Research Article

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Global existence and boundedness in a two-species chemotaxis system with nonlinear diffusion

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Abstract: This paper is concerned with a chemotaxis system

$$\begin{cases} u_t = \Delta u^m - \nabla \cdot (\chi_1(w)u\nabla w) + \mu_1 u(1 - u - a_1 v), & x \in \Omega, \ t > 0, \\ v_t = \Delta v^n - \nabla \cdot (\chi_2(w)v\nabla w) + \mu_2 v(1 - a_2 u - v), & x \in \Omega, \ t > 0, \\ w_t = \Delta w - (\alpha u + \beta v)w, & x \in \Omega, \ t > 0, \end{cases}$$

under homogeneous Neumann boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary, where $\mu_1, \mu_2 > 0$, $a_1, a_2 > 0$, $\alpha, \beta > 0$, and the chemotactic sensitivity function $\chi_i \in C^1([0, \infty))$, $\chi_i' \geq 0$. It is proved that for any large initial data, for any $m, n > 1$, the system admits a global weak solution, which is uniformly bounded.

Keywords: chemotaxis system, nonlinear diffusion, boundedness, logistic source**MSC 2020:** 92C17, 35K55, 35Q92

1 Introduction

Chemotaxis refers to the effect of chemical substances in the environment on the movement of species. This can lead to strict directional movement or partial orientation and partial tumbling movement. The movement to higher chemical concentrations is called positive chemotaxis, and the movement to lower chemical concentrations is called negative chemotaxis. Chemotaxis is an important means of cellular communication. After the pioneering work of Keller and Segel [1], a number of works concerning on the classical Keller-Segel model and its variations are investigated.

This paper is devoted to making development for the following two species chemotaxis system with nonlinear diffusion and consumption of chemoattractant

$$\begin{cases} u_t = \Delta u^m - \nabla \cdot (u\chi_1(w)\nabla w) + \mu_1 u(1 - u - a_1 v), & x \in \Omega, \ t > 0, \\ v_t = \Delta v^n - \nabla \cdot (v\chi_2(w)\nabla w) + \mu_2 v(1 - a_2 u - v), & x \in \Omega, \ t > 0, \\ w_t = \Delta w - (\alpha u + \beta v)w, & x \in \Omega, \ t > 0, \\ (\nabla u^m - u\chi_1(w)\nabla w) \cdot \nu = (\nabla v^n - v\chi_2(w)\nabla w) \cdot \nu = \nabla w \cdot \nu = 0, & x \in \partial\Omega, \ t > 0, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), w(x, 0) = w_0(x), & x \in \Omega, \end{cases} \quad (1)$$

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where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, and $m, n > 1$, $\mu_1, \mu_2 > 0$, $a_1, a_2 > 0$, $\alpha, \beta > 0$ are positive constants and ν is the outward normal vector to $\partial\Omega$. The functions $u = u(x, t)$ and $v = v(x, t)$ denote, respectively, the unknown population density of two species, and $w = w(x, t)$ represents the concentration of the chemoattractant. $\chi_i(w)$ ($i = 1, 2$) is the sensitivity function of aggregation induced by the concentration changes of chemoattractant, $\mu_1 u(1 - u - a_1 v)$ and $\mu_2 v(1 - a_2 u - v)$ ($\mu_i > 0$) are both the proliferation and death of bacteria according to a generalized logistic law and $-(\alpha u + \beta v)w$ denotes the consumption of chemoattractant.

In order to better understand model (1), we recall some previous contributions in this direction. Consider the following chemotaxis model with consumption of chemoattractant

$$\begin{cases} u_t = \nabla \cdot (D(u) \nabla u) - \chi \nabla \cdot (u \nabla w), & x \in \Omega, \quad t > 0, \\ w_t = \Delta w - u w, & x \in \Omega, \quad t > 0, \\ (\nabla D(u) - \chi u \cdot \nabla w) \cdot \nu = \nabla w \cdot \nu = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x, 0) = u_0(x), \quad w(x, 0) = w_0(x), & x \in \Omega, \end{cases} \quad (2)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $\chi > 0$ is a parameter referred to as chemosensitivity. When $D(u) \equiv 1$, in [2], Tao and Winkler proved that problem (2) possesses global bounded smooth solutions in the spatially two-dimensional setting, whereas in the three-dimensional counterpart, at least global weak solutions can be constructed, which eventually become smooth and bounded. When the nonlinear nonnegative function $D(u) \geq D_0(u + 1)^{m+1}$ ($D_0 > 0$), it has been shown in [3] that system (2) admits a unique global classical solution that is uniformly bounded when $m > \frac{1}{2}$ in the case $N = 1$ and $m > 2 - \frac{2}{N}$ in the case $N \geq 2$. Later, Zheng and Wang [4] improved this result to $m > 2 - \frac{6}{N+4}$ when $N \geq 3$.

If the reproduction and death of species themselves are taken into account, some logistic type sources will be added to the first equation of (2). For instance, when $D(u) \equiv 1$, in the three-dimensional case, Zheng et al. proved that the system (2) with a logistic type source $\mu u(1 - u)$ admits a unique global classical solution if the initial data of w are small in [5]. In arbitrary N -dimensional bounded smooth domain, Lankeit and Wang obtained the global bounded classical solutions of (2) for any large initial data in [6] when μ is appropriately large, and they also proved the existence of global weak solutions for any large μ . In a bounded domain $\Omega \subset \mathbb{R}^3$, when $D(u) \geq C_D(u + 1)^{m-1}$ for all $u \geq 0$ with some $C_D > 0$, Zheng [7] studied the issue of boundedness to solutions of (2) without any restriction on the space dimension, if

$$m > \begin{cases} 1 - \frac{\mu}{\chi(1 + \lambda_0 \|w_0\|_{L^\infty(\Omega)} 2^3)}, & \text{if } N \leq 2, \\ 1, & \text{if } N \geq 3, \end{cases}$$

where λ_0 is a positive constant which is corresponding to the maximal Sobolev regularity, then for any sufficiently smooth initial data there exists a classical solution which is global in time and bounded.

Also, multi-species chemotaxis systems have been extensively studied by many authors. When the two species have effect on each other, the system involved Lotka-Volterra competitive kinetics

$$\begin{cases} u_t = d_1 \Delta u - \nabla \cdot (u \chi_1(w) \nabla w) + \mu_1 u(1 - u - a_1 v), & x \in \Omega, \quad t > 0, \\ v_t = d_2 \Delta v - \nabla \cdot (v \chi_2(w) \nabla w) + \mu_2 v(1 - a_2 u - v), & x \in \Omega, \quad t > 0, \\ w_t = d_3 \Delta w - (\alpha u + \beta v) w, & x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad w(x, 0) = w_0(x), & x \in \Omega, \end{cases} \quad (3)$$

has been proposed to describe the evolution of two competing species that react on a single chemoattractant. Here u , v , and w are represented as model (1), the chemotactic function $\chi_i(w)$ ($i = 1, 2$) is smooth. In [8], it is proved that the corresponding initial-boundary value problem possesses a unique global bounded classical solution. Moreover, the authors there also proved the asymptotic stabilization of solution. When $-(\alpha u + \beta v)w$ in (3) is replaced by $-w + \alpha u + \beta v$, this model has been extensively studied by many authors. In the two-dimensional case, Bai and Winkler [9] obtained global existence of solution to the system

if $\chi_i(w) = \chi_i$ are positive constants. Moreover, they also considered asymptotic behavior of solutions to the system: when $a_1, a_2 \in (0, 1)$, $u(\cdot, t) \rightarrow \frac{1-a_1}{1-a_1a_2}$, $v(\cdot, t) \rightarrow \frac{1-a_2}{1-a_1a_2}$, $w \rightarrow \frac{a(1-a_1)+\beta(1-a_2)}{1-a_1a_2}$ in $L^\infty(\Omega)$ as $t \rightarrow \infty$; when $a_1 \geq 1 > a_2 > 0$, $u(\cdot, t) \rightarrow 0$, $v(\cdot, t) \rightarrow 1$, $w \rightarrow \beta$ in $L^\infty(\Omega)$ as $t \rightarrow \infty$. In the three-dimensional case, Lin and Mu [10] obtained similar results if μ_1 and μ_2 are sufficiently large.

When the two species have no effect on each other, the competitive kinetics terms $\mu_1 u(1-u-a_1 v)$ and $\mu_2 v(1-a_2 u-v)$ will be replaced by $\mu_1 u(1-u)$ and $\mu_2 v(1-v)$ in system (3):

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \chi_1(w) \nabla w) + \mu_1 u(1-u), & x \in \Omega, \quad t > 0, \\ v_t = \Delta v - \nabla \cdot (v \chi_2(w) \nabla w) + \mu_2 v(1-v), & x \in \Omega, \quad t > 0, \\ w_t = d \Delta w + h(u, v, w), & x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial \Omega, \quad t > 0, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), w(x, 0) = w_0(x), & x \in \Omega, \end{cases} \quad (4)$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$. Negreanu and Tello [11,12] proved global existence and asymptotic behavior of solutions to the above system when $0 \leq d < 1$. This result was later improved by Mizukami and Yokota, and they removed the restriction of $0 \leq d < 1$ in [13].

