Research Article

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Solution of integral equations via coupled fixed point theorems in $\mathfrak{F}$ -complete metric spaces

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Abstract: The concept of coupled $\mathfrak{F}$ -orthogonal contraction mapping is introduced in this paper, and some coupled fixed point theorems in orthogonal metric spaces are proved. The obtained results generalize and extend some of the well-known results in the literature. An example is presented to support our results. Furthermore, we apply our result to obtain the existence theorem for a common solution of the integral equations:

$$\begin{cases} \zeta(v) = \delta(v) + \int_0^{\mathfrak{M}} \Xi(v, \beta) \Omega(\beta, \zeta(\beta), \xi(\beta)) d\beta, & v \in [0, \mathcal{H}], \\ \xi(v) = \delta(v) + \int_0^{\mathfrak{M}} \Xi(v, \beta) \Omega(\beta, \xi(\beta), \zeta(\beta)) d\beta, & v \in [0, \mathcal{H}], \end{cases}$$

where

- (a) $\delta : \mathfrak{M} \rightarrow \mathbb{R}$ and $\Omega : \mathfrak{M} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous;
- (b) $\Xi : \mathfrak{M} \times \mathfrak{M}$ is continuous and measurable at $\beta \in \mathfrak{M}$, $\forall v \in \mathfrak{M}$;
- (c) $\Xi(v, \beta) \geq 0$, $\forall v, \beta \in \mathfrak{M}$ and $\int_0^{\mathcal{H}} \Xi(v, \beta) d\beta \leq 1$, $\forall v \in \mathfrak{M}$.

Keywords: orthogonal set, orthogonal metric space, orthogonal continuous, orthogonal preserving, orthogonal $\mathfrak{F}$ -contraction, coupled fixed point

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1 Introduction and preliminaries

The concept of an orthogonal set has many applications in several branches in mathematics and it has many types of orthogonality. Eshaghi Gordji et al. [1] introduced the new concept of orthogonality in metric

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spaces and proved the fixed point result for contraction mappings in metric spaces endowed with the new orthogonality. Eshaghi Gordji and Habibi [2] proved fixed point theory in generalized orthogonal metric space. Sawangsup et al. [3] introduced the new concept of orthogonal $\mathfrak{F}$ -contraction mappings and proved the fixed point theorems on orthogonal-complete metric spaces. The (orthogonal) contractive-type mappings have been studied and important results have been obtained by many authors [4–16].

In this paper, we introduce the new concept coupled $\mathfrak{F}$ -orthogonal contraction mapping and prove the coupled fixed point theorem on orthogonal complete metric space.

Throughout this paper, we denote by $\mathfrak{W}$, $\mathbb{R}^+$ and ι the non-void set, the set of positive real numbers and the set of positive integers, respectively.

First, we recall the concept of a control function which was introduced by Wardowski [17]. Let $\mathfrak{J}$ denote the family of all functions $\mathfrak{F} : \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfying the following properties:

($\mathfrak{F}_1$) $\mathfrak{F}$ is strictly increasing;

($\mathfrak{F}_2$) for each sequence $\{\kappa_i\}$ of positive numbers, we have

$$\lim_{i \rightarrow \infty} \kappa_i = 0 \Leftrightarrow \lim_{i \rightarrow \infty} \mathfrak{F}(\kappa_i) = -\infty;$$

($\mathfrak{F}_3$) there exists $i \in (0, 1)$ such that $\lim_{\kappa \rightarrow 0^+} \kappa^i \mathfrak{F}(\kappa) = 0$.

Example 1.1. [17] Let $\mathfrak{F} : \mathbb{R}^+ \rightarrow \mathbb{R}$ be defined by $\mathfrak{F}(\iota) = \ln \iota$ for $\iota > 0$. Then $\mathfrak{F} \in \mathfrak{J}$.

Definition 1.1. [8] Let $\mathfrak{W}$ be a nonempty set and $\mathfrak{L} : \mathfrak{W} \times \mathfrak{W} \rightarrow \mathfrak{W}$ be a mapping. A point $(u, v) \in \mathfrak{W} \times \mathfrak{W}$ is said to be a coupled fixed point of $\mathfrak{L}$ if $\mathfrak{L}(u, v) = u$ and $\mathfrak{L}(v, u) = v$.

Ozkan [9] proved the following theorem.

Theorem 1.2. [9] Let $(\mathfrak{W}, \mathfrak{d})$ be a complete metric space and $\mathfrak{L} : \mathfrak{W} \times \mathfrak{W} \rightarrow \mathfrak{W}$ be a mapping. If there exist $\mathfrak{F} \in \mathfrak{J}$ and $\ell \in (0, \infty)$ such that the following condition holds

$$\mathfrak{d}(\mathfrak{L}(a, b), \mathfrak{L}(u, v)) > 0 \Rightarrow \ell + \mathfrak{F}(\mathfrak{d}(\mathfrak{L}(a, b), \mathfrak{L}(u, v))) \leq \mathfrak{F}(\kappa \mathfrak{d}(a, u) + \iota \mathfrak{d}(b, v))$$

for all $a, b, u, v \in \mathfrak{W}$, $\kappa, \iota \in \mathbb{R}^+$, $\mathfrak{L}(a, b) \neq \mathfrak{L}(u, v)$, where $\kappa + \iota < 1$, then $\mathfrak{L}$ has a unique coupled fixed point.

