Research Article

Chen Yang* and Xiaolin Zhu

The concept of cone b -Banach space and fixed point theorems

<https://doi.org/10.1515/math-2021-0068>

received August 18, 2020; accepted June 30, 2021

Abstract: In this article, the concepts of cone b -norm and cone b -Banach space are given. Some new fixed point theorems in cone b -Banach spaces are established. The new results improve some fixed point theorems in cone Banach spaces. Furthermore, we also investigate the uniqueness of fixed points.

Keywords: fixed point, uniqueness, cone b -norm, cone b -Banach space

MSC 2020: 47H10, 54H25, 55M20

1 Introduction and preliminaries

Recently, there have been some papers studying new concepts of a metric which generalize the classical notion, see for example [1–9]. As a generalization of metric spaces, Bakhtin [10] gave a definition of b -metric spaces in 1989 and then the author obtained the contraction mapping principle in this new space. It is obviously a generalization of Banach contraction principle of metric spaces. From then, many people paid much attention to fixed points in such spaces, see [11–18] for instance.

The theory of cones in ordered normed spaces has a lot of applications in applied mathematics, for example, see [19–32]. Some scholars introduced K -metric and K -normed spaces [19,20] by applying ordered Banach spaces. Huang and Zhang [33] re-introduced such spaces and gave a name of cone metric spaces, and they also used some properties of cone to define convergent and Cauchy sequences. Afterward, many scholars studied fixed points in cone metric spaces and their related theories, see [34–36]. Note also that some scholars not only studied the existence of fixed point theorems in ordered abstract spaces but also applied it to the discussion of the solution of matrix equations, see [37,38].

To extend b -metric spaces and cone metric spaces, the concept of cone b -metric space (CbMS) has been introduced by Hussain and Shah [39] in 2001. They also present some topological properties in such spaces. Many authors are inspired by the research of Hussain and Shah [39], and then obtain different fixed point theorems and common fixed point theorems for multiple operators on these new spaces, see [40–46].

Inspired by the above works and to extend previous results in the literature, in this article, we define new concepts: cone b -norm and cone b -Banach space, and give some new fixed point theorems in cone b -Banach spaces, which improve and complement some known results in the literature.

The following concepts and results are needed in the sequel.

A real Banach space (i.e., a complete normed vector space), say $(E, \|\cdot\|)$, together with a positive cone $P \subseteq E$, gives rise to a partial ordering defined as $x \leq y$ if and only if $y - x \in P$. θ is the zero element of E .

* **Corresponding author: Chen Yang**, Shanxi Vocational University of Engineering Science and Technology, Taiyuan, Shanxi 030619, P. R. China, e-mail: yangchen0809@126.com

Xiaolin Zhu: School of Mathematical Sciences, Shanxi University, Taiyuan 030006, Shanxi, P. R. China

The cone P is called normal if there is a constant $K > 0$ such that $\theta \leq x \leq y$ implies $\|x\| \leq K\|y\|$, for all $x, y \in E$. A cone P is called to be solid if $\text{int}P \neq \emptyset$. We write $x \ll y$ for $y - x \in \text{int}P$, where $\text{int}P$ stands for the interior of P .

A subset P of E is called a cone if and only if:

1. P is closed, nonempty, and $P \neq \emptyset$;
2. $a, b \in \mathbf{R}$, $a, b \geq 0$, and $x, y \in P$ imply $ax + by \in P$;
3. $P \cap (-P) = \emptyset$.

In our following considerations, E is a real Banach space, P is a solid cone of E , and $\leq$ is a partial ordering induced by P .

Definition 1.1. (see [39]) Let $s \geq 1$ be a given real number. Suppose that X is a nonempty set. We call a mapping $\rho : X \times X \rightarrow E$ a cone b -metric if it satisfies:

- (i) $\theta < \rho(x, y)$ with $x \neq y$ and $\rho(x, y) = \theta$ if and only if $x = y$;
- (ii) $\rho(x, y) = \rho(y, x)$;
- (iii) $\rho(x, y) \leq s[\rho(x, z) + \rho(z, y)]$,

for all $x, y, z \in X$. The pair (X, ρ) , we say it a CbMS.

Definition 1.2. (see [39]) If (X, ρ) is a CbMS, $P \subset E$ is a solid cone, $x \in X$, and $\{x_n\}_{n \geq 1}$ is a sequence in X . Then

- (i) $\{x_n\}$ is called a Cauchy sequence, if for every $c \in E$ with $\theta \ll c$, there is an N such that for $n, m > N$, $\rho(x_n, x_m) \ll c$;
- (ii) $\{x_n\}$ converges to $x \in X$, if for every $c \in E$ with $\theta \ll c$, there is an N such that for $n > N$, $\rho(x_n, x) \ll c$ for some fixed $x \in X$;
- (iii) (X, ρ) is complete if every Cauchy sequence in X is convergent in X .

Lemma 1.3. (see [39]) If (X, ρ) is a CbMS, then one has the following properties:

- (p1) If $x \leq y$ and $y \ll z$, then $x \ll z$.
- (p2) If $x \ll y$ and $y \ll z$, then $x \ll z$.
- (p3) If $\theta \leq x \ll c$ for each $c \in \text{int}P$, then $x = \theta$.
- (p4) If $c \in \text{int}P$, $\theta \leq x_n$ and $x_n \rightarrow \theta$, then there exists n_0 such that $x_n \ll c$ for all $n > n_0$.
- (p5) Let $\theta \ll c$, if $\theta \leq \rho(x_n, x) \leq y_n$ and $y_n \rightarrow \theta$, then eventually $\rho(x_n, x) \ll c$, where $x \in X$ and $\{x_n\}_{n \geq 1}$ is a sequence in X .
- (p6) If $\theta \leq x_n \leq y_n$ and $x_n \rightarrow x$, $y_n \rightarrow y$, then $x \leq y$, for each cone P .
- (p7) If $x \leq \lambda x$ where $x \in P$ and $0 \leq \lambda < 1$, then $x = \theta$.

