two finite groups have the same order type and one of them is solvable, then the other is solvable. In this article, we continue this work and prove that if G is a finite group which has $4p^2q$ elements of maximal order, where p, q are primes and $7 \leq p \leq q$, then either G is solvable or G has a section who is isomorphic to one of $L_2(7)$, $L_2(8)$ or $U_3(3)$.

Keywords: finite groups, solvable groups, the order of elements

MSC 2020: 05C25

1 Introduction

In this paper, all groups discussed are finite. By $\pi_e(G)$ we denote the set of orders of elements of G . k denotes the largest number of $\pi_e(G)$ and n the number of cyclic subgroups of order k in G . Let l be a positive integer and $\pi(l)$ denote the set of prime divisors of l , especially, $\pi(G) = \pi(|G|)$. $M_l(G)$ denotes the set of elements of order l in G , and so $M(G)$ denotes the set of elements with maximal order in G . The Euler function of a positive integer x is denoted by $\varphi(x)$. $\Gamma(G)$ denotes the prime graph of G , is a graph whose vertex set is $\pi(G)$, and two distinct vertices p and q are joined by an edge if $pq \in \pi_e(G)$. We denote the number of connected components of $\Gamma(G)$ as $t(G)$, the connected components of $\Gamma(G)$ as $\{\pi_i = \pi_i(G), i = 1, \dots, t(G)\}$, and we always assume that 2 is in $\pi_1(G)$ if $2 \mid |G|$. We also use $P_r(G)$ to denote a Sylow r -subgroup of G , where $r \in \pi(G)$. All unexplained symbols and notations are standard (cf. [1]).

The topic to study solvability of finite groups is one of the most important topics of finite group theory. The topic in this article is related to the following Thompson's conjecture [2, Problem 12.37]:

Conjecture 1.1. Let G_1 and G_2 be finite groups. Assume that $|M_\ell(G_1)| = |M_\ell(G_2)|$ with $\ell = 1, 2, 3, \dots$. If G_1 is solvable, then G_2 is solvable.

In 1963, professor Feit and J. G. Thompson proved the famous odd order theorem. It states that any finite odd order groups are solvable. For even order groups, J. G. Thompson put forward the famous Thompson's conjecture in a letter to professor Wujie Shi, in his letter, professor J. G. Thompson pointed out that if the Conjecture is right, then we can study the solvability of the same order type groups to distinguish the solvability of finite groups. So it is important to prove that Thompson's conjecture in the study of the

* **Corresponding author: Yanxiong Yan**, School of Mathematics and Statistics, Southwest University, Beibei, Chongqing, 400715, P. R. China, e-mail: 2003yyx@163.com

Sanbiao Tan: School of Mathematics and Statistics, Southwest University, Chongqing, 400715, P. R. China, e-mail: 1107354233@qq.com

Guiyun Chen: School of Mathematics and Statistics, Southwest University, Chongqing, 400715, P. R. China, e-mail: gychen1963@163.com

solvability of finite groups. However, up to now, there is no way to draw a general result. It is well known that the elements of maximal or minimal order of finite group can be used to describe the structure of the given group. Some authors studied if G is solvable while the number of elements of maximal order is specially restricted. For example, in [3] authors proved that G is solvable if $|M(G)| = 2pqr$, where p, q and r are primes and $3 < p < q < r$; in [4] authors proved that G is solvable if $|M(G)| = 6pq$, where p, q are primes and $13 < p < q$; in [5] authors proved that G is solvable if $|M(G)| = 18p$, where p is a prime; and in [6] authors gave a complete classification of the finite group with $|M(G)| = 30$, showing that they are solvable. If G_1 and G_2 satisfy the hypothesis of Thompson's conjecture, then G_1 and G_2 have the same number of elements of maximal order. Hence, if we proved G is solvable for special $|M(G)|$, then for all groups G_1 and G_2 with $|M_\ell(G_1)| = |M_\ell(G_2)|$ equals this number are solvable. Hence, this study may give useful information to Thompson's conjecture. The aim of the present paper is to study if a finite group having exactly $4p^2q$ elements of the maximal order, we prove the following theorem:

Theorem 1.2. *Let G be a finite group having exactly $4p^2q$ elements of maximal order, where p, q are primes and $q \geq p \geq 7$, then one of the following holds:*

- (1) *If $q = p > 7$, then G is solvable;*
- (2) *If $q > p > 7$, while $2p + 1$, $2q + 1$ and $2p^2 + 1$ are not primes, then G is solvable;*
- (3) *If $p = 7$, then either G is solvable or G has a section who is isomorphic to $L_2(7)$, $L_2(8)$ or $U_3(3)$. In particular, G is a $\{2, 3, 7\}$ -group.*

2 Preliminaries

We collect in this section some known results and elementary facts as well as some technical lemmas.