Assuming that the random movement of species is nonlinearly enhanced at large densities, several works addressed a porous medium-type diffusion chemotaxis model (see, e.g. [14–17]). Inspired by the aforementioned works, in this paper, we consider the two species chemotaxis system (1) with nonlinear diffusion. The purpose of this paper is to obtain global existence and uniform boundedness of weak solution in a three-dimensional setting.

Throughout this paper, we assume that

$$\begin{cases} u_0, v_0, w_0 \in C^{2+\alpha}(\bar{\Omega}), \\ u_0, v_0, w_0 \geq 0, \\ \left. \frac{\partial u_0}{\partial \nu} \right|_{\partial \Omega} = \left. \frac{\partial v_0}{\partial \nu} \right|_{\partial \Omega} = \left. \frac{\partial w_0}{\partial \nu} \right|_{\partial \Omega} = 0. \end{cases} \quad (5)$$

The chemotactic sensitivity function χ_i ($i = 1, 2$) satisfies the following conditions:

$$\chi_i \in C^1([0, \infty)), \quad \chi_i' \geq 0. \quad (6)$$

Now, we state the main results of this paper as follows.

Theorem 1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Assume that (5) and (6) hold, and $m, n > 1$, $a_1, a_2 > 0$, $\alpha, \beta > 0$. Then for any $\mu_i > 0$ ($i = 1, 2$), system (1) possesses a nonnegative weak solution (u, v, w) with $u \in A_1$, $v \in A_2$, $w \in A_3$, where for any $p > 1$,*

$$\begin{cases} A_1 = \left\{ u \in L^\infty(\Omega \times (0, \infty)); \nabla u^m \in L^\infty((0, \infty); L^2(\Omega)); (u^{\frac{m+1}{2}})_t, \nabla u^{\frac{m+1}{2}} \in L^2_{\text{loc}}([0, \infty); L^2(\Omega)) \right\}, \\ A_2 = \left\{ v \in L^\infty(\Omega \times (0, \infty)); \nabla v^n \in L^\infty((0, \infty); L^2(\Omega)); (v^{\frac{n+1}{2}})_t, \nabla v^{\frac{n+1}{2}} \in L^2_{\text{loc}}([0, \infty); L^2(\Omega)) \right\}, \\ A_3 = \{ w \in L^\infty((0, \infty); W^{1,\infty}(\Omega)); w_t, \Delta w \in L^p_{\text{loc}}([0, \infty); L^p(\Omega)) \}, \end{cases}$$

such that

$$\sup_{t \in (0, \infty)} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)}) \leq C, \quad (7)$$

where C depends only on $\mu_1, \mu_2, u_0, v_0, w_0$.

The rest of this paper is organized as follows. In Section 2, we introduce the conception of the weak solution and summarize some basic definitions and useful lemmas. In Section 3, we shall first establish the existence of global classical solutions to the regularized problems and second show the convergence of the solution of regularized problems and thus obtain the proof of Theorem 1.

2 Some preliminaries

We first give some notations, which will be used throughout this paper.

Notations: $Q_\tau(t) = \Omega \times (t, t + \tau)$, $Q_T := Q_T(0) = \Omega \times (0, T)$. $W_p^{m,k}(Q_T) = \{u; D^\alpha u, D_t^r u \in L^p(Q_T), \|u\|_{W_p^{m,k}(Q_T)} < \infty\}$, where $|\alpha| \leq m$, $r \leq k$, $m = 0, 1, 2, \dots, k \in \{0, 1\}$, $1 \leq p < \infty$, and $\|u\|_{W_p^{m,k}(Q_T)} = \left(\iint_{Q_T} \left(\sum_{|\alpha| \leq m} |D^\alpha u|^p + \sum_{r \leq k} |D_t^r u|^p \right) dx dt \right)^{1/p}$.

Next, we introduce the definition of weak solutions.

Definition 1. (u, v, w) is called a weak solution of (1) on Q_T , if $u \geq 0$, $v \geq 0$, $w \geq 0$, with

$$\begin{aligned} u &\in L^2(Q_T), & \nabla u^m &\in L^2(Q_T), \\ v &\in L^2(Q_T), & \nabla v^n &\in L^2(Q_T), \\ w &\in W_2^{1,0}(Q_T), \end{aligned}$$

for any $T > 0$, and

$$-\iint_{Q_T} u \varphi_t dx dt - \int_{\Omega} u(x, 0) \varphi(x, 0) dx + \iint_{Q_T} (\nabla u^m - \chi_1(w) u \nabla w) \nabla \varphi dx dt = \mu_1 \iint_{Q_T} u(1 - u - a_1 v) \varphi dx dt, \quad (8)$$

$$-\iint_{Q_T} v \varphi_t dx dt - \int_{\Omega} v(x, 0) \varphi(x, 0) dx + \iint_{Q_T} (\nabla v^n - \chi_2(w) v \nabla w) \nabla \varphi dx dt = \mu_2 \iint_{Q_T} v(1 - a_2 u - v) \varphi dx dt, \quad (9)$$

$$-\iint_{Q_T} w \varphi_t dx dt - \int_{\Omega} w(x, 0) \varphi(x, 0) dx + \iint_{Q_T} \nabla w \nabla \varphi dx dt + \iint_{Q_T} (\alpha u + \beta v) w \varphi dx dt = 0 \quad (10)$$

hold for any $\varphi \in C^\infty(\overline{Q_T})$ with $\frac{\partial \varphi}{\partial \nu} \Big|_{\partial \Omega} = 0$, $\varphi(x, T) = 0$.

Before going further, we give some lemmas, which will be used later. We firstly list the following lemma, a proof of which can be found in [18] (see also [19]).

Lemma 1. Let $T > 0$, $\tau \in (0, T)$, $a > 0$, $b > 0$, and suppose that $y : [0, T] \rightarrow [0, \infty)$ is absolutely continuous such that

$$y'(t) + ay(t) \leq h(t), \quad \text{for } t \in [0, T], \quad (11)$$

where $h \geq 0$, $h(t) \in L_{\text{loc}}^1([0, T])$, and

$$\int_{t-\tau}^t h(s) ds \leq b, \quad \text{for } t \in [\tau, T]. \quad (12)$$

Then

$$y(t) \leq \max \left\{ y(0) + b, \frac{b}{a\tau} + 2b \right\}, \quad \text{for } t \in [0, T]. \quad (13)$$

By [20,21], we have the following two lemmas.

Lemma 2. Let $T > 0$, $\tau \in (0, T)$, $\sigma \geq 0$, $a > 0$, $b \geq 0$, and suppose that $f : [0, T] \rightarrow [0, \infty)$ is absolutely continuous, and satisfies

$$f'(t) + af^{1+\sigma}(t) \leq h(t), \quad t \in \mathbb{R}, \quad (14)$$

where $h \geq 0$, $h(t) \in L^1_{\text{loc}}([0, T])$ and

$$\int_{t-\tau}^t h(s) ds \leq b, \quad \text{for all } t \in [\tau, T].$$

Then

$$\sup_{t \in (0, T)} f(t) + a \sup_{t \in (\tau, T)} \int_{t-\tau}^t f^{1+\sigma}(s) ds \leq b + 2 \max \left\{ f(0) + b + a\tau, \frac{b}{a\tau} + 1 + 2b + 2a\tau \right\}. \quad (15)$$

Lemma 3. Assume that $p > 1$, $u_0 \in W^{2,p}(\Omega)$, and $f \in L^p_{\text{loc}}((0, +\infty); L^p(\Omega))$ satisfying

$$\sup_{t \in (\tau, +\infty)} \int_{t-\tau}^t \|f\|_{L^p(\Omega)}^p ds \leq A$$

with some $A > 0$, where $\tau > 0$ is a fixed constant. Then the following system

$$\begin{cases} u_t - \Delta u + u = f(x, t), \\ \frac{\partial u}{\partial \nu} \Big|_{\partial \Omega} = 0, \\ u(x, 0) = u_0(x) \end{cases} \quad (16)$$

has a unique solution u with $u \in L^p_{\text{loc}}((0, +\infty); W^{2,p}(\Omega))$, $u_t \in L^p_{\text{loc}}((0, +\infty); L^p(\Omega))$, and

$$\sup_{t \in (\tau, +\infty)} \int_{t-\tau}^t (\|u\|_{W^{2,p}(\Omega)}^p + \|u_t\|_{L^p(\Omega)}^p) ds \leq AM \frac{e^{p\tau}}{e^{\frac{p}{2}\tau} - 1} + Me^{\frac{p}{2}\tau} \|u_0\|_{W^{2,p}(\Omega)}^p, \quad (17)$$

where M is a constant independent of τ .

