Research Article

Huseyin Budak, Hasan Kara, Muhammad Aamir Ali, Sundas Khan, and Yuming Chu*

Fractional Hermite-Hadamard-type inequalities for interval-valued co-ordinated convex functions

<https://doi.org/10.1515/math-2021-0067>

received November 7, 2020; accepted July 5, 2021

Abstract: In this work, we introduce the notions about the Riemann-Liouville fractional integrals for interval-valued functions on co-ordinates. We also establish Hermite-Hadamard and some related inequalities for co-ordinated convex interval-valued functions by applying the newly defined fractional integrals. The results of the present paper are the extension of several previously published results.

Keywords: fractional integrals, Hermite-Hadamard inequality, interval-valued functions

MSC 2020: 26D10, 26D15, 26A51

1 Introduction

The Hermite-Hadamard inequality discovered by Hermite and Hadamard (see, e.g., [1], [2, p. 137]) is one of the most well-established inequalities in the theory of convex functions having a geometrical interpretation and many applications. The Hermite-Hadamard inequality states that if $f : I \rightarrow \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}. \quad (1)$$

Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it can be easily done by using Jensen's inequality. Hermite-Hadamard inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been studied (see, for example, [3–26] and references therein).

On the other hand, interval analysis is a particular case of set-valued analysis, which is the study of sets in the spirit of mathematical analysis and general topology. It was introduced as an attempt to handle the

* **Corresponding author: Yuming Chu**, Institute for Advanced Study Honoring Chen Jian Gong, Hangzhou Normal University, Hangzhou 311121, China, e-mail: chuyuming2005@126.com

Huseyin Budak: Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey, e-mail: hsyn.budak@gmail.com

Hasan Kara: Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey, e-mail: hasan64kara@gmail.com

Muhammad Aamir Ali: Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing, 210023, China, e-mail: mahr.muhammad.aamir@gmail.com

Sundas Khan: Department of Mathematics, GC Women University, Sialkot, Pakistan, e-mail: sundaskhan818@gmail.com

interval uncertainty that appears in many mathematical or computer models of some deterministic real-world phenomena. An old example of interval enclosure is Archimede's method which is related to computing of the circumference of a circle. In 1966, the first book related to interval analysis was given by Moore, who is known as the first user of intervals in computational mathematics [27]. After his book, several scientists started to investigate the theory and applications of interval arithmetic. Nowadays, because of its applications, interval analysis is a useful tool in various areas which are interested intensely in uncertain data. You can see applications in computer graphics, experimental and computational physics, error analysis, robotics, and many others.

In addition, several important inequalities (Hermite-Hadamard, Ostrowski, etc.) have been studied for interval-valued functions in recent years. In [28,29], Chalco-Cano et al. obtained Ostrowski-type inequalities for interval-valued functions by using Hukuhara derivative for interval-valued functions. However, inequalities were studied for more general set-valued maps. In [30–32], the authors proved different variants of the Hermite-Hadamard inequalities for interval-valued functions.

The purpose of this paper is to complete the Riemann-Liouville integrals for interval-valued functions and to obtain Hermite-Hadamard inequalities via these integrals. Furthermore, Hermite-Hadamard-type inequalities are given using these integrals.

2 Preliminaries

In this section, we recall some basic definitions, results, notions, and properties, which are used throughout the paper. We denote $\mathbb{R}_I^+$ the family of all positive intervals of $\mathbb{R}$. The Hausdorff distance between $[\underline{X}, \overline{X}]$ and $[\underline{Y}, \overline{Y}]$ is defined as

$$d([\underline{X}, \overline{X}], [\underline{Y}, \overline{Y}]) = \max\{|\underline{X} - \underline{Y}|, |\overline{X} - \overline{Y}|\}.$$

The $(\mathbb{R}_I, d)$ is a complete metric space. For more details and basic notations on interval-valued functions, see [33,34].

In [27], Moore introduced the Riemann integral for interval-valued functions. The set of all Riemann integrable interval-valued functions and real-valued functions on $[a, b]$ are denoted by $\mathcal{IR}_{([a,b])}$ and $\mathcal{R}_{([a,b])}$, respectively. The following theorem gives a relation between (IR) -integrable and Riemann integrable (R -integrable) (see [33, p. 131]):

Theorem 1. Let $F : [a, b] \rightarrow \mathbb{R}_I$ be an interval-valued function such that $F(t) = [\underline{F}(t), \overline{F}(t)]$. $F \in \mathcal{IR}_{([a,b])}$ if and only if $\underline{F}(t), \overline{F}(t) \in \mathcal{R}_{([a,b])}$, and

$$(IR) \int_a^b F(t)dt = \left[(R) \int_a^b \underline{F}(t)dt, (R) \int_a^b \overline{F}(t)dt \right].$$

In [34,35], Zhao et al. introduced a class of convex interval-valued functions as follows:

Definition 1. Let $h : J \subset \mathbb{R} \rightarrow \mathbb{R}$ be a positive function. We say that $F : I \subset \mathbb{R} \rightarrow \mathbb{R}_I^+$ is a h -convex interval-valued function, if for all $x, y \in I$ and $t \in (0, 1)$, we have

$$h(t)F(x) + h(1-t)F(y) \subseteq F(tx + (1-t)y). \quad (2)$$

With $SX(h, [a, b], \mathbb{R}_I^+)$ we will show the set of all h -convex interval-valued functions.

The usual notion of convex interval-valued function corresponds to relation (2) with $h(t) = t$, see [30]. Furthermore, if we take $h(t) = t^s$ in (2), then Definition 1 gives the convex interval-valued function defined by Breckner, see [36].

In [34], Zhao et al. obtained the following Hermite-Hadamard inequality for h -convex interval-valued functions:

Theorem 2. [34] Let $F : [a, b] \rightarrow \mathbb{R}_I^+$ be an interval-valued function such that $F(t) = [\underline{F}(t), \overline{F}(t)]$ and $F \in \mathcal{IR}_{([a,b])}$, $h : [0, 1] \rightarrow \mathbb{R}$ be a non-negative function and $h(\frac{1}{2}) \neq 0$. If $F \in SX(h, [a, b], \mathbb{R}_I^+)$, then

$$\frac{1}{2h(\frac{1}{2})}F\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}(IR) \int_a^b F(x)dx \supseteq [F(a) + F(b)] \int_0^1 h(t)dt. \quad (3)$$

Remark 1.

(i) If $h(t) = t$, then (3) reduces to the following result:

$$F\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}(IR) \int_a^b F(x)dx \supseteq \frac{F(a) + F(b)}{2}, \quad (4)$$

which is obtained by [30].

(ii) If $h(t) = t^s$, then (3) reduces to the following result:

$$2^{s-1}F\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}(IR) \int_a^b F(x)dx \supseteq \frac{F(a) + F(b)}{s+1},$$

which is obtained by [37].

In [38], Budak et al. gave the fractional version of Hermite-Hadamard inequalities for convex interval-valued functions as follows:

Theorem 3. If $F : [a, b] \rightarrow \mathbb{R}_I^+$ is a convex interval-valued function such that $F(t) = [\underline{F}(t), \overline{F}(t)]$ and $\alpha > 0$, then we have

$$F\left(\frac{a+b}{2}\right) \supseteq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a+}^\alpha F(b) + J_{b-}^\alpha F(a)] \supseteq \frac{F(a) + F(b)}{2}. \quad (5)$$

Theorem 4. If $F, G : [a, b] \rightarrow \mathbb{R}_I^+$ are two convex interval-valued functions such that $F(t) = [\underline{F}(t), \overline{F}(t)]$ and $G(t) = [\underline{G}(t), \overline{G}(t)]$, then for $\alpha > 0$ we have

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a+}^\alpha F(b)G(b) + J_{b-}^\alpha F(a)G(a)] \supseteq \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)M(a, b) + \frac{\alpha}{(\alpha+1)(\alpha+2)}N(a, b) \quad (6)$$

and

$$\begin{aligned} 2F\left(\frac{a+b}{2}\right)G\left(\frac{a+b}{2}\right) &\supseteq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a+}^\alpha F(b)G(b) + J_{b-}^\alpha F(a)G(a)] \\ &\quad + \frac{\alpha}{(\alpha+1)(\alpha+2)}M(a, b) + \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)N(a, b), \end{aligned} \quad (7)$$

where

$$M(a, b) = F(a)G(a) + F(b)G(b) \quad \text{and} \quad N(a, b) = F(a)G(b) + F(b)G(a).$$

3 Fractional integral of interval-valued functions

In this section, we introduce the notions of fractional double integral for interval-valued functions and recall some basic definitions of interval-valued integrals. We also give the definition of interval-valued convex functions on co-ordinates. In the sequel of the paper, $\Delta = [a, b] \times [c, d]$.

In [39], Lupulescu defined the following interval-valued left-sided Riemann-Liouville fractional integral.