On the other hand, Eshaghi Gordji et al. [1] introduced the basic concept as follows:

Definition 1.2. [1] Let $\mathfrak{W} \neq \emptyset$ and $\vdash \subseteq \mathfrak{W} \times \mathfrak{W}$ be a binary relation. If $\vdash$ satisfies the following condition:

$$\exists \mathfrak{w}_0 \in \mathfrak{W} : (\forall \mathfrak{w} \in \mathfrak{W}, \mathfrak{w} \vdash \mathfrak{w}_0) \quad \text{or} \quad (\forall \mathfrak{w} \in \mathfrak{W}, \mathfrak{w}_0 \vdash \mathfrak{w}),$$

then it is called an orthogonal set (briefly $\mathfrak{O}$ -set). We denote this $\mathfrak{O}$ -set by $(\mathfrak{W}, \vdash)$.

Example 1.3. [1] Let $\mathfrak{W} = [0, \infty)$ and define $\mathfrak{w} \vdash u$ if $\mathfrak{w}u \in \{\mathfrak{w}, u\}$. Then, by setting $\mathfrak{w}_0 = 0$ or $\mathfrak{w}_0 = 1$, $(\mathfrak{W}, \vdash)$ is an $\mathfrak{O}$ -set.

Definition 1.3. [1] Let $(\mathfrak{W}, \vdash)$ be an $\mathfrak{O}$ -set. A sequence $\{\mathfrak{w}_i\}$ is called an orthogonal sequence (briefly, $\mathfrak{O}$ -sequence) if

$$(\forall \iota \in \iota, \mathfrak{w}_\iota \vdash \mathfrak{w}_{\iota+1}) \quad \text{or} \quad (\forall \iota \in \iota, \mathfrak{w}_{\iota+1} \vdash \mathfrak{w}_\iota).$$

Definition 1.4. [1] A triplet $(\mathfrak{W}, \vdash, \mathfrak{d})$ is called an orthogonal metric space if $(\mathfrak{W}, \vdash)$ is an $\mathfrak{O}$ -set and $(\mathfrak{W}, \mathfrak{d})$ is a metric space.

Definition 1.5. Let $(\mathfrak{W}, \vdash)$ be an $\mathfrak{O}$ -set. A mapping $\mathfrak{L} : \mathfrak{W} \times \mathfrak{W} \rightarrow \mathfrak{W}$ is said to be $\vdash$ -preserving if $\mathfrak{L}(\mathfrak{w}, u) \vdash \mathfrak{L}(s, r)$ whenever $\mathfrak{w} \vdash s$ and $u \vdash r$. Also, $\mathfrak{L} : \mathfrak{W} \times \mathfrak{W} \rightarrow \mathfrak{W}$ is said to be weakly $\vdash$ -preserving if $\mathfrak{L}(\mathfrak{w}, u) \vdash \mathfrak{L}(s, r)$ or $\mathfrak{L}(s, r) \vdash \mathfrak{L}(\mathfrak{w}, u)$ whenever $\mathfrak{w} \vdash s$ and $u \vdash r$.

2 Main results

In this section, inspired by Theorem 1.2 and an orthogonal set, we introduce a new coupled $\mathfrak{F}$ -orthogonal contraction mapping and prove some coupled fixed point theorem for this contraction mapping in an orthogonal metric space.

Definition 2.1. Let $(\mathfrak{W}, \vdash, \mathfrak{d})$ be an orthogonal metric space. A mapping $\mathfrak{L} : \mathfrak{W} \times \mathfrak{W} \rightarrow \mathfrak{W}$ is said to be a coupled $\mathfrak{F}$ -orthogonal contraction mapping on $(\mathfrak{W}, \vdash, \mathfrak{d})$ if there are $\mathfrak{F} \in \mathcal{I}$ and $\ell > 0$ such that for all $\mathfrak{w}, \mathfrak{u}, \mathfrak{s}, \mathfrak{r} \in \mathfrak{W}$ with $\mathfrak{w} \vdash \mathfrak{s}$ and $\mathfrak{u} \vdash \mathfrak{r}$ [$\mathfrak{d}(\mathfrak{L}(\mathfrak{w}, \mathfrak{u}), \mathfrak{L}(\mathfrak{s}, \mathfrak{r}))$] > 0 ,

$$\ell + \mathfrak{F}(\mathfrak{d}(\mathfrak{L}(\mathfrak{w}, \mathfrak{u}), \mathfrak{L}(\mathfrak{s}, \mathfrak{r})) \leq \mathfrak{F}(a\mathfrak{d}(\mathfrak{w}, \mathfrak{s}) + b\mathfrak{d}(\mathfrak{u}, \mathfrak{r}))),$$

where $a + b = 1$, $a, b \geq 0$.

Theorem 2.1. Let $(\mathfrak{W}, \vdash, \mathfrak{d})$ be an $\mathfrak{O}$ -complete metric space with an orthogonal element $\mathfrak{w}_0$ and $\mathfrak{L} : \mathfrak{W} \times \mathfrak{W} \rightarrow \mathfrak{W}$ be a mapping. Suppose that there exist $\mathfrak{F} \in \mathcal{I}$ and $\ell > 0$ such that

- (i) $\mathfrak{L}$ is $\vdash$ -preserving;
- (ii) $\mathfrak{L}$ is a coupled $\mathfrak{F}$ -orthogonal contraction mapping.