3 Global existence and boundedness of weak solution

The degeneracy at $u = 0$ of system (1) results in the failure of the classical parabolic regularity theory. To overcome this difficulty, we shall first consider the following regularized version:

$$\begin{cases} u_{\varepsilon t} = \Delta \left((u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} u_{\varepsilon} \right) - \nabla \cdot (u_{\varepsilon} \chi_1(w_{\varepsilon}) \nabla w_{\varepsilon}) + \mu_1 u_{\varepsilon} (1 - u_{\varepsilon} - a_1 v_{\varepsilon}), & x \in \Omega, \quad t > 0, \\ v_{\varepsilon t} = \Delta \left((v_{\varepsilon}^2 + \varepsilon)^{\frac{n-1}{2}} v_{\varepsilon} \right) - \nabla \cdot (v_{\varepsilon} \chi_2(w_{\varepsilon}) \nabla w_{\varepsilon}) + \mu_2 v_{\varepsilon} (1 - a_2 u_{\varepsilon} - v_{\varepsilon}), & x \in \Omega, \quad t > 0, \\ w_{\varepsilon t} = \Delta w_{\varepsilon} - (\alpha u_{\varepsilon} + \beta v_{\varepsilon}) w_{\varepsilon}, & x \in \Omega, \quad t > 0, \\ \frac{\partial u_{\varepsilon}}{\partial \nu} = \frac{\partial v_{\varepsilon}}{\partial \nu} = \frac{\partial w_{\varepsilon}}{\partial \nu} = 0, & x \in \partial \Omega, \quad t > 0, \\ u_{\varepsilon}(x, 0) = u_0(x), \quad v_{\varepsilon}(x, 0) = v_0(x), \quad w_{\varepsilon}(x, 0) = w_0(x), & x \in \Omega. \end{cases} \quad (18)$$

We begin with the local existence of classical solutions to system (18), the proof of which is similar (refer to, e.g., [17, 22–25], for the details).

Lemma 4. Suppose that (5) and (6) hold. Then there exists $T_{\max, \varepsilon} \in (0, +\infty]$ and a unique classical solution $(u_{\varepsilon}, v_{\varepsilon}, w_{\varepsilon}) \in C^{2+\alpha, 1+\alpha}(\bar{\Omega} \times [0, T_{\max, \varepsilon}))$ solving the system (18) in the classical sense with $u_{\varepsilon}, v_{\varepsilon}, w_{\varepsilon} \geq 0$ in $\Omega \times (0, T_{\max, \varepsilon})$ and if $T_{\max, \varepsilon} < \infty$, then

$$\|u_{\varepsilon}(\cdot, t)\|_{L^{\infty}(\Omega)} + \|v_{\varepsilon}(\cdot, t)\|_{L^{\infty}(\Omega)} + \|w_{\varepsilon}(\cdot, t)\|_{W^{1, \infty}(\Omega)} \rightarrow \infty \quad \text{as } t \rightarrow T_{\max, \varepsilon}.$$

In what follows, we shall show that for each ε , the solution $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ is actually global in time, that is, we have

Proposition 1. *Suppose that (5) and (6) hold. Then for any $\varepsilon > 0$, (18) has a global classical solution $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ which is, furthermore, bounded, and*

$$\sup_{t \in (0, \infty)} \int_{\Omega} |\nabla u_\varepsilon^m|^2 + \sup_{t \in (0, \infty)} \left\| u_\varepsilon^{\frac{m+1}{2}} \right\|_{W_2^{1,1}(Q_1(t))} \leq M_1, \quad (19)$$

$$\sup_{t \in (0, \infty)} \int_{\Omega} |\nabla v_\varepsilon^n|^2 + \sup_{t \in (0, \infty)} \left\| v_\varepsilon^{\frac{n+1}{2}} \right\|_{W_2^{1,1}(Q_1(t))} \leq M_2, \quad (20)$$

$$\sup_{t \in (0, \infty)} \|w_\varepsilon\|_{W_p^{2,1}(Q_1(t))} \leq M_3, \quad \text{for any } p > 1, \quad (21)$$

where M_i ($i = 1, 2, 3$) are some constants independent of ε .

Next, we give some estimates of $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$. Take $\tau = \min\{1, T_{\max, \varepsilon}\}$. It is easy to see that $\tau \leq 1$.

Lemma 5. *Suppose that (5) and (6) hold. Let $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ be the classical solution of (18). Then*

$$\|w_\varepsilon\|_{L^\infty(\Omega)} \leq \|w_0\|_{L^\infty(\Omega)}, \quad t \in (0, T_{\max, \varepsilon}), \quad (22)$$

$$\sup_{t \in (0, T_{\max, \varepsilon})} \|u_\varepsilon(\cdot, t)\|_{L^1(\Omega)} + \sup_{t \in (\tau, T_{\max, \varepsilon})} \int_{t-\tau}^t \int_{\Omega} u_\varepsilon^2 dx \leq C_1, \quad (23)$$

$$\sup_{t \in (0, T_{\max, \varepsilon})} \|v_\varepsilon(\cdot, t)\|_{L^1(\Omega)} + \sup_{t \in (\tau, T_{\max, \varepsilon})} \int_{t-\tau}^t \int_{\Omega} v_\varepsilon^2 dx \leq C_2, \quad (24)$$

$$\sup_{t \in (\tau, T_{\max, \varepsilon})} \int_{t-\tau}^t \|w_\varepsilon\|_{W^{2,2}(\Omega)}^2 ds \leq C_3, \quad (25)$$

where C_i ($i = 1, 2, 3$) are independent of $T_{\max, \varepsilon}$, τ , and ε .

Proof. By comparison principle, it is easy to see that (22) holds. Integrating the first equation of (18) and using Young's inequality, we obtain

$$\frac{d}{dt} \int_{\Omega} u_\varepsilon dx + \mu_1 \int_{\Omega} u_\varepsilon^2 dx \leq \mu_1 \int_{\Omega} u_\varepsilon dx \leq \frac{\mu_1}{2} \int_{\Omega} u_\varepsilon^2 dx + \frac{1}{2} \mu_1 |\Omega|. \quad (26)$$

We get by (26) and Hölder's inequality that

$$\frac{d}{dt} \int_{\Omega} u_\varepsilon dx + \frac{\mu_1}{2|\Omega|} \left(\int_{\Omega} u_\varepsilon dx \right)^2 \leq \frac{1}{2} \mu_1 |\Omega|.$$

Due to $\int_{t-\tau}^t |\Omega| ds = |\Omega| \tau \leq |\Omega|$ for all $t \in (\tau, T_{\max, \varepsilon})$, here $\tau \leq 1$, then by Lemma 2, there exists $C_1 > 0$ such that

$$\sup_{t \in (0, T_{\max, \varepsilon})} \|u_\varepsilon(\cdot, t)\|_{L^1(\Omega)} + \sup_{t \in (\tau, T_{\max, \varepsilon})} \int_{t-\tau}^t \int_{\Omega} u_\varepsilon^2 dx \leq C_1 \left(|\Omega|, \mu_1, \int_{\Omega} u_0 dx \right).$$

By the same way, there exists $C_2 > 0$ such that (24) holds. From the third equation of (18), we have

$$w_{\varepsilon t} - \Delta w_\varepsilon + w_\varepsilon = w_\varepsilon - (\alpha u_\varepsilon + \beta v_\varepsilon) w_\varepsilon.$$

By (22), (23), (24), and Minkowski's inequality, we have

$$\sup_{t \in (\tau, T_{\max, \varepsilon})} \int_{t-\tau}^t \|w_\varepsilon - (\alpha u_\varepsilon + \beta v_\varepsilon) w_\varepsilon\|_{L^2(\Omega)}^2 ds \leq A(\|w_0\|_{L^\infty(\Omega)}, \alpha, \beta, C_1, C_2).$$

Using Lemma 3, there exists $C_3 > 0$ such that

$$\sup_{t \in (\tau, T_{\max, \varepsilon})} \int_{t-\tau}^t \|w_\varepsilon\|_{W^{2,2}(\Omega)}^2 ds \leq AM \frac{e^{2\tau}}{e^\tau - 1} + Me^\tau \|w_0\|_{W^{2,2}(\Omega)}^2 \leq AMe^2 + Me^2 \|w_0\|_{W^{2,2}(\Omega)}^2 = C_3,$$