Definition 2. Let $F : [a, b] \rightarrow \mathbb{R}_I$ be an interval-valued function such that $F(t) = [\underline{F}(t), \overline{F}(t)]$ and let $\alpha > 0$. The interval-valued left-sided Riemann-Liouville fractional integral of a function f is defined by

$$\mathcal{J}_{a+}^{\alpha} F(x) = \frac{1}{\Gamma(\alpha)} (IR) \int_a^x (x-s)^{\alpha-1} F(s) ds, \quad x > a,$$

where Γ is the Euler Gamma function.

Based on the definition of Lupulescu, Budak et al. in [38] gave the definition of interval-valued right-sided Riemann-Liouville fractional integral of the function by

$$\mathcal{J}_b^{\alpha} F(x) = \frac{1}{\Gamma(\alpha)} (IR) \int_x^b (s-x)^{\alpha-1} F(s) ds, \quad x < b,$$

where Γ is the Euler Gamma function.

Theorem 5. If $F : [a, b] \rightarrow \mathbb{R}_I$ is an interval-valued function such that $F(t) = [\underline{F}(t), \overline{F}(t)]$, then we have

$$\mathcal{J}_{a+}^{\alpha} \overline{F}(x) = [I_{a+}^{\alpha} \underline{F}(x), I_{a+}^{\alpha} \overline{F}(x)]$$

and

$$\mathcal{J}_b^{\alpha} \underline{F}(x) = [I_b^{\alpha} \underline{F}(x), I_b^{\alpha} \overline{F}(x)].$$

Now we recall the concept of interval-valued double integral given by Zhao et al. in [40]:

Theorem 6. [40] Let $F : \Delta = [a, b] \times [c, d] \rightarrow \mathbb{R}_I$. Then F is called ID-integrable on Δ with ID-integral $U = (ID) \iint_{\Delta} F(t, s) dA$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(S(F, P, \delta, \Delta)) < \varepsilon$$

for any $P \in \mathcal{P}(\delta, \Delta)$. The collection of all ID-integrable functions on Δ will be denoted by $ID_{(\Delta)}$. For more details about the notations used here, one can read [40].

Theorem 7. [40] Let $\Delta = [a, b] \times [c, d]$. If $F : \Delta \rightarrow \mathbb{R}_I$ is ID-integrable on Δ , then we have

$$(ID) \iint_{\Delta} F(s, t) dA = (IR) \int_a^b (IR) \int_c^d F(s, t) ds dt.$$

Example 1. Let $F : \Delta = [0, 1] \times [1, 2] \rightarrow \mathbb{R}_I^+$ be defined by

$$F(s, t) = [st, s + t],$$

then $F(s, t)$ is integrable on Δ and $(ID) \iint_{\Delta} F(s, t) dA = \left[\frac{3}{4}, 2 \right]$.

By applying the concepts of Lupulescu [39] and Zhao et al. [40] about interval-valued integrals, we can define interval-valued Riemann-Liouville double fractional integral of the function $F(x, y)$ by

Definition 3. Let $F \in L_1([a, b] \times [c, d])$. The Riemann-Liouville integrals $\mathcal{J}_{a+, c+}^{\alpha, \beta}$, $\mathcal{J}_{a+, d-}^{\alpha, \beta}$, $\mathcal{J}_{b-, c+}^{\alpha, \beta}$, and $\mathcal{J}_{b-, d-}^{\alpha, \beta}$ of order $\alpha, \beta > 0$ with $a, c \geq 0$ are defined by

$$\begin{aligned}\mathcal{J}_{a+, c+}^{\alpha, \beta} F(x, y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} (IR) \int_a^x \int_c^y (x-t)^{\alpha-1} (y-s)^{\beta-1} F(t, s) ds dt, \quad x > a, \ y > c, \\ \mathcal{J}_{a+, d-}^{\alpha, \beta} F(x, y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} (IR) \int_a^x \int_y^d (x-t)^{\alpha-1} (s-y)^{\beta-1} F(t, s) ds dt, \quad x > a, \ y > d, \\ \mathcal{J}_{b-, c+}^{\alpha, \beta} F(x, y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} (IR) \int_x^b \int_c^y (t-x)^{\alpha-1} (y-s)^{\beta-1} F(t, s) ds dt, \quad x < b, \ y > c, \\ \mathcal{J}_{b-, d-}^{\alpha, \beta} F(x, y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} (IR) \int_x^b \int_y^d (t-x)^{\alpha-1} (s-y)^{\beta-1} F(t, s) ds dt, \quad x < b, \ y < d,\end{aligned}$$

respectively.

Definition 4. [41] A function $F : \Delta \rightarrow \mathbb{R}_I^+$ is said to be interval-valued co-ordinated convex function, if the following inequality holds

$$F(tx + (1-t)y, su + (1-s)w) \supseteq tF(x, u) + t(1-s)F(x, w) + s(1-t)F(y, u) + (1-s)(1-t)F(y, w),$$

for all $(x, y), (u, w) \in \Delta$ and $s, t \in [0, 1]$.

Lemma 1. [41] A function $F : \Delta \rightarrow \mathbb{R}_I^+$ is interval-valued convex on co-ordinates if and only if there exist two functions $F_x : [c, d] \rightarrow \mathbb{R}_I^+$, $F_x(w) = F(x, w)$ and $F_y : [a, b] \rightarrow \mathbb{R}_I^+$, $F_y(u) = F(y, u)$ are interval-valued convex.

It is easy to prove that an interval-valued convex function is interval-valued co-ordinated convex but the converse may not be true. For this we can see the following example.

Example 2. An interval-valued function $f : [0, 1]^2 \rightarrow \mathbb{R}_I^+$ defined as $F(x, y) = [xy, (6 - e^x)(6 - e^y)]$ is interval-valued convex on co-ordinates but is not interval-valued convex on $[0, 1]^2$.

Proposition 1. [41] If $F, G : \Delta \rightarrow \mathbb{R}_I^+$ are two interval-valued co-ordinated convex functions on Δ and $\alpha \geq 0$, then $F + G$ and αF are interval-valued co-ordinated convex functions.

Proposition 2. [41] If $F, G : \Delta \rightarrow \mathbb{R}_I^+$ are two interval-valued co-ordinated convex functions on Δ , then (FG) is interval-valued co-ordinated convex function on Δ .

4 Main results

In this section, we establish Hermite-Hadamard integral inequalities for interval-valued co-ordinated convex functions by applying interval-valued double fractional integral. We also present inequalities of Hermite-Hadamard-type for the product of interval-valued co-ordinated convex functions.

Theorem 8. If $F : \Delta \rightarrow \mathbb{R}_I^+$ is an interval-valued co-ordinated convex function on Δ such that $F(t) = [\underline{F}(t), \overline{F}(t)]$, then the following inequalities hold:

$$\begin{aligned}
 F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\supseteq \frac{\Gamma(\alpha+1)}{4(b-a)^\alpha} \left[\mathcal{J}_{a^+}^\alpha F\left(b, \frac{c+d}{2}\right) + \mathcal{J}_b^\alpha F\left(a, \frac{c+d}{2}\right) \right] \\
 &\quad + \frac{\Gamma(\beta+1)}{4(d-c)^\beta} \left[\mathcal{J}_{c^+}^\beta F\left(\frac{a+b}{2}, d\right) + \mathcal{J}_d^\beta F\left(\frac{a+b}{2}, c\right) \right] \\
 &\supseteq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} [\mathcal{J}_{a^+,c}^{\alpha,\beta} F(b, d) + \mathcal{J}_{a^+,d}^{\alpha,\beta} F(b, c) + \mathcal{J}_{b^-,c}^{\alpha,\beta} F(a, d) + \mathcal{J}_{b^-,d}^{\alpha,\beta} F(a, c)] \\
 &\supseteq \frac{\Gamma(\alpha+1)}{8(b-a)^\alpha} [\mathcal{J}_{a^+}^\alpha F(b, c) + \mathcal{J}_{a^+}^\alpha F(b, d) + \mathcal{J}_b^\alpha F(a, c) + \mathcal{J}_b^\alpha F(a, d)] \\
 &\quad + \frac{\Gamma(\beta+1)}{4(d-c)^\beta} [\mathcal{J}_{c^+}^\beta F(a, d) + \mathcal{J}_{c^+}^\beta F(b, d) + \mathcal{J}_d^\beta F(a, c) + \mathcal{J}_d^\beta F(b, c)] \\
 &\supseteq \frac{F(a, c) + F(a, d) + F(b, c) + F(b, d)}{4}.
 \end{aligned} \tag{8}$$

Proof. Since F is an interval-valued co-ordinated convex function on Δ , it follows that the mapping, $G_y : [a, b] \rightarrow \mathbb{R}_I^+$, $G_y := F(x, y)$ is an interval-valued convex on $[a, b]$ for all $x \in [a, b]$. From inequality (5), we have:

$$G_y\left(\frac{a+b}{2}\right) \supseteq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [\mathcal{J}_{a^+}^\alpha G_y(b) + \mathcal{J}_b^\alpha G_y(a)] \supseteq \frac{G_y(a) + G_y(b)}{2}.$$