Lemma 2.1. [7, Lemma 2.2] *Suppose that G has m cyclic subgroups of order ℓ , then $|M_\ell(G)| = m\varphi(\ell)$ where $\ell > 2$. In particular, $|M(G)| = n\varphi(k)$.*

Lemma 2.2. [7, Theorem 1.1] *Let G be a finite group. If $|M(G)| = \varphi(k)$, then G is super-solvable.*

Definition 2.3. Suppose $1 \neq H \leq G$. Then H is called an r -total subgroup of G if $H = \langle a | a \in M_r(H) \rangle$ and $\text{Inn}(\mathbf{N}_G(H))$ acts transitively on $M_r(H)$.

Lemma 2.4. [7, Lemma 3.1] *Let $1 \neq H \leq G$, $H = \langle a | a \in M_r(H) \rangle$. If $|M_r(H)| = \alpha \in \pi(\mathbf{N}_G(H)/\mathbf{C}_G(H))$, then H is an r -total subgroup of G . Moreover, if $\alpha = 2$, then $\text{Inn}(\mathbf{N}_G(H))$ acts on $M_r(H)$ transitively and doubly.*

Lemma 2.5. [7, Theorem 5.1.1] *Let G be a finite group. Suppose $r \in \pi_e(G)$ and $r = r_1 r_2$ with $1 < r_1 < r$ and $(r_1, r_2) = 1$. Let $a \in M_{r_1}(G)$. If $\text{Inn}(G)$ acts transitively on $M_{r_1}(G)$, then $|M_r(G)| = |M_{r_1}(G)| |M_{r_2}(\mathbf{C}_G(a))|$.*

Lemma 2.6. [8, 9] *Let G be a simple K_3 -group. Then G is isomorphic to one of the groups: A_5 , A_6 , $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$ and $U_4(2)$.*

For convenience, we list the information of all K_3 -simple groups in Table 1.

Lemma 2.7. [10, Theorem 1] *Let G be a simple K_4 -group with $3 \notin \pi(G)$. Then $G \cong S_2(8)(2^6 \cdot 5 \cdot 7 \cdot 13)$ or $G \cong S_2(32)(2^{10} \cdot 5^2 \cdot 31 \cdot 41)$.*

Lemma 2.8. [11, Theorem 1] *If $\pi_e(G) = \{1, 2, 3, 5, 6\}$, then G is solvable.*

Lemma 2.9. [12, Theorem A] *Let G be a finite group whose prime graph has more than one component, then one of the following holds:*

- (i) *G is a Frobenius group or 2-Frobenius group;*
- (ii) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1 -groups and K/H is a simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group and $|G/K| |\text{Out}(K/H)|$.*

Table 1: The $|G|$ and $|\text{Out}(G)|$ of all simple K_3 -groups

G	$ G $	$ \text{Out}(G) $
A_5	$2^2 \cdot 3 \cdot 5$	2
A_6	$2^3 \cdot 3^2 \cdot 5$	4
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3
$L_2(17)$	$2^4 \cdot 3^2 \cdot 17$	2
$L_3(3)$	$2^4 \cdot 3^3 \cdot 13$	2
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2
$U_4(2)$	$2^6 \cdot 3^4 \cdot 5$	2

Lemma 2.10. [13, Theorem 2] *Let G be a 2-Frobenius group of even order, then $t(G) = 2$, and G has normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = \pi_2$, $\pi(H) \cup \pi(G/K) = \pi_1$, $|G/K| \mid |\text{Out}(K/H)|$, G/K and K/H are cyclic. In particular, $|G/K| < |K/H|$, G is solvable.*

Lemma 2.11. [13, Theorem 1] *Let G be a Frobenius group of even order with H and K its Frobenius kernel and Frobenius complement, respectively. Then $t(G) = 2$ and $T(G) = \{\pi(H), \pi(K)\}$. Furthermore, one of the following holds:*

- (i) *If $2 \in \pi(H)$, then Sylow subgroups of K are cyclic;*
- (ii) *If $2 \in \pi(K)$, then group H is abelian. Suppose K is solvable, then every Sylow subgroups of K of odd order is cyclic and a Sylow 2-subgroup of K is either cyclic or generalized quaternion group. If K is not solvable, then K has a normal subgroup K_0 with $|K : K_0| \leq 2$ such that $K_0 \cong Z \times SL(2, 5)$, where the Sylow subgroups of Z are cyclic and $(30, |Z|) = 1$.*

3 Proof of Theorem 1.2

With this preparation, we are ready to embark on the proof of Theorem 1.2.