Then $\mathfrak{L}$ has a unique coupled fixed point.

Proof. Since $(\mathfrak{W}, \vdash)$ is an $\mathfrak{O}$ -set, there exists $\mathfrak{w}_0 \in \mathfrak{W}$ such that

$$(\forall \mathfrak{w} \in \mathfrak{W}, \mathfrak{w} \vdash \mathfrak{w}_0) \quad \text{or} \quad (\forall \mathfrak{w} \in \mathfrak{W}, \mathfrak{w}_0 \vdash \mathfrak{w}),$$

and there exists $\mathfrak{z}_0 \in \mathfrak{W}$ such that

$$(\forall \mathfrak{w} \in \mathfrak{W}, \mathfrak{w} \vdash \mathfrak{z}_0) \quad \text{or} \quad (\forall \mathfrak{w} \in \mathfrak{W}, \mathfrak{z}_0 \vdash \mathfrak{w}).$$

It follows that $\mathfrak{w}_0 \vdash \mathfrak{L}(\mathfrak{w}_0, \mathfrak{z}_0)$ or $\mathfrak{L}(\mathfrak{w}_0, \mathfrak{z}_0) \vdash \mathfrak{w}_0$ and $\mathfrak{z}_0 \vdash \mathfrak{L}(\mathfrak{z}_0, \mathfrak{w}_0)$ or $\mathfrak{L}(\mathfrak{z}_0, \mathfrak{w}_0) \vdash \mathfrak{z}_0$.

Let

$$\mathfrak{w}_1 := \mathfrak{L}(\mathfrak{w}_0, \mathfrak{z}_0), \quad \mathfrak{w}_2 := \mathfrak{L}(\mathfrak{w}_1, \mathfrak{z}_1) = \mathfrak{L}^2(\mathfrak{w}_0, \mathfrak{z}_0), \quad \dots, \quad \mathfrak{w}_{t+1} := \mathfrak{L}(\mathfrak{w}_t, \mathfrak{z}_t) = \mathfrak{L}^{t+1}(\mathfrak{w}_0, \mathfrak{z}_0)$$

and

$$\mathfrak{z}_1 := \mathfrak{L}(\mathfrak{z}_0, \mathfrak{w}_0), \quad \mathfrak{z}_2 := \mathfrak{L}(\mathfrak{z}_1, \mathfrak{w}_1) = \mathfrak{L}^2(\mathfrak{z}_0, \mathfrak{w}_0), \quad \dots, \quad \mathfrak{z}_{t+1} := \mathfrak{L}(\mathfrak{z}_t, \mathfrak{w}_t) = \mathfrak{L}^{t+1}(\mathfrak{z}_0, \mathfrak{w}_0).$$

If $\mathfrak{w}_t = \mathfrak{w}_{t+1}$, $\mathfrak{z}_t = \mathfrak{z}_{t+1}$ for any $t \in \iota \cup \{0\}$, then it is clear that $(\mathfrak{w}_t, \mathfrak{z}_t)$ is a coupled fixed point of $\mathfrak{L}$. Assume that $\mathfrak{w}_t \neq \mathfrak{w}_{t+1}$ or $\mathfrak{z}_t \neq \mathfrak{z}_{t+1}$ for all $t \in \iota \cup \{0\}$. Then we have

$$\mathfrak{d}(\mathfrak{L}(\mathfrak{w}_t, \mathfrak{z}_t), \mathfrak{L}(\mathfrak{w}_{t+1}, \mathfrak{z}_{t+1})) > 0 \quad \text{or} \quad \mathfrak{d}(\mathfrak{L}(\mathfrak{z}_t, \mathfrak{w}_t), \mathfrak{L}(\mathfrak{z}_{t+1}, \mathfrak{w}_{t+1})) > 0$$

for all $t \in \iota \cup \{0\}$. Since $\mathfrak{L}$ is $\vdash$ -preserving, we have

$$\mathfrak{w}_t \vdash \mathfrak{w}_{t+1} \quad \text{or} \quad \mathfrak{w}_{t+1} \vdash \mathfrak{w}_t$$

and

$$\mathfrak{z}_t \vdash \mathfrak{z}_{t+1} \quad \text{or} \quad \mathfrak{z}_{t+1} \vdash \mathfrak{z}_t$$

for all $t \in \iota \cup \{0\}$. This implies that $\{\mathfrak{w}_t\}$ and $\{\mathfrak{z}_t\}$ are $\mathfrak{O}$ -sequences. Since $\mathfrak{L}$ is a coupled $\mathfrak{F}$ -orthogonal contraction mapping, we have

$$\ell + \mathfrak{F}(\mathfrak{d}(\mathfrak{w}_t, \mathfrak{w}_{t+1})) = \ell + \mathfrak{F}(\mathfrak{d}(\mathfrak{L}(\mathfrak{w}_{t-1}, \mathfrak{z}_{t-1}), \mathfrak{L}(\mathfrak{w}_t, \mathfrak{z}_t))) \leq \mathfrak{F}(a\mathfrak{d}(\mathfrak{w}_{t-1}, \mathfrak{w}_t) + b\mathfrak{d}(\mathfrak{z}_{t-1}, \mathfrak{z}_t)), \quad \forall t \in \iota$$

and

$$\ell + \mathfrak{F}(\mathfrak{d}(\mathfrak{z}_t, \mathfrak{z}_{t+1})) = \ell + \mathfrak{F}(\mathfrak{d}(\mathfrak{L}(\mathfrak{z}_{t-1}, \mathfrak{w}_{t-1}), \mathfrak{L}(\mathfrak{z}_t, \mathfrak{w}_t))) \leq \mathfrak{F}(a\mathfrak{d}(\mathfrak{z}_{t-1}, \mathfrak{z}_t) + b\mathfrak{d}(\mathfrak{w}_{t-1}, \mathfrak{w}_t)), \quad \forall t \in \iota.$$