That can be written as,

$$F\left(\frac{a+b}{2}, y\right) \supseteq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [\mathcal{J}_{a^+}^\alpha F(b, y) + \mathcal{J}_b^\alpha F(a, y)] \supseteq \frac{F(a, y) + F(b, y)}{2}.$$

That is,

$$F\left(\frac{a+b}{2}, y\right) \supseteq \frac{\alpha}{2(b-a)^\alpha} \left[(IR) \int_a^b (b-t)^{\alpha-1} F(t, y) dt + (IR) \int_a^b (t-a)^{\alpha-1} F(t, y) dt \right] \supseteq \frac{F(a, y) + F(b, y)}{2}. \tag{9}$$

Multiplying the both sides of inequality (28) by $\frac{\beta(d-y)^{\beta-1}}{2(d-c)^\beta}$ and integrating the resultant one with respect to y over $[c, d]$, we have

$$\begin{aligned}
 &\frac{\beta}{2(d-c)^\beta} (IR) \int_c^d F\left(\frac{a+b}{2}, y\right) (d-y)^{\beta-1} dy \\
 &\supseteq (IR) \int_a^b \int_c^d (b-t)^{\alpha-1} (d-y)^{\beta-1} F(t, y) dy dt + (IR) \int_a^b \int_c^d (t-a)^{\alpha-1} (d-y)^{\beta-1} F(t, y) dy dt \\
 &\supseteq \frac{1}{2} \left[\frac{\beta}{2(d-c)^\beta} (IR) \int_c^d F(a, y) (d-y)^{\beta-1} dy + \frac{\beta}{2(d-c)^\beta} (IR) \int_c^d F(b, y) (d-y)^{\beta-1} dy \right].
 \end{aligned} \tag{10}$$

Again, multiplying the both sides of inequality (28) by $\frac{\beta(y-c)^{\beta-1}}{2(d-c)^\beta}$ and integrating the resultant one with respect to y over $[c, d]$, we have

$$\begin{aligned}
 &\frac{\beta}{2(d-c)^\beta} (IR) \int_c^d F\left(\frac{a+b}{2}, y\right) (y-c)^{\beta-1} dy \\
 &\supseteq \frac{\alpha\beta}{4(b-a)^\alpha(d-c)^\beta} \left[(IR) \int_a^b \int_c^d (b-t)^{\alpha-1} (y-c)^{\beta-1} F(t, y) dy dt + (IR) \int_a^b \int_c^d (t-a)^{\alpha-1} (y-c)^{\beta-1} F(t, y) dy dt \right]
 \end{aligned} \tag{11}$$

$$\geq \frac{1}{2} \left[\frac{\beta}{2(d-c)^\beta} (IR) \int_c^d F(a, y)(y-c)^{\beta-1} dy + \frac{\beta}{2(d-c)^\beta} (IR) \int_c^d F(b, y)(y-c)^{\beta-1} dy \right].$$

Moreover, inequality (10) can be written as

$$\begin{aligned} \frac{\Gamma(\beta+1)}{2(d-c)^\beta} \mathcal{J}_{c^+}^\beta F\left(\frac{a+b}{2}, d\right) &\geq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} [\mathcal{J}_{a^+,c^+}^{\alpha,\beta} F(b, d) + \mathcal{J}_{b^-,c^+}^{\alpha,\beta} F(a, d)] \\ &\geq \frac{\Gamma(\beta+1)}{4(d-c)^\beta} \mathcal{J}_{c^+}^\beta F(a, d) + \mathcal{J}_{c^+}^\beta F(b, d) \end{aligned} \quad (12)$$

and inequality (11) can be written as

$$\begin{aligned} \frac{\Gamma(\beta+1)}{2(d-c)^\beta} \mathcal{J}_d^\beta F\left(\frac{a+b}{2}, c\right) &\geq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} [\mathcal{J}_{a^+,d}^{\alpha,\beta} F(b, c) + \mathcal{J}_{b^-,d}^{\alpha,\beta} F(a, c)] \\ &\geq \frac{\Gamma(\beta+1)}{4(d-c)^\beta} \mathcal{J}_d^\beta F(a, c) + \mathcal{J}_d^\beta F(b, c). \end{aligned} \quad (13)$$

Similarly, $H_x : [c, d] \rightarrow \mathbb{R}_I^+$, $H_x(y) := F(x, y)$ is an interval-valued convex function on $[c, d]$ and $y \in [a, b]$, we get

$$\begin{aligned} \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \mathcal{J}_{a^+}^\alpha F\left(b, \frac{c+d}{2}\right) &\geq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} [\mathcal{J}_{a^+,c^+}^{\alpha,\beta} F(b, d) + \mathcal{J}_{a^+,d}^{\alpha,\beta} F(b, c)] \\ &\geq \frac{\Gamma(\alpha+1)}{4(b-a)^\alpha} \mathcal{J}_{a^+}^\alpha F(b, c) + \mathcal{J}_{a^+}^\alpha F(b, d) \end{aligned} \quad (14)$$

and

$$\begin{aligned} \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \mathcal{J}_b^\alpha F\left(a, \frac{c+d}{2}\right) &\geq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} [\mathcal{J}_{b^-,c^+}^{\alpha,\beta} F(a, d) + \mathcal{J}_{b^-,d}^{\alpha,\beta} F(a, c)] \\ &\geq \frac{\Gamma(\alpha+1)}{4(b-a)^\alpha} [\mathcal{J}_{b^+}^\alpha F(a, c) + \mathcal{J}_{b^+}^\alpha F(a, d)]. \end{aligned} \quad (15)$$

Summing inequalities (12), (13), (14), and (15), we obtain second, third, and fourth inequalities of Theorem 8.

From (5), we have

$$F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \geq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[\mathcal{J}_{a^+}^\alpha F\left(b, \frac{c+d}{2}\right) + \mathcal{J}_b^\alpha F\left(a, \frac{c+d}{2}\right) \right] \quad (16)$$

$$F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \geq \frac{\Gamma(\beta+1)}{2(d-c)^\beta} \left[\mathcal{J}_{c^+}^\beta F\left(\frac{a+b}{2}, d\right) + \mathcal{J}_d^\beta F\left(\frac{a+b}{2}, c\right) \right]. \quad (17)$$

By adding (16) and (17) and using Theorem 9, we have the first inequality in (8),

$$\begin{aligned} F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\geq \frac{\Gamma(\alpha+1)}{4(b-a)^\alpha} \left[\mathcal{J}_{a^+}^\alpha F\left(b, \frac{c+d}{2}\right) + \mathcal{J}_b^\alpha F\left(a, \frac{c+d}{2}\right) \right] \\ &\quad + \frac{\Gamma(\beta+1)}{4(d-c)^\beta} \left[\mathcal{J}_{c^+}^\beta F\left(\frac{a+b}{2}, d\right) + \mathcal{J}_d^\beta F\left(\frac{a+b}{2}, c\right) \right]. \end{aligned}$$

At the end, again from (3) and Theorem 9, we have

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[\mathcal{J}_{a^+}^\alpha F(b, c) + \mathcal{J}_b^\alpha F(a, c) \right] \geq \frac{F(a, c) + F(b, c)}{2}, \quad (18)$$

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[\mathcal{J}_{a^+}^\alpha F(b, d) + \mathcal{J}_b^\alpha F(a, d) \right] \geq \frac{F(a, d) + F(b, d)}{2}, \quad (19)$$

$$\frac{\Gamma(\beta+1)}{2(d-c)^\beta} \mathcal{J}_{c^+}^\beta F(a, d) + \mathcal{J}_d^\beta F(a, c) \geq \frac{F(a, c) + F(a, d)}{2}, \quad (20)$$

$$\frac{\Gamma(\beta+1)}{2(d-c)^\beta} \mathcal{J}_{c^+}^\beta F(b, d) + \mathcal{J}_d^\beta F(b, c) \geq \frac{F(b, c) + F(b, d)}{2}. \quad (21)$$

If we add (18), (19), (20), and (21) and then multiplying by $\frac{1}{4}$, we have the last inequality of Theorem 8.

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{8(b-a)^\alpha} [\mathcal{J}_{a^+}^\alpha F(b, c) + \mathcal{J}_{a^+}^\alpha F(b, d) + \mathcal{J}_b^\alpha F(a, c) + \mathcal{J}_b^\alpha F(a, d)] \\ & + \frac{\Gamma(\beta+1)}{4(d-c)^\beta} [\mathcal{J}_{c^+}^\beta F(a, d) + \mathcal{J}_{c^+}^\beta F(b, d) + \mathcal{J}_d^\beta F(a, c) + \mathcal{J}_d^\beta F(b, c)] \\ & \geq \frac{F(a, c) + F(a, d) + F(b, c) + F(b, d)}{4}, \end{aligned}$$

and the proof is completed. $\square$

Remark 2. If we choose $\alpha = \beta = 1$ in Theorem 8, then we have

$$\begin{aligned} F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) & \geq \frac{1}{2} \left[\frac{1}{b-a} (IR) \int_a^b F\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c} (IR) \int_c^d F\left(\frac{a+b}{2}, y\right) dy \right] \\ & \geq \frac{1}{(b-a)(d-c)} (IR) \int_a^b \int_c^d F(x, y) dy dx \\ & \geq \frac{1}{4} \left[\frac{1}{b-a} (IR) \int_a^b [F(x, c) + F(x, d)] dx + \frac{1}{d-c} (IR) \int_c^d [F(a, y) + F(b, y)] dy \right] \\ & \geq \frac{F(a, c) + F(a, d) + F(b, c) + F(b, d)}{4}, \end{aligned}$$

which is proved by Zhao et al. in [41].