Proof of Theorem 1.2. We fix an element

$$x \in G \text{ with } o(x) \text{ the largest element of } \pi_e(G).$$

Write $o(x) = k$. Then $|\mathbf{N}_G(\langle x \rangle) : \mathbf{C}_G(\langle x \rangle)| \mid \varphi(k)$. We use n to denote the number of cyclic subgroups of order k in G . The following equality is always true:

$$|G| = |G : \mathbf{N}_G(\langle x \rangle)| |\mathbf{N}_G(\langle x \rangle) : \mathbf{C}_G(\langle x \rangle)| |\mathbf{C}_G(\langle x \rangle)|. \quad (1)$$

By Lemma 2.1, if $n\varphi(k) = 4p^2q$, then all the possibilities for n , $\varphi(k)$ and k are in Table 2.

In what follows, we prove the main theorem case by case according to all the possible values of k in Table 2.

Case 1. If $n = 1$ and $\varphi(k) = 4p^2q$. By Lemma 2.2, then we can get G is super-solvable.

Case 2. If $n = 2$ and $\varphi(k) = 2p^2q$. By Table 2, we have $k \in \{r, 2r\}$ with prime $r = 2p^2q + 1$. Then we have $\mathbf{C}_G(\langle x \rangle)$ is a r -group or $\{2, r\}$ -group and $|\mathbf{N}_G(\langle x \rangle) : \mathbf{C}_G(\langle x \rangle)| \mid 2p^2q$. Hence, by the formula (1), we get $\pi(G) \subseteq \{2, p, q, r\}$. Since $r > q \geq p \geq 7$, then by Lemmas 2.6 and 2.7, we can conclude that there exists no such normal series of G : $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is isomorphic to one of some simple K_3 -groups or simple K_4 -groups. Hence, G is solvable.

Table 2: Values of n , k and $\varphi(k)$

n	$\varphi(k)$	k
1	$4p^2q$	k
2	$2p^2q$	$r, 2r$ with prime $r = 2p^2q + 1$
p	$4pq$	(1) $r, 2r$ with prime $r = 4pq + 1$, (2) $3r, 4r, 6r$ with prime $r = 2pq + 1$, (3) q^2 with prime $q = 4p + 1$
q	$4p^2$	(1) $r, 2r$ with prime $r = 4p^2 + 1$, (2) $3r, 4r, 6r$ with prime $r = 2p^2 + 1$
p^2	$4q$	(1) $r, 2r$ with prime $r = 4q + 1$, (2) $3r, 4r, 6r$ with prime $r = 2q + 1$
pq	$4p$	(1) $r, 2r$ with prime $r = 4p + 1$, (2) $3r, 4r, 6r$ with prime $r = 2p + 1$
$2p$	$2pq$	(1) $r, 2r$ with prime $r = 2pq + 1$, (2) q^2 with prime $q = 2p + 1$
$2pq$	$2p$	$r, 2r$ with prime $r = 2p + 1$
$2p^2$	$2q$	$r, 2r$ with prime $r = 2q + 1$
p^2q	4	5, 8, 10, 12
$2p^2q$	2	3, 4, 6
$4p^2q$	1	2

Case 3. If $n = p$ and $\varphi(k) = 4pq$. By Table 2, we have $k \in \{r, 2r\}$ with prime $r = 4pq + 1$, $k \in \{3r, 4r, 6r\}$ with prime $r = 2pq + 1$ or $k = q^2$ with prime $q = 4p + 1$. We separate the discussion into three subcases.

Subcase 3.1. Let $k = r$ or $2r$ with prime $r = 4pq + 1$, then $\mathbf{C}_G(\langle x \rangle)$ is a r -group or $\{2, r\}$ -group. Suppose that $|\mathbf{C}_G(\langle x \rangle)| = 2^\alpha \cdot r^\beta$, where $\alpha \geq 0, \beta \geq 1$ and $t = |G : \mathbf{N}_G(\langle x \rangle)|$. Then we have $|G : \mathbf{N}_G(\langle x \rangle)| = t \leq p$, $|\mathbf{N}_G(\langle x \rangle) : \mathbf{C}_G(\langle x \rangle)| = 4pq$ and we also can assert that $\pi(t)$ is contained in $\{2, p, q, r\}$. Otherwise, there exists a prime $r_1 \in \pi(t)$ and $r_1 \notin \{2, p, q, r\}$, then $r_1 \in \pi(G)$. On the other hand, we can find another element $x_1 \in M(G)$ such that $r_1 \notin \pi(|G : \mathbf{N}_G(\langle x_1 \rangle)|)$. In fact, if $r_1 \in \pi(|G : \mathbf{N}_G(\langle x \rangle)|)$ for each element $x \in M(G)$, then $r_1 \in \pi(p)$. So $r_1 = p$ for p is a prime, a contradiction. Since $r > q \geq p \geq 7$, then by Lemmas 2.6 and 2.7, we can conclude that there exists no such normal series of G : $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is isomorphic to one of some simple K_3 -groups or simple K_4 -groups. Therefore, G is solvable.

