Research Article

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**Power moments of automorphic $L$-functions related to Maass forms for $SL_3(\mathbb{Z})$**

https://doi.org/10.1515/math-2021-0076
received September 30, 2020; accepted July 12, 2021

**Abstract:** Let $f$ be a self-dual Hecke-Maass eigenform for the group $SL_3(\mathbb{Z})$. For $\frac{1}{2} < \sigma < 1$ fixed we define $m(\sigma) \geq 2$ as the supremum of all numbers $m$ such that

$$\int_{1}^{T} |L(s, f)|^m dt \ll_{f, \varepsilon} T^{1+\varepsilon},$$

where $L(s, f)$ is the Godement-Jacquet $L$-function related to $f$. In this paper, we first show the lower bound of $m(\sigma)$ for $\frac{2}{3} < \sigma < 1$. Then we establish asymptotic formulas for the second, fourth and sixth powers of $L(s, f)$ as applications.

**Keywords:** power moments, $L$-function, automorphic form

MSC 2020: 11F03, 11F66

1 Introduction

Let $f$ be a self-dual Hecke-Maass eigenform for the group $SL_3(\mathbb{Z})$ of type $\nu = (\alpha, \beta)$. Then the Langlands’ parameters for $f$ are

$$\mu_j(1) = \alpha + 2\beta - 1, \quad \mu_j(2) = \alpha - \beta, \quad \mu_j(3) = 1 - 2\alpha - \beta.$$  

It is known that $f$ has the following Fourier-Whittaker expansion:

$$f(z) = \sum_{\gamma \in U_2(Z) \setminus SL_2(Z)} \sum_{m \geq 1} \sum_{n = 0} A_f(m, n) \frac{\gamma(n)}{|m| |n|} W_f\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} z, \nu, \psi_{4,1}\right),$$

where $U_2 = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$, $W_f(z, \nu, \psi_{4,1})$ is the Jacquet-Whittaker function, $\psi_{4,1}$ is a character of $U_2(\mathbb{R})$, $M = \text{diag}(m|n|, m, 1)$ and $A_f(m, n)$ are the Fourier coefficients of $f$. The function $W_f(z, \nu, \psi_{4,1})$ represents an exponential decay in $\gamma_1$ and $\gamma_2$ for

$$z = \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_1 \gamma_2 \\ \gamma_2 \\ 1 \end{pmatrix}.$$
From Kim and Sarnak [1] and Sarnak [2] we know that
\[ A_f(m, n) \ll |mn|^\frac{1}{3} + \epsilon. \]

From [3], the Rankin-Selberg theory shows that
\[ \sum_{m,n \leq N} |A_f(m, n)|^2 \ll_f N. \]

Due to \( A_f(m, n) = A_f(n, m) \), then
\[ \sum_{m,n \leq N} |A_f(m, n)|^2 \ll_f N \tag{1.1} \]
also holds, where \( \tilde{f} \) is the contragredient form of \( f \). According to these estimates, we have
\[ \sum_{m \leq N} \frac{|A_f(1, 1)|^2}{m} \ll \log N, \quad \sum_{n \leq N} \frac{|A_f(1, n)|^2}{n} \ll \log N. \tag{1.2} \]

As in [4] and [5], the Godement-Jacquet \( L \)-function associated with \( f \) is defined as
\[ L(s, f) = \prod_{n=1}^{\infty} \frac{A_f(1, n)}{n^s}, \quad \text{for } \Re s > 1. \]

This \( L \)-function has a standard functional equation and analytic continuation to an entire function on complex plane \( \mathbb{C} \). Due to the fact that \( f \) is a Hecke eigenform, the Fourier coefficients are multiplicative and the \( L \)-function has an Euler product (see [5, pp. 173–174]), for \( \Re s > 1 \),
\[ L(s, f) = \prod_p (1 - A_f(1, p)p^{-s} + A_f(p, 1)p^{-2s} - p^{-3s})^{-1}. \]

Then the \( L \)-function associated with the dual Maass form \( \tilde{f} \) takes the form
\[ L(s, \tilde{f}) = \sum_{n=1}^{\infty} \frac{A_f(n, 1)}{n^s} = \prod_p (1 - A_f(1, p)p^{-s} + A_f(p, 1)p^{-2s} - p^{-3s})^{-1}. \]