Remark 3. If $\underline{F}(t) = \overline{F}(t)$ in Theorem 8, then Theorem 8 reduces to the result of Sarikaya [42, Theorem 4].

Theorem 9. If $F, G : \Delta \rightarrow \mathbb{R}_I^+$ are two interval-valued co-ordinated convex functions on Δ such that $F(t) = [\underline{F}(t), \overline{F}(t)]$ and $G(t) = [\underline{G}(t), \overline{G}(t)]$, then the following inequality holds:

$$\begin{aligned} & \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} \\ & \times [\mathcal{J}_{a^+, c^+}^{\alpha, \beta} F(b, d)G(b, d) + \mathcal{J}_{a^+, d^-}^{\alpha, \beta} F(b, c)G(b, c) + \mathcal{J}_{b^-, c^+}^{\alpha, \beta} F(a, d)G(a, d) + \mathcal{J}_{b^-, d^-}^{\alpha, \beta} F(a, c)G(a, c)] \\ & \geq \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) K(a, b, c, d) \\ & + \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) \frac{\alpha}{(\alpha+1)(\alpha+2)} L(a, b, c, d) \\ & + \frac{\beta}{(\beta+1)(\beta+2)} \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) M(a, b, c, d) \\ & + \frac{\beta}{(\beta+1)(\beta+2)} \frac{\alpha}{(\alpha+1)(\alpha+2)} N(a, b, c, d), \end{aligned} \quad (22)$$

where

$$K(a, b, c, d) = F(a, c)G(a, c) + F(b, c)G(b, c) + F(a, d)G(a, d) + F(b, d)G(b, d),$$

$$\begin{aligned} L(a, b, c, d) &= F(a, c)G(b, c) + F(b, c)G(a, c) + F(a, d)G(b, d) + F(b, d)G(a, d), \\ M(a, b, c, d) &= F(a, c)G(a, d) + F(b, c)G(b, d) + F(a, d)G(a, c) + F(b, d)G(b, c), \end{aligned}$$

and

$$N(a, b, c, d) = F(a, c)G(b, d) + F(b, c)G(a, d) + F(a, d)G(b, c) + F(b, d)G(a, c).$$

Proof. Since F and G are interval-valued co-ordinated convex functions on Δ , if we define the mappings $F_x : [c, d] \rightarrow \mathbb{R}_I^+$, $F_x(y) = F(x, y)$, and $G_x : [c, d] \rightarrow \mathbb{R}_I^+$, $G_x(y) = G(x, y)$, then $F_x(y)$ and $G_x(y)$ are convex functions on $[c, d]$ for all $x \in [a, b]$. If we apply inequality (6) for the convex functions $F_x(y)$ and $G_x(y)$, then we have

$$\begin{aligned} \frac{\Gamma(\beta+1)}{2(d-c)^\beta} [J_{c+}^\beta F_x(d)G_x(d) + J_{d-}^\beta F_x(c)G_x(c)] &\geq \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) [F_x(c)G_x(c) + F_x(d)G_x(d)] \\ &\quad + \frac{\beta}{(\beta+1)(\beta+2)} [F_x(c)G_x(d) + F_x(d)G_x(c)]. \end{aligned} \quad (23)$$

That is,

$$\begin{aligned} &\frac{\beta}{2(d-c)^\beta} \left[(IR) \int_c^d (d-y)^{\beta-1} F(x, y) G(x, y) dy + (IR) \int_c^d (y-a)^{\beta-1} F(x, y) G(x, y) dy \right] \\ &\geq \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) [F(x, c)G(x, c) + F(x, d)G(x, d)] \\ &\quad + \frac{\beta}{(\beta+1)(\beta+2)} [F(x, c)G(x, d) + F(x, d)G(x, c)]. \end{aligned} \quad (24)$$

Multiplying inequality (24) by $\frac{\alpha}{2(b-a)^\alpha} (b-x)^{\alpha-1}$ and integrating the resultant one with respect to x over $[a, b]$, we obtain

$$\begin{aligned} &\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} [\mathcal{J}_{a+,c+}^{\alpha,\beta} F(b, d)G(b, d) + \mathcal{J}_{a+,d-}^{\alpha,\beta} F(b, c)G(b, c)] \\ &\geq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) [\mathcal{J}_{a+}^\alpha F(b, c)G(b, c) + \mathcal{J}_{a+}^\alpha F(b, d)G(b, d)] \\ &\quad + \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \frac{\beta}{(\beta+1)(\beta+2)} [\mathcal{J}_{a+}^\alpha F(b, c)G(b, d) + \mathcal{J}_{a+}^\alpha F(b, d)G(b, c)]. \end{aligned} \quad (25)$$

Similarly, multiplying inequality (24) by $\frac{\alpha}{2(b-a)^\alpha} (x-a)^{\alpha-1}$ and integrating the resultant one with respect to x on $[a, b]$, we have

$$\begin{aligned} &\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} [\mathcal{J}_{b-,c+}^{\alpha,\beta} F(a, d)G(a, d) + \mathcal{J}_{b-,d-}^{\alpha,\beta} F(a, c)G(a, c)] \\ &\geq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) [\mathcal{J}_{b-}^\alpha F(a, c)G(a, c) + \mathcal{J}_{b-}^\alpha F(a, d)G(a, d)] \\ &\quad + \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \frac{\beta}{(\beta+1)(\beta+2)} [\mathcal{J}_{b-}^\alpha F(a, c)G(a, d) + \mathcal{J}_{b-}^\alpha F(a, d)G(a, c)]. \end{aligned} \quad (26)$$

From (25) and (26), we get

$$\begin{aligned} &\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{2(b-a)^\alpha(d-c)^\beta} \\ &\quad \times [\mathcal{J}_{a+,c+}^{\alpha,\beta} f(b, d)g(b, d) + \mathcal{J}_{a+,d-}^{\alpha,\beta} F(b, c)G(b, c) + \mathcal{J}_{b-,c+}^{\alpha,\beta} F(a, d)G(a, d) + \mathcal{J}_{b-,d-}^{\alpha,\beta} F(a, c)G(a, c)] \end{aligned} \quad (27)$$

$$\begin{aligned}
&\geq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) [\mathcal{J}_{a+}^\alpha F(b, c)G(b, c) + \mathcal{J}_{b-}^\alpha F(a, c)G(a, c)] \\
&\quad + \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) [\mathcal{J}_{a+}^\alpha F(b, d)G(b, d) + \mathcal{J}_{b-}^\alpha F(a, d)G(a, d)] \\
&\quad + \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \frac{\beta}{(\beta+1)(\beta+2)} [\mathcal{J}_{a+}^\alpha F(b, c)G(b, d) + \mathcal{J}_{b-}^\alpha F(a, c)G(a, d)] \\
&\quad + \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \frac{\beta}{(\beta+1)(\beta+2)} [\mathcal{J}_{a+}^\alpha F(b, d)G(b, c) + \mathcal{J}_{b-}^\alpha F(a, d)G(a, c)].
\end{aligned}$$

For each term of the right hand side of (27), by inequality (6), we have

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [\mathcal{J}_{a+}^\alpha F(b, c)G(b, c) + \mathcal{J}_{b-}^\alpha F(a, c)G(a, c)] \\
&\quad \geq \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) [F(a, c)G(a, c) + F(b, c)G(b, c)] + \frac{\alpha}{(\alpha+1)(\alpha+2)} [F(a, c)G(b, c) + F(b, c)G(a, c)],
\end{aligned} \tag{28}$$

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [\mathcal{J}_{a+}^\alpha F(b, d)G(b, d) + \mathcal{J}_{b-}^\alpha F(a, d)G(a, d)] \\
&\quad \geq \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) [F(a, d)G(a, d) + F(b, d)G(b, d)] + \frac{\alpha}{(\alpha+1)(\alpha+2)} [F(a, d)G(b, d) + F(b, d)G(a, d)],
\end{aligned} \tag{29}$$

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [\mathcal{J}_{a+}^\alpha F(b, c)G(b, d) + \mathcal{J}_{b-}^\alpha F(a, c)G(a, d)] \\
&\quad \geq \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) [F(a, c)G(a, d) + F(b, c)G(b, d)] + \frac{\alpha}{(\alpha+1)(\alpha+2)} [F(a, c)G(b, d) + F(b, c)G(a, d)],
\end{aligned} \tag{30}$$

and

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [\mathcal{J}_{a+}^\alpha F(b, d)G(b, c) + \mathcal{J}_{b-}^\alpha F(a, d)G(a, c)] \\
&\quad \geq \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) [F(a, d)G(a, c) + F(b, d)G(b, c)] + \frac{\alpha}{(\alpha+1)(\alpha+2)} [F(a, d)G(b, c) + F(b, d)G(a, c)].
\end{aligned} \tag{31}$$

If we substitute (28)–(31) in (27), we obtain the desired result (22). $\square$

Remark 4. If we choose $\alpha = \beta = 1$ in Theorem 9, then we have

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t, s) ds dt \geq \frac{1}{9} K(a, b, c, d) + \frac{1}{18} [L(a, b, c, d) + M(a, b, c, d)] + \frac{1}{36} N(a, b, c, d),$$

which is proved by Zhao et al. in [41].