Subcase 3.2. Let $k \in \{3r, 4r, 6r\}$ with prime $r = 2pq + 1$.

If $k = 3r$, then by the formula (1), we have $\pi(G) \subseteq \{2, 3, p, q, r\}$. Suppose that $|G| = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot p^{\alpha_3} \cdot q^{\alpha_4} \cdot r^{\alpha_5}$, where $\alpha_1, \alpha_3, \alpha_4 \geq 0, \alpha_2, \alpha_5 \geq 1$. If $\alpha_5 \geq 2$, then G has at least $2(r^2 - 1) = 8pq(pq + 1)$ elements of order $3r$. Since $q \geq p$, then $8pq(pq + 1) > 4p^2q$, a contradiction. So $\alpha_5 = 1$ and $|G| = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot p^{\alpha_3} \cdot q^{\alpha_4} \cdot r$, where $\alpha_1, \alpha_3, \alpha_4 \geq 0, \alpha_2 \geq 1$.

If $p = q$, then $\pi(G) \subseteq \{2, 3, p, r\}$. Suppose G is non-solvable, then there exists a section W of G such that W is isomorphic to a simple K_3 -group or simple K_4 -group. If W is isomorphic to a simple K_3 -group, then by Lemma 2.6 we have $p = 7, 13$ or 17 . Hence, $r = 2pq + 1 = 99, 339$ or 579 . Clearly, $99, 339$ and 579 are not primes, a contradiction. If W is isomorphic to the one of simple K_4 -groups. By Lemma 2.7 and [10, Theorem 2], W is isomorphic to one of following simple K_4 -groups:

- (1) $L_3(7), L_3(8), L_3(17), G_2(3), U_3(7), U_3(8)$ or ${}^3D_4(2)$;
- (2) $L_2(t)$, where r is the prime number such that $t^2 - 1 = 2^a \cdot 3^b \cdot v^c$ for $a \geq 1, b \geq 1, c \geq 1$ and prime number $v > 3$;
- (3) $L_2(2^m)$, where the number $m \geq 2$ is such that the number $2^m - 1 = u$ is prime and $2^m + 1 = 3t^b$; moreover, $t > 3$ is prime number and $b \geq 1$;
- (4) $L_2(3^m)$, where the positive integers $m \geq 2$ and $3^m + 1 = 4t, 3^m - 1 = 2u^c$ or $3^m + 1 = 4t^b, 3^m - 1 = 2u$. Moreover, the positive integer u and t are odd prime numbers, $b \geq 1$ and $c \geq 1$.

Assume that $W \cong L_3(7)$, then $p = 7$ and $r = 19$. But $r = 19 \neq 99 = 2pq + 1$, a contradiction. Arguing as above, we can get that W is not isomorphic to the one of the simple K_4 -groups in (1).

Assume that $W \cong L_2(t)$, then we have $t = r, u = p$ and equation

$$2p^2(r + 1) = 2^a \cdot 3^b \cdot p^c \quad \text{and} \quad 2p^2 + 2 = r + 1 = 2^{a-1} \cdot 3^b \cdot p^{c-2} \quad (*)$$

If $c \geq 3$, then $p|2$, a contradiction. Hence, $c = 2$. Noting that $p \geq 7$ and $r = 2p^2 + 1$, then we can easily know that there is no solution to the equation (*), a contradiction. Therefore, $W \neq L_2(t)$.

Assume that $W \cong L_2(2^m)$. Now, if $u = p$ and $t = r$, then $2 = 3r^b - p = 3(2p^2 + 1)^b - p \geq 6p^2 - p + 3 > 3$, a contradiction. If $u = r$ and $t = p$, then $2 = 3p^b - (2p^2 + 1)$, so $3p^b - 3 = 2p^2$. Thus, we have $3|2p^2$, and hence $3|p$, again a contradiction. Therefore, $W \neq L_2(2^m)$.