(25) is proved. The proof is complete. $\square$

Lemma 6. Suppose that (5) and (6) hold. Let $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ be the classical solution of (18). Then for any $k = 1, 2, 3, \dots$,

$$\sup_{t \in (0, T_{\max, \varepsilon})} \|u_\varepsilon(\cdot, t)\|_{L^{2m^{k-1}}(\Omega)}^{2m^{k-1}} + \sup_{t \in (\tau, T_{\max, \varepsilon})} \int_{t-\tau}^t \left(\int_\Omega u_\varepsilon^{m+2m^k-4} |\nabla u_\varepsilon|^2 dx + \int_\Omega u_\varepsilon^{2m^k} dx \right) ds \leq C_k, \quad (27)$$

$$\sup_{t \in (0, T_{\max, \varepsilon})} \|v_\varepsilon(\cdot, t)\|_{L^{2n^{k-1}}(\Omega)}^{2n^{k-1}} + \sup_{t \in (\tau, T_{\max, \varepsilon})} \int_{t-\tau}^t \left(\int_\Omega v_\varepsilon^{n+2n^k-4} |\nabla v_\varepsilon|^2 dx + \int_\Omega v_\varepsilon^{2n^k} dx \right) ds \leq C'_k, \quad (28)$$

$$\sup_{t \in (\tau, T_{\max, \varepsilon})} \int_{t-\tau}^t \left(\|w_\varepsilon\|_{W^{2,2m^k}(\Omega)}^{2m^k} + \|w_{\varepsilon t}\|_{L^{2m^k}(\Omega)}^{2m^k} \right) ds \leq \widetilde{C}_k, \quad (29)$$

$$\sup_{t \in (\tau, T_{\max, \varepsilon})} \int_{t-\tau}^t \left(\|w_\varepsilon\|_{W^{2,2n^k}(\Omega)}^{2n^k} + \|w_{\varepsilon t}\|_{L^{2n^k}(\Omega)}^{2n^k} \right) ds \leq \widetilde{C}'_k, \quad (30)$$

where $C_k, C'_k, \widetilde{C}_k, \widetilde{C}'_k$ depend on k and all of them are independent of $T_{\max, \varepsilon}, \tau$ and ε .

Proof. Multiplying the first equation of (18) by u_ε^r for any $r > 0$ and integrating over Ω by parts,

$$\begin{aligned} & \frac{1}{r+1} \frac{d}{dt} \int_\Omega u_\varepsilon^{r+1} dx + r \int_\Omega u_\varepsilon^{m+r-2} |\nabla u_\varepsilon|^2 dx + \mu_1 \int_\Omega u_\varepsilon^{r+2} dx \\ & \leq r \int_\Omega \chi_1(w_\varepsilon) u_\varepsilon^r \nabla u_\varepsilon \nabla w_\varepsilon dx + \mu_1 \int_\Omega u_\varepsilon^{r+1} dx \\ & \leq \frac{r}{2} \int_\Omega u_\varepsilon^{m+r-2} |\nabla u_\varepsilon|^2 dx + \frac{\mu_1}{2} \int_\Omega u_\varepsilon^{r+2} dx + \frac{r}{2} \int_\Omega \chi_1^2(w_\varepsilon) u_\varepsilon^{r+2-m} |\nabla w_\varepsilon|^2 dx + C_1, \quad t \in (0, T_{\max, \varepsilon}). \end{aligned} \quad (31)$$

It is easy to see that

$$\begin{aligned} & \frac{1}{r+1} \frac{d}{dt} \int_\Omega u_\varepsilon^{r+1} dx + \frac{r}{2} \int_\Omega u_\varepsilon^{m+r-2} |\nabla u_\varepsilon|^2 dx + \frac{\mu_1}{2} \int_\Omega u_\varepsilon^{r+2} dx \\ & \leq \frac{r}{2} \int_\Omega \chi_1^2(w_\varepsilon) u_\varepsilon^{r+2-m} |\nabla w_\varepsilon|^2 dx + C_1, \quad t \in (0, T_{\max, \varepsilon}). \end{aligned} \quad (32)$$

By Gagliardo-Nirenberg inequality and (22), for any $k = 0, 1, 2, \dots$, we have

$$\begin{aligned} \int_{t-\tau}^t \|\nabla w_\varepsilon\|_{L^{4m^k}(\Omega)}^{4m^k} ds &\leq C_2 \int_{t-\tau}^t (\|\Delta w_\varepsilon\|_{L^{2m^k}(\Omega)}^{4m^k \alpha} \|w_\varepsilon\|_{L^\infty(\Omega)}^{4m^k(1-\alpha)} + \|w_\varepsilon\|_{L^\infty(\Omega)}^{4m^k}) ds \\ &\leq C_3 \left(1 + \int_{t-\tau}^t \|\Delta w_\varepsilon\|_{L^{2m^k}(\Omega)}^{2m^k} ds \right), \quad t \in (\tau, T_{\max, \varepsilon}), \end{aligned} \quad (33)$$

where $\alpha = \frac{1}{2}$, C_3 depends on m, k, Ω , and $\|w_0\|_{L^\infty(\Omega)}$. Inserting $r = 2(m-1)$ in (32), by (6) and (22), we infer from Young's inequality that in $(0, T_{\max, \varepsilon})$

$$\begin{aligned} &\frac{1}{2m-1} \frac{d}{dt} \int_{\Omega} u_\varepsilon^{2m-1} dx + (m-1) \int_{\Omega} u_\varepsilon^{3m-4} |\nabla u_\varepsilon|^2 dx + \frac{\mu_1}{2} \int_{\Omega} u_\varepsilon^{2m} dx \\ &\leq (m-1) \int_{\Omega} \chi_1^2(w_\varepsilon) u_\varepsilon^m |\nabla w_\varepsilon|^2 dx + C_1 \leq \frac{\mu_1}{4} \int_{\Omega} u_\varepsilon^{2m} dx + C_4 \int_{\Omega} |\nabla w_\varepsilon|^4 dx + C_1, \end{aligned} \quad (34)$$

it implies

$$\frac{d}{dt} \int_{\Omega} u_\varepsilon^{2m-1} dx + \int_{\Omega} u_\varepsilon^{3m-4} |\nabla u_\varepsilon|^2 dx + \int_{\Omega} u_\varepsilon^{2m} dx + \int_{\Omega} u_\varepsilon^{2m-1} dx \leq C_5 \int_{\Omega} |\nabla w_\varepsilon|^4 dx + C_5,$$

then, taking $k = 0$ in (33) and using (25), we infer from Lemma 1 that

$$\sup_{t \in (0, T_{\max, \varepsilon})} \|u_\varepsilon\|_{L^{2m-1}(\Omega)}^{2m-1} + \sup_{t \in (\tau, T_{\max, \varepsilon})} \int_{t-\tau}^t \left(\int_{\Omega} u_\varepsilon^{3m-4} |\nabla u_\varepsilon|^2 dx + \int_{\Omega} u_\varepsilon^{2m} dx \right) ds \leq C.$$

By the same way, we have

$$\sup_{t \in (0, T_{\max, \varepsilon})} \|v_\varepsilon\|_{L^{2n-1}(\Omega)}^{2n-1} + \sup_{t \in (\tau, T_{\max, \varepsilon})} \int_{t-\tau}^t \left(\int_{\Omega} v_\varepsilon^{3n-4} |\nabla v_\varepsilon|^2 dx + \int_{\Omega} v_\varepsilon^{2n} dx \right) ds \leq C'.$$

Next, we use recursive method to prove (27) and (28). Assume that $i \in \mathbb{N}^+$

$$\sup_{t \in (0, T_{\max, \varepsilon})} \|u_\varepsilon\|_{L^{2m^i-1}(\Omega)}^{2m^i-1} + \sup_{t \in (\tau, T_{\max, \varepsilon})} \int_{t-\tau}^t \left(\int_{\Omega} u_\varepsilon^{m+2m^i-4} |\nabla u_\varepsilon|^2 dx + \int_{\Omega} u_\varepsilon^{2m^i} dx \right) ds \leq C_i, \quad (35)$$

$$\sup_{t \in (0, T_{\max, \varepsilon})} \|v_\varepsilon\|_{L^{2n^i-1}(\Omega)}^{2n^i-1} + \sup_{t \in (\tau, T_{\max, \varepsilon})} \int_{t-\tau}^t \left(\int_{\Omega} v_\varepsilon^{n+2n^i-4} |\nabla v_\varepsilon|^2 dx + \int_{\Omega} v_\varepsilon^{2n^i} dx \right) ds \leq C'_i. \quad (36)$$

Due to $u, v \geq 0$, and the third equation of (18), using Lemma 3, (22), we derive

$$\sup_{t \in (\tau, T_{\max, \varepsilon})} \int_{t-\tau}^t \left(\|w_\varepsilon\|_{W^{2,2m^i}(\Omega)}^{2m^i} + \|w_{\varepsilon t}\|_{L^{2m^i}(\Omega)}^{2m^i} \right) ds \leq \tilde{C}_i, \quad (37)$$

$$\sup_{t \in (\tau, T_{\max, \varepsilon})} \int_{t-\tau}^t \left(\|w_\varepsilon\|_{W^{2,2n^i}(\Omega)}^{2n^i} + \|w_{\varepsilon t}\|_{L^{2n^i}(\Omega)}^{2n^i} \right) ds \leq \tilde{C}'_i. \quad (38)$$