We write \( s = \sigma + it \) and suppose that \( \frac{1}{2} < \sigma < 1 \) is fixed. Let \( m(\sigma) \geq 2 \) be the supremum of all numbers \( m \geq 2 \) such that
\[ \int_{1}^{T} |L(s, f)|^m \, dt \ll_{f, \epsilon} T^{1+\epsilon}, \tag{1.3} \]
where the \( \ll \)-constant may depend on \( L(s, f) \) and \( \epsilon \). Naturally, we want to seek lower bounds for \( m(\sigma) \), which occurs frequently in applications. In the cases of full modular group \( SL_2(\mathbb{Z}) \) and the congruence group, many scholars have obtained lot of results (e.g., see [6–25], etc.).

In this paper, we focus our attention on the Hecke-Maass eigenforms for the group \( SL_3(\mathbb{Z}) \). In this situation, for one thing, we do not know whether the Ramanujan conjecture is true; for another, the square and fourth mean value estimates of \( L(s, f) \) are weaker than ones over \( SL_2(\mathbb{Z}) \). Our results are as follows.

**Theorem 1.** Let \( m(\sigma) \) for each \( \frac{2}{3} < \sigma < 1 \) be defined by (1.3). Then we have
\[ m(\sigma) \geq \frac{4(3 - 2\sigma)}{5(4 - 3\sigma)(1 - \sigma)}. \tag{1.4} \]

From Theorem 1 we can get the following corollary immediately.
Corollary. We have
\[ m\left(\frac{2}{3}\right) \geq 2, \quad m\left(\frac{97 - \sqrt{769}}{90}\right) \geq 3, \ldots, \quad m\left(\frac{103 - \sqrt{349}}{90}\right) \geq 12, \ldots. \]

Remark. Due to the fact that \( L(s, f) \) is an \( L \)-function of degree 3, then Perelli’s mean value theorem \([26]\) shows that, for \( \frac{1}{2} \leq \sigma \leq 1 \) and \( T \geq 1 \) uniformly,
\[ \int_1^T |L(\sigma + it, f)|^2 dt \ll T^{\max(3(1-\sigma),1) + \epsilon}, \]
which implies
\[ \int_1^T |L(\sigma + it, f)|^2 dt \ll T^{1+\epsilon} \quad \left( \frac{2}{3} \leq \sigma \leq 1 \right). \]

Thus, we restrict the range of \( \sigma \) in Theorem 1 into \( \frac{2}{3} < \sigma < 1 \).

As applications of Theorem 1, we can establish the asymptotic formulas for the second, fourth and sixth powers of \( L(s, f) \).

Theorem 2. For any \( \epsilon > 0 \) and \( \sigma \) fixed, we have
\[ \int_1^T |L(\sigma + it, f)|^2 dt = T \sum_{n=1}^{\infty} |A_f(1, n)|^2 n^{-2\sigma} + O\left(T^{\frac{2\sigma - 1}{2} + \epsilon}\right), \quad (1.5) \]
\[ \int_1^T |L(\sigma + it, f)|^4 dt = T \sum_{n=1}^{\infty} |A_f(1) A_f(1, n)|^2 n^{-2\sigma} + O\left(T^{\frac{2\sigma - 1}{2} + \epsilon}\right), \quad (1.6) \]
\[ \int_1^T |L(\sigma + it, f)|^6 dt = T \sum_{n=1}^{\infty} |A_f(1) A_f(1) A_f(1, n)|^2 n^{-2\sigma} + O\left(T^{\frac{2\sigma - 1}{2} + \epsilon}\right), \quad (1.7) \]
where \( A_f(1, n) = \sum_{m \equiv n \mod 1} A_f(1, m) A_f(1, l) \) is the Dirichlet convolution of \( A_f(1, n) \) with itself. The asymptotic formulas (1.5), (1.6) and (1.7) follow for \( \frac{2}{3} < \sigma < 1 \), \( \frac{31 - \sqrt{469}}{30} < \sigma < 1 \) and \( \frac{101 - \sqrt{3601}}{90} < \sigma < 1 \), respectively.

Notation. Throughout this paper, the letter \( \epsilon \) stands for a sufficiently small positive number, and the value of \( \epsilon \) may change from statement to statement.