Assume that $W \cong L_2(3^m)$ satisfies the equation $3^m + 1 = 4t$ and $3^m - 1 = 2u^c$ where the positive integers u and t are odd prime numbers and $c \geq 1$. Now, if $u = p$ and $t = r$, then $2 = 4r - 2p^c$, that is, $4p^2 = p^c - 1$. Obviously, this equation has no solutions. If $u = r$ and $t = p$, then $2 = 4p - 2r^c \leq 4p - 4p^2 - 2 < 0$, a contradiction. Assume that $W \cong L_2(3^m)$ satisfies the equation $3^m + 1 = 4t^b$ and $3^m - 1 = 2u$ where the positive integer u and t are odd prime numbers and $b \geq 1$. Now, if $u = p$ and $t = r$, then $2 = 4r^b - 2u = 4(2p^2 + 1) - 2p \geq 8p^2 - 2p + 4 > 4$, a contradiction. If $u = r$ and $t = p$, then $2 = 4p^b - 2r = 4p^b - 4p^2 - 2$, i.e., $p^b - p^2 = 1$, which also leads to a contradiction. Therefore, $W \neq L_2(3^m)$.

Let $p < q$. Suppose that R is a Sylow r -subgroup in $\mathbf{C}_G(\langle x \rangle)$. Then $R \text{ char } \mathbf{C}_G(\langle x \rangle) \trianglelefteq \mathbf{N}_G(\langle x \rangle)$, namely, $R \trianglelefteq \mathbf{N}_G(\langle x \rangle)$ and $\mathbf{N}_G(\langle x \rangle) \leq \mathbf{N}_G(R)$. Since $|G : \mathbf{C}_G(\langle x \rangle)| = |G : \mathbf{N}_G(R)| |\mathbf{N}_G(R) : \mathbf{C}_G(\langle x \rangle)|$ and $|G : \mathbf{N}_G(\langle x \rangle)| \leq p$. By Sylow theorem, we have $|G : \mathbf{N}_G(R)| = 1$, then $R \trianglelefteq G$. So $\pi(G/R) \subseteq \{2, 3, p, q\}$. Suppose G/R is non-solvable. Then G/R has a section U such that U is isomorphic to a simple K_3 -group or simple K_4 -group. Similar argument as above, we again arrive at a contradiction.

If $k = 4r$, then by the formula (1) we have $\pi(G) \subseteq \{2, p, q, r\}$. Since $q \geq p \geq 7$ and $r > 7$, we conclude that G is solvable by Lemmas 2.6 and 2.7.

If $k = 6r$, then by the formula (1) we have $\pi(G) \subseteq \{2, 3, p, q, r\}$. Arguing as above in the case $k = 3r$, we can prove G is solvable.

Subcase 3.3. Let $k = q^2$ with prime $q = 4p + 1$. We have $|\mathbf{N}_G(\langle x \rangle) : \mathbf{C}_G(\langle x \rangle)| \mid 4pq$ and $\mathbf{C}_G(\langle x \rangle)$ is a q -group. Then by the formula (1), we can get $\pi(G) \subseteq \{2, p, q\}$. Therefore, we can deduce that G is solvable by Lemma 2.6.

Case 4. If $n = q$ and $\varphi(k) = 4p^2$. We separate the discussion into two subcases.

Subcase 4.1. If $p = q$, then by Table 2 we have $k \in \{r, 2r\}$ with prime $r = 4p^2 + 1$ or $k \in \{3r, 4r, 6r\}$ with prime $r = 2p^2 + 1$. Let $k = r$ or $2r$ with prime $r = 4p^2 + 1$. Thus by the formula (1), we have $\pi(G) \subseteq \{2, p, r\}$. Since $r > p \geq 7$, then by Lemma 2.6, we get G is solvable. If $k \in \{3r, 4r, 6r\}$ with prime $r = 2p^2 + 1$, then $\pi(G) \subseteq \{2, 3, p, r\}$. Reasoning as in Subcase 3.2, we also can get G is solvable.

Subcase 4.2. If $p < q$, according to our assumption that $2p^2 + 1$ is not a prime, then by Table 2 we have $k = r$ or $k = 2r$ with prime $r = 4p^2 + 1$. By the formula (1), we get $\pi(G) \subseteq \{2, p, q, r\}$. Since $r > p$ and $q > p \geq 7$, then by Lemmas 2.6 and 2.7, we can conclude that there exists no such normal series of G : $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is isomorphic to one of some simple K_3 -groups or simple K_4 -groups. Therefore, G is solvable.

Case 5. If $n = p^2$ and $\varphi(k) = 4q$. We separate the discussion into two subcases.