Definition 1.4. (see [35]) If X is a vector space over $\mathbf{R}$, cone $P \subset E$, and a mapping $\|\cdot\|_E : X \rightarrow E$ satisfies

- (i) $\|x\|_E \geq \theta$ for all $x \in X$ and $\|x\|_E = \theta$ if and only if $x = \theta$;
- (ii) $\|x + y\|_E \leq \|x\|_E + \|y\|_E$ for $x, y \in X$;
- (iii) $\|kx\|_E = |k|\|x\|_E$ for $k \in \mathbf{R}$,

then $\|\cdot\|_E$ is said a cone norm on X , and the pair $(X, \|\cdot\|_E)$ is said a cone normed space (CNS). Let $\rho(x, y) = \|x - y\|_E$, then every CNS is a cone metric space (CMS).

Now we give some new concepts about b -metric spaces.

Definition 1.5. Let $1 \leq s \leq 2$ be a given real number. Assume X is a vector space over $\mathbf{R}$, cone $P \subset E$. If $\|\cdot\|_P : X \rightarrow E$ satisfies

- (i) $\|x\|_P \geq \theta$ for $x \in X$ and $\|x\|_P = \theta$ if and only if $x = \theta$;
- (ii) $\|x + y\|_P \leq s[\|x\|_P + \|y\|_P]$ for $x, y \in X$;
- (iii) $\|kx\|_P = |k|^s \|x\|_P$ for $k \in \mathbf{R}$,

then we call $\|\cdot\|_P$ a cone b -norm on X , and the pair $(X, \|\cdot\|_P)$, we call it a cone b -normed space (CbNS). Evidently, each CbNS is a CbMS. In fact, we only need to set $\rho(x, y) = \|x - y\|_P$.

Definition 1.6. $(X, \|\cdot\|_P)$ is a CbNS, $P \subset E$ is a solid cone, $x \in X$, and $\{x_n\}_{n \geq 1}$ is a sequence in X . Then

- (i) we say that $\{x_n\}_{n \geq 1}$ converges to x if for every $c \in E$ with $\theta \ll c$, there is a natural number N satisfying $\|x_n - x\|_P \ll c$ for each $n \geq N$. We denote $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$;
- (ii) we say that $\{x_n\}_{n \geq 1}$ is a Cauchy sequence if for every $c \in E$ with $\theta \ll c$, there exists a natural number N satisfying $\|x_n - x_m\|_P \ll c$ for all $n, m \geq N$;
- (iii) we say that $(X, \|\cdot\|_P)$ is complete if every Cauchy sequence is convergent.

Naturally, we call a complete CbNS to be a cone b -Banach space. Every cone b -Banach space is a CbMS complete.

From (p3) and (p4) in Lemma 1.3, we can prove the following lemma.

Lemma 1.7. Suppose $(X, \|\cdot\|_P)$ is a CbNS, P is a solid cone, $x \in X$, and $\{x_n\}_{n \geq 1}$ is a sequence in X . Then the following statements hold:

- (i) $\|x_n - x\|_P \rightarrow \theta (n \rightarrow \infty)$ if and only if $\{x_n\}$ converges to x .
- (ii) $\|x_n - x_m\|_P \rightarrow \theta (n, m \rightarrow \infty)$ if and only if $\{x_n\}$ is a Cauchy sequence.

Proof. First, we prove (i). If $\|x_n - x\|_P \rightarrow \theta (n \rightarrow \infty)$, then for any $c \in \text{int}P$, $\theta \leq \|x_n - x\|_P$, and $\|x_n - x\|_P \rightarrow \theta$, by (p4), one can obtain that there exists n_0 such that for $n > n_0$ we have $\|x_n - x\|_P \ll c$, then $\{x_n\}$ converges to x .

In turn, if $\{x_n\}$ converges to x , using (p3) we get that for any $c \in E$, $\theta \ll c (c \in \text{int}P)$, $\theta \leq \|x_n - x\|_P \ll c$, then $\|x_n - x\|_P = \theta$.

The proof of (ii) is similar to (i). On one hand, if $\|x_n - x_m\|_P \rightarrow \theta (n, m \rightarrow \infty)$, then for any $c \in \text{int}P$, $\theta \leq \|x_n - x_m\|_P$, and $\|x_n - x_m\|_P \rightarrow \theta$, by (p4), one can obtain that there exists N such that for all $n, m > N$ we have $\|x_n - x_m\|_P \ll c$, then $\{x_n\}$ is a Cauchy sequence.

On the other hand, if $\{x_n\}$ is a Cauchy sequence, then for any $c \in E$, $\theta \ll c (c \in \text{int}P)$, $\theta \leq \|x_n - x_m\|_P \ll c$, by (p3) we can obtain that $\|x_n - x_m\|_P = \theta$. $\square$

Example 1.8. Let $X = \mathbf{R}^2$, $E = \mathbf{R}^2$, and $P = \{(x, y) \in E : x \geq 0, y \geq 0\}$, we define $\|(x, y)\|_P = (|x|^2, |y|^2)$. Then $(X, \|\cdot\|_P)$ is a cone b -Banach space.

Proof. First, by the definition of $\|(x, y)\|_P$, we get

$$\|(x, y)\|_P = (|x|^2, |y|^2) \geq \theta,$$

for all $u = (x, y) \in X$. It is obvious that $\|(x, y)\|_P = \theta$ if and only if $(x, y) = \theta$.