By the property $(\mathfrak{F}_1)$, we obtain

$$\mathfrak{d}(\mathfrak{w}_t, \mathfrak{w}_{t+1}) < a\mathfrak{d}(\mathfrak{w}_{t-1}, \mathfrak{w}_t) + b\mathfrak{d}(\mathfrak{z}_{t-1}, \mathfrak{z}_t)$$

and

$$\mathfrak{d}(\mathfrak{z}_l, \mathfrak{z}_{l+1}) < a\mathfrak{d}(\mathfrak{z}_{l-1}, \mathfrak{z}_l) + b\mathfrak{d}(\mathfrak{w}_{l-1}, \mathfrak{w}_l).$$

Set

$$h_{l-1} := \mathfrak{d}(\mathfrak{w}_l, \mathfrak{w}_{l+1}) + \mathfrak{d}(\mathfrak{z}_l, \mathfrak{z}_{l+1}).$$

Then we have

$$h_{l-1} < (a + b)\mathfrak{d}(\mathfrak{w}_{l-1}, \mathfrak{w}_l) + (a + b)\mathfrak{d}(\mathfrak{z}_{l-1}, \mathfrak{z}_l) = (a + b)h_{l-2}, \quad \forall l \in \iota.$$

Since $a + b = 1$, we get $h_{l-1} < h_{l-2}$, $\forall l \in \iota$. Consequently,

$$\ell + \mathfrak{F}(h_{l-1}) \leq \mathfrak{F}(h_{l-2}), \quad \forall l \in \iota.$$

Therefore, we get

$$\mathfrak{F}(h_l) \leq \mathfrak{F}(h_{l-1}) - \ell \leq \dots \leq \mathfrak{F}(h_0) - \iota\ell, \quad \forall l \in \iota. \quad (2.1)$$

As $\iota \rightarrow \infty$, we obtain $\lim_{\iota \rightarrow \infty} \mathfrak{F}(h_l) = -\infty$.

By the property $(\mathfrak{F}_2)$, we have

$$\lim_{\iota \rightarrow \infty} h_l = 0.$$

By the property $(\mathfrak{F}_3)$, there exists $k \in (0, \infty)$ such that

$$\lim_{\iota \rightarrow \infty} h_l^k \mathfrak{F}(h_l) = 0.$$

By (2.1), we have

$$h_l^k \mathfrak{F}(h_l) - h_l^k \mathfrak{F}(h_0) \leq -h_l^k \iota \ell \leq 0. \quad (2.2)$$

Taking the limit as $n \rightarrow \infty$ in (2.2), we get

$$\lim_{\iota \rightarrow \infty} \iota h_l^k = 0.$$

Then there exists $\iota_1 \in \iota$ such that $\iota h_l^k \leq 1$ for all $\iota \geq \iota_1$ and so

$$h_l \leq \frac{1}{\iota^{\frac{1}{k}}} \quad (2.3)$$

for all $\iota \geq \iota_1$. Now, we claim that $\{\mathfrak{w}_l\}_{l=1}^{\infty}$ and $\{\mathfrak{z}_l\}_{l=1}^{\infty}$ are Cauchy $\mathfrak{D}$ -sequences. Using (2.3), we obtain that, for all $m > \iota \geq \iota_1$,

$$\begin{aligned} \mathfrak{d}(\mathfrak{w}_m, \mathfrak{w}_l) + \mathfrak{d}(\mathfrak{z}_m, \mathfrak{z}_l) &\leq \mathfrak{d}(\mathfrak{w}_m, \mathfrak{w}_{m-1}) + \mathfrak{d}(\mathfrak{z}_m, \mathfrak{z}_l) + \dots + \mathfrak{d}(\mathfrak{w}_{l-1}, \mathfrak{w}_l) + \mathfrak{d}(\mathfrak{z}_{l-1}, \mathfrak{z}_l) \\ &= h_{m-1} + h_{m-2} + \dots + h_l \\ &= \sum_{i=l}^{m-1} h_i \leq \sum_{i=1}^{\infty} h_i \leq \sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