Subcase 5.1. If $p = q$, then by Table 2 we have $k \in \{r, 2r\}$ with prime $r = 4q + 1$ or $k \in \{3r, 4r, 6r\}$ with prime $r = 2q + 1$. Let $k = r$ or $2r$ with prime $r = 4q + 1$. Thus by the formula (1), we have $\pi(G) \subseteq \{2, p, r\}$. Since $r > p \geq 7$, then by Lemma 2.6, we get G is solvable. If $k \in \{3r, 4r, 6r\}$ with prime $r = 2q + 1$, then $\pi(G) \subseteq \{2, 3, p, r\}$. Reasoning as in Subcase 3.2, we can get G is solvable.

Subcase 5.2. If $p < q$, according to our assumption that $2q + 1$ is not a prime, then by Table 2 one has that $k = r$ or $k = 2r$ with prime $r = 4q + 1$. By the formula (1), we have $\pi(G) \subseteq \{2, p, q, r\}$. Since $r > p$ and $q > p \geq 7$, then by Lemmas 2.6 and 2.7, we can conclude that G is solvable.

Case 6. If $n = pq$ and $\varphi(k) = 4p$. We also separate the discussion into two subcases.

Subcase 6.1. If $p = q$, then by Table 2 we have $k \in \{r, 2r\}$ with prime $r = 2p + 1$ or $k \in \{3r, 4r, 6r\}$ with prime $r = 2p + 1$. Let $k = r$ or $2r$ with prime $r = 4p + 1$. Thus by the formula (1) we have $\pi(G) \subseteq \{2, p, r\}$. Since $r > p \geq 7$, then by Lemma 2.6, we get G is solvable. If $k \in \{3r, 4r, 6r\}$ with prime $r = 2p + 1$, then $\pi(G) \subseteq \{2, 3, p, r\}$. Reasoning as in Subcase 3.2, we can show that G is solvable.

Subcase 6.2. If $p < q$, according to our assumption that $2p + 1$ is not a prime, then by Table 2 we have $k = r$ or $k = 2r$ with prime $r = 4p + 1$. By the formula (1), we have $\pi(G) \subseteq \{2, p, q, r\}$. Since $r > p$ and $q > p \geq 7$, then by Lemmas 2.6 and 2.7, we can conclude that G is solvable.

Case 7. If $n = 2p$ and $\varphi(k) = 2pq$. If $p = q$, by Table 2 we have $k \in \{r, 2r\}$ with prime $r = 2p^2 + 1$. Thus by the formula (1), we have $\pi(G) \subseteq \{2, p, r\}$. Since $r > p \geq 7$, then by Lemma 2.6 we have G is solvable. If $p < q$, by our assumption that $2p + 1$ is not a prime, then by Table 2 we have $k = r$ or $k = 2r$ with prime $r = 2pq + 1$. By the formula (1), we have $\pi(G) \subseteq \{2, p, q, r\}$. Since $r > q > p \geq 7$, then by Lemmas 2.6 and 2.7, we can conclude that G is solvable.

Case 8. If $n = 2pq$ and $\varphi(k) = 2p$. If $p < q$, by our assumption that $2p + 1$ is not a prime. Then we only need to consider the case $p = q$. By Table 2, we have $k = r$ or $k = 2r$ with prime $r = 2p + 1$. Therefore, $\pi(G) \subseteq \{2, p, r\}$. Since $r > p \geq 7$. By Lemma 2.6, we know that G is solvable.

By using the same argument as above, we can prove that G is solvable if $n = 2p^2$ and $\varphi(k) = 2q$.

Case 9. If $|M(G)| = 4p^2q$ and $\varphi(k) = 4$. By Table 2, k can only be equal to one of 5, 8, 10 or 12.

If $k = 8$, then $\pi(G) \subseteq \{2, p, q\}$. Since $p, q \geq 7$, then we know that G is solvable by Lemma 2.6.

If $k = 5$ or 10, then $\pi(G) \subseteq \{2, 5, p, q\}$. Since the maximal order of G is 12, thus the prime divisors of $|G|$ are not more than 11. Then G is one of a $\{2, 5\}$ -group, $\{2, 5, 7\}$ -group, $\{2, 5, 11\}$ -group or $\{2, 5, 7, 11\}$ -group. By Lemmas 2.6 and 2.7, it is easy to know that G is solvable.

If $k = 12$, then $\pi(G) \subseteq \{2, 3, p, q\}$. If G is a $\{2, 3\}$ -group or a $\{2, 3, 11\}$ -group, then G is solvable. Hence, we only need to consider that G is a $\{2, 3, 7\}$ -group or $\{2, 3, 7, 11\}$ -group.