Taking $r = 2m^{i+1} - 2$ in (32), we can obtain in $t \in (0, T_{\max, \varepsilon})$

$$\begin{aligned} &\frac{1}{2m^{i+1}-1} \frac{d}{dt} \int_{\Omega} u_\varepsilon^{2m^{i+1}-1} dx + (m^{i+1}-1) \int_{\Omega} u_\varepsilon^{m+2m^{i+1}-4} |\nabla u_\varepsilon|^2 dx + \frac{\mu_1}{2} \int_{\Omega} u_\varepsilon^{2m^{i+1}} dx \\ &\leq (m^{i+1}-1) \int_{\Omega} \chi_1^2(w_\varepsilon) u_\varepsilon^{2m^{i+1}-m} |\nabla w_\varepsilon|^2 dx + C_1 \leq C_6 \int_{\Omega} |\nabla w_\varepsilon|^{4m^i} dx + \frac{\mu_1}{4} \int_{\Omega} u_\varepsilon^{2m^{i+1}} dx + C_1, \end{aligned} \quad (39)$$

it implies

$$\frac{d}{dt} \int_{\Omega} u_{\varepsilon}^{2m^{i+1}-1} dx + \int_{\Omega} u_{\varepsilon}^{m+2m^{i+1}-4} |\nabla u_{\varepsilon}|^2 dx + \int_{\Omega} u_{\varepsilon}^{2m^{i+1}} dx + \int_{\Omega} u_{\varepsilon}^{2m^{i+1}-1} dx \leq C_7 \int_{\Omega} |\nabla w_{\varepsilon}|^{4m^i} dx + C_7.$$

Combining (33) and (37) with Lemma 1, a direct calculation shows

$$\sup_{t \in (0, T_{\max, \varepsilon})} \|u_{\varepsilon}(\cdot, t)\|_{L^{2m^{i+1}-1}(\Omega)}^{2m^{i+1}-1} + \sup_{t \in (\tau, T_{\max, \varepsilon})} \int_{\tau}^t \left(\int_{\Omega} u_{\varepsilon}^{m+2m^{i+1}-4} |\nabla u_{\varepsilon}|^2 dx + \int_{\Omega} u_{\varepsilon}^{2m^{i+1}} dx \right) ds \leq C_{i+1}.$$

Arguing similarly as above, we see that

$$\sup_{t \in (0, T_{\max, \varepsilon})} \|v_{\varepsilon}(\cdot, t)\|_{L^{2n^{i+1}-1}(\Omega)}^{2n^{i+1}-1} + \sup_{t \in (\tau, T_{\max, \varepsilon})} \int_{\tau}^t \left(\int_{\Omega} v_{\varepsilon}^{n+2n^{i+1}-4} |\nabla v_{\varepsilon}|^2 dx + \int_{\Omega} v_{\varepsilon}^{2n^{i+1}} dx \right) ds \leq C'_{i+1}. \quad (40)$$

The proof is complete. $\square$

Lemma 7. Suppose that (5) and (6) hold. Let $(u_{\varepsilon}, v_{\varepsilon}, w_{\varepsilon})$ be the classical solution of (18). Then

$$\sup_{t \in (0, T_{\max, \varepsilon})} (\|u_{\varepsilon}(\cdot, t)\|_{L^{\infty}(\Omega)} + \|v_{\varepsilon}(\cdot, t)\|_{L^{\infty}(\Omega)} + \|w_{\varepsilon}(\cdot, t)\|_{W^{1, \infty}(\Omega)}) \leq C, \quad (41)$$

where C is independent of $T_{\max, \varepsilon}$ and ε .

Proof. We take k appropriately large such that $2m^k \geq 10$ in Lemma 6. Then, by t -anisotropic embedding theorem [26], there exists C_0 such that

$$\|w_{\varepsilon}\|_{C^{\frac{3}{2}, \frac{3}{4}}(Q_{\tau}(t))} \leq C_0 \sup_{t \in (\tau, T_{\max, \varepsilon})} \|w_{\varepsilon}\|_{W^{2,1}_0(Q_{\tau}(t))}, \quad (42)$$

where $Q_{\tau}(t) = \Omega \times (t - \tau, t)$, it means

$$\sup_{t \in (0, T_{\max, \varepsilon})} \|w_{\varepsilon}\|_{W^{1, \infty}(\Omega)} \leq C'_0. \quad (43)$$

Multiplying the first equation of (18) by ru_{ε}^{r-1} with $r \geq 3m$, integrating it over Ω , and using (6), (22), (43),

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} u_{\varepsilon}^r dx + r(r-1) \int_{\Omega} u_{\varepsilon}^{m+r-3} |\nabla u_{\varepsilon}|^2 dx + r\mu_1 \int_{\Omega} u_{\varepsilon}^{r+1} dx + \int_{\Omega} u_{\varepsilon}^r dx \\ & \leq r(r-1) \int_{\Omega} \chi_1(w_{\varepsilon}) u_{\varepsilon}^{r-1} \nabla u_{\varepsilon} \nabla w_{\varepsilon} dx + r\mu_1 \int_{\Omega} u_{\varepsilon}^r dx + \int_{\Omega} u_{\varepsilon}^r dx \\ & \leq \frac{r(r-1)}{2} \int_{\Omega} (u_{\varepsilon}^{m+r-3} |\nabla u_{\varepsilon}|^2 + \chi_1^2(w_{\varepsilon}) u_{\varepsilon}^{r+1-m} |\nabla w_{\varepsilon}|^2) dx + (r\mu_1 + 1) \int_{\Omega} u_{\varepsilon}^r dx \\ & \leq \frac{r(r-1)}{2} \int_{\Omega} u_{\varepsilon}^{m+r-3} |\nabla u_{\varepsilon}|^2 dx + C_1 r^2 \int_{\Omega} u_{\varepsilon}^{r+1-m} dx + \frac{r\mu_1}{2} \int_{\Omega} u_{\varepsilon}^{r+1} dx + C_1, \quad t \in (0, T_{\max, \varepsilon}). \end{aligned}$$

It is easy to see that

$$\frac{d}{dt} \int_{\Omega} u_{\varepsilon}^r dx + \int_{\Omega} \left| \nabla u_{\varepsilon}^{\frac{r+m-1}{2}} \right|^2 dx + \frac{r\mu_1}{2} \int_{\Omega} u_{\varepsilon}^{r+1} dx + \int_{\Omega} u_{\varepsilon}^r dx \leq C_1 r^2 \int_{\Omega} u_{\varepsilon}^{r+1-m} dx + C_1, \quad t \in (0, T_{\max, \varepsilon}). \quad (44)$$

Using the Gagliardo-Nirenberg interpolation inequality and Young's inequality, we can derive in $(0, T_{\max, \varepsilon})$

$$\begin{aligned}
C_1 r^2 \|u_\varepsilon\|_{L^{r+1-m}(\Omega)}^{r+1-m} &= C_1 r^2 \left\| u_\varepsilon^{\frac{r+m-1}{2}} \right\|_{L^{\frac{2(r+1-m)}{r+m-1}}(\Omega)}^{\frac{2(r+1-m)}{r+m-1}} \\
&\leq C_2 r^2 \left\| \nabla u_\varepsilon^{\frac{r+m-1}{2}} \right\|_{L^2(\Omega)}^{\frac{2s(r+1-m)}{r+m-1}} \left\| u_\varepsilon^{\frac{r+m-1}{2}} \right\|_{L^{\frac{r}{m+r-1}}(\Omega)}^{\frac{2(1-s)(r+1-m)}{r+m-1}} + C_3 r^2 \left\| u_\varepsilon^{\frac{r+m-1}{2}} \right\|_{L^{\frac{r}{m+r-1}}(\Omega)}^{\frac{2(r+1-m)}{r+m-1}} \\
&\leq \frac{1}{2} \left\| \nabla u_\varepsilon^{\frac{r+m-1}{2}} \right\|_{L^2(\Omega)}^2 + C_4 r^{\frac{6(m-1)+5r}{6(m-1)+r}} \|u_\varepsilon\|_{L^{\frac{r}{6m+r-6}}(\Omega)}^{\frac{r(2m+r-2)}{6m+r-6}} + C_3 r^2 \|u_\varepsilon\|_{L^{\frac{r}{2}}(\Omega)}^{r+1-m} \\
&\leq \frac{1}{2} \left\| \nabla u_\varepsilon^{\frac{r+m-1}{2}} \right\|_{L^2(\Omega)}^2 + C_4 r^5 \|u_\varepsilon\|_{L^{\frac{r}{6m+r-6}}(\Omega)}^{\frac{r(2m+r-2)}{6m+r-6}} + C_3 r^2 \|u_\varepsilon\|_{L^{\frac{r}{2}}(\Omega)}^{r+1-m},
\end{aligned}$$

where $s = \frac{m+r-1}{r+1-m} \cdot \frac{-3r-6+6m}{-5r+6-6m} \in (0, 1)$, taking the above inequality into (44), in $(0, T_{\max, \varepsilon})$

$$\frac{d}{dt} \int_{\Omega} u_\varepsilon^r dx + \int_{\Omega} u_\varepsilon^r dx \leq C_4 r^5 \|u_\varepsilon\|_{L^{\frac{r}{6m+r-6}}(\Omega)}^{\frac{r(2m+r-2)}{6m+r-6}} + C_3 r^2 \|u_\varepsilon\|_{L^{\frac{r}{2}}(\Omega)}^{r+1-m} + C_1. \quad (45)$$