Next, according to

$$\begin{aligned} (x + y)^p &\leq x^p + y^p \quad (x, y \geq 0, 0 < p \leq 1), \\ (x + y)^p &\leq 2^{p-1}(x^p + y^p) \quad (x, y \geq 0, p \geq 1). \end{aligned}$$

We see for any $(x_1, y_1), (x_2, y_2) \in X$, it follows that

$$\begin{aligned} \|(x_1, y_1) + (x_2, y_2)\|_P &= \|(x_1 + x_2, y_1 + y_2)\|_P \\ &= (|x_1 + x_2|^2, |y_1 + y_2|^2) \\ &\leq 2(|x_1|^2 + |x_2|^2, |y_1|^2 + |y_2|^2) \\ &= 2(|x_1|^2, |y_1|^2) + 2(|x_2|^2, |y_2|^2) \\ &= 2\|(x_1, y_1)\|_P + 2\|(x_2, y_2)\|_P. \end{aligned}$$

For any $u = (x, y) \in X$, $k \in \mathbf{R}$, we have

$$\|ku\|_P = \|k(x, y)\|_P = \|(kx, ky)\|_P = (|kx|^2, |ky|^2) = |k|^2(|x|^2, |y|^2) = |k|^2\|u\|_P.$$

Thus, from Definition 1.5, $(X, \|\cdot\|_P)$ is a CbNS with the coefficient $s = 2 > 1$.

Finally, we suppose $\{u_n\} \in X$ is a Cauchy sequence in X , where $u_n = (x_n, y_n)$, $u_m = (x_m, y_m)$. By Definition 1.6 and Lemma 1.3, we deduce that

$$\|u_n - u_m\|_P = \|(x_n, y_n) - (x_m, y_m)\|_P = (|x_n - x_m|^2, |y_n - y_m|^2) \rightarrow (0, 0).$$

Therefore,

$$|x_n - x_m| \rightarrow 0, \quad |y_n - y_m| \rightarrow 0,$$

which means that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in $\mathbf{R}$. Since $\mathbf{R}$ is complete, there are $x, y \in \mathbf{R}$, such that $x_n \rightarrow x (n \rightarrow \infty)$ and $y_n \rightarrow y (n \rightarrow \infty)$. Let $u = (x, y) \in X$, then

$$\|u_n - u\|_P = (|x_n - x|^2, |y_n - y|^2) \rightarrow (0, 0).$$

By Lemma 1.3 (p4), one can obtain that $\{u_n\}$ converges to u in X , that is, $(X, \|\cdot\|_P)$ is a cone b -Banach space. $\square$

Remark 1.9. The new definition of cone b -Banach space generalizes cone Banach space. Clearly, a cone b -Banach space with $s = 1$ is exactly a cone Banach space. The above example is also sufficient to support the cone b -Banach space in Definition 1.5.

2 Main results

In our following discussions, we let $X = (X, \|\cdot\|_P)$ be a cone b -Banach space, P be a solid cone, and S be an operator defined on $D \subset X$.

Theorem 2.1. Let X be a cone b -Banach space with the coefficient $1 \leq s \leq 2$ and the b -norm $\|x\|_P = \rho(x, \theta)$, $D \subset X$ is closed and convex, $S : D \rightarrow D$ is a map, satisfying

$$\rho(x, Sx) + \rho(y, Sy) \leq q\rho(x, y), \quad \forall x, y \in D, \quad (2.1)$$

where $q \geq 0$ and $\rho(\cdot, \cdot)$ is a cone b -metric, then S has at least one fixed point.

Proof. When $q = 0$, the conclusion holds. So we only consider the case $q > 0$. Let $x = y$ in (2.1), we get that

$$2\rho(x, Sx) \leq q\rho(x, x) = \theta. \quad (2.2)$$

Then $\rho(x, Sx) = \theta$, that is $Sx = x$ for all $x \in D$. Thus, every x in D is a fixed point of S . Next we assume that $x \neq y$ and S is not an identity mapping on D . First, we prove that condition (2.1) implies that $q \geq 1$. Otherwise, $q < 1$. Suppose x is not a fixed point of S and let $y = Sx$, then we get that

$$\rho(x, Sx) + \rho(Sx, S^2x) \leq q\rho(x, Sx) < \rho(x, Sx). \quad (2.3)$$

Thus, $\rho(Sx, S^2x) < \theta$. This is not true. Thus, $q \geq 1$.

When $q = 1$, substituting $y = Sx$ in (2.1) implies that

$$\rho(x, Sx) + \rho(Sx, S^2x) \leq \rho(x, Sx). \quad (2.4)$$

Thus, $\rho(Sx, S^2x) = \theta$, that is $Sx = S^2x$, so Sx is a fixed point of S .

When $q > 1$, we introduce a sequence $\{x_n\}$, defined by:

$$x_{n+1} = \frac{ax_n + bSx_n}{a + b}, \quad n = 0, 1, 2, \dots, \quad (2.5)$$

where $x_0 \in D$ is arbitrary and $a, b > 0$ which satisfy $\frac{1}{q} \leq \frac{b^s}{(a+b)^s} < \frac{2}{q}$. Since D is a convex set, $\{x_n\} \subset D$.