Corollary 1. If we choose $G(x, y) = [1, 1]$ for all $(x, y) \in \Delta$ in Theorem 9, then we have the following inequality:

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} [\mathcal{J}_{a+,c+}^{\alpha,\beta} F(b, d) + \mathcal{J}_{a+,d-}^{\alpha,\beta} F(b, c) + \mathcal{J}_{b-,c+}^{\alpha,\beta} F(a, d) + \mathcal{J}_{b-,d-}^{\alpha,\beta} F(a, c)] \\
&\quad \geq \frac{1}{4} [F(a, c) + F(b, c) + F(a, d) + F(b, d)].
\end{aligned}$$

Remark 5. If $\underline{F}(t) = \overline{F}(t)$ in Theorem 9, then Theorem 9 reduces to the result of Budak and Sarikaya [43, Theorem 2.1].

Theorem 10. If $F, G : \Delta \rightarrow \mathbb{R}_I^+$ are two interval-valued co-ordinated convex functions on Δ such that $F(t) = [F(t), \bar{F}(t)]$ and $G(t) = [\underline{G}(t), \bar{G}(t)]$, then the following Hermite-Hadamard-type inequality holds

$$\begin{aligned}
 & 4F\left(\frac{a+b}{2}, \frac{c+d}{2}\right)G\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
 & \supseteq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} \\
 & \quad \times [\mathcal{J}_{a^+, c^+}^{\alpha, \beta} F(b, d)G(b, d) + \mathcal{J}_{a^+, d^-}^{\alpha, \beta} F(b, c)G(b, c) + \mathcal{J}_{b^-, c^+}^{\alpha, \beta} F(a, d)G(a, d) + \mathcal{J}_{b^-, d^-}^{\alpha, \beta} F(a, c)G(a, c)] \\
 & \quad + \left[\frac{\alpha}{2(\alpha+1)(\alpha+2)} + \frac{\beta}{(\beta+1)(\beta+2)} \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \right] K(a, b, c, d) \\
 & \quad + \left[\frac{1}{2} \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) + \frac{\alpha}{(\alpha+1)(\alpha+2)} \frac{\beta}{(\beta+1)(\beta+2)} \right] L(a, b, c, d) \\
 & \quad + \left[\frac{1}{2} \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) + \frac{\alpha}{(\alpha+1)(\alpha+2)} \frac{\beta}{(\beta+1)(\beta+2)} \right] M(a, b, c, d) \\
 & \quad + \left[\frac{1}{4} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \frac{\beta}{(\beta+1)(\beta+2)} \right] N(a, b, c, d),
 \end{aligned} \tag{32}$$

where $K(a, b, c, d)$, $L(a, b, c, d)$, $M(a, b, c, d)$, and $N(a, b, c, d)$ are defined as in Theorem 9.

Proof. Since F and G are interval-valued co-ordinated convex functions on Δ , by inequality (7), we have

$$\begin{aligned}
 & 2F\left(\frac{a+b}{2}, \frac{c+d}{2}\right)G\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
 & \supseteq \frac{\alpha}{2(b-a)^\alpha} \left[(IR) \int_a^b (b-x)^{\alpha-1} F\left(x, \frac{c+d}{2}\right) G\left(x, \frac{c+d}{2}\right) dx + (IR) \int_a^b (x-a)^{\alpha-1} F\left(x, \frac{c+d}{2}\right) G\left(x, \frac{c+d}{2}\right) dx \right] \\
 & \quad + \frac{\alpha}{(\alpha+1)(\alpha+2)} \left[F\left(a, \frac{c+d}{2}\right) G\left(a, \frac{c+d}{2}\right) + F\left(b, \frac{c+d}{2}\right) G\left(b, \frac{c+d}{2}\right) \right] \\
 & \quad + \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \left[F\left(a, \frac{c+d}{2}\right) G\left(b, \frac{c+d}{2}\right) + F\left(b, \frac{c+d}{2}\right) G\left(a, \frac{c+d}{2}\right) \right]
 \end{aligned} \tag{33}$$

and

$$\begin{aligned}
 & 2F\left(\frac{a+b}{2}, \frac{c+d}{2}\right)G\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
 & \supseteq \frac{\beta}{2(d-c)^\beta} \left[(IR) \int_c^d (d-y)^{\beta-1} F\left(\frac{a+b}{2}, y\right) G\left(\frac{a+b}{2}, y\right) dy + (IR) \int_c^d (y-c)^{\beta-1} F\left(\frac{a+b}{2}, y\right) G\left(\frac{a+b}{2}, y\right) dy \right] \\
 & \quad + \frac{\beta}{(\beta+1)(\beta+2)} \left[F\left(\frac{a+b}{2}, c\right) G\left(\frac{a+b}{2}, c\right) + F\left(\frac{a+b}{2}, d\right) G\left(\frac{a+b}{2}, d\right) \right] \\
 & \quad + \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) \left[F\left(\frac{a+b}{2}, c\right) G\left(\frac{a+b}{2}, d\right) + F\left(\frac{a+b}{2}, d\right) G\left(\frac{a+b}{2}, c\right) \right].
 \end{aligned} \tag{34}$$

From (33) and (34), we get

$$\begin{aligned}
 & 8F\left(\frac{a+b}{2}, \frac{c+d}{2}\right)G\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
 & \supseteq \frac{\alpha}{2(b-a)^\alpha} \left[\int_a^b (b-x)^{\alpha-1} 2F\left(x, \frac{c+d}{2}\right) G\left(x, \frac{c+d}{2}\right) dx + (IR) \int_a^b (x-a)^{\alpha-1} 2F\left(x, \frac{c+d}{2}\right) G\left(x, \frac{c+d}{2}\right) dx \right]
 \end{aligned} \tag{35}$$

$$\begin{aligned}
& \times \frac{\beta}{2(d-c)^\beta} \left[(IR) \int_c^d (d-y)^{\beta-1} 2F\left(\frac{a+b}{2}, y\right) G\left(\frac{a+b}{2}, y\right) dy + \int_c^d (y-c)^{\beta-1} 2F\left(\frac{a+b}{2}, y\right) G\left(\frac{a+b}{2}, y\right) dy \right] \\
& + \frac{\alpha}{(\alpha+1)(\alpha+2)} \left[2F\left(a, \frac{c+d}{2}\right) G\left(a, \frac{c+d}{2}\right) + 2F\left(b, \frac{c+d}{2}\right) G\left(b, \frac{c+d}{2}\right) \right] \\
& + \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \left[2F\left(a, \frac{c+d}{2}\right) G\left(b, \frac{c+d}{2}\right) + 2F\left(b, \frac{c+d}{2}\right) G\left(a, \frac{c+d}{2}\right) \right] \\
& + \frac{\beta}{(\beta+1)(\beta+2)} \left[2F\left(\frac{a+b}{2}, c\right) G\left(\frac{a+b}{2}, c\right) + 2F\left(\frac{a+b}{2}, d\right) G\left(\frac{a+b}{2}, d\right) \right] \\
& + \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) \left[2F\left(\frac{a+b}{2}, c\right) G\left(\frac{a+b}{2}, d\right) + 2F\left(\frac{a+b}{2}, d\right) G\left(\frac{a+b}{2}, c\right) \right].
\end{aligned}$$