By (27), we can take $k^* > 0$ appropriately large such that $2m^{k^*} - 1 > 3m$, then

$$\sup_{t \in (0, T_{\max, \varepsilon})} \|u_\varepsilon\|_{L^{2m^{k^*}-1}(\Omega)} \leq C_5.$$

Taking $r_j = 2r_{j-1} = 2^j r_0$, $r_0 = 2m^{k^*} - 1$, $M_j = \max\{1, \sup_{t \in (0, T_{\max, \varepsilon})} \|u_\varepsilon\|_{L^{r_j}(\Omega)}\}$, then by (45), we have

$$\begin{aligned}
M_j &\leq (C_3 + C_4 + C_1)^{\frac{1}{j}} r_j^{\frac{5}{j}} M_{j-1} \\
&= (C_3 + C_4 + C_1)^{\frac{1}{2^j r_0}} r_0^{\frac{5}{2^j r_0}} 2^{\frac{5j}{2^j r_0}} M_{j-1} \\
&\leq (C_3 + C_4 + C_1)^{\sum_{k=1}^j \frac{1}{2^k r_0}} 2^{\sum_{k=1}^j \frac{5}{2^k r_0}} M_0 \leq C,
\end{aligned}$$

where C is independent of j . Letting $j \rightarrow \infty$, $\sup_{t \in (0, T_{\max, \varepsilon})} \|u_\varepsilon\|_{L^\infty(\Omega)} \leq C$ is obtained. By the same way, $\sup_{t \in (0, T_{\max, \varepsilon})} \|v_\varepsilon\|_{L^\infty(\Omega)} \leq C$ is obtained. Then (41) is complete.

Proof of Proposition 1. By Lemmas 4 and 7, for any $\varepsilon > 0$, $T_{\max, \varepsilon} = +\infty$, the system (18) has a global classical solution which is, furthermore, bounded. Namely, there exists C that is independent of ε , $T_{\max, \varepsilon}$ such that

$$\sup_{t \in (0, \infty)} (\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|w_\varepsilon(\cdot, t)\|_{W^{1, \infty}(\Omega)}) \leq C. \quad (46)$$

Meanwhile, due to $T_{\max, \varepsilon} = +\infty$, it means that $\tau = 1$.

Multiplying the first equation of (18) by u_ε and integrating it over Ω , we have

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} u_\varepsilon^2 dx &= \int_{\Omega} u_\varepsilon u_{\varepsilon t} \\
&= \int_{\Omega} u_\varepsilon \Delta((u_\varepsilon^2 + \varepsilon)^{\frac{m-1}{2}} u_\varepsilon) - \int_{\Omega} u_\varepsilon \nabla \cdot (u_\varepsilon \chi_1(w_\varepsilon) \nabla w_\varepsilon) + \mu_1 \int_{\Omega} u_\varepsilon^2 (1 - u_\varepsilon - a_1 v_\varepsilon) \\
&= - \int_{\Omega} \nabla u_\varepsilon \nabla (u_\varepsilon^2 + \varepsilon)^{\frac{m-1}{2}} u_\varepsilon + \int_{\Omega} \chi_1(w_\varepsilon) u_\varepsilon \nabla w_\varepsilon \nabla u_\varepsilon + \mu_1 \int_{\Omega} u_\varepsilon^2 (1 - u_\varepsilon - a_1 v_\varepsilon) \\
&= - \int_{\Omega} \nabla u_\varepsilon \nabla (u_\varepsilon^2 + \varepsilon)^{\frac{m-1}{2}} u_\varepsilon + \frac{1}{2} \int_{\Omega} \chi_1(w_\varepsilon) \nabla w_\varepsilon \nabla u_\varepsilon^2 + \mu_1 \int_{\Omega} u_\varepsilon^2 (1 - u_\varepsilon - a_1 v_\varepsilon) \\
&= - \int_{\Omega} \nabla u_\varepsilon \nabla (u_\varepsilon^2 + \varepsilon)^{\frac{m-1}{2}} u_\varepsilon - \frac{1}{2} \int_{\Omega} \chi_1'(w_\varepsilon) u_\varepsilon^2 |\nabla w_\varepsilon|^2 - \frac{1}{2} \int_{\Omega} \chi_1(w_\varepsilon) u_\varepsilon^2 \Delta w_\varepsilon + \mu_1 \int_{\Omega} u_\varepsilon^2 (1 - u_\varepsilon - a_1 v_\varepsilon) \\
&\leq - \int_{\Omega} \nabla u_\varepsilon \nabla (u_\varepsilon^2 + \varepsilon)^{\frac{m-1}{2}} u_\varepsilon - \frac{1}{2} \int_{\Omega} \chi_1(w_\varepsilon) u_\varepsilon^2 \Delta w_\varepsilon + \mu_1 \int_{\Omega} u_\varepsilon^2 (1 - u_\varepsilon - a_1 v_\varepsilon)
\end{aligned}$$

$$\begin{aligned} &\leq -(m-1) \int_{\Omega} (u_{\varepsilon}^2 + \varepsilon)^{\frac{m-3}{2}} u_{\varepsilon}^2 |\nabla u_{\varepsilon}|^2 - \int_{\Omega} (u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} |\nabla u_{\varepsilon}|^2 + \frac{1}{4} \chi_1^2 (\|w_0\|_{L^\infty(\Omega)}) \int_{\Omega} u_{\varepsilon}^4 + \frac{1}{4} \int_{\Omega} |\Delta w_{\varepsilon}|^2 \\ &\quad + \mu_1 \int_{\Omega} u_{\varepsilon}^2 (1 - u_{\varepsilon} - a_1 v_{\varepsilon}), \quad t \in (0, \infty), \end{aligned}$$

by (46), we have

$$\frac{d}{dt} \int_{\Omega} u_{\varepsilon}^2 + \int_{\Omega} (u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} |\nabla u_{\varepsilon}|^2 \leq C_1 (1 + \int_{\Omega} |\Delta w_{\varepsilon}|^2), \quad t \in (0, +\infty).$$

By (25) and Lemma 1, it implies

$$\sup_{t \in (1, \infty)} \int_{t-1}^t \int_{\Omega} (u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} |\nabla u_{\varepsilon}|^2 \leq C_2, \quad (47)$$

where C_2 is independent of ε . Multiplying the first equation of (18) by $\frac{\partial((u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} u_{\varepsilon})}{\partial t}$, and integrating it over Ω ,

$$\begin{aligned} \int_{\Omega} u_{\varepsilon t} \frac{\partial((u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} u_{\varepsilon})}{\partial t} &= \int_{\Omega} \left((m-1)(u_{\varepsilon}^2 + \varepsilon)^{\frac{m-3}{2}} u_{\varepsilon}^2 + (u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} \right) \left| \frac{\partial u_{\varepsilon}}{\partial t} \right|^2 \\ &= \int_{\Omega} \Delta((u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} u_{\varepsilon}) \frac{\partial((u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} u_{\varepsilon})}{\partial t} - \int_{\Omega} \nabla \cdot (u_{\varepsilon} \chi_1(w_{\varepsilon}) \nabla w_{\varepsilon}) \frac{\partial((u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} u_{\varepsilon})}{\partial t} \\ &\quad + \mu_1 \int_{\Omega} u_{\varepsilon} (1 - u_{\varepsilon} - a_1 v_{\varepsilon}) \frac{\partial((u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} u_{\varepsilon})}{\partial t} \\ &= - \int_{\Omega} \nabla \cdot ((u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} u_{\varepsilon}) \frac{\partial \nabla \cdot ((u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} u_{\varepsilon})}{\partial t} - \int_{\Omega} \nabla \cdot (u_{\varepsilon} \chi_1(w_{\varepsilon}) \nabla w_{\varepsilon}) \frac{\partial((u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} u_{\varepsilon})}{\partial t} \\ &\quad + \mu_1 \int_{\Omega} u_{\varepsilon} (1 - u_{\varepsilon} - a_1 v_{\varepsilon}) \frac{\partial((u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} u_{\varepsilon})}{\partial t}, \quad t \in (0, +\infty), \end{aligned}$$