Notice that

$$x_n - Sx_n = \frac{a+b}{b} \left(x_n - \frac{ax_n + bSx_n}{a+b} \right) = \frac{a+b}{b} (x_n - x_{n+1}), \quad (2.6)$$

which is equivalent to

$$\rho(x_n, Sx_n) = \|x_n - Sx_n\|_p = \left\| \frac{a+b}{b} (x_n - x_{n+1}) \right\|_p = \left(\frac{a+b}{b} \right)^s \rho(x_n, x_{n+1}), \quad n = 0, 1, 2, \dots \quad (2.7)$$

Taking into account the equation (2.7) and the condition (2.1), we have

$$\rho(x_{n-1}, Sx_{n-1}) + \rho(x_n, Sx_n) = \left(\frac{a+b}{b} \right)^s \rho(x_{n-1}, x_n) + \left(\frac{a+b}{b} \right)^s \rho(x_n, x_{n+1}) \leq q\rho(x_{n-1}, x_n). \quad (2.8)$$

Therefore, $\rho(x_n, x_{n+1}) \leq k_1 \rho(x_{n-1}, x_n)$, where $k_1 = \frac{qb^s}{(a+b)^s} - 1$. Since $\frac{1}{q} \leq \frac{b^s}{(a+b)^s} < \frac{2}{q}$, we know $0 \leq k_1 < 1$. Then, we get

$$\rho(x_{n+1}, x_n) \leq k_1 \rho(x_n, x_{n-1}) \leq \dots \leq k_1^n \rho(x_1, x_0).$$

For any $m \geq 1$, $p \geq 1$, it follows that

$$\begin{aligned} \rho(x_{m+p}, x_m) &\leq s[\rho(x_{m+p}, x_{m+p-1}) + \rho(x_{m+p-1}, x_m)] \\ &\leq s\rho(x_{m+p}, x_{m+p-1}) + s^2[\rho(x_{m+p-1}, x_{m+p-2}) + \rho(x_{m+p-2}, x_m)] \\ &\leq s\rho(x_{m+p}, x_{m+p-1}) + s^2\rho(x_{m+p-1}, x_{m+p-2}) + s^3\rho(x_{m+p-2}, x_{m+p-3}) \\ &\quad + \dots + s^{p-1}\rho(x_{m+2}, x_{m+1}) + s^{p-1}\rho(x_{m+1}, x_m) \\ &\leq sk_1^{m+p-1}\rho(x_1, x_0) + s^2k_1^{m+p-2}\rho(x_1, x_0) + s^3k_1^{m+p-3}\rho(x_1, x_0) \\ &\quad + \dots + s^{p-1}k_1^{m+1}\rho(x_1, x_0) + s^{p-1}k_1^m\rho(x_1, x_0) \\ &= (sk_1^{m+p-1} + s^2k_1^{m+p-2} + s^3k_1^{m+p-3} + \dots + s^{p-1}k_1^{m+1})\rho(x_1, x_0) + s^{p-1}k_1^m\rho(x_1, x_0) \\ &= \frac{sk_1^{m+p}[(sk_1^{-1})^{p-1} - 1]}{s - k_1}\rho(x_1, x_0) + s^{p-1}k_1^m\rho(x_1, x_0) \\ &\leq \frac{s^pk_1^{m+1}}{s - k_1}\rho(x_1, x_0) + s^{p-1}k_1^m\rho(x_1, x_0). \end{aligned}$$

Let $\theta \ll c$ be given. Note that $\frac{s^pk_1^{m+1}}{s - k_1}\rho(x_1, x_0) + s^{p-1}k_1^m\rho(x_1, x_0) \rightarrow \theta$, as $m \rightarrow \infty$. According to Lemma 1.3 (p4), we find $m_0 \in \mathbb{N}$, such that

$$\frac{s^pk_1^{m+1}}{s - k_1}\rho(x_1, x_0) + s^{p-1}k_1^m\rho(x_1, x_0) \ll c,$$

for each $m > m_0$. Therefore,

$$\rho(x_{m+p}, x_m) \leq \frac{s^pk_1^{m+1}}{s - k_1}\rho(x_1, x_0) + s^{p-1}k_1^m\rho(x_1, x_0) \ll c,$$

for all $m > m_0$ and any p . Thus by Lemma 1.3 (p1), we can get that Cauchy sequence $\{x_n\}$ converges to $z \in D$. Considering the following inequality

$$\rho(z, Sx_n) \leq s[\rho(z, x_n) + \rho(x_n, Sx_n)] = s\rho(z, x_n) + s\left(\frac{a+b}{b}\right)^s \rho(x_n, x_{n+1}), \quad (2.9)$$

and from Lemma 1.3 (p5)

$$Sx_n \rightarrow z. \quad (2.10)$$

Combining (2.7) with (2.1), we substitute $x = z$ and $y = x_n$ into (2.1) to get that

$$\rho(z, Sz) + \rho(x_n, Sx_n) = \rho(z, Sz) + \left(\frac{a+b}{b}\right)^s \rho(x_n, x_{n+1}) \leq q\rho(z, x_n). \quad (2.11)$$

Thus when $n \rightarrow \infty$, from Lemma 1.3, we can get $d(z, Sz) \leq \theta$, that is, $Sz = z$. $\square$

Corollary 2.2. Let X be a cone Banach space with the norm $\|x\|_E = \rho(x, \theta)$, $D \subset X$ is closed and convex, $S : D \rightarrow D$ is a map, satisfying

$$\rho(x, Sx) + \rho(y, Sy) \leq q\rho(x, y), \quad \forall x, y \in D,$$

where $q \geq 0$, then S has at least one fixed point.

Remark 2.3. If $s = 1$ in Theorem 2.1, we can obtain Corollary 2.2 which improves and extends Theorem 2.4 in [35]. We do not need the conditions: cone P is normal and $2 \leq q \leq 4$, and we only require $q \geq 0$. If $1 \leq s \leq 2$ in Theorem 2.1, we extend this fixed point theorem to our newly defined cone b -Banach space. According to the proof of Theorem 2.1, we can easily get the following inference.

Corollary 2.4. Let X be a cone b -Banach space with the coefficient $1 \leq s \leq 2$ and the b -norm $\|x\|_P = \rho(x, \theta)$, $D \subset X$ is closed and convex, $S : D \rightarrow D$ is a map, satisfying

$$\rho(x, Sx) + \rho(y, Sy) \leq q\rho(x, y), \quad \forall x, y \in D,$$

where $q \geq 0$ and $x \neq y$. Then (i) $q \geq 1$; (ii) S has at least one fixed point.