2 Some lemmas

In order to prove Theorems 1 and 2, we first introduce some lemmas.

Lemma 2.1. Let \( T \leq t \leq 2T \) and \( k \geq 1 \) be a fixed integer. Then for \( \frac{1}{2} < \sigma < 1 \), we have
\[ |L(\sigma + it, f)|^k \ll 1 + \log T \int_{-\log^2 T}^{\log^2 T} \left| L(\sigma - \frac{1}{\log T} + it + iv, f) \right|^k e^{-\epsilon v} dv. \]
Proof. The proof of this lemma is similar to [27, Lemma 7.1], and we just need to use the following functional equation:

\[ G_\nu(s)L(s, f) = \bar{G}_\nu(1-s)L(1-s, \bar{f}), \]

where

\[ G_\nu(s) = \pi^{-\frac{3\nu}{2}} \left( \frac{s + 1 - 2\alpha - \beta}{2} \right) \Gamma \left( \frac{s + 1 + 2\beta}{2} \right), \]

\[ \bar{G}_\nu(s) = \pi^{-\frac{3\nu}{2}} \left( \frac{s + 1 - 2\alpha + \beta}{2} \right) \Gamma \left( \frac{s - 1 + 2\alpha + \beta}{2} \right), \]

in place of the functional equation of \( \zeta(s) \).

\[ \square \]

Lemma 2.2. For \( m = m(\sigma) \),

\[ \int_1^T |L(\sigma + it, f)|^{m(\sigma)} dt \ll T^{1+\varepsilon} \]  (2.1)

is equivalent to

\[ \sum_{r \leq R} |L(\sigma + it, f)|^{m(\sigma)} \ll T^{1+\varepsilon}, \]  (2.2)

where

\[ t_r \in [T, 2T] \text{ for } r = 1, \ldots, R; \quad |t_r - t_s| \geq \log^4 T \text{ for } 1 \leq r \neq s \leq R. \]  (2.3)

Proof. Let

\[ L(\sigma + it_m, f) = \max_{t \in \mathbb{R}} |L(\sigma + it, f)|, \quad I_m = [2T - m \log^4 T, 2T - (m - 1) \log^4 T], \]

where \( m = 1, 2, \ldots, \lfloor T \log^{-4} T \rfloor \). Denote by \( \{t_r\} \) either of the sets \( \{t_{2m}\} \) or \( \{t_{2m-1}\} \). Then the \( t_r \)'s satisfy (2.3) and

\[ \int_1^T |L(\sigma + it, f)|^{m(\sigma)} dt \ll \sum_{m=1}^{\lfloor T \log^{-4} T \rfloor} \int_{2T - m \log^4 T}^{2T - (m - 1) \log^4 T} |L(\sigma + it_m, f)|^{m(\sigma)} dt \]

\[ \ll \sum_{m=1}^{\lfloor T \log^{-4} T \rfloor} |L(\sigma + it_m, f)|^{m(\sigma)} \log^4 T \]

\[ \ll T^{1+\varepsilon}. \]

And then replacing \( T \) by \( \frac{T}{2}, \frac{T}{2^2}, \ldots \) and adding we can get (2.1). On the other hand, by Lemma 2.1, we have

\[ \sum_{r \leq R} |L(\sigma + it, f)|^{m(\sigma)} dt \ll R + \log T \sum_{r \leq R} \int_{r - \log^2 T}^{r + \log^2 T} \left| L \left( \sigma - \frac{1}{\log T} + it + iv, f \right) \right|^{m(\sigma)} dv \]

\[ \ll R + \log T \sum_{r \leq R} \int_{r T \log^2 T}^{2T + \log^2 T} \left| L \left( \sigma - \frac{1}{\log T} + it, f \right) \right|^{m(\sigma)} dt \]

\[ \ll T \log^{-4} T + \log T \int_1^T \left| L \left( \sigma - \frac{1}{\log T} + it, f \right) \right|^{m(\sigma)} dt \]

\[ \ll T^{1+\varepsilon}, \]

which implies (2.1). \( \square \)
Lemma 2.3. We suppose that \( \frac{1}{2} < \sigma < 1 \) is fixed and
\[
R \ll T^{1+\varepsilon}V^{-m(\sigma)},
\]
where for \( t \), defined by (2.3) we have
\[
|L(\sigma + it, f)| \geq V \geq T^r (r = 1, 2, \ldots, R),
\]
which is equivalent to
\[
\sum_{r \leq R} |L(\sigma + it_r, f)|^{m(\sigma)} \ll T^{1+\varepsilon}.
\]