Since $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{k}}} < \infty$, $\{\mathfrak{w}_l\}_{l=1}^{\infty}$ and $\{\mathfrak{z}_l\}_{l=1}^{\infty}$ are Cauchy $\mathfrak{D}$ -sequences. Since $\mathfrak{W}$ is $\mathfrak{D}$ -complete, there exist $\mathfrak{w}, \mathfrak{z} \in \mathfrak{W}$ such that $\lim_{\iota \rightarrow \infty} \mathfrak{w}_l = \mathfrak{w}$ and $\lim_{\iota \rightarrow \infty} \mathfrak{z}_l = \mathfrak{z}$. By the property of $\mathfrak{D}$ -metric, we obtain

$$\begin{aligned} \mathfrak{d}(\mathfrak{L}(\mathfrak{w}, \mathfrak{z}), \mathfrak{w}) &\leq \mathfrak{d}(\mathfrak{L}(\mathfrak{w}, \mathfrak{z}), \mathfrak{w}_{l+1}) + \mathfrak{d}(\mathfrak{w}_{l+1}, \mathfrak{w}), \\ \mathfrak{d}(\mathfrak{L}(\mathfrak{w}, \mathfrak{z}), \mathfrak{w}) - \mathfrak{d}(\mathfrak{w}_{l+1}, \mathfrak{w}) &\leq \mathfrak{d}(\mathfrak{L}(\mathfrak{w}, \mathfrak{z}), \mathfrak{w}_{l+1}). \end{aligned} \quad (2.4)$$

By choice of $\mathfrak{w}$ and $\mathfrak{z}$, we have

$$\mathfrak{w} \vdash \mathfrak{w}_l \quad \text{or} \quad \mathfrak{w}_l \vdash \mathfrak{w}$$

and

$$\mathfrak{z} \vdash \mathfrak{z}_l \quad \text{or} \quad \mathfrak{z}_l \vdash \mathfrak{z}.$$

Since $\mathcal{L}$ is a coupled $\mathfrak{F}$ -orthogonal contraction mapping, we get

$$\mathfrak{F}(\mathfrak{d}(\mathcal{L}(\mathfrak{w}, \mathfrak{z}), \mathcal{L}(\mathfrak{w}_l, \mathfrak{z}_l))) < \ell + \mathfrak{F}(\mathfrak{d}(\mathcal{L}(\mathfrak{w}, \mathfrak{z}), \mathcal{L}(\mathfrak{w}_l, \mathfrak{z}_l))) \leq \mathfrak{F}(a\mathfrak{d}(\mathfrak{w}, \mathfrak{w}_l) + b\mathfrak{d}(\mathfrak{z}, \mathfrak{z}_l)).$$

By the property of $(\mathfrak{F}_1)$, we have

$$\mathfrak{d}(\mathcal{L}(\mathfrak{w}, \mathfrak{z}), \mathcal{L}(\mathfrak{w}_l, \mathfrak{z}_l)) < a\mathfrak{d}(\mathfrak{w}, \mathfrak{w}_l) + b\mathfrak{d}(\mathfrak{z}, \mathfrak{z}_l). \quad (2.5)$$

From (2.4) and (2.5), we obtain

$$\mathfrak{d}(\mathcal{L}(\mathfrak{w}, \mathfrak{z}), \mathfrak{w}) - \mathfrak{d}(\mathfrak{w}_{l+1}, \mathfrak{w}) \leq a\mathfrak{d}(\mathfrak{w}, \mathfrak{w}_l) + b\mathfrak{d}(\mathfrak{z}, \mathfrak{z}_l).$$

As $l \rightarrow \infty$, we get

$$\lim_{l \rightarrow \infty} \mathfrak{d}(\mathcal{L}(\mathfrak{w}, \mathfrak{z}), \mathfrak{w}) = 0.$$

Therefore, $\mathcal{L}(\mathfrak{w}, \mathfrak{z}) = \mathfrak{w}$.

Similarly, we can prove that $\mathcal{L}(\mathfrak{z}, \mathfrak{w}) = \mathfrak{z}$. Then $(\mathfrak{w}, \mathfrak{z})$ is a coupled fixed point of $\mathcal{L}$. Assume that $(\mathfrak{w}^*, \mathfrak{z}^*)$ is another coupled fixed point of $\mathcal{L}$ such that $(\mathfrak{w}, \mathfrak{z}) \neq (\mathfrak{w}^*, \mathfrak{z}^*)$. Then $\mathfrak{d}(\mathcal{L}(\mathfrak{w}, \mathfrak{z}), \mathcal{L}(\mathfrak{w}^*, \mathfrak{z}^*)) = \mathfrak{d}(\mathfrak{w}, \mathfrak{w}^*) > 0$ and $\mathfrak{d}(\mathcal{L}(\mathfrak{z}, \mathfrak{w}), \mathcal{L}(\mathfrak{z}^*, \mathfrak{w}^*)) = \mathfrak{d}(\mathfrak{z}, \mathfrak{z}^*) > 0$. Since $\mathcal{L}$ is $\vdash$ -preserving, we get

$$\mathfrak{w} \vdash \mathfrak{w}^* \quad \text{or} \quad \mathfrak{w}^* \vdash \mathfrak{w}$$

and

$$\mathfrak{z} \vdash \mathfrak{z}^* \quad \text{or} \quad \mathfrak{z}^* \vdash \mathfrak{z}.$$

Since $\mathcal{L}$ is a coupled $\mathfrak{F}$ -orthogonal contraction mapping, we get

$$\mathfrak{F}(\mathfrak{d}(\mathfrak{w}, \mathfrak{w}^*)) = \mathfrak{F}(\mathfrak{d}(\mathcal{L}(\mathfrak{w}, \mathfrak{z}), \mathcal{L}(\mathfrak{w}^*, \mathfrak{z}^*))) \leq \mathfrak{F}(a\mathfrak{d}(\mathfrak{w}, \mathfrak{w}^*) + b\mathfrak{d}(\mathfrak{z}, \mathfrak{z}^*)) - \ell \quad (2.6)$$

and

$$\mathfrak{F}(\mathfrak{d}(\mathfrak{z}, \mathfrak{z}^*)) = \mathfrak{F}(\mathfrak{d}(\mathcal{L}(\mathfrak{z}, \mathfrak{w}), \mathcal{L}(\mathfrak{z}^*, \mathfrak{w}^*))) \leq \mathfrak{F}(a\mathfrak{d}(\mathfrak{z}, \mathfrak{z}^*) + b\mathfrak{d}(\mathfrak{w}, \mathfrak{w}^*)) - \ell. \quad (2.7)$$