Corollary 2.5. Let X be a cone Banach space with the norm $\|x\|_E = \rho(x, \theta)$, $D \subset X$ is closed and convex, $S : D \rightarrow D$ is a map, satisfying

$$\rho(x, Sx) + \rho(y, Sy) \leq q\rho(x, y), \quad \forall x, y \in D,$$

where $q \geq 0$ and $x \neq y$. Then (i) $q \geq 1$; (ii) S has at least one fixed point.

Theorem 2.6. Let X be a cone b -Banach space with the coefficient $1 \leq s \leq 2$ and the b -norm $\|x\|_P = \rho(x, \theta)$, $D \subset X$ is closed and convex, $S : D \rightarrow D$ is a map, satisfying

$$\rho(y, Sx) + \rho(x, Sy) \leq p\rho(x, y), \quad \forall x, y \in D, \quad (2.12)$$

where $p \geq 0$. Then S has at least one fixed point. Moreover, when $0 \leq p < 2$, S has an unique fixed point.

Proof. If $x = y$ in (2.12), we have $2\rho(x, Sx) \leq p\rho(x, x)$. One can obtain $\rho(x, Sx) = \theta$, then $Sx = x$ for all $x \in D$. Thus, every x in D is a fixed point of S . Next we consider the case $x \neq y$. From the triangle inequality (iii) in Definition 1.1, we have

$$\rho(x, Sx) + \rho(y, Sy) \leq sp(x, y) + sp(y, Sx) + sp(y, x) + sp(x, Sy). \quad (2.13)$$

By (2.12),

$$\rho(x, Sx) + \rho(y, Sy) \leq 2sp(x, y) + spp(x, y) = s(2 + p)\rho(x, y), \quad p \geq 0. \quad (2.14)$$

Thus, letting $q = s(2 + p)$ implies that

$$\rho(x, Sx) + \rho(y, Sy) \leq q\rho(x, y), \quad q \geq 2. \quad (2.15)$$

Hence, by Theorem 2.1, S has at least one fixed point.

When $0 \leq p < 2$, we show the uniqueness of fixed points. Assume that there are $x, y \in D$ such that $Sx = x$ and $Sy = y$. By (2.12), we have

$$2\rho(x, y) \leq p\rho(x, y), \quad (2.16)$$

that is, $\rho(x, y) \leq \frac{p}{2}\rho(x, y)$, since $0 \leq \frac{p}{2} < 1$, then by Lemma 1.3 (p7), we get $\rho(x, y) = \theta$. It is clear that S has an unique fixed point. $\square$

Corollary 2.7. Let X be a cone Banach space with the norm $\|x\|_E = \rho(x, \theta)$, $D \subset X$ is closed and convex, $S : D \rightarrow D$ is a map, satisfying

$$\rho(y, Sx) + \rho(x, Sy) \leq p\rho(x, y), \quad \forall x, y \in D,$$

where $p \geq 0$. Then S has at least one fixed point. Especially, when $0 \leq p < 2$, S has an unique fixed point.

Remark 2.8. When $s = 1$ in Theorem 2.6, our result exists in cone Banach space, that is Corollary 2.7. Clearly, Corollary 2.7 amends and improves Theorem 2.5 in [35] and we particularly discuss the uniqueness of fixed points. When $1 < s \leq 2$, the condition is in cone b -Banach space, we extend this fixed point theorems to our newly defined cone b -Banach space.

Theorem 2.9. Let X be a cone b -Banach space with the coefficient $1 \leq s \leq 2$ and the b -norm $\|x\|_P = \rho(x, \theta)$, $D \subset X$ is closed and convex, $S : D \rightarrow D$ is a map, satisfying

$$\rho(Sx, Sy) + \rho(x, Sx) + \rho(y, Sy) \leq r\rho(x, y), \quad \forall x, y \in D, \quad (2.17)$$

where $r > 1$. Then S has at least one fixed point.

Proof. If $x = y$, by (2.17) we have $\rho(x, Sx) = \theta$, then $Sx = x$ for all $x \in D$. That is, S is an identity mapping on D . Next we consider the case $x \neq y$. Introduce a sequence $\{x_n\}$ defined by (2.5) and the following equalities

$$x_n - Sx_n = \frac{ax_{n-1} + bSx_{n-1}}{a+b} - Sx_{n-1} = \frac{a}{a+b}(x_{n-1} - Sx_{n-1}), \quad (2.18)$$

$$\rho(x_n, Sx_{n-1}) = \|x - Sx_{n-1}\|_P = \left(\frac{a}{a+b}\right)^s \|x - Sx_{n-1}\|_P = \left(\frac{a}{a+b}\right)^s \rho(x_{n-1}, Sx_{n-1}), \quad (2.19)$$

hold. The s -triangle inequality implies

$$\rho(x_n, Sx_n) - s\rho(x_n, Sx_{n-1}) \leq s\rho(Sx_{n-1}, Sx_n). \quad (2.20)$$

Then by (2.19) and (2.7) we have

$$\left(\frac{a+b}{b}\right)^s \rho(x_n, x_{n+1}) - s\left(\frac{a}{b}\right)^s \rho(x_{n-1}, x_n) \leq s\rho(Sx_{n-1}, Sx_n). \quad (2.21)$$