**Proof.** We suppose that (2.6) is true and let \( \{t_1, \ldots, t_r, \} \) be the subset of \( \{t_i\} \) such that
\[
|L(\sigma + ib_j, f)| \geq V (j = 1, \ldots, R).
\]
Then from (2.6) we have
\[
R_i V^{m(\sigma)} \leq \sum_{r \leq R} |L(\sigma + it_r, f)|^{m(\sigma)} \ll T^{1+\varepsilon},
\]
thus for \( R_i = R \), (2.4) holds.

Inversely, we let (2.4) hold and denote by \( \{t_1, \ldots, t_r, V\} \) those of the points \( t_i, \ldots, t_r \) for which
\[
V \leq |L(\sigma + it_{v,j}, f)| \leq 2V (j = 1, \ldots, R(V)).
\]
For each \( V \), we have \( O(\log T) \) choices. And from the following Lemma 2.6, we take \( V = T^{\frac{\log V}{2}} \), \( V = 2^{-\frac{\log V}{2}} \), \( V = 2^{-\frac{\log V}{2}} \), \( \ldots \). Then we can obtain
\[
\sum_{r \leq R} L(\sigma + it_r, f)^{m(\sigma)}dt \ll RT^\varepsilon + \sum_{V \leq R(V)} 2V^{m(\sigma)} \ll RT^\varepsilon + \sum_{V} T^{1+\varepsilon} \ll T^{1+\varepsilon}.
\]
\( \Box \)

**Lemma 2.4.** Let \( t_1 < \cdots < t_r \) be real numbers such that \( t_r \in [T, 2T] \) for \( r = 1, \ldots, R \); \( |t_r - t_s| \geq \log^4 T \) for \( 1 \leq r \neq s \leq R \). If
\[
T^\varepsilon \ll V \leq \sum_{M \ll 2M} a(n) n^{-\sigma - it_r},
\]
where \( \sum_{n \leq M} |a(n)|^2 \ll M^{1+\varepsilon} \) for \( 1 \ll M \ll T^C \) \((C > 0)\), then we have
\[
R \ll T^{\varepsilon(M^{-2a}V^{-2} + TV^{-f(\sigma)})},
\]
where
\[
f(\sigma) = \begin{cases} 
\frac{2}{3 - 4\sigma}, & \text{if } \frac{1}{2} < \sigma < \frac{2}{3}, \\
\frac{10}{7 - 8\sigma}, & \text{if } \frac{2}{3} < \sigma < \frac{11}{14}, \\
\frac{34}{15 - 16\sigma}, & \text{if } \frac{11}{14} < \sigma < \frac{13}{15}, \\
\frac{98}{31 - 32\sigma}, & \text{if } \frac{13}{15} < \sigma < \frac{57}{62}, \\
\frac{5}{1 - \sigma}, & \text{if } \frac{57}{62} < \sigma < 1 - \varepsilon.
\end{cases}
\]

**Proof.** We can get this lemma by following a similar argument to [6, Lemma 8.2] replacing \( a(n) \ll M^\varepsilon \) by \( \sum_{n \leq M} |a(n)|^2 \ll M^{1+\varepsilon} \). \( \Box \)
Lemma 2.5. [27, Theorem 5.2] Let $a_1, \ldots, a_N$ be arbitrary complex numbers. Then
\[
\int_0^T \left| \sum_{n \leq N} a_n e^{int} \right|^2 dt = T \sum_{n \leq N} |a_n|^2 + O\left(\sum_{n \leq N} n|a_n|^2\right).
\]
and the above formula remains also valid if $N = \infty$, provided that the series on the right hand side of the aforementioned formula converge.