then

$$\begin{aligned} &\int_{\Omega} \nabla \cdot ((u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} u_{\varepsilon}) \frac{\partial \nabla \cdot ((u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} u_{\varepsilon})}{\partial t} + \int_{\Omega} (u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} \left| \frac{\partial u_{\varepsilon}}{\partial t} \right|^2 \\ &= \frac{1}{2} \frac{d}{dt} \int_{\Omega} \left| \nabla \cdot ((u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} u_{\varepsilon}) \right|^2 + \int_{\Omega} (u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} \left| \frac{\partial u_{\varepsilon}}{\partial t} \right|^2 \\ &\leq - \int_{\Omega} \nabla \cdot (u_{\varepsilon} \chi_1(w_{\varepsilon}) \nabla w_{\varepsilon}) \frac{\partial((u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} u_{\varepsilon})}{\partial t} + \mu_1 \int_{\Omega} u_{\varepsilon} (1 - u_{\varepsilon} - a_1 v_{\varepsilon}) \frac{\partial((u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} u_{\varepsilon})}{\partial t} \\ &\leq m^2 \int_{\Omega} |\nabla \cdot (u_{\varepsilon} \chi_1(w_{\varepsilon}) \nabla w_{\varepsilon})|^2 (u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} + \frac{1}{2} \int_{\Omega} (u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} \left| \frac{\partial u_{\varepsilon}}{\partial t} \right|^2 + \mu_1^2 m^2 \int_{\Omega} u_{\varepsilon}^2 (1 - u_{\varepsilon} - a_1 v_{\varepsilon})^2 (u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} \\ &\leq C_3 \left(\int_{\Omega} \chi_1^2(w_{\varepsilon}) |\nabla u_{\varepsilon}|^2 |\nabla w_{\varepsilon}|^2 (u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} + \int_{\Omega} |\chi_1'(w_{\varepsilon})|^2 |\nabla w_{\varepsilon}|^4 u_{\varepsilon}^2 (u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} + \int_{\Omega} \chi_1^2(w_{\varepsilon}) u_{\varepsilon}^2 |\Delta w_{\varepsilon}|^2 (u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} \right) \\ &\quad + \frac{1}{2} \int_{\Omega} (u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} \left| \frac{\partial u_{\varepsilon}}{\partial t} \right|^2 + \mu_1^2 m^2 \int_{\Omega} u_{\varepsilon}^2 (1 - u_{\varepsilon} - a_1 v_{\varepsilon})^2 (u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} \\ &\leq C_4 \left(1 + \int_{\Omega} |\Delta w_{\varepsilon}|^2 + \int_{\Omega} (u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} |\nabla u_{\varepsilon}|^2 \right) + \frac{1}{2} \int_{\Omega} (u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} \left| \frac{\partial u_{\varepsilon}}{\partial t} \right|^2, \quad t \in (0, +\infty). \end{aligned}$$

It implies

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \left| \nabla \left((u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} u_{\varepsilon} \right) \right|^2 + \int_{\Omega} (u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} \left| \frac{\partial u_{\varepsilon}}{\partial t} \right|^2 + \int_{\Omega} \left| \nabla \left((u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} u_{\varepsilon} \right) \right|^2 \\ & \leq C_5 \left(1 + \int_{\Omega} |\Delta w_{\varepsilon}|^2 + \int_{\Omega} (u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} |\nabla u_{\varepsilon}|^2 \right), \quad t \in (0, +\infty). \end{aligned}$$

By (47), we infer from Lemma 1 that

$$\sup_{t \in (0, +\infty)} \int_{\Omega} \left| \nabla \left((u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} u_{\varepsilon} \right) \right|^2 + \sup_{t \in (1, +\infty)} \int_{t-1}^t \int_{\Omega} (u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} \left| \frac{\partial u_{\varepsilon}}{\partial t} \right|^2 \leq C,$$

which implies (19). By the same way, we can obtain (20). The estimate of (21) can be easily obtained from (46) and Lemma 3. $\square$

Next, we can use this to show Theorem 1.

Proof of Theorem 1. According to Proposition 1, for any $T > 0$, we have subsequence of $(u_{\varepsilon}, v_{\varepsilon}, w_{\varepsilon})$ (for simplicity, we still denote subsequence by $(u_{\varepsilon}, v_{\varepsilon}, w_{\varepsilon})$) and (u, v, w) such that

$$\begin{aligned} u_{\varepsilon} & \xrightarrow{*} u \quad \text{and} \quad v_{\varepsilon} \xrightarrow{*} v, \quad \text{in } L^{\infty}(Q_T), \\ u_{\varepsilon}^{\frac{m+1}{2}} & \rightharpoonup u^{\frac{m+1}{2}} \quad \text{and} \quad v_{\varepsilon}^{\frac{n+1}{2}} \rightharpoonup v^{\frac{n+1}{2}}, \quad \text{in } W_2^{1,1}(Q_T), \\ w_{\varepsilon} & \rightharpoonup w, \quad \text{in } W_p^{2,1}(Q_T), \quad \text{for any } p \in (1, \infty). \end{aligned}$$

By the Aubin-Lions theorem, (19) and (20), we have

$$u_{\varepsilon}^{\frac{m+1}{2}} \rightarrow u^{\frac{m+1}{2}} \quad \text{and} \quad v_{\varepsilon}^{\frac{n+1}{2}} \rightarrow v^{\frac{n+1}{2}}, \quad \text{in } C([0, T]; L^2(\Omega)),$$

then $u_{\varepsilon} \rightarrow u$, $v_{\varepsilon} \rightarrow v$ a.e. in Q_T . Using (46) and Lebesgue's dominated convergence theorem,

$$u_{\varepsilon} \rightarrow u \quad \text{and} \quad v_{\varepsilon} \rightarrow v, \quad \text{in } C([0, T]; L^p(\Omega)), \quad \text{for any } p \in (1, \infty).$$

By (21) and $W_p^{2,1}(Q_T) \hookrightarrow C^{2-\frac{5}{p}, 1-\frac{5}{2p}}(Q_T)$ for any $p > \frac{5}{2}$, then

$$w_{\varepsilon} \rightarrow w, \quad \text{in } C^{2-\frac{5}{p}, 1-\frac{5}{2p}}(Q_T).$$

And (7) holds. Due to $u_{\varepsilon} \rightarrow u$ in $L^p(Q_T)$, $\nabla w_{\varepsilon} \rightharpoonup \nabla w$ in $L^p(Q_T)$ for any $p > 1$, (6), we have

$$\chi_1(w_{\varepsilon}) u_{\varepsilon} \nabla w_{\varepsilon} \rightharpoonup \chi_1(w) u \nabla w, \quad \text{in } L^p(Q_T), \quad \text{for any } p \in (1, \infty).$$

By the same way, we also have

$$\chi_2(w_{\varepsilon}) v_{\varepsilon} \nabla w_{\varepsilon} \rightharpoonup \chi_2(w) v \nabla w \quad \text{in } L^p(Q_T), \quad \text{for any } p \in (1, \infty).$$

We fix $\varphi \in C^{\infty}(\overline{Q_T})$ with $\frac{\partial \varphi}{\partial \nu} \Big|_{\partial \Omega} = 0$ and $\varphi(x, T) = 0$, multiplying the equation in (18) by φ and integrating by parts, we obtain

$$\begin{aligned} & - \iint_{Q_T} u_{\varepsilon} \varphi_t dx dt - \int_{\Omega} u_0 \varphi(x, 0) dx - \iint_{Q_T} (u_{\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} u_{\varepsilon} \Delta \varphi dx dt \\ & = \iint_{Q_T} \chi_1(w_{\varepsilon}) u_{\varepsilon} \nabla w_{\varepsilon} \nabla \varphi dx dt + \iint_{Q_T} \mu_1 u_{\varepsilon} (1 - u_{\varepsilon} - a_1 v_{\varepsilon}) \varphi dx dt, \\ & - \iint_{Q_T} v_{\varepsilon} \varphi_t dx dt - \int_{\Omega} v_0 \varphi(x, 0) dx - \iint_{Q_T} (v_{\varepsilon}^2 + \varepsilon)^{\frac{n-1}{2}} v_{\varepsilon} \Delta \varphi dx dt \\ & = \iint_{Q_T} \chi_2(w_{\varepsilon}) v_{\varepsilon} \nabla w_{\varepsilon} \nabla \varphi dx dt + \iint_{Q_T} \mu_2 v_{\varepsilon} (1 - a_2 u_{\varepsilon} - v_{\varepsilon}) \varphi dx dt, \end{aligned}$$

$$-\iint_{Q_T} w_\varepsilon \varphi_t dx dt - \int_{\Omega} w_0 \varphi(x, 0) dx + \iint_{Q_T} \nabla w_\varepsilon \nabla \varphi dx dt + \iint_{Q_T} (\alpha u_\varepsilon + \beta v_\varepsilon) w_\varepsilon \varphi dx dt = 0.$$