Substituting $x = x_{n-1}$ and $y = x_n$ in (2.17) and considering (2.7) and (2.21), we can obtain

$$\frac{1}{s}\left(\frac{a+b}{b}\right)^s \rho(x_n, x_{n+1}) - \left(\frac{a}{b}\right)^s \rho(x_{n-1}, x_n) + \left(\frac{a+b}{b}\right)^s \rho(x_{n-1}, x_n) + \left(\frac{a+b}{b}\right)^s \rho(x_n, x_{n+1}) \leq r\rho(x_{n-1}, x_n), \quad (2.22)$$

and thus $\rho(x_n, x_{n+1}) \leq k_2 \rho(x_{n-1}, x_n)$, where $k_2 = \frac{rsb^s + sa^s - s(a+b)^s}{(1+s)(a+b)^s}$. We can find a, b such that $\frac{(a+b)^s + a^s}{b^s} < r < \frac{(1+2s)(a+b)^s}{sb^s} + \left(\frac{a}{b}\right)^s$. Then $0 < k_2 < 1$. Hence, Cauchy sequence $\{x_n\}$ converges to some $z \in D$. Since $\{Sx_n\}$ also converges to z as in the proof of Theorem 2.1, the inequality (2.17) yields that $\rho(Sz, z) + \rho(z, Sz) \leq \theta$, that is, $Sz = z$. $\square$

Corollary 2.10. Let X be a cone Banach space with the norm $\|x\|_E = \rho(x, \theta)$, $D \subset X$ is closed and convex, $S : D \rightarrow D$ is a map, satisfying

$$\rho(Sx, Sy) + \rho(x, Sx) + \rho(y, Sy) \leq r\rho(x, y), \quad \forall x, y \in D,$$

where $r > 1$. Then S has at least one fixed point.

Remark 2.11. If $s = 1$ in Theorem 2.9, we can obtain Corollary 2.10. In Corollary 2.10, we only require $r > 1$, which improves and extends Theorem 2.6 in [35] where $2 \leq r < 5$, and generalize this theorem from cone Banach space to our newly defined cone b -Banach space with $1 \leq s \leq 2$.

Theorem 2.12. Let X be a cone b -Banach space with the coefficient $1 \leq s \leq 2$ and the b -norm $\|x\|_P = \rho(x, \theta)$, $D \subset X$ is closed and convex. If there exist constants h, l, m , and $S : D \rightarrow D$ be a mapping which satisfies the conditions

$$h < 0, \quad hs + l > 0, \quad 3lhs + m(hs + l) > 0, \quad (2.23)$$

$$h\rho(Sx, Sy) + l(\rho(x, Sx) + \rho(y, Sy)) \leq m\rho(x, y), \quad \forall x, y \in D. \quad (2.24)$$

Then S has at least one fixed point.

Proof. Introduce a sequence x_n defined by (2.5). Substituting $x = x_{n-1}$ and $y = x_n$ in (2.24) implies that

$$hs\left(\frac{a+b}{b}\right)^s \rho(x_n, x_{n+1}) + hs\left(\frac{a}{b}\right)^s \rho(x_{n-1}, x_n) + l\left(\frac{a+b}{b}\right)^s \rho(x_{n-1}, x_n) + l\left(\frac{a+b}{b}\right)^s \rho(x_n, x_{n+1}) \leq m\rho(x_{n-1}, x_n), \quad (2.25)$$

for all h, l, m that satisfy (2.23). Recall from (2.7) that

$$\rho(x_{n-1}, Sx_{n-1}) = \left(\frac{a+b}{b}\right)^s \rho(x_{n-1}, x_n), \quad \rho(x_n, Sx_n) = \left(\frac{a+b}{b}\right)^s \rho(x_n, x_{n+1}). \quad (2.26)$$

Since $h < 0$, the inequality $\rho(Sx_{n-1}, Sx_n) \leq s\rho(x_n, Sx_n) + s\rho(x_n, Sx_{n-1})$ is equivalent to

$$hs(\rho(x_n, Sx_n) + \rho(x_n, Sx_{n-1})) \leq h\rho(Sx_{n-1}, Sx_n). \quad (2.27)$$

Substituting $x = x_{n-1}$ and $y = x_n$ in (2.24) together with (2.26), (2.27) and (2.29), one can obtain (2.25).

By (2.25), we can get $\rho(x_n, x_{n+1}) \leq k_3 \rho(x_{n-1}, x_n)$, where $k_3 = \frac{-(hs + 2^{s-1}l)\left(\frac{a}{b}\right)^s - l2^{s-1} + m}{(hs + l)\left(\frac{a+b}{b}\right)^s}$. We can find a, b such that $\frac{-l2^s + m - hs2^{s-1} - 1}{l2^s + hs + hs2^{s-1}} < \left(\frac{a}{b}\right)^s \leq \frac{m - ls^{s-1}}{hs + 2^{s-1}l}$. Then $0 \leq k_3 < 1$. Thus, Cauchy sequence $\{x_n\}$ converges to some $z \in D$. We claim that the inequality (2.24) for $x = z$ and $y = x_n$, one can obtain

$$hsp(Sz, z) + lp(z, Sz) \leq \theta \quad \text{as } n \rightarrow \infty. \quad (2.28)$$

That is $Sz = z$ as $hs + l > 0$. □

Remark 2.13. In Theorem 2.12, we generalize Theorem 2.7 in [35] from cone Banach space to our newly defined cone b -Banach space.

Corollary 2.14. Let X be a cone Banach space with the norm $\|x\|_E = \rho(x, \theta)$, $D \subset X$ is closed and convex. If there exist constants h, l, m and $S : D \rightarrow D$ be a mapping which satisfies the conditions

$$h < 0, \quad h + l > 0, \quad m - h - 2l > 0, \quad (2.29)$$

$$h\rho(Sx, Sy) + l(\rho(x, Sx) + \rho(y, Sy)) \leq m\rho(x, y), \quad \forall x, y \in D. \quad (2.30)$$

Then S has at least one fixed point.