By the property of $(\mathfrak{F}_1)$, (2.6) and (2.7), we get

$$\mathfrak{d}(\mathfrak{w}, \mathfrak{w}^*) \leq a\mathfrak{d}(\mathfrak{w}, \mathfrak{w}^*) + b\mathfrak{d}(\mathfrak{z}, \mathfrak{z}^*)$$

and

$$\mathfrak{d}(\mathfrak{z}, \mathfrak{z}^*) \leq a\mathfrak{d}(\mathfrak{z}, \mathfrak{z}^*) + b\mathfrak{d}(\mathfrak{w}, \mathfrak{w}^*).$$

Thus, we have

$$\mathfrak{d}(\mathfrak{w}, \mathfrak{w}^*) + \mathfrak{d}(\mathfrak{z}, \mathfrak{z}^*) \leq (a + b)(\mathfrak{d}(\mathfrak{w}, \mathfrak{w}^*) + \mathfrak{d}(\mathfrak{z}, \mathfrak{z}^*)).$$

Since $a + b = 1$, we obtain

$$\mathfrak{d}(\mathfrak{w}, \mathfrak{w}^*) + \mathfrak{d}(\mathfrak{z}, \mathfrak{z}^*) = 0.$$

Therefore, $(\mathfrak{w}, \mathfrak{z}) = (\mathfrak{w}^*, \mathfrak{z}^*)$, which is a contradiction. So $\mathcal{L}$ has a unique coupled fixed point. $\square$

In Theorem 2.1, if $a = 1$, $b = 0$, then we obtain the following corollary.

Corollary 2.2. *Let $(\mathfrak{W}, \vdash, \mathfrak{d})$ be an $\mathfrak{D}$ -complete metric space with an orthogonal element $\mathfrak{w}_0$ and $\mathcal{L} : \mathfrak{W} \times \mathfrak{W} \rightarrow \mathfrak{W}$ be a mapping. Suppose that there exist $\mathfrak{F} \in \mathfrak{I}$ and $\ell > 0$ such that*

- (i) $\mathcal{L}$ is $\vdash$ -preserving;
- (ii) for all $\mathfrak{w}, \mathfrak{u}, \mathfrak{s}, \mathfrak{r} \in \mathfrak{W}$ with $\mathfrak{w} \vdash \mathfrak{u}$ and $\mathfrak{s} \vdash \mathfrak{r}$ [$\mathfrak{d}(\mathcal{L}(\mathfrak{w}, \mathfrak{u}), \mathcal{L}(\mathfrak{s}, \mathfrak{r})) > 0$,

$$\ell + \mathfrak{F}(\mathfrak{d}(\mathcal{L}(\mathfrak{w}, \mathfrak{u}), \mathcal{L}(\mathfrak{s}, \mathfrak{r}))) \leq \mathfrak{F}(\mathfrak{d}(\mathfrak{w}, \mathfrak{s})).$$

Then $\mathcal{L}$ has a unique coupled fixed point.

In Theorem 2.1, $a = b = \frac{1}{2}$, then we obtain the following corollary.

Corollary 2.3. Let $(\mathfrak{W}, \vdash, \mathfrak{d})$ be an $\mathfrak{D}$ -complete metric space with an orthogonal element $\mathfrak{w}_0$ and $\mathfrak{L} : \mathfrak{W} \times \mathfrak{W} \rightarrow \mathfrak{W}$ be a mapping. Suppose that there exist $\mathfrak{F} \in \mathfrak{I}$ and $\ell > 0$ such that

- (i) $\mathfrak{L}$ is $\vdash$ -preserving;
- (ii) for all $\mathfrak{w}, \mathfrak{u}, \mathfrak{s}, \mathfrak{r} \in \mathfrak{W}$ with $\mathfrak{w} \vdash \mathfrak{u}$ and $\mathfrak{s} \vdash \mathfrak{r}$ [$\mathfrak{d}(\mathfrak{L}(\mathfrak{w}, \mathfrak{u}), \mathfrak{L}(\mathfrak{s}, \mathfrak{r})) > 0$,

$$\ell + \mathfrak{F}(\mathfrak{d}(\mathfrak{L}(\mathfrak{w}, \mathfrak{u}), \mathfrak{L}(\mathfrak{s}, \mathfrak{r}))) \leq \mathfrak{F}(\mathfrak{d}(\mathfrak{w}, \mathfrak{s}) + \mathfrak{d}(\mathfrak{u}, \mathfrak{r}))].$$

Then $\mathfrak{L}$ has a unique coupled fixed point.