Lemma 2.6. [28, Corollary 1.2] Let $\frac{1}{2} \leq \sigma \leq 1$ be fixed, we have
\[
|L(\sigma + it, f)| \ll |t|^{\frac{1}{2}(1-\sigma)} \epsilon.
\]

Lemma 2.7. For any $\epsilon > 0$, we have
\[
\int_0^T \left| L\left(\frac{2}{3} + it, f\right)\right|^2 dt \ll T^{1+\epsilon},
\]
\[
\int_0^T \left| L\left(\frac{2}{3} + it, f\right)\right|^{\delta} dt \ll T^{\frac{\delta}{1+\epsilon}}.
\]

Proof. The first result is a general result of Perelli [26], which we can also get from Lemma 2.5 with $m = 3$ and $\sigma = \frac{2}{3}$ in Liu and Zhang [29]. From Lemma 2.6 and the first result, we can easily get the second result. \qed

Lemma 2.8. For $t$ satisfying (2.3), we have
\[
\sum_{r \in R} \left| L\left(\frac{2}{3} + it, f\right)\right|^2 dt \ll T^{1+\epsilon},
\]
\[
\sum_{r \in R} \left| L\left(\frac{2}{3} + it, f\right)\right|^{\delta} dt \ll T^{\frac{\delta}{1+\epsilon}}.
\]

Proof. Following a similar argument of Lemma 2.2, with the help of Lemma 2.7 we can obtain this lemma. \qed

Lemma 2.9. [27, Lemma 8.3] Let $F(s)$ be regular in the region $\mathcal{D} : \alpha \leq \sigma \leq \beta, t \geq 1$ and let $F(s) \ll e^{C^2}$ for $s \in \mathcal{D}$. Then for any fixed $q > 0$ and $\alpha < \gamma < \beta$, we have
\[
\int_2^T |F(\gamma + it)|^{\beta} dt \ll \left(\int_1^{2T} |F(\alpha + it)|^{\beta} dt + 1\right)^{\frac{\beta + \gamma - \alpha}{\beta + \gamma}} \left(\int_1^{2T} |F(\beta + it)|^{\beta} dt + 1\right)^{\frac{\beta + \gamma - \beta}{\beta + \gamma}}.
\]

In the following two lemmas, though the definitions of $\varphi_{\gamma}(m)$ and $\psi_{\gamma}(n)$ are different from ones in Lemmas 2.11 and 2.12 of [18], we still can get these two lemmas by following similar arguments, respectively.

Lemma 2.10. Let $\varphi_{\gamma}(n)$ be the arithmetic function generated by $L(s,f)^k$, that is
\[
\varphi_{\gamma}(n) = A_f \ast \cdots \ast A_f(1,n).
\]
Then we have
\[
\sum_{n \leq x} \varphi_{\gamma}(n) \ll x^{1+\epsilon}, \quad \sum_{n \leq x} \varphi^2_{\gamma}(n) \ll x^{1+\epsilon}.
\]
Lemma 2.11. Let $0 < \delta < \frac{1}{2}$ be a fixed constant and

$$\psi_k(n) = \begin{cases} 
\varphi_{2k}(n) - \sum_{m=1}^{\infty} \varphi_k(m) \varphi_k(l), & T < n \leq T^2, \\
\varphi_{2k}(n), & n > T^2.
\end{cases}$$

Then we have

$$\sum_{n \leq T} \psi_k^2(n) n^{2-2\delta} = O(1).$$

3 Proofs of Theorems 1 and 2

3.1 Proof of Theorem 1

In this section, we restrict the range of $\sigma$ into $\frac{2}{3} < \sigma < 1$ and shall give lower bounds for $m(\sigma)$ by establishing formulas of type $R \ll T^{1+\varepsilon}V^{-m(\sigma)}$.

Recalling Mellin’s formula

$$e^{-x} = (2\pi)^{-1} \int_{2-i\infty}^{2+i\infty} \Gamma(\omega) x^{-\omega} d\omega (x > 0). \quad (3.1)$$

Taking $x = \frac{n}{T}$ and multiplying (3.1) by $A_f(1, n_1)A_f(1, n_2)n_1^{-s}n_2^{-s}$, where $n = n_1n_2$ and summing over $n$, we can obtain

$$\sum_{n=1}^{\infty} \sum_{n=n_1n_2} \left( \sum_{n_1, n_2} A_f(1, n_1)A_f(1, n_2) \right) e^{-\frac{n}{T}} = (2\pi)^{-1} \int_{2-i\infty}^{2+i\infty} Y^{\omega} \Gamma(\omega) L(s + \omega, f)^2 d\omega. \quad (3.2)$$