Since the mappings $F_x : [c, d] \rightarrow \mathbb{R}_I^+$, $F_x(y) = F(x, y)$ and $G_x : [c, d] \rightarrow \mathbb{R}_I^+$, $G_x(y) = G(x, y)$ are convex interval-valued, by applying inequality (7), we have

$$\begin{aligned}
2F\left(a, \frac{c+d}{2}\right) G\left(a, \frac{c+d}{2}\right) & \supseteq \frac{\Gamma(\beta+1)}{2(d-c)^\beta} [\mathcal{J}_{c^+}^\beta F(a, d) G(a, d) + \mathcal{J}_{d^-}^\beta F(a, c) G(a, c)] \\
& + \frac{\beta}{(\beta+1)(\beta+2)} [F(a, c) G(a, c) + F(a, d) G(a, d)] \\
& + \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) [F(a, c) G(a, d) + F(a, d) G(a, c)],
\end{aligned} \tag{36}$$

$$\begin{aligned}
2F\left(b, \frac{c+d}{2}\right) G\left(b, \frac{c+d}{2}\right) & \supseteq \frac{\Gamma(\beta+1)}{2(d-c)^\beta} [\mathcal{J}_{c^+}^\beta F(b, d) G(b, d) + \mathcal{J}_{d^-}^\beta F(b, c) G(b, c)] \\
& + \frac{\beta}{(\beta+1)(\beta+2)} [F(b, c) G(b, c) + F(b, d) G(b, d)] \\
& + \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) [F(b, c) G(b, d) + F(b, d) G(b, c)],
\end{aligned} \tag{37}$$

$$\begin{aligned}
2F\left(a, \frac{c+d}{2}\right) G\left(b, \frac{c+d}{2}\right) & \supseteq \frac{\Gamma(\beta+1)}{2(d-c)^\beta} [\mathcal{J}_{c^+}^\beta F(a, d) G(b, d) + \mathcal{J}_{d^-}^\beta F(a, c) G(b, c)] \\
& + \frac{\beta}{(\beta+1)(\beta+2)} [F(a, c) G(b, c) + F(a, d) G(b, d)] \\
& + \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) [F(a, c) G(b, d) + F(a, d) G(b, c)],
\end{aligned} \tag{38}$$

and

$$\begin{aligned}
2F\left(b, \frac{c+d}{2}\right) G\left(a, \frac{c+d}{2}\right) & \supseteq \frac{\Gamma(\beta+1)}{2(d-c)^\beta} [\mathcal{J}_{c^+}^\beta F(b, d) G(a, d) + \mathcal{J}_{d^-}^\beta F(b, c) G(a, c)] \\
& + \frac{\beta}{(\beta+1)(\beta+2)} [F(b, c) G(a, c) + F(b, d) G(a, d)] \\
& + \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) [F(b, c) G(a, d) + F(b, d) G(a, c)].
\end{aligned} \tag{39}$$

Similarly, since the mappings $F_y : [a, b] \rightarrow \mathbb{R}_I^+$, $F_y(x) = F(x, y)$, and $G_y : [a, b] \rightarrow \mathbb{R}_I^+$, $G_y(x) = G(x, y)$ are convex interval-valued, by applying inequality (7), we have

$$\begin{aligned}
2F\left(\frac{a+b}{2}, c\right) G\left(\frac{a+b}{2}, c\right) & \supseteq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [\mathcal{J}_{a^+}^\alpha F(b, c) G(b, c) + \mathcal{J}_{b^-}^\alpha F(a, c) G(a, c)] \\
& + \frac{\alpha}{(\alpha+1)(\alpha+2)} [F(a, c) G(a, c) + F(b, c) G(b, c)] \\
& + \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) [F(a, c) G(b, c) + F(b, c) G(a, c)],
\end{aligned} \tag{40}$$

$$\begin{aligned}
2F\left(\frac{a+b}{2}, d\right)G\left(\frac{a+b}{2}, d\right) &\supseteq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [\mathcal{J}_{a+}^\alpha F(b, d)G(b, d) + \mathcal{J}_{b-}^\alpha F(a, d)G(a, d)] \\
&\quad + \frac{\alpha}{(\alpha+1)(\alpha+2)} [F(a, d)G(a, d) + F(b, d)G(b, d)] \\
&\quad + \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right) [F(a, d)G(b, d) + F(b, d)G(a, d)],
\end{aligned} \tag{41}$$

$$\begin{aligned}
2F\left(\frac{a+b}{2}, c\right)G\left(\frac{a+b}{2}, d\right) &\supseteq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [\mathcal{J}_{a+}^\alpha F(b, c)G(b, d) + \mathcal{J}_{b-}^\alpha F(a, c)G(a, d)] \\
&\quad + \frac{\alpha}{(\alpha+1)(\alpha+2)} [F(a, c)G(a, d) + F(b, c)G(b, d)] \\
&\quad + \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right) [F(a, c)G(b, d) + F(b, c)G(a, d)]
\end{aligned} \tag{42}$$

and

$$\begin{aligned}
2F\left(\frac{a+b}{2}, d\right)G\left(\frac{a+b}{2}, c\right) &\supseteq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [\mathcal{J}_{a+}^\alpha F(b, d)G(b, c) + \mathcal{J}_{b-}^\alpha F(a, d)G(a, c)] \\
&\quad + \frac{\alpha}{(\alpha+1)(\alpha+2)} [F(a, d)G(a, c) + F(b, d)G(b, c)] \\
&\quad + \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)}\right) [F(a, d)G(b, c) + F(b, d)G(a, c)].
\end{aligned} \tag{43}$$

On the other hand, by applying inequality (7), we get

$$\begin{aligned}
&\frac{\alpha}{2(b-a)^\alpha} (IR) \int_a^b (b-x)^{\alpha-1} 2F\left(x, \frac{c+d}{2}\right) G\left(x, \frac{c+d}{2}\right) dx \\
&\supseteq \frac{\alpha\beta}{4(b-a)^\alpha(d-c)^\beta} \left[(IR) \int_a^b \int_c^d (b-x)^{\alpha-1} (d-y)^{\beta-1} F(x, y) G(x, y) dy dx \right. \\
&\quad \left. + (IR) \int_a^b \int_c^d (b-x)^{\alpha-1} (y-c)^{\beta-1} F(x, y) G(x, y) dy dx \right] \\
&\quad + \frac{\beta}{(\beta+1)(\beta+2)} \frac{\alpha}{2(b-a)^\alpha} (IR) \int_a^b (b-x)^{\alpha-1} [F(x, c)G(x, c) + F(x, d)G(x, d)] dx \\
&\quad + \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)}\right) \frac{\alpha}{2(b-a)^\alpha} (IR) \int_a^b (b-x)^{\alpha-1} [F(x, c)G(x, d) + F(x, d)G(x, c)] dx \\
&= \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} [\mathcal{J}_{a+,c+}^{\alpha,\beta} F(b, d)G(b, d) + \mathcal{J}_{a+,d-}^{\alpha,\beta} F(b, c)G(b, c)] \\
&\quad + \frac{\beta}{(\beta+1)(\beta+2)} \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [\mathcal{J}_{a+}^\alpha F(b, c)G(b, c) + \mathcal{J}_{a+}^\alpha F(b, d)G(b, d)] \\
&\quad + \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)}\right) \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [\mathcal{J}_{a+}^\alpha F(b, c)G(b, d) + \mathcal{J}_{a+}^\alpha F(b, d)G(b, c)].
\end{aligned} \tag{44}$$

Similarly, we also have

$$\begin{aligned}
&\frac{\alpha}{2(b-a)^\alpha} (IR) \int_a^b (x-a)^{\alpha-1} 2F\left(x, \frac{c+d}{2}\right) G\left(x, \frac{c+d}{2}\right) dx \\
&\supseteq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} [\mathcal{J}_{b-,c+}^{\alpha,\beta} F(a, d)G(a, d) + \mathcal{J}_{b-,d-}^{\alpha,\beta} F(a, c)G(a, c)]
\end{aligned} \tag{45}$$

$$\begin{aligned}
& + \frac{\beta}{(\beta+1)(\beta+2)} \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [\mathcal{J}_{b-}^\alpha F(a, c)G(a, c) + \mathcal{J}_{b-}^\alpha F(a, d)G(a, d)] \\
& + \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [\mathcal{J}_{b-}^\alpha F(a, c)G(a, d) + \mathcal{J}_{b-}^\alpha F(a, d)G(a, c)], \\
& \frac{\beta}{2(d-c)^\beta} (IR) \int_c^d (d-y)^{\beta-1} 2F\left(\frac{a+b}{2}, y\right) G\left(\frac{a+b}{2}, y\right) dy \\
& \geq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} [\mathcal{J}_{a+,c+}^{\alpha,\beta} F(b, d)G(b, d) + \mathcal{J}_{b-,c+}^{\alpha,\beta} F(a, d)G(a, d)] \\
& + \frac{\alpha}{(\alpha+1)(\alpha+2)} \frac{\Gamma(\beta+1)}{2(d-c)^\beta} [\mathcal{J}_{c+}^\beta F(a, d)G(a, d) + \mathcal{J}_{c+}^\beta F(b, d)G(b, d)] \\
& + \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \frac{\Gamma(\beta+1)}{2(d-c)^\beta} [\mathcal{J}_{c+}^\beta F(a, d)G(b, d) + \mathcal{J}_{c+}^\beta F(b, d)G(a, d)],
\end{aligned} \tag{46}$$

and

$$\begin{aligned}
& \frac{\beta}{2(d-c)^\beta} \int_c^d (y-c)^{\beta-1} 2F\left(\frac{a+b}{2}, y\right) G\left(\frac{a+b}{2}, y\right) dy \\
& \geq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} [\mathcal{J}_{a+,d-}^{\alpha,\beta} F(b, c)G(b, c) + \mathcal{J}_{b-,d-}^{\alpha,\beta} F(a, c)G(a, c)] \\
& + \frac{\alpha}{(\alpha+1)(\alpha+2)} \frac{\Gamma(\beta+1)}{2(d-c)^\beta} [\mathcal{J}_{d-}^\beta F(a, c)G(a, c) + \mathcal{J}_{d-}^\beta F(b, c)G(b, c)] \\
& + \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \frac{\Gamma(\beta+1)}{2(d-c)^\beta} [\mathcal{J}_{d-}^\beta F(a, c)G(b, c) + \mathcal{J}_{d-}^\beta F(b, c)G(a, c)].
\end{aligned} \tag{47}$$