If G is a $\{2, 3, 7\}$ -group, then $p = 7$. Write $|G| = 2^\alpha \cdot 3^\beta \cdot 7^\gamma$, where $\alpha, \beta, \gamma \geq 1$. Since $k = 12$, then the prime graph of G is disconnected and $t(G) = 2$. By Lemma 2.9, we get G can only be isomorphic to one of the following holds:

- (1) G is a Frobenius group or 2-Frobenius group;
- (2) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1 -groups, K/H is simple group and H is nilpotent, where $2 \in \pi_1$. In particular, $|G/K| \mid |\text{Out}(K/H)|$.

If G is a 2-Frobenius group, then by Lemma 2.10, we have G is solvable. If G is a Frobenius group with H the Frobenius kernel and K the Frobenius complement, respectively. Then it follows from Lemma 2.11 that $T(G) = \{\pi(H), \pi(K)\}$. Thus, H is either a $\{2, 3\}$ -Hall subgroup or 7-Hall subgroup, and so H is solvable. Then G/H is a 7-group or a $\{2, 3\}$ -group, and so G/H is solvable. Hence, G is solvable. Therefore, we suppose that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1 -groups and K/H is simple, where π_1 is the prime graph component containing 2, H is nilpotent and $|G/K| \mid |\text{Out}(K/H)|$. Noting that G is a $\{2, 3, 7\}$ -group. Then by Lemma 2.6, we have K/H is isomorphic to one of $L_2(7)$, $L_2(8)$ or $U_3(3)$.

If G is $\{2, 3, 7, 11\}$ -group, then we have $p = 7$ and $q = 11$. Suppose that $|G| = 2^u \cdot 3^v \cdot 7 \cdot 11$, where $u, v \geq 1$. Since $t(G) = 3$, then G is not a Frobenius group or 2-Frobenius group by Lemmas 2.10 and 2.11. Hence, by Lemma 2.9 we get G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1 -groups and K/H is simple, where π_1 is the prime graph component containing 2, H is a nilpotent group and $|G/K| \mid |\text{Out}(K/H)|$. If K/H is a simple K_4 -group, then $p = 7$, $q = 11$ and at least satisfies one of the equations (2), (3) or (4) in Subcase 3.2. However, it is easy to see this is impossible. Therefore, K/H can only be isomorphic to a simple K_3 -group. By Lemma 2.6, we have K/H is isomorphic to one of $L_2(7)$, $L_2(8)$ or $U_3(3)$. Then by $|G/K| \mid |\text{Out}(K/H)|$ we have $11 \mid |H|$, and so G has a normal subgroup N of order 11. Now let x be an element order of 7 acts on N through conjugation, it follows that $77 \in \pi_e(G)$, a contradiction.

Case 10. If $|M(G)| = 4p^2q$ and $\varphi(k) = 2$. Suppose that G is non-solvable. Then $5 \in \pi(G)$ and so $k = 6$. By Lemma 2.8, we have $\pi_e(G) = \{1, 2, 3, 4, 5, 6\}$. Then by [14, Lemma 4], G is isomorphic to one of S_5 , S_6 or $G/O_2(G) \cong A_5$, where $|O_2(G)| = 2^{4m}$ and $\exp(P_2(G))|4$.

If $G \cong S_5$, then $|M(G)| = 20 \neq 4p^2q$, a contradiction.

If $G \cong S_6$, then $|M(G)| = 240 \neq 4p^2q$, again a contradiction.

Assume that $G/O_2(G) \cong A_5$, where $|O_2(G)| = 2^{4m}$ and $\exp(P_2(G))|4$. Then $|P_3(G)| = 3$. We have $\mathbf{N}_G(P_3(G))/\mathbf{C}_G(P_3(G)) \leq \text{Aut}(P_3(G))$. If $\mathbf{N}_G(P_3(G)) = \mathbf{C}_G(P_3(G))$, then G is obviously solvable, which leads to a contradiction. So we have that $|\mathbf{N}_G(P_3(G))/\mathbf{C}_G(P_3(G))| = 2$. Let $r = 3$, $\alpha = 2$. Then by Lemma 2.4, we know $P_3(G)$ is an r -total subgroup of G . Furthermore, by Sylow's theorem, $\text{Inn}(G)$ acts transitively on $M_3(G)$. Let $x \in P_3(G)$, $k = 3 \times 2$, $r_1 = 3$ and $r_2 = 2$. Then $|M(G)| = |M_3(G)||M_2(\mathbf{C}_G(x))|$ by Lemma 2.5. So $|M(G)| = 2n_1|M_2(\mathbf{C}_G(x))| = 4p^2q$, where n_1 is the number of Sylow 3-subgroups of G . Moreover, $P_3(G)$ is not normal in G for $\pi(|G : P_3(G)|) = \{2, 5\}$, which means that $n_1 > 1$. Hence, n_1 can only be one of $2, p, q, 2p, 2q, pq, 2pq, p^2, 2p^2, p^2q$ or $2p^2q$. Noting that $n_1|G$ and $n_1 \equiv 1 \pmod{3}$. If $n_1 = 2$, then $n_1 \equiv 2 \pmod{3}$, a contradiction. If $n_1 = p, q, 2p, 2q, pq, 2pq, p^2, 2p^2, p^2q$ or $2p^2q$, then there exists an element of order p or q , which also leads to a contradiction. Thus, G is solvable.