Shifting the line of integration in (3.2) to $\Re \omega = \frac{2}{3} - \sigma$, we encounter a simple pole at $\omega = 0$ with residue $L(s, f)^2$ and get, as $Y \to \infty$,

$$\sum_{n \leq Y \log^2 T} \left( \sum_{n=n_1n_2} A_f(1, n_1)A_f(1, n_2) \right) e^{-\frac{n}{T}} + o(1) = L(s, f)^2 + (2\pi)^{-1} \int_{\Re \omega = \frac{2}{3} - \sigma} Y^{\omega} \Gamma(\omega) L(s + \omega, f)^2 d\omega. \quad (3.3)$$

The integral part of (3.3) for which $3\omega \leq \log^2 T$ is $o(1)$ as $T \to \infty$ by Stirling’s formula. Then let $s = \sigma + it$, and thus for each $t$, we have

$$L(\sigma + it, f)^2 \ll 1 + \left| \sum_{n \leq Y \log^2 T} \left( \sum_{n=n_1n_2} A_f(1, n_1)A_f(1, n_2) \right) e^{-\frac{n}{T}} \right| + \int_{-\log^2 T}^{\log^2 T} Y^{\frac{2}{3} + it + iv} \left| \left( \frac{2}{3} + it + iv \right)^{\frac{2}{3} - \sigma} \right|^2 e^{-v} dv. \quad (3.4)$$

Combining (2.5) with (3.4), we can obtain

$$V^2 \ll \left| \sum_{n \leq Y \log^2 T} \left( \sum_{n=n_1n_2} A_f(1, n_1)A_f(1, n_2) \right) e^{-\frac{n}{T}} \right| \ll \log T \max_{M \leq \frac{2}{3} Y \log^2 T} \left| \sum_{M < n \leq 2M} \left( \sum_{n=n_1n_2} A_f(1, n_1)A_f(1, n_2) \right) e^{-\frac{n}{T}} \right|.$$  \quad (3.5)
\[ V^2 \ll Y^{2-\sigma} \left| L \left( \frac{2}{3} + it', f \right) \right|^2, \quad (3.6) \]

where \( V \gg T^\epsilon \) and \( t' \) is defined as

\[
\left| L \left( \frac{2}{3} + it', f \right) \right| = \max_{-\log^2 T \leq y \leq \log^2 T} \left| L \left( \frac{2}{3} + it + iv, f \right) \right|.
\]

For convenience, denote by \( R'_1 \) and \( R'_2 \) those points which satisfy (3.5) and (3.6), respectively.

Recalling (1.1), we know that Lemma 2.4 is valid. We first consider \( R'_1 \). By Lemma 2.4, we have

\[
R'_1 \ll \log Y \times T \langle M^2 - 2\sigma V^{-4} + TV^{-2\sigma(\sigma)} \rangle \ll T \langle Y^{2-2\sigma}V^{-4} + TV^{-2\sigma(\sigma)} \rangle.
\]

While for \( R'_2 \), by Lemma 2.8, Hölder’s inequality and (3.6), we can obtain

\[
R'_2 \ll Y^{2-\sigma}V^{-2} \sum_{r \in \cal{R}_1} \left| L \left( \frac{2}{3} + it', f \right) \right|^2 \ll Y^{2-\sigma}V^{-2}T^{1+\epsilon}.
\]

For (3.8), if we take \( Y = V^{-\frac{1}{2\sigma}}T^\frac{1}{2\sigma} \), then we have

\[
R \ll R'_1 + R'_2 \ll T \langle Y^{2-2\sigma}V^{-4} + TV^{-2\sigma(\sigma)} + Y^{2-\sigma}V^{-2}T^{1+\epsilon} \rangle \ll T \langle V^{-\frac{1}{2\sigma}}T^\frac{1}{2\sigma} + TV^{-2\sigma(\sigma)} \rangle.
\]