Letting $\varepsilon \rightarrow 0$, we can get

$$\begin{aligned} & -\iint_{Q_T} u \varphi_t dx dt - \int_{\Omega} u_0 \varphi(x, 0) dx - \iint_{Q_T} u^m \Delta \varphi dx dt = \iint_{Q_T} \chi_1(w) u \nabla w \nabla \varphi dx dt + \iint_{Q_T} \mu_1 u (1 - u - a_1 v) \varphi dx dt, \\ & -\iint_{Q_T} v \varphi_t dx dt - \int_{\Omega} v_0 \varphi(x, 0) dx - \iint_{Q_T} v^n \Delta \varphi dx dt = \iint_{Q_T} \chi_2(w) v \nabla w \nabla \varphi dx dt + \iint_{Q_T} \mu_2 v (1 - a_2 u - v) \varphi dx dt, \\ & -\iint_{Q_T} w \varphi_t dx dt - \int_{\Omega} w_0 \varphi(x, 0) dx + \iint_{Q_T} \nabla w \nabla \varphi dx dt + \iint_{Q_T} (\alpha u + \beta v) w \varphi dx dt = 0. \end{aligned}$$

By $\nabla u^m \in L^2(Q_T)$, $\nabla v^n \in L^2(Q_T)$ we obtain

$$\begin{aligned} & -\iint_{Q_T} u \varphi_t dx dt - \int_{\Omega} u_0 \varphi(x, 0) dx + \iint_{Q_T} (\nabla u^m - \chi_1(w) u \nabla w) \nabla \varphi dx dt = \iint_{Q_T} \mu_1 u (1 - u - a_1 v) \varphi dx dt, \\ & -\iint_{Q_T} v \varphi_t dx dt - \int_{\Omega} v_0 \varphi(x, 0) dx + \iint_{Q_T} (\nabla v^n - \chi_2(w) v \nabla w) \nabla \varphi dx dt = \iint_{Q_T} \mu_2 v (1 - a_2 u - v) \varphi dx dt, \end{aligned}$$

which means (u, v, w) with $u \in A_1$, $v \in A_2$, $w \in A_3$ is a global weak solution of (1). $\square$

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References

- [1] E. F. Keller and L. A. Segel, *Initiation of slime mold aggregation viewed as an instability*, J. Theoret. Biol. **26** (1970), no. 3, 399–415, DOI: [https://doi.org/10.1016/0022-5193\(70\)90092-5](https://doi.org/10.1016/0022-5193(70)90092-5).
- [2] Y. Tao and M. Winkler, *Eventual smoothness and stabilization of large-data solutions in a three-dimensional chemotaxis system with consumption of chemoattractant*, J. Differ. Equ. **252** (2012), no. 3, 2520–2543, DOI: <https://doi.org/10.1016/j.jde.2011.07.010>.
- [3] L. Wang, C. Mu, and S. Zhou, *Boundedness in a parabolic-parabolic chemotaxis system with nonlinear diffusion*, Z. Angew. Math. Phys. **65** (2014), no. 6, 1137–1152, DOI: <https://doi.org/10.1007/s00033-013-0375-4>.
- [4] J. Zheng and Y. Wang, *A note on global existence to a higher-dimensional quasilinear chemotaxis system with consumption of chemoattractant*, Discrete Contin. Dyn. Syst. Ser. B **22** (2017), no. 2, 669–686, DOI: <https://doi.org/10.3934/dcdsb.2017032>.
- [5] P. Zheng and C. Mu, *Global existence of solutions for a fully parabolic chemotaxis system with consumption of chemoattractant and logistic source*, Math. Nachr. **288** (2015), no. 5, 710–720, DOI: <https://doi.org/10.1002/mana.201300105>.
- [6] J. Lankeit and Y. Wang, *Global existence, boundedness and stabilization in a high dimensional chemotaxis system with consumption*, Discrete Contin. Dyn. Syst. **37** (2017), no. 12, 6099–6121, DOI: <https://doi.org/10.3934/dcds.2017262>.
- [7] J. Zheng, *Global solvability and boundedness in the N-dimensional quasilinear chemotaxis model with logistic source and consumption of chemoattractant*, arxiv:1801.01774v1, (2018).
- [8] L. Wang, C. Mu, X. Hu, and P. Zheng, *Boundedness and asymptotic stability of solutions to a two species chemotaxis system with consumption of chemoattractant*, J. Differ. Equ. **264** (2018), no. 5, 3369–3401, DOI: <https://doi.org/10.1016/j.jde.2017.11.019>.
- [9] X. Bai and M. Winkler, *Equilibration in a fully parabolic two species chemotaxis system with competitive kinetics*, Indian Univ. Math. J. **65** (2016), no. 2, 553–583.

- [10] K. Lin and C. Mu, *Convergence of global and bounded solutions of a two species chemotaxis model with a logistic source*, Discrete Contin. Dyn. Syst. Ser. B **22** (2017), no. 6, 2233–2260, DOI: <https://doi.org/10.3934/dcdsb.2017094>.
- [11] M. Negreanu and I. Tello, *Asymptotic stability of a two species chemotaxis system with non-diffusive chemoattractant*, J. Differ. Equ. **258** (2015), no. 5, 1592–1617, DOI: <https://doi.org/10.1016/j.jde.2014.11.009>.
- [12] M. Negreanu and I. Tello, *On a two species chemotaxis model with slow chemical diffusion*, SIAM J. Math. Anal. **46** (2014), no. 6, 3761–3781, DOI: <https://doi.org/10.1137/140971853>.
- [13] M. Mizukami and T. Yokota, *Global existence and asymptotic stability of solutions to a two species chemotaxis system with any chemical diffusion*, J. Differ. Equ. **261** (2016), no. 5, 2650–2669, DOI: <https://doi.org/10.1016/j.jde.2016.05.008>.
- [14] X. Li, Y. Wang, and Z. Xiang, *Global existence and boundedness in a 2D Keller-Segel-Stokes system with nonlinear diffusion and rotational flux*, Commun. Math. Sci. **14** (2016), no. 7, 1889–1910, DOI: <https://doi.org/10.4310/CMS.2016.v14.n7.a5>.
- [15] Y. Wang, *Global solvability in a two-dimensional self-consistent chemotaxis-Navier-Stokes system*, Discrete Contin. Dyn. Syst. Ser. S **13** (2020), no. 2, 329–349, DOI: <https://doi.org/10.3934/dcdss.2020019>.
- [16] Y. Wang and Z. Xiang, *Global existence and boundedness in a higher-dimensional quasilinear chemotaxis system*, Zeitschrift für angewandte Mathematik und Physik **66** (2015), 3159–3179, DOI: <https://doi.org/10.1007/s00033-015-0557-3>.
- [17] Y. Wang and L. Zhao, *A 3D self-consistent chemotaxis-fluid system with nonlinear diffusion*, J. Differ. Equ. **269** (2020), no. 1, 148–179, DOI: <https://doi.org/10.1016/j.jde.2019.12.002>.
- [18] C. Stinner, C. Surulescu, and M. Winkler, *Global weak solutions in a PDE-ODE system modeling multiscale cancer cell invasion*, SIAM J. Math. Anal. **46** (2014), no. 3, 1969–2007, DOI: <https://doi.org/10.1137/13094058X>.
- [19] Y. Wang, M. Winkler, and Z. Xiang, *Global classical solutions in a two-dimensional chemotaxis-Navier-Stokes system with subcritical sensitivity*, Ann. Sc. Norm. Super. Pisa, Cl. Sci. **18** (2018), no. 2, 421–466.
- [20] C. Jin, *Boundedness and global solvability to a chemotaxis model with nonlinear diffusion*, J. Differ. Equ. **263** (2017), no. 9, 5759–5772, DOI: <https://doi.org/10.1016/j.jde.2017.06.034>.
- [21] C. Jin, *Global classical solution and boundedness to a chemotaxis-haptotaxis model with re-establishment mechanisms*, Bull. London Math. Soc. **50** (2018), no. 4, 598–618, DOI: <https://doi.org/10.1112/blms.12160>.
- [22] Y. Tao and M. Winkler, *Global existence and boundedness in a Keller-Segel-Stokes model with arbitrary porous medium diffusion*, Discrete Contin. Dyn. Syst. **32** (2012), no. 5, 1901–1914, DOI: <https://doi.org/10.3934/dcds.2012.32.1901>.
- [23] M. Winkler, *Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source*, Comm. Partial Differ. Equ. **35** (2010), no. 8, 1516–1537, DOI: <https://doi.org/10.1080/03605300903473426>.
- [24] M. Winkler, *Chemotaxis with logistic source: very weak global solutions and their boundedness properties*, J. Math. Anal. Appl. **348** (2008), no. 2, 708–729, DOI: <https://doi.org/10.1016/j.jmaa.2008.07.071>.
- [25] M. Winkler, *Global large-data solutions in a Chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops*, Comm. Partial Differ. Equ. **37** (2012), no. 2, 319–351, DOI: <https://doi.org/10.1080/03605302.2011.591865>.
- [26] Z. Wu, J. Yin, and C. Wang, *Elliptic and Parabolic Equations*, World Scientific Publishing Co. Pvt. Ltd, Singapore, 2006.