Case 11. If $|M(G)| = 4p^2q$ and $\varphi(k) = 1$. In this case, we have $k = 2$, and so G is an elementary abelian 2-group. Clearly, the number of largest order elements in G is $2^m - 1$ for some positive integer m , a contradiction.

This completes the proof of Theorem 1.2. □

Acknowledgement: The authors would like to thank the referee for valuable comments.

Funding information: This work was partially supported by Natural Science Foundation of China (Grant Nos 12071376 and 11971391), Natural Science Foundation Project of CQ CSTC (Nos cstc2021jcyj-msxm0103), Fundamental Research Funds for the Central Universities (Nos XDJK2019C116 and XDJK2019B030) and Teaching Reform Project of Southwest University (No. 2018JY061).

Conflict of interest: Authors state no conflict of interest.

References

- [1] D. Gorenstein, *Finite Groups*, Harper & Row Press, New York, 1968.
- [2] V. D. Mazurov and E. I. Khukhro, *The Kourovka Notebook: Unsolved Problems in Group Theory*, 17th edn., Institute of Mathematics, Russian Academy of Sciences, Siberian Division, 2010.
- [3] B. Asadian and N. Ahanjideh, *Finite groups with $2pqr$ elements of the maximal order*, Ukrainian Math. J. **69** (2018), 1479–1484, DOI: <https://doi.org/10.1007/s11253-018-1447-6>.
- [4] G. Y. Chen and W. J. Shi, *Finite groups with 30 elements of maximal order*, Appl. Categ. Structures **16** (2008), 239–247, DOI: <https://doi.org/10.1007/s10485-007-9067-6>.
- [5] Y. Y. Jiang, *A theorem of finite groups with $18p$ having maximal order*, Algebra Colloq. **15** (2008), no. 2, 317–329, DOI: <https://doi.org/10.1142/S1005386708000291>.
- [6] Y. Xu, J. J. Gao, and H. L. Hou, *Finite groups with $6pq$ elements of the largest order*, Ital. J. Pure Appl. Math. **31** (2013), 277–284.
- [7] C. Yang, *Finite groups based on the numbers of elements of maximal order*, Ann. Math. **14A** (1993), no. 5, 561–567. (in Chinese)
- [8] M. Herzog, *On finite simple groups of order divisible by three primes only*, J. Algebra **10** (1968), no. 3, 383–388, DOI: [https://doi.org/10.1016/0021-8693\(68\)90088-4](https://doi.org/10.1016/0021-8693(68)90088-4).
- [9] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.
- [10] W. J. Shi, *On simple K_4 -groups*, Chin. Sci. Bull. **36** (1991), no. 7, 1281–1283. (in Chinese)
- [11] C. Yang, S. G. Wang, and X. J. Zhao, *On finite groups with $\pi_e(G) = \{1, 2, 3, 5, 6\}$* , J. Eng. Math. **17** (2000), 105–108. (in Chinese)

- [12] J. S. Williams, *Prime graph components of finite group*, J. Algebra **69** (1981), no. 2, 487–513, DOI: [https://doi.org/10.1016/0021-8693\(81\)90218-0](https://doi.org/10.1016/0021-8693(81)90218-0).
- [13] G. Y. Chen, *On Frobenius groups and 2-Frobenius groups*, J. Southwest China Normal Univ. (Natural Science Edition) **20** (1995), no. 5, 485–487.
- [14] R. Brandl and S. Wujie, *Finite groups whose elements order are consecutive integers*, J. Algebra **143** (1991), no. 2, 388–400, DOI: [https://doi.org/10.1016/0021-8693\(91\)90271-9](https://doi.org/10.1016/0021-8693(91)90271-9).