Proof. Introduce a sequence $\{x_n\}$ as (2.5). Substituting $x = x_{n-1}$ and $y = x_n$ in (2.24) implies that

$$h\left(\frac{a+b}{b}\right) \rho(x_n, x_{n+1}) + h\left(\frac{a}{b}\right) \rho(x_{n-1}, x_n) + l\left(\frac{a+b}{b}\right) (\rho(x_{n-1}, x_n) + \rho(x_n, x_{n+1})) \leq m\rho(x_{n-1}, x_n), \quad (2.31)$$

for all h, l, m that satisfy (2.29). Recall from (2.7) that

$$\rho(x_{n-1}, Sx_{n-1}) = \left(\frac{a+b}{b}\right) \rho(x_{n-1}, x_n), \quad \rho(x_n, Sx_n) = \left(\frac{a+b}{b}\right) \rho(x_n, x_{n+1}). \quad (2.32)$$

Since $h < 0$, it is clear that $\rho(Sx_{n-1}, Sx_n) \leq \rho(x_n, Sx_n) + \rho(x_n, Sx_{n-1})$ is equivalent to

$$h(\rho(x_n, Sx_n) + \rho(x_n, Sx_{n-1})) \leq h\rho(Sx_{n-1}, Sx_n). \quad (2.33)$$

Substituting $x = x_{n-1}$ and $y = x_n$ in (2.30) together with (2.32), (2.33) and (2.19), we can get (2.31). By (2.31), we can obtain $\rho(x_n, x_{n+1}) \leq k_4 \rho(x_{n-1}, x_n)$, where $k_4 = \frac{-(h+l)\left(\frac{a}{b}\right) - 1 + m}{(h+l)\left(\frac{a}{b} + 1\right)}$. We can choose a, b such that $\frac{m-h-2l}{2(h+l)} < \left(\frac{a}{b}\right) \leq \frac{m-l}{h+l}$. Then $0 \leq k_4 < 1$. Thus, Cauchy sequence $\{x_n\}$ converges to some $z \in D$. Replacing $x = z$ and $y = x_n$ in (2.30), we get

$$h\rho(Sz, z) + lp(z, Sz) \leq \theta, \quad n \rightarrow \infty. \quad (2.34)$$

Condition (2.34) is equivalent to saying that $Sz = z$ as $h + l > 0$. □

Remark 2.15. According to Theorem 2.7 in [35], we get Corollary 2.14 after making some modifications and using a different method. So these two theorems are different.

Acknowledgments: The authors are grateful to the referees for many detailed comments and thoughtful suggestions that have helped improve this paper substantially.

Conflict of interest: No potential conflict of interest was reported by the authors.

References

- [1] H. Lakzian and B. Samet, *Fixed points for (ψ, φ) -weakly contractive mappings in generalized metric spaces*, Appl. Math. Lett. **25** (2012), 902–906.
- [2] F. Vetro and S. Radenović, *Some results of Perov type in rectangular cone metric spaces*, J. Fixed Point Theory Appl. **20** (2018), 41.
- [3] H. Kunze, D. LaTorre, F. Mendivil, and E. R. Vrscaj, *Generalized fractal transforms and self-similar objects in cone metric spaces*, Comput. Math. Appl. **64** (2012), 1761–1769.
- [4] M. Abbas and B. E. Rhoades, *Fixed and periodic point results in cone metric spaces*, Appl. Math. Lett. **22** (2009), no. 4, 511–515.
- [5] R. George, S. Radenovic, K. P. Reshma, and S. Shukla, *Rectangular b -metric space and contraction principles*, J. Nonlinear Sci. **8** (2015), 1005–1013.
- [6] A. Aliouche and C. Simpson, *Fixed points and lines in 2-metric spaces*, Adv. Math. **229** (2012), 668–690.
- [7] H. P. Huang, S. Radenović, and G. Deng, *A sharp generalization on cone b -metric space over Banach algebra*, J. Nonlinear Sci. Appl. **10** (2017), 429–435.
- [8] P. Das and L. K. Dey, *Fixed point of contractive mappings in generalized metric spaces*, Math. Slovaca **59** (2009), no. 4, 499–504.
- [9] A. Azam and A. Muhammad, *Kannan fixed point theorem on generalized metric spaces*, J. Nonlinear Sci. Appl. **1** (2008), 45–48.
- [10] I. A. Bakhtin, *The contraction mapping principle in almost metric spaces*, Funct. Anal. **30** (1989), 26–37.
- [11] O. Yamaod and W. Sintunavarat, *Fixed point theorems for $(\alpha, \beta) - (\psi, \varphi)$ -contractive mappings in b -metric spaces with some numerical results and applications*, J. Nonlinear Sci. Appl. **9** (2016), 22–33.
- [12] H. P. Huang, G. Deng, and R. Stojan, *Fixed point theorems in b -metric spaces with applications to differential equations*, J. Fixed Point Theory Appl. **20** (2018), no. 1, 52.
- [13] R. Miculescu and A. Mihail, *New fixed point theorems for set-valued contractions in b -metric spaces*, J. Fixed Point Theory Appl. **19** (2017), no. 3, 2153–2163.
- [14] C. Chifu and G. Petrusel, *Fixed points for multivalued contractions in b -metric spaces with applications to fractals*, Taiwanese J. Math. **18** (2014), 1365–1375.
- [15] H. H. Alsulami, E. Karapinar, and V. Rakočević, *Ćirić type nonunique fixed point theorems on b -metric spaces*, Filomat **31** (2017), no. 11, 3147–3156.
- [16] E. Ameer, H. Aydi, M. Arshad, H. Alsamir, and M. S. Noorani, *Hybrid multivalued type contraction mappings in α_K -complete partial b -metric spaces and applications*, Symmetry **11** (2019), no. 1, 86.
- [17] E. Karapinar, S. Czerwik, and H. Aydi, *(α, ψ) -Meir-Keeler contraction mappings in generalized b -metric spaces*, J. Funct. Spaces **2018** (2018), 3264620.
- [18] H. Afshari, H. Aydi, and E. Karapinar, *On generalized α - ψ -Geraghty contractions on b -metric spaces*, Georgian Math. J. **27** (2020), 9–21.
- [19] L. V. Kantorovich, *The majorant principle and Newton's method*, Dokl. Akad. Nauk. SSSR(NS) **76** (1951), 17–20.
- [20] J. S. Vandergraft, *Newton's method for convex operators in partially ordered spaces*, SIAM J. Numer. Anal. Appl. **4** (1967), 406–432.
- [21] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, 1985.
- [22] W. Wang and X. Guo, *Eigenvalue problem for fractional differential equations with nonlinear integral and disturbance parameter in boundary conditions*, Bound. Value Probl. **17** (2016), no. 6, 2530–2538.
- [23] H. Wang and L. Zhang, *The solution for a class of sum operator equation and its application to fractional differential equation boundary value problems*, Bound. Value Probl. **2015** (2015), 203.
- [24] C. Zhai and L. Zhang, *New fixed point theorems for mixed monotone operators and local existence-uniqueness of positive solutions for nonlinear boundary value problems*, J. Math. Anal. Appl. **382** (2011), 594–614.
- [25] C. Zhai, C. Yang, and X. Zhang, *Positive solutions for nonlinear operator equations and several classes of applications*, Math. Z. **266** (2010), 43–63.