Example 2.4. Let $\mathfrak{W} = \mathbb{R}$ and $\mathfrak{d}(\mathfrak{u}, \mathfrak{v}) = |\mathfrak{u} - \mathfrak{v}|$ for all $\mathfrak{u}, \mathfrak{v} \in \mathfrak{W}$. Define a relation $\vdash$ on $\mathfrak{K}$ by

$$\mathfrak{u} \vdash \mathfrak{v} \text{ iff } \mathfrak{u}, \mathfrak{v} \geq 0.$$

Then $(\mathfrak{W}, \vdash, \mathfrak{d})$ is an $\mathfrak{D}$ -complete metric space. Define a mapping $\mathfrak{L} : \mathfrak{W} \times \mathfrak{W} \rightarrow \mathfrak{W}$ by $\mathfrak{L}(\mathfrak{u}, \mathfrak{v}) = \frac{\mathfrak{u} + \mathfrak{v}}{e^\ell}$ for $\ell > 0$. It is easy to see that $\mathfrak{L}$ is $\vdash$ -preserving. Next, let $\mathfrak{F} \in \mathfrak{I}$ be a function defined by $\mathfrak{F}(c) = \ln c$ for $c > 0$. Then for all $\mathfrak{u}, \mathfrak{v}, \mathfrak{w}, \mathfrak{s} \in \mathfrak{W}$, $\mathfrak{L}(\mathfrak{u}, \mathfrak{v}) \neq \mathfrak{L}(\mathfrak{w}, \mathfrak{s})$ and $a = \frac{5}{2}$, we get

$$\begin{aligned} \ell + \mathfrak{F}(\mathfrak{d}(\mathfrak{L}(\mathfrak{u}, \mathfrak{v}), \mathfrak{L}(\mathfrak{w}, \mathfrak{s}))) &= \ell + \ln \left(\left| \frac{\mathfrak{u} + \mathfrak{v}}{e^\ell} - \frac{\mathfrak{w} + \mathfrak{s}}{e^\ell} \right| \right) \\ &\leq \ell + \ln \left(\left| \frac{\mathfrak{u} - \mathfrak{w}}{e^\ell} \right| + \left| \frac{\mathfrak{v} - \mathfrak{s}}{e^\ell} \right| \right) \\ &= \ell + \ln(|\mathfrak{u} - \mathfrak{w}| + |\mathfrak{v} - \mathfrak{s}|) - \ln e^\ell \\ &= \mathfrak{F}(\mathfrak{d}(\mathfrak{u}, \mathfrak{w}) + \mathfrak{d}(\mathfrak{v}, \mathfrak{s})). \end{aligned}$$

Therefore, all the conditions of Corollary 2.3 are satisfied and so $\mathfrak{L}$ has a unique coupled fixed point $(0, 0) \in \mathbb{R} \times \mathbb{R}$.

3 Application

Let $\mathfrak{M} = [0, \mathcal{H}]$. Let $\mathfrak{W} = C(\mathfrak{M}, \mathbb{R})$ be the set of all real valued continuous functions with domain $\mathfrak{M}$. Consider the integral equations

$$\begin{cases} \zeta(\mathfrak{v}) = \mathfrak{d}(\mathfrak{v}) + \int_0^{\mathfrak{M}} \Xi(\mathfrak{v}, \beta) \Omega(\beta, \zeta(\beta), \xi(\beta)) d\beta, & \mathfrak{v} \in [0, \mathcal{H}], \\ \xi(\mathfrak{v}) = \mathfrak{d}(\mathfrak{v}) + \int_0^{\mathfrak{M}} \Xi(\mathfrak{v}, \beta) \Omega(\beta, \xi(\beta), \zeta(\beta)) d\beta, & \mathfrak{v} \in [0, \mathcal{H}], \end{cases} \quad (3.1)$$

where

- (a) $\mathfrak{d} : \mathfrak{M} \rightarrow \mathbb{R}$ and $\Omega : \mathfrak{M} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous;
- (b) $\Xi : \mathfrak{M} \times \mathfrak{M}$ is continuous and measurable at $\beta \in \mathfrak{M}$, $\forall \mathfrak{v} \in \mathfrak{M}$;
- (c) $\Xi(\mathfrak{v}, \beta) \geq 0$, $\forall \mathfrak{v}, \beta \in \mathfrak{M}$ and $\int_0^{\mathcal{H}} \Xi(\mathfrak{v}, \beta) d\beta \leq 1$, $\forall \mathfrak{v} \in \mathfrak{M}$.

Theorem 3.1. Assume that the conditions (a)–(c) hold. Suppose that there exists $\mathfrak{r} > 0$ such that

$$|\Omega(\mathfrak{v}, \zeta(\mathfrak{v}), \xi(\mathfrak{v})) - \Omega(\mathfrak{v}, \xi(\mathfrak{v}), \zeta(\mathfrak{v}))| \leq e^{-\mathfrak{r}} |\zeta(\mathfrak{v}) - \xi(\mathfrak{v})|$$

for each $\mathfrak{v} \in \mathfrak{M}$ and for all $\zeta, \xi \in C(\mathfrak{M}, \mathbb{R})$. Then the integral equation (3.1) has a unique solution in $C(\mathfrak{M}, \mathbb{R})$.