For (3.9), if we take \( Y = T^\frac{\sigma}{2} \), then we have

\[
R \ll R'_1 + R'_2 \ll T \langle Y^{2-2\sigma}V^{-4} + TV^{-2\sigma(\sigma)} + Y^{2-\sigma}V^{-2}T^\frac{\sigma}{2} \rangle \ll T \langle V^{-\frac{1}{2}}T^{\frac{\sigma}{2}} + TV^{-2\sigma(\sigma)} \rangle.
\]

Therefore, combining (10.10) with (3.11) we have

\[
R \ll T \langle TV^{-2\sigma(\sigma)} + V^{-\frac{1}{2}}T^\frac{\sigma}{2} + V^{-2\sigma(\sigma)} + TV^{-2\sigma(\sigma)} \rangle.
\]

We assume that the second and the third terms in (1.12) do not exceed \( TV^{-x} \) and \( TV^{-y} \), for values \( x \) and \( y \) which can be determined by Lemma 2.6, then we can obtain

\[
x \lesssim \frac{4(3 - 2\sigma)}{5(1 - \sigma)(4 - 3\sigma)}, \quad y \lesssim \frac{7 - 3\sigma}{5(1 - \sigma)}.
\]

Thus, we have

\[
R \ll T^{1+\epsilon}V^{-z}
\]

with

\[
z = \min \left( 2f(\sigma), \frac{4(3 - 2\sigma)}{5(1 - \sigma)(4 - 3\sigma)}, \frac{7 - 3\sigma}{5(1 - \sigma)} \right).
\]

For \( \frac{2}{3} < \sigma \leq 1 - \epsilon \), we always have

\[
\frac{4(3 - 2\sigma)}{5(1 - \sigma)(4 - 3\sigma)} < \frac{7 - 3\sigma}{5(1 - \sigma)}.
\]

Recalling the value of \( f(\sigma) \) in Lemma 2.4, we can take

\[
z = \frac{4(3 - 2\sigma)}{5(1 - \sigma)(4 - 3\sigma)}, \quad \frac{2}{3} < \sigma \leq 1 - \epsilon.
\]

Thus, we complete the proof of Theorem 1.
3.2 Proof of Theorem 2

In this section, we give the proof of Theorem 2 by following a similar argument to [6, Theorem 2]. Let \( \alpha_k^* \) denote the infimum of all numbers \( \sigma \) for which

\[
\int_1^T |L(\sigma + it, f)|^{2k} dt \ll T^{1+\varepsilon}
\]

holds for any \( \varepsilon > 0 \), where \( k \geq 1 \) is a fixed integer, \( \frac{1}{2} \leq \alpha_k^* < 1 \).

Writing \( s = \sigma + it \), we have

\[
\int_1^T |L(\sigma + it, f)|^{2k} dt = \int_1^T \left| \sum_{n \leq T} \varphi_k(n)n^{-\sigma-it} \right|^2 dt + O \left( \int_1^T \left| L(\sigma + it, f) - \left( \sum_{n \leq T} \varphi_k(n)n^{-\sigma-it} \right) \right|^2 dt \right) \quad (3.13)
\]

where \( \varphi_k(n) \) is given by Lemma 2.10.

Combining Abel's summation formula with Lemmas 2.5 and 2.10, we can obtain

\[
\int_1^T \left| \sum_{n \leq T} \varphi_k(n)n^{-\sigma-it} \right|^2 dt = T \sum_{n \leq T} \varphi_k^2(n)n^{-2\sigma} + O \left( \sum_{n \leq T} \varphi_k^2(n)n^{-1-2\sigma} \right) = T \sum_{n=1}^{\infty} \varphi_k^2(n)n^{-2\sigma} + O(T^{2-2\sigma+\varepsilon}) \quad (3.14)
\]

Let

\[
F(\sigma + it, f) = L^{2k}(\sigma + it, f) - \left( \sum_{n \leq T} \varphi_k(n)n^{-\sigma-it} \right)^2.
\]

And applying Lemma 2.9 with \( q = 1, \alpha = \alpha_k^* + \delta, \beta = 1 + \delta, \gamma = \sigma, \) where \( 0 < \delta < \frac{1}{2} \) is a fixed constant which may be chosen arbitrarily small, for fixed \( k \) we have

\[
\frac{\beta - \sigma}{\beta - \alpha} = \frac{1 + \delta - \alpha}{1 - \alpha_k^*} \leq \frac{1 - \sigma}{1 - \alpha_k^*} + \delta^2
\]

and

\[
\frac{\sigma - \alpha}{\beta - \alpha} = \frac{\sigma - \alpha_k^* - \delta}{1 - \alpha_k^*} \leq \frac{\sigma - \alpha_k^*}{1 - \alpha_k^*}.
\]