If we substitute (36)–(47) in (35), we get

$$\begin{aligned}
& 8F\left(\frac{a+b}{2}, \frac{c+d}{2}\right) G\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
& \geq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(2b-a)^\alpha(d-c)^\beta} [\mathcal{J}_{a+,c+}^{\alpha,\beta} F(b, d)G(b, d) + \mathcal{J}_{a+,d-}^{\alpha,\beta} F(b, c)G(b, c) \\
& + \mathcal{J}_{b-,c+}^{\alpha,\beta} F(a, d)G(a, d) + \mathcal{J}_{b-,d-}^{\alpha,\beta} F(a, c)G(a, c)] \\
& + \frac{2\alpha}{(\alpha+1)(\alpha+2)} \left\{ \frac{\Gamma(\beta+1)}{2(d-c)^\beta} [\mathcal{J}_{c+}^\beta F(a, d)G(a, d) + \mathcal{J}_{c+}^\beta F(b, d)G(b, d)] \right. \\
& + \left. \frac{\Gamma(\beta+1)}{2(d-c)^\beta} [\mathcal{J}_{d-}^\beta F(a, c)G(a, c) + \mathcal{J}_{d-}^\beta F(b, c)G(b, c)] \right\} \\
& + 2\left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \left\{ \frac{\Gamma(\beta+1)}{2(d-c)^\beta} [\mathcal{J}_{c+}^\beta F(a, d)G(b, d) + \mathcal{J}_{c+}^\beta F(b, d)G(a, d)] \right. \\
& + \left. \frac{\Gamma(\beta+1)}{2(d-c)^\beta} [\mathcal{J}_{d-}^\beta F(a, c)G(b, c) + \mathcal{J}_{d-}^\beta F(b, c)G(a, c)] \right\} \\
& + \frac{2\beta}{(\beta+1)(\beta+2)} \left\{ \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [\mathcal{J}_{a+}^\alpha F(b, c)G(b, c) + \mathcal{J}_{a+}^\alpha F(b, d)G(b, d)] \right. \\
& + \left. \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [\mathcal{J}_{b-}^\alpha F(a, c)G(a, c) + \mathcal{J}_{b-}^\alpha F(a, d)G(a, d)] \right\} \\
& + 2\left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) \left\{ \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [\mathcal{J}_{a+}^\alpha F(b, c)G(b, d) + \mathcal{J}_{a+}^\alpha F(b, d)G(b, c)] \right. \\
& + \left. \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [\mathcal{J}_{b-}^\alpha F(a, c)G(a, d) + \mathcal{J}_{b-}^\alpha F(a, d)G(a, c)] \right\}
\end{aligned} \tag{48}$$

$$\begin{aligned}
& + \frac{2\alpha}{(\alpha+1)(\alpha+2)} \frac{\beta}{(\beta+1)(\beta+2)} K(a, b, c, d) + \frac{2\alpha}{(\alpha+1)(\alpha+2)} \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) M(a, b, c, d) \\
& + \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \frac{2\beta}{(\beta+1)(\beta+2)} L(a, b, c, d) \\
& + 2 \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) N(a, b, c, d).
\end{aligned}$$

Using inequality (6), we have

$$\begin{aligned}
& \frac{\Gamma(\beta+1)}{2(d-c)^\beta} [\mathcal{J}_{c+}^\beta F(a, d)G(a, d) + \mathcal{J}_{c+}^\beta F(b, d)G(b, d)] + \frac{\Gamma(\beta+1)}{2(d-c)^\beta} [\mathcal{J}_{d-}^\beta F(a, c)G(a, c) + \mathcal{J}_{d-}^\beta F(b, c)G(b, c)] \\
& \geq \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) K(a, b, c, d) + \frac{\beta}{(\beta+1)(\beta+2)} M(a, b, c, d),
\end{aligned} \quad (49)$$

$$\begin{aligned}
& \frac{\Gamma(\beta+1)}{2(d-c)^\beta} [\mathcal{J}_{c+}^\beta F(a, d)G(b, d) + \mathcal{J}_{c+}^\beta F(b, d)G(a, d)] + \frac{\Gamma(\beta+1)}{2(d-c)^\beta} [\mathcal{J}_{d-}^\beta F(a, c)G(b, c) + \mathcal{J}_{d-}^\beta F(b, c)G(a, c)] \\
& \geq \left(\frac{1}{2} - \frac{\beta}{(\beta+1)(\beta+2)} \right) L(a, b, c, d) + \frac{\beta}{(\beta+1)(\beta+2)} N(a, b, c, d),
\end{aligned} \quad (50)$$

$$\begin{aligned}
& \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [\mathcal{J}_{a+}^\alpha F(b, c)G(b, c) + \mathcal{J}_{a+}^\alpha F(b, d)G(b, d)] + \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [\mathcal{J}_{b-}^\alpha F(a, c)G(a, c) + \mathcal{J}_{b-}^\alpha F(a, d)G(a, d)] \\
& \geq \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) K(a, b, c, d) + \frac{\alpha}{(\alpha+1)(\alpha+2)} L(a, b, c, d),
\end{aligned} \quad (51)$$

and

$$\begin{aligned}
& \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [\mathcal{J}_{a+}^\alpha F(b, c)G(b, d) + \mathcal{J}_{a+}^\alpha F(b, d)G(b, c)] + \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [\mathcal{J}_{b-}^\alpha F(a, c)G(a, d) + \mathcal{J}_{b-}^\alpha F(a, d)G(a, c)] \\
& \geq \left(\frac{1}{2} - \frac{\alpha}{(\alpha+1)(\alpha+2)} \right) M(a, b, c, d) + \frac{\alpha}{(\alpha+1)(\alpha+2)} N(a, b, c, d).
\end{aligned} \quad (52)$$

If we substitute (49)–(52) in (48), and divide the resulting inequality by 2, then we obtain the desired result (32). This completes the proof. $\square$

Remark 6. If we choose $\alpha = \beta = 1$ in Theorem 10, then we have

$$\begin{aligned}
4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) G\left(\frac{a+b}{2}, \frac{c+d}{2}\right) & \geq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d F(t, s)G(t, s)dsdt + \frac{5}{36}K(a, b, c, d) \\
& + \frac{7}{36}[L(a, b, c, d) + M(a, b, c, d)] + \frac{2}{9}N(a, b, c, d),
\end{aligned}$$

given by Zhao et al. in [41].

Corollary 2. If we choose $G(x, y) = [1, 1]$ in Theorem 10, then we have the following inequality:

$$\begin{aligned}
4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) & \geq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta} [\mathcal{J}_{a+,c+}^{\alpha,\beta} F(b, d) + \mathcal{J}_{a+,d-}^{\alpha,\beta} F(b, c) + \mathcal{J}_{b-,c+}^{\alpha,\beta} F(a, d) + \mathcal{J}_{b-,d-}^{\alpha,\beta} F(a, c)] \\
& + \frac{3}{4}[F(a, c) + F(a, d) + F(b, c) + F(b, d)].
\end{aligned}$$

Remark 7. If $\underline{F}(t) = \overline{F}(t)$ in Theorem 10, then Theorem 10 reduces to the result of Budak and Sarikaya [43, Theorem 2.2].

5 Conclusion

In this paper, we presented ideas about interval-valued fractional integrals on co-ordinates. We have used newly described fractional integrals to prove some new Hermite-Hadamard-type inequalities for co-ordinated convex interval-valued functions. It is a fascinating and novel problem that the future researchers may find similar inequalities for various types of convexity in their study.

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of interest: Authors state no conflict of interest.

Data availability statement: Data sharing is not applicable to this paper as no data sets were generated or analyzed during the current study.