Proof. Let $\mathfrak{W} = \{w \in C(I, \mathbb{R}) : w(\beta) > 0 \text{ for all } \beta \in I\}$. Define the orthogonality relation $\vdash$ on $\mathfrak{W}$ by

$$\zeta \vdash \xi \Leftrightarrow \zeta(\beta)\xi(\beta) \geq \zeta(\beta) \quad \text{or} \quad \zeta(\beta)\xi(\beta) \geq \xi(\beta) \quad \text{for all } \beta \in I.$$

Define a mapping $\mathfrak{d} : \mathfrak{W} \times \mathfrak{W} \rightarrow [0, \infty)$ by

$$\mathfrak{d}(\zeta, \xi) = \sup_{\beta \in I} |\zeta(\beta) - \xi(\beta)|$$

for all $\zeta, \xi \in \mathfrak{W}$. Thus, $(\mathfrak{W}, \mathfrak{d})$ is a $\mathfrak{D}$ -complete metric space. Define $\mathfrak{L} : C(\mathfrak{M}, \mathbb{R}) \times C(\mathfrak{M}, \mathbb{R}) \rightarrow C(\mathfrak{M}, \mathbb{R})$ by

$$\mathfrak{L}(\zeta, \xi)(\mathfrak{v}) = \mathfrak{d}(\mathfrak{v}) + \int_0^{\mathfrak{M}} \Xi(\mathfrak{v}, \beta) \Omega(\beta, \zeta(\beta), \xi(\beta)) d\beta, \quad \mathfrak{v} \in [0, \mathcal{H}].$$

For each $\zeta, \xi \in \mathfrak{W}$ with $\zeta \vdash \xi$ and $\mathfrak{v} \in I$, we have

$$\mathfrak{L}(\zeta, \xi)(\mathfrak{v}) = \mathfrak{d}(\mathfrak{v}) + \int_0^{\mathfrak{M}} \Xi(\mathfrak{v}, \beta) \Omega(\beta, \zeta(\beta), \xi(\beta)) d\beta \geq 1.$$

It follows that $[\mathfrak{L}(\zeta, \xi)(\mathfrak{v})][\mathfrak{L}(\xi, \zeta)(\mathfrak{v})] \geq \mathfrak{L}(\zeta, \xi)(\mathfrak{v})$ and so $\mathfrak{L}(\zeta, \xi)(\mathfrak{v}) \vdash \mathfrak{L}(\xi, \zeta)(\mathfrak{v})$. Then $\mathfrak{L}$ is $\vdash$ -preserving.

Let $\zeta, \xi \in \mathfrak{W}$ with $\zeta \vdash \xi$. Suppose that $\mathfrak{L}(\zeta, \xi)(\mathfrak{v}) \neq \mathfrak{L}(\xi, \zeta)(\mathfrak{v})$. For every $\mathfrak{v} \in [0, \mathcal{H}]$, we have

$$\begin{aligned} |\mathfrak{L}(\zeta, \xi)(\mathfrak{v}) - \mathfrak{L}(\xi, \zeta)(\mathfrak{v})| &= \left| \int_0^{\mathcal{H}} \Xi(\mathfrak{v}, \beta) (\Omega(\beta, \zeta(\beta), \xi(\beta)) - \Omega(\beta, \xi(\beta), \zeta(\beta))) d\beta \right| \\ &\leq \int_0^{\mathcal{H}} \Xi(\mathfrak{v}, \beta) |\Omega(\beta, \zeta(\beta), \xi(\beta)) - \Omega(\beta, \xi(\beta), \zeta(\beta))| d\beta \\ &\leq \int_0^{\mathcal{H}} \Xi(\mathfrak{v}, \beta) e^{-|\zeta(\mathfrak{v}) - \xi(\mathfrak{v})|} d\beta \\ &\leq e^{-|\zeta(\mathfrak{v}) - \xi(\mathfrak{v})|} \int_0^{\mathcal{H}} \Xi(\mathfrak{v}, \beta) d\beta \\ &\leq e^{-|\zeta(\mathfrak{v}) - \xi(\mathfrak{v})|}, \end{aligned}$$

which implies that

$$\mathfrak{d}(\mathfrak{L}(\zeta, \xi), \mathfrak{L}(\xi, \zeta)) \leq e^{-\mathfrak{d}(\zeta, \xi)}.$$

Therefore,

$$\mathfrak{r} + \ln(\mathfrak{d}(\mathfrak{L}(\zeta, \xi), \mathfrak{L}(\xi, \zeta))) \leq \ln(\mathfrak{d}(\zeta, \xi)).$$

Taking $\mathfrak{F}(\mathfrak{v}) = \ln(\mathfrak{v})$, we obtain

$$\mathfrak{r} + \mathfrak{F}(\mathfrak{d}(\mathfrak{L}(\zeta, \xi), \mathfrak{L}(\xi, \zeta))) \leq \mathfrak{F}(\mathfrak{d}(\zeta, \xi))$$

for all $\zeta, \xi \in \mathfrak{W}$. Therefore, by Corollary 2.2, $\mathfrak{L}$ has a unique coupled fixed point. Hence, there is a unique solution for (3.1). $\square$

4 Open problem

In this paper, we started the study of coupled $\mathfrak{J}$ -orthogonal contraction and established coupled fixed point results in $\mathfrak{D}$ -complete metric space. Recently, Khalehghli, Rahimi and Eshaghi Gordji [18,19] introduced R -metric spaces and obtained a generalization of Banach fixed point theorem. It is an interesting open

problem to study the relation R instead of orthogonal relation and obtain coupled fixed point results on R -complete metric spaces.

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