Recalling the definition of \( \alpha_k^* \), by Lemma 2.5 we have

\[
\int_1^{2T} |F(\alpha + it, f)| dt \leq \int_1^{2T} |L(\sigma_k^* + \delta + it, f)|^{2k} dt + \int_1^{2T} \left| \sum_{n \leq T} \varphi_k(n)n^{-\sigma_k^* - \delta-it} \right|^2 dt \ll T^{1+\delta} + T^{2-2\sigma+\varepsilon} \ll T^{1+\delta}.
\]

Moreover,

\[
F(\beta + it, f) = \sum_{n=1}^{\infty} \varphi_{2k}(n)n^{1-\delta-it} - \left( \sum_{n \leq T} \varphi_k(n)n^{-1-\delta-it} \right)^2 = \sum_{n \leq T} \psi_k(n)n^{-1-\delta-it},
\]

where \( \psi_k(n) \) is given by Lemma 2.11.

By Lemma 2.5, Lemma 2.10 and Hölder’s inequality, we can obtain

\[
\int_1^{2T} |F(\beta + it, f)| dt \ll T^\delta \left( \int_1^{2T} \left| \sum_{n \leq T} \psi_k(n)n^{-1-\delta-it} \right|^2 dt \right)^{\frac{1}{2}} \ll T^\delta.
\]
Thus, Lemma 2.9 shows
\[
\int_1^{2T} |F(\sigma + it, f)| dt \ll T^{1+\delta}\left(\frac{1+\delta}{1-\delta}+\delta^4\right)^{\frac{\sigma-\sigma_k^*}{2-2\sigma_k^*}}.
\]

Note that
\[
(1 + \delta)\left(\frac{1 - \sigma}{1 - \sigma_k^*} + \delta^2\right) + \frac{\sigma - \sigma_k^*}{2 - 2\sigma_k^*} \leq \frac{2 - \sigma - \sigma_k^*}{2 - 2\sigma_k^*} + \epsilon
\]
holds for any \(\epsilon > 0\) if \(\delta = \delta(\epsilon)\) is sufficiently small. Noting that for the exponent of the \(O\)-term in (3.14), we have
\[
2 - 2\sigma < \frac{2 - \sigma - \sigma_k^*}{2 - 2\sigma_k^*} < 1.
\]

Thus,
\[
\int_1^{T} |L(\sigma + it, f)|^{2k} dt = T \sum_{n=1}^{\infty} \varphi_k^2(n)n^{-2\sigma} + R(k, \sigma; T),
\]
and for fixed \(\sigma\) satisfying \(\sigma_k^* < \sigma < 1\), we have
\[
R(k, \sigma; T) \ll T^{\frac{2 - \sigma - \sigma_k^*}{2 - 2\sigma_k^*} + \epsilon}.
\]

From Theorem 1 we have
\[
\int_1^{T} \left|L\left(\frac{2}{3} + it, f\right)\right|^6 dt \ll T^{1+\epsilon},
\]
\[
\int_1^{T} \left|L\left(\frac{33 - \sqrt{69}}{30} + it, f\right)\right|^4 dt \ll T^{1+\epsilon},
\]
\[
\int_1^{T} \left|L\left(\frac{101 - \sqrt{481}}{90} + it, f\right)\right|^6 dt \ll T^{1+\epsilon}.
\]

Recalling the definition of \(\sigma_k^*\), we can take \(\sigma_1^* = \frac{2}{3}, \sigma_2^* = \frac{33 - \sqrt{69}}{30}\) and \(\sigma_3^* = \frac{101 - \sqrt{481}}{90}\), from which we can obtain Theorem 2 immediately.

**Acknowledgment:** The authors are greatly indebted to the reviewers for very beneficial suggestions and comments which led to essential improvement of the original version of this paper. This work was supported by the National Natural Science Foundation of China (Grant Nos. 11771256 and 11801328).

**Conflict of interest:** Authors state no conflict of interest.

**References**