- [26] C. Zhai and J. Ren, *Some properties of sets, fixed point theorems in ordered product spaces and applications to a nonlinear system of fractional differential equations*, Topol. Method. Nonl. Anal. **49** (2017), no. 2, 625–645.
- [27] C. Zhai and L. Xu, *Properties of positive solutions to a class of four-point boundary value problem of Caputo fractional differential equations with a parameter*, Commun. Nonlinear Sci. Numer. Simulat. **19** (2014), 2820–2827.
- [28] L. Zhang, B. Ahmad, G. Wang, and R. P. Agarwal, *Nonlinear fractional integro-differential equations on unbounded domains in a Banach space*, J. Comput. App. Math. **249** (2013), 51–56.
- [29] X. Zhang, L. Liu, and Y. Zou, *Fixed-point theorems for systems of operator equations and their applications to the fractional differential equations*, J. Funct. Spaces **2018** (2018), 7469868.
- [30] X. Guo, G. Zhang, and H. Li, *Fixed point theorems for Meir-Keeler condensing nonself-mappings with an application*, J. Fixed Point Theory Appl. **20** (2018), 33.
- [31] Z. Zhao, *Fixed points of τ - ϕ -convex operators and applications*, Appl. Math. Lett. **23** (2010), no. 5, 561–566.
- [32] Y. Zou and G. He, *A fixed point theorem for systems of nonlinear operator equations and applications to (p_1, p_2) -Laplacian system*, Mediterr. J. Math. **15** (2018), 74.
- [33] L. G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **332** (2007), 1467–1475.
- [34] M. Abbas and G. Jungck, *Common fixed point result for noncommuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl. **341** (2008), 416–420.
- [35] E. Karapinar, *Fixed point theorems in cone Banach spaces*, Fixed Point Theory Appl. **2009** (2009), 609281.
- [36] Sh. Rezapour and R. Hambarani, *Some notes on the paper “Cone metric spaces and fixed point theorems of contractive mappings”*, J. Math. Anal. Appl. **345** (2008), 719–724.
- [37] J. J. Nieto and R. Rodríguez-López, *Contractive mappings theorems in partially ordered sets and applications to ordinary differential equations*, Order **22** (2005), no. 3, 223–239.
- [38] D. O’Regan and A. Petrusel, *Fixed point theorems for generalized contractions in ordered metric spaces*, J. Math. Anal. Appl. **341** (2008), 1241–1252.
- [39] N. Hussain and M. H. Shah, *KKM mappings in cone b-metric spaces*, Comput. Math. Appl. **62** (2011), no. 4, 1677–1684.
- [40] M. H. Shah, S. Simić, N. Hussain, A. Sretenović, and S. Radenović, *Common fixed points for occasionally weakly compatible pairs on cone metric type spaces*, J. Comput. Anal. Appl. **14** (2012), no. 2, 290–297.
- [41] H. P. Huang and S. Y. Xu, *Fixed point theorems of contractive mappings in cone b-metric spaces and applications*, Fixed Point Theory Appl. **2013** (2013), 112.
- [42] Z. M. Fadail and A. G. B. Ahmad, *Coupled coincidence point and common coupled fixed point results in cone b-metric spaces*, Fixed Point Theory Appl. **2013** (2013), 177.
- [43] A. Amini-Harandi, *Fixed point theory for quasi-contraction maps in b-metric spaces*, Appl. Math. Lett. **24** (2011), no. 11, 1791–1794.
- [44] H. Aydi, A. Felhi, and S. Sahmim, *On common fixed points for (α, ψ) -contractions and generalized cyclic contractions in b-metric-like spaces and consequences*, J. Nonlinear Sci. Appl. **9** (2016), 2492–2510.
- [45] X. Y. Lv and Y. Q. Feng, *Some fixed point theorems for Reich type contraction in generalized metric spaces*, J. Math. Anal. **9** (2018), no. 5, 80–88.
- [46] A. Petrusel, G. Petrusel, and J. C. Yao, *Fixed point and coincidence point theorems in b-metric spaces with applications*, Appl. Anal. Discrete Math. **11** (2017), 199–215.