References

- [1] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.
- [2] J. E. Pečarić, F. Proschan, and Y. L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Boston, 1992.
- [3] T.-H. Zhao, Z.-Y. He, and Y.-M. Chu, *Sharp bounds for the weighted Hölder mean of the zero-balanced generalized complete elliptic integrals*, Comput. Methods Funct. Theory **21** (2021), no. 3, 413–426.
- [4] M.-K. Wang, H.-H. Chu, and Y.-M. Chu, *On the approximation of some special functions in Ramanujan's generalized modular equation with signature 3*, Ramanujan J. **56** (2021), no. 1, 1–22.
- [5] P. Agarwal, M. Kadakal, I. Iscan, and Y.-M. Chu, *Better approaches for n -times differentiable convex functions*, Mathematics **8** (2020), no. 6, 950.
- [6] M. A. Ali, H. Budak, Z. Zhang, and H. Yildirim, *Some new Simpson's type inequalities for co-ordinated convex functions in quantum calculus*, Math. Meth. Appl. Sci. **44** (2021), 4515–4540.
- [7] M. A. Ali, H. Budak, M. Abbas, and Y.-M. Chu, *Quantum Hermite-Hadamard-type inequalities for functions with convex absolute values of second q^b -derivatives*, Adv. Differ. Equ. **2021** (2021), 7.
- [8] M. A. Ali, M. Abbas, H. Budak, P. Agarwal, G. Murtaza, and Y.-M. Chu, *New quantum boundaries for quantum Simpson's and quantum Newton's type inequalities for preinvex functions*, Adv. Differ. Equ. **2021** (2021), 64.
- [9] M. A. Ali, Y.-M. Chu, H. Budak, A. Akkurt, and H. Yildirim, *Quantum variant of Montgomery identity and Ostrowski-type inequalities for the mappings of two variables*, Adv. Differ. Equ. **2021** (2021), 25.
- [10] M. A. Ali, N. Alp, H. Budak, Y.-M. Chu, and Z. Zhang, *On some new quantum midpoint type inequalities for twice quantum differentiable convex functions*, Open Math. **19** (2021), 427–439.
- [11] M. A. Ali, H. Budak, A. Akkurt, and Y.-M. Chu, *Quantum Ostrowski type inequalities for twice quantum differentiable functions in quantum calculus*, Open Math. **19** (2021), 440–449.
- [12] S.-S. Zhou, S. Rashid, M. A. Noor, K. I. Noor, F. Safdar, and Y.-M. Chu, *New Hermite-Hadamard type inequalities for exponentially convex functions and applications*, AIMS Math. **5** (2020), 6874–6901.
- [13] H. Budak, S. Erden, and M. A. Ali, *Simpson and Newton type inequalities for convex functions via newly defined quantum integrals*, Math. Meth. Appl. Sci. **44** (2020), 378–390.
- [14] S.-B. Chen, S. Rashid, M. A. Noor, R. Ashraf, and Y.-M. Chu, *A new approach on fractional calculus and probability density functions*, AIMS Math. **5** (2020), 7041–7054.
- [15] H. Budak, M. A. Ali, and M. Tarhanaci, *Some new quantum Hermite-Hadamard-like inequalities for coordinated convex functions*, J. Optim. Theory Appl. **186** (2020), 899–910.
- [16] Y.-X. Li, M. A. Ali, H. Budak, M. Abbas, and Y.-M. Chu, *A new generalization of some quantum integral inequalities for quantum differentiable convex functions*, Adv. Differ. Equ. **2021** (2021), 225.
- [17] F. Chen and S. Wu, *Several complementary inequalities to inequalities of Hermite-Hadamard-type for s -convex functions*, J. Nonlinear Sci. Appl. **9** (2016), 705–716.

- [18] S. B. Chen, H. Jahanshahi, O. Alhadji Abba, J. E. Solis-Perez, S. Bekiros, J. F. Gomez-Aguilar, et al., *The effect of market confidence on a financial system from the perspective of fractional calculus: numerical investigation and circuit realization*, Chaos Solitons Fractals. **140** (2020), 110223.
- [19] H.-H. Chu, T.-H. Zhao, and Y.-M. Chu, *Sharp bounds for the Toader mean of order 3 in terms of arithmetic, quadratic and contraharmonic means*, Math. Slovaca **70** (2020), no. 5, 1097–1112.
- [20] T.-H. Zhao, M.-K. Wang, and Y.-M. Chu, *Concavity and bounds involving generalized elliptic integral of the first kind*, J. Math. Inequal. **15** (2021), no. 2, 701–724.
- [21] T.-H. Zhao, M.-K. Wang, and Y.-M. Chu, *Monotonicity and convexity involving generalized elliptic integral of the first kind*, Rev. R. Acad. Cienc. Exactas, Fis. Nat. Ser. A Mat. RACSAM **115** (2021), no. 2, 46.
- [22] Y.-X. Li, A. Rauf, M. Naeem, M. A. Binyamin, and A. Aslam, *Valency-based topological properties of linear hexagonal chain and hammer-like benzenoid*, Complexity **2021** (2021), 9939469.
- [23] M. Tomar, P. Agarwal, and J. Choi, *Hermite-Hadamard-type inequalities for generalized convex functions on fractal sets style*, Bol. Soc. Parana. Mat. **38** (2020), no. 1, 101–116.
- [24] T.-H. Zhao, Z.-H. Shen, and Y.-M. Chu, *Sharp power mean bounds for the lemniscate type means*, Rev. R. Acad. Cienc. Exactas, Fis. Nat. Ser. A Mat. RACSAM **115** (2021), no. 4, 175.
- [25] S.-B. Chen, S. Rashid, Z. Hammouch, M. A. Noor, R. Ashraf, and Y.-M. Chu, *Integral inequalities via Raina's fractional integrals operator with respect to a monotone function*, Adv. Differ. Equ. **2020** (2020), 647.
- [26] D. Zhao, M. A. Ali, A. Kashuri, and H. Budak, *Generalized fractional integral inequalities of Hermite-Hadamard-type for harmonically convex functions*, Adv. Differ. Equ. **2020** (2020), no. 1, 1–14.
- [27] R. E. Moore, *Interval Analysis*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1966.
- [28] Y. Chalco-Cano, A. Flores-Franulic, and H. Roman-Flores, *Ostrowski type inequalities for interval-valued functions using generalized Hukuhara derivative*, Comput. Appl. Math. **31** (2012), 457–472.
- [29] Y. Chalco-Cano, W. A. Lodwick, and W. Condori-Equice, *Ostrowski type inequalities and applications in numerical integration for interval-valued functions*, Soft Comput. **19** (2015), 3293–3300.
- [30] E. Sadowska, *Hadamard inequality and a refinement of Jensen inequality for set-valued functions*, Results Math. **32** (1997), 332–337.
- [31] F. C. Mitroi, N. Kazimierz, and W. Szymon, *Hermite-Hadamard inequalities for convex set-valued functions*, Demonstr. Math. **46** (2013), no. 4, 655–662.
- [32] K. Nikodem, J. L. Sánchez, and L. Sánchez, *Jensen and Hermite-Hadamard inequalities for strongly convex set-valued maps*, Math. Aeterna **4** (2014), no. 8, 979–987.
- [33] R. E. Moore, R. B. Kearfott, and M. J. Cloud, *Introduction to Interval Analysis*, vol. 110, SIAM, Philadelphia, 2009.
- [34] D. Zhao, T. An, G. Ye, and W. Liu, *New Jensen and Hermite-Hadamard-type inequalities for h -convex interval-valued functions*, J. Inequal. Appl. **2018** (2018), 302.
- [35] D. Zhao, G. Ye, W. Liu, and D. F. M. Torres, *Some inequalities for interval-valued functions on time scales*, Soft Comput. **23** (2019), 6005–6015.
- [36] W. W. Breckner, *Continuity of generalized convex and generalized concave set-valued functions*, Rev. Anal. Numér. Théor. Approx. **22** (1993), 39–51.
- [37] R. Osuna-Gómez, M. D. Jiménez-Gamero, Y. Chalco-Cano, and M. A. Rojas-Medar, *Hadamard and Jensen inequalities for s -convex fuzzy processes*, in: M. López-Díaz, M. Á. Gil, P. Grzegorzewski, O. Hryniewicz, and J. Lawry (eds), Soft Methodology and Random Information Systems, Springer, Berlin, 2004, pp. 645–652.
- [38] H. Budak, T. Tunç, and M. Z. Sarikaya, *Fractional Hermite-Hadamard-type inequalities for interval-valued functions*, Proc. Amer. Math. Soc. **148** (2020), 705–718.
- [39] V. Lupulescu, *Fractional calculus for interval-valued functions*, Fuzzy Sets and Systems **265** (2015), 63–85.
- [40] D. Zhao, T. An, G. Ye, and W. Liu, *Chebyshev type inequalities for interval-valued functions*, Fuzzy Sets and Systems **396** (2020), 82–101.
- [41] D. Zhao, M. A. Ali, G. Murtaza, and Z. Zhang, *On the Hermite-Hadamard inequalities for interval-valued coordinated convex functions*, Adv. Differ. Equ. **2020** (2020), 570.
- [42] M. Z. Sarikaya, *On the Hermite-Hadamard-type inequalities for co-ordinated convex function via fractional integrals*, Integr. Transforms Special Funt. **25** (2014), no. 2, 134–147.
- [43] H. Budak and M. Z. Sarikaya, *Hermite-Hadamard-type inequalities for products of two co-ordinated convex mappings via fractional integrals*, Int. J. Appl. Math. Stat. **58** (2019), no. 4, 